Maximal perimeter and maximal width of a convex small polygon

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Abstract

A small polygon is a polygon of unit diameter. The maximal perimeter and the maximal width of a convex small polygon with $n = 2^s$ sides are unknown when $s \geq 4$. In this paper, we construct a family of convex small n-gons, $n = 2^s$ with $s \geq 4$, and show that their perimeters and their widths are within $O(1/n^8)$ and $O(1/n^5)$ of the maximal perimeter and the maximal width, respectively. From this result, it follows that Mossinghoff's conjecture on the diameter graph of a convex small 2^s-gon with maximal perimeter is not true when $s \geq 4$.

Keywords Planar geometry, polygons, isodiametric problems, maximal perimeter, maximal width

1 Introduction

Let $\mathtt{P}$ be a convex polygon. The *diameter* of $\mathtt{P}$ is the maximum distance between pairs of its vertices. The polygon $\mathtt{P}$ is *small* if its diameter equals one. The diameter graph of a small polygon is defined as the graph with the vertices of the polygon, and an edge between two vertices exists only if the distance between these vertices equals one. Diameter graphs of some convex small polygons are represented in Figure 1, Figure 2, and Figure 3. The solid lines illustrate pairs of vertices which are unit distance apart. The height associated to a side of $\mathtt{P}$ is defined as the maximum distance between a vertex of $\mathtt{P}$ and the line containing the side. The minimum height for all sides is the *width* of the polygon $\mathtt{P}$.

When $n = 2^s$ with integer $s \geq 4$, both the maximal perimeter and the maximal width of a convex small n-gon are unknown. However, tight bounds can be obtained analytically. It is well known that, for an integer $n \geq 3$, the value $2n \sin \frac{\pi}{2n}$ [1, 2] is an upper bound on the perimeter $L(\mathtt{P}_n)$ of a convex small n-gon $\mathtt{P}_n$ and the value $\cos \frac{\pi}{2n}$ [3] an upper bound on its width $W(\mathtt{P}_n)$. Recently, the author [4] constructed a family of convex small n-gons, for $n = 2^s$ with $s \geq 2$, whose perimeters and widths differ from the upper bounds $2n \sin \frac{\pi}{2n}$ and $\cos \frac{\pi}{2n}$ by just $O(1/n^6)$ and $O(1/n^4)$, respectively. By constrast, both the perimeter and the width of a regular small n-gon differ by $O(1/n^2)$ when $n \geq 4$ is even. In the present paper, we further tighten lower bounds on the maximal perimeter and the maximal width. Thus, our main result is the following:

Theorem 1. *Suppose $n = 2^s$ with integer $s \geq 4$. Let $\overline{L}_n := 2n \sin \frac{\pi}{2n}$ denote an upper bound on the perimeter $L(\mathtt{P}_n)$ of a convex small n-gon $\mathtt{P}_n$, and $\overline{W}_n := \cos \frac{\pi}{2n}$ denote an upper bound on its width*

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$W(\mathtt{P}_n)$. *Then there exists a convex small n-gon $\mathtt{D}_n$ such that*

$$L(\mathtt{D}_n) = 2n \sin\frac{\pi}{2n} \cos\left(\frac{1}{2}\arctan\left(\tan\frac{2\pi}{n}\tan\frac{\pi}{n}\right) - \frac{1}{2}\arcsin\left(\frac{\sin(2\pi/n)\sin(\pi/n)}{\sqrt{4\sin^2(\pi/n)+\cos(4\pi/n)}}\right)\right),$$

$$W(\mathtt{D}_n) = \cos\left(\frac{\pi}{2n} + \frac{1}{2}\arctan\left(\tan\frac{2\pi}{n}\tan\frac{\pi}{n}\right) - \frac{1}{2}\arcsin\left(\frac{\sin(2\pi/n)\sin(\pi/n)}{\sqrt{4\sin^2(\pi/n)+\cos(4\pi/n)}}\right)\right),$$

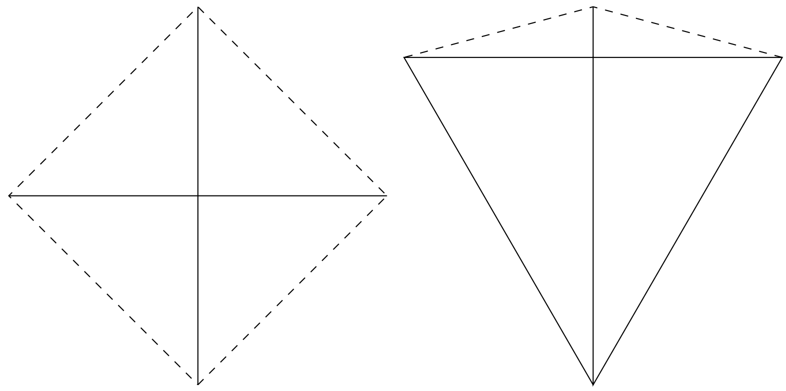
and

$$\overline{L}_n - L(\mathtt{D}_n) = \frac{\pi^9}{8n^8} + O\left(\frac{1}{n^{10}}\right),$$

$$\overline{W}_n - W(\mathtt{D}_n) = \frac{\pi^5}{4n^5} + O\left(\frac{1}{n^7}\right).$$

For all $n = 2^s$ and $s \geq 4$, the diameter graph of the n-gon $\mathtt{D}_n$ has a cycle of length $3n/4 - 1$ plus $n/4 + 1$ pendant edges. In 2006, Mossinghoff [5] conjectured that, when $n = 2^s$ and $s \geq 2$, the diameter graph of a convex small n-gon of maximal perimeter has a cycle of length $n/2 + 1$ plus $n/2 - 1$ additional pendant edges, and that is verified for $s = 2$ and $s = 3$. However, the conjecture is no longer true for $s \geq 4$ as the perimeter of $\mathtt{D}_n$ exceeds that of the optimal n-gon obtained by Mossinghoff.

The remainder of this paper is organized as follows. Section 2 recalls principal results on the maximal perimeter and the maximal width of convex small polygons. The proof of Theorem 1 is given in Section 3. We maximize the perimeter and obtain polygons with even larger perimeters in Section 4. We conclude the paper in Section 5.



(a) $(\mathtt{R}_4, 2.828427, 0.707107)$ (b) $(\mathtt{B}_4, 3.035276, 0.866025)$

Figure 1: Two convex small 4-gons $(\mathtt{P}_4, L(\mathtt{P}_4), W(\mathtt{P}_4))$: (a) Regular 4-gon; (b) 4-gon of maximal perimeter and maximal width [3, 6]

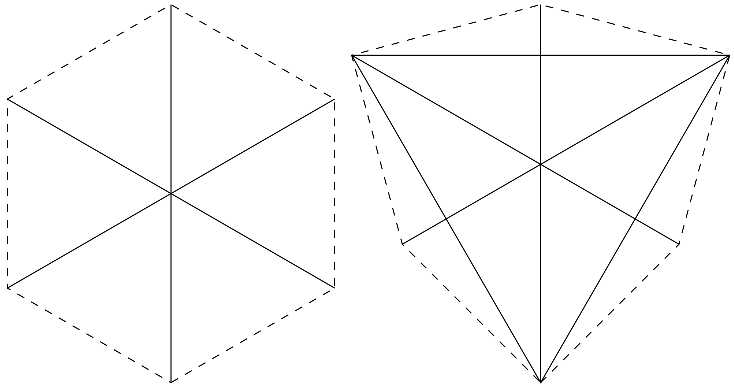
2 Perimeters and widths of convex small polygons

Let $L(\mathtt{P})$ denote the perimeter of a polygon $\mathtt{P}$ and $W(\mathtt{P})$ its width. For a given integer $n \geq 3$, let $\mathtt{R}_n$ denote the regular small n-gon. We have

$$L(\mathtt{R}_n) = \begin{cases} 2n\sin\frac{\pi}{2n} & \text{if } n \text{ is odd,} \\ n\sin\frac{\pi}{n} & \text{if } n \text{ is even,} \end{cases}$$

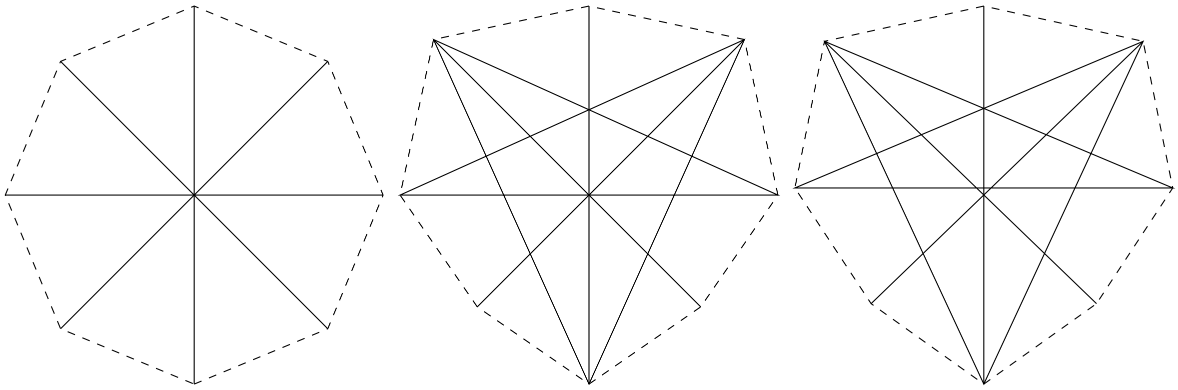
and

$$W(\mathtt{R}_n) = \begin{cases} \cos\frac{\pi}{2n} & \text{if } n \text{ is odd,} \\ \cos\frac{\pi}{n} & \text{if } n \text{ is even.} \end{cases}$$



(a) $(\mathtt{R}_6, 3, 0.866025)$ (b) $(\mathtt{R}_{3,6}, 3.105829, 0.965926)$

Figure 2: Two convex small 6-gons $(\mathtt{P}_6, L(\mathtt{P}_6), W(\mathtt{P}_6))$: (a) Regular 6-gon; (b) Reinhardt 6-gon [1]



(a) $(\mathtt{R}_8, 3.061467, 0.923880)$ (b) $(\mathtt{B}_8, 3.121062, 0.977609)$ (c) $(\mathtt{V}_8, 3.121147, 0.976410)$

Figure 3: Three convex small 8-gons $(\mathtt{P}_8, L(\mathtt{P}_8), W(\mathtt{P}_8))$: (a) Regular 8-gon; (b) An 8-gon of maximal width [7]; (c) 8-gon of maximal perimeter [8]

When n has an odd factor m, consider the family of convex equilateral small n-gons constructed as follows:

1. Transform the regular small m-gon $\mathtt{R}_m$ into a Reuleaux m-gon by replacing each edge by a circle's arc passing through its end vertices and centered at the opposite vertex;
2. Add at regular intervals $n/m - 1$ vertices within each arc;
3. Take the convex hull of all vertices.

These n-gons are denoted $\mathtt{R}_{m,n}$ and

$$L(\mathtt{R}_{m,n}) = 2n \sin \frac{\pi}{2n},$$
$$W(\mathtt{R}_{m,n}) = \cos \frac{\pi}{2n}.$$

The 6-gon $\mathtt{R}_{3,6}$ is illustrated in Figure 2b.

Theorem 2 (Reinhardt [1], Datta [2])**.** *For all $n \geq 3$, let L_n^* denote the maximal perimeter among all convex small n-gons and $\overline{L}_n := 2n \sin \frac{\pi}{2n}$.*

- *When n has an odd factor m, $L_n^* = \overline{L}_n$ is achieved by finitely many equilateral n-gons [9–11], including $\mathtt{R}_{m,n}$. The optimal n-gon $\mathtt{R}_{m,n}$ is unique if m is prime and $n/m \leq 2$.*

- *When $n = 2^s$ with $s \geq 2$, $L(\mathtt{R}_n) < L_n^* < \overline{L}_n$.*

When $n = 2^s$, the maximal perimeter L_n^* is only known for $s \leq 3$. Tamvakis [6] determined that $L_4^* = 2 + \sqrt{6} - \sqrt{2}$, and this value is only achieved by $\mathtt{B}_4$, represented in Figure 1b. Audet, Hansen, and Messine [8] proved that $L_8^* = 3.1211471340\ldots$, which is only achieved by $\mathtt{V}_8$, represented in Figure 3c.

Theorem 3 (Bezdek and Fodor [3])**.** *For all $n \geq 3$, let W_n^* denote the maximal width among all convex small n-gons and let $\overline{W}_n := \cos\frac{\pi}{2n}$.*

- *When n has an odd factor, $W_n^* = \overline{W}_n$ is achieved by a convex small n-gon with maximal perimeter $L_n^* = \overline{L}_n$.*
- *When $n = 2^s$ with integer $s \geq 2$, $W(\mathtt{R}_n) < W_n^* < \overline{W}_n$.*

When $n = 2^s$, as the maximal perimeter L_n^*, the maximal width W_n^* is known for $s \leq 3$. Bezdek and Fodor [3] proved that $W_4^* = \frac{1}{2}\sqrt{3}$, and this value is achieved by infinitely many convex small 4-gons, including that of maximal perimeter $\mathtt{B}_4$. Audet, Hansen, Messine, and Ninin [7] found that $W_8^* = \frac{1}{4}\sqrt{10 + 2\sqrt{7}}$, which is also achieved by infinitely many convex small 8-gons, including $\mathtt{B}_8$ represented in Figure 3b.

For $n = 2^s$ with $s \geq 4$, tight lower bounds on the maximal perimeter and the maximal width can be obtained analytically. The author [4] constructed a family of convex small n-gons $\mathtt{B}_n$, for $n = 2^s$ with $s \geq 2$, such that

$$
\begin{aligned}
L(\mathtt{B}_n) &= 2n \sin\frac{\pi}{2n} \cos\left(\frac{\pi}{2n} - \frac{1}{2}\arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right)\right), \\
W(\mathtt{B}_n) &= \cos\left(\frac{\pi}{n} - \frac{1}{2}\arcsin\left(\frac{1}{2}\sin\frac{2\pi}{n}\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{L}_n - L(\mathtt{B}_n) &= \frac{\pi^7}{32n^6} + O\left(\frac{1}{n^8}\right), \\
\overline{W}_n - W(\mathtt{B}_n) &= \frac{\pi^4}{8n^4} + O\left(\frac{1}{n^6}\right).
\end{aligned}
$$

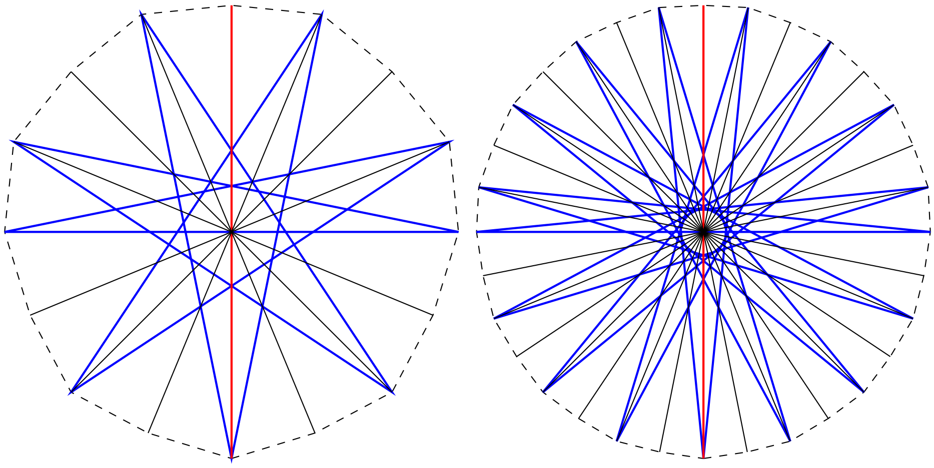
By contrast,

$$
\begin{aligned}
\overline{L}_n - L(\mathtt{R}_n) &= \frac{\pi^3}{8n^2} + O\left(\frac{1}{n^4}\right), \\
\overline{W}_n - W(\mathtt{R}_n) &= \frac{3\pi^2}{8n^2} + O\left(\frac{1}{n^4}\right)
\end{aligned}
$$

for all even $n \geq 4$. Note that $L(\mathtt{B}_4) = L_4^*$, $W(\mathtt{B}_4) = W_4^*$, and $W(\mathtt{B}_8) = W_8^*$. The hexadecagon $\mathtt{B}_{16}$ and the triacontadigon $\mathtt{B}_{32}$ are illustrated in Figure 4. The diameter graph of $\mathtt{B}_n$ in Figure 4 has the vertical edge as axis of symmetry and can be described by a cycle of length $n/2+1$, plus $n/2-1$ additional pendant edges, arranged so that all but two particular vertices of the cycle have a pendant edge.

3 Proof of Theorem 1

For any $n = 2^s$ where $s \geq 4$ is an integer, consider a convex small n-gon $\mathtt{P}_n$ having the following diameter graph: a $3n/4-1$-length cycle $\mathtt{v}_0 - \mathtt{v}_1 - \ldots - \mathtt{v}_k - \ldots - \mathtt{v}_{\frac{3n}{8}-1} - \mathtt{v}_{\frac{3n}{8}} - \ldots - \mathtt{v}_{\frac{3n}{4}-k-1} - \ldots - \mathtt{v}_{\frac{3n}{4}-2} - \mathtt{v}_0$ plus $n/4+1$ pendant edges $\mathtt{v}_0 - \mathtt{v}_{\frac{3n}{4}-1}$, $\mathtt{v}_{3j-2} - \mathtt{v}_{\frac{3n}{4}+j-1}$, $j = 1, 2, \ldots, n/4$, as



(a) $(\mathtt{B}_{16}, 3.136543, 0.994996)$ (b) $(\mathtt{B}_{32}, 3.140331, 0.998784)$

Figure 4: Best prior polygons $(\mathtt{B}_n, L(\mathtt{B}_n), W(\mathtt{B}_n))$: (a) Hexadecagon $\mathtt{B}_{16}$; (b) Triacontadigon $\mathtt{B}_{32}$

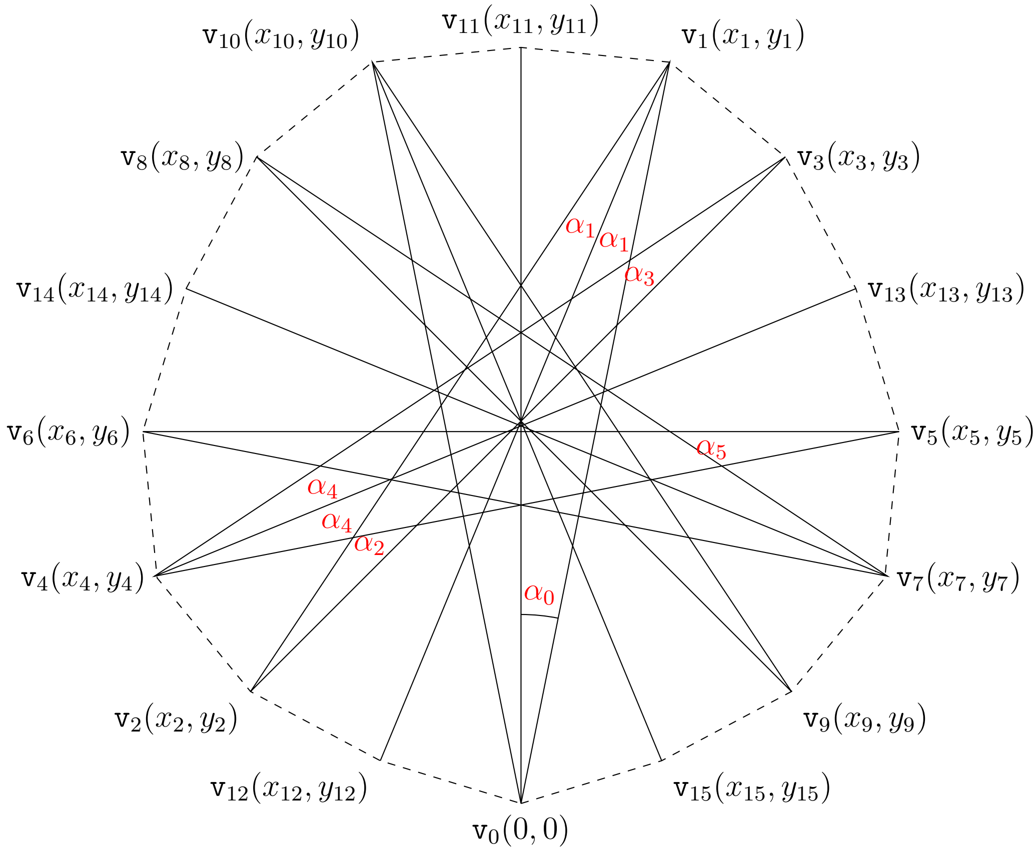


Figure 5: Definition of variables: Case of $n = 16$ vertices

illustred in Figure 5. We assume that $\mathtt{P}_n$ has the edge $\mathtt{v}_0 - \mathtt{v}_{\frac{3n}{4}-1}$ as axis of symmetry, and for all $j = 1, 2, \ldots, n/4$, the pendant edge $\mathtt{v}_{3j-2} - \mathtt{v}_{\frac{3n}{4}+j-1}$ bissects the angle formed at the vertex $\mathtt{v}_{3j-2}$ by the edge $\mathtt{v}_{3j-2} - \mathtt{v}_{3j-1}$ and the edge $\mathtt{v}_{3j-2} - \mathtt{v}_{3j-3}$.

For $k = 0, 1, \ldots, 3n/8 - 1$, let $c_k = 2$ if $k = 3j - 2$, and $c_k = 1$ otherwise. Then let $c_0\alpha_0$ denote the angle formed at the vertex $\mathtt{v}_0$ by the edge $\mathtt{v}_0 - \mathtt{v}_1$ and the edge $\mathtt{v}_0 - \mathtt{v}_{\frac{3n}{4}-1}$, and for all $k = 1, 2, \ldots, 3n/8 - 1$, $c_k\alpha_k$ the angle formed at the vertex $\mathtt{v}_k$ by the edge $\mathtt{v}_k - \mathtt{v}_{k+1}$ and the edge $\mathtt{v}_k - \mathtt{v}_{k-1}$. Since $\mathtt{P}_n$ is symmetric, we have

$$\sum_{j=1}^{n/8} \alpha_{3j-3} + 2\alpha_{3j-2} + \alpha_{3j-1} = \frac{\pi}{2}, \tag{1}$$

and

$$L(\mathtt{P}_n) = \sum_{j=0}^{n/8} 4\sin\frac{\alpha_{3j-3}}{2} + 8\sin\frac{\alpha_{3j-2}}{2} + 4\sin\frac{\alpha_{3j-1}}{2}, \tag{2a}$$

$$W(\mathtt{P}_n) = \min_{k=0,1,\ldots,3n/8-1} \cos\frac{\alpha_k}{2}. \tag{2b}$$

We use cartesian coordinates to describe the n-gon $\mathtt{P}_n$, assuming that a vertex $\mathtt{v}_k$, $k = 0, 1, \ldots, n-1$, is positioned at abscissa x_k and ordinate y_k. Placing the vertex $\mathtt{v}_0$ at the origin, we set $x_0 = y_0 = 0$. We also assume that $\mathtt{P}_n$ is in the half-plane $y \geq 0$.

Place the vertex $\mathtt{v}_{\frac{3n}{4}-1}$ at $(0, 1)$ in the plane. We have

$$x_{\frac{3n}{8}-1} = \sum_{k=1}^{3n/8-1} (-1)^{k-1} \sin\left(\sum_{i=0}^{k-1} c_i\alpha_i\right) = -x_{\frac{3n}{8}}, \tag{3a}$$

$$y_{\frac{3n}{8}-1} = \sum_{k=1}^{3n/8-1} (-1)^{k-1} \cos\left(\sum_{i=0}^{k-1} c_i\alpha_i\right) = y_{\frac{3n}{8}}. \tag{3b}$$

Since the edge $\mathtt{v}_{\frac{3n}{8}-1} - \mathtt{v}_{\frac{3n}{8}}$ is horizontal and $\|\mathtt{v}_{\frac{3n}{8}-1} - \mathtt{v}_{\frac{3n}{8}}\| = 1$, we also have

$$x_{\frac{3n}{8}-1} = 1/2 = -x_{\frac{3n}{8}}. \tag{4}$$

Now, suppose $\alpha_k = \frac{\pi}{n} + (-1)^k\delta$ with $|\delta| < \frac{\pi}{n}$ for all $k = 0, 1, \ldots, 3n/8 - 1$. Then (1) is verified and (2) becomes

$$L(\mathtt{P}_n) = 2n\sin\frac{\pi}{2n}\cos\frac{\delta}{2}, \tag{5a}$$

$$W(\mathtt{P}_n) = \cos\left(\frac{\pi}{2n} + \frac{|\delta|}{2}\right). \tag{5b}$$

Coordinates $(x_{\frac{3n}{8}-1}, y_{\frac{3n}{8}-1})$ in (3) are given by

$$\begin{aligned} x_{\frac{3n}{8}-1} = &\sum_{j=1}^{n/8}(-1)^{j-1}\left(\sin\left(\frac{(4j-3)\pi}{n} - (-1)^j\delta\right) - \sin\left(\frac{(4j-1)\pi}{n} + (-1)^j\delta\right)\right) \\ &+ \sum_{j=1}^{n/8-1} (-1)^{j-1}\sin\frac{4j\pi}{n} = \frac{\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n}}{2\cos\frac{2\pi}{n}} - \frac{\sin\frac{\pi}{n}\cos\delta}{\cos\frac{2\pi}{n}} + \frac{\cos\frac{\pi}{n}\sin\delta}{\sin\frac{2\pi}{n}}, \end{aligned} \tag{6a}$$

$$\begin{aligned} y_{\frac{3n}{8}-1} = &\sum_{j=1}^{n/8}(-1)^{j-1}\left(\cos\left(\frac{(4j-3)\pi}{n} - (-1)^j\delta\right) - \cos\left(\frac{(4j-1)\pi}{n} + (-1)^j\delta\right)\right) \\ &+ \sum_{j=1}^{n/8-1} (-1)^{j-1}\cos\frac{4j\pi}{n} = \frac{\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n}}{2\cos\frac{2\pi}{n}} - \frac{\sin\frac{\pi}{n}\cos\delta}{\cos\frac{2\pi}{n}} - \frac{\cos\frac{\pi}{n}\sin\delta}{\sin\frac{2\pi}{n}}. \end{aligned} \tag{6b}$$

From (4) and (6a), we deduce that

$$\frac{\sin\frac{\pi}{n}\cos\delta}{\cos\frac{2\pi}{n}} - \frac{\cos\frac{\pi}{n}\sin\delta}{\sin\frac{2\pi}{n}} = \frac{\sin\frac{2\pi}{n}}{2\cos\frac{2\pi}{n}}.$$

This equation has a solution $\delta_0(n)$ satisfying

$$\begin{aligned}\delta_0(n) &= \arctan\left(\tan\frac{2\pi}{n}\tan\frac{\pi}{n}\right) - \arcsin\left(\frac{\sin(2\pi/n)\sin(\pi/n)}{\sqrt{4\sin^2(\pi/n)+\cos(4\pi/n)}}\right)\\ &= \frac{\pi^4}{n^4} + \frac{19\pi^6}{12n^6} + O\left(\frac{1}{n^8}\right).\end{aligned}$$

Let $\mathtt{D}_n$ denote the n-gon obtained by setting $\delta = \delta_0(n)$. We have, from (5),

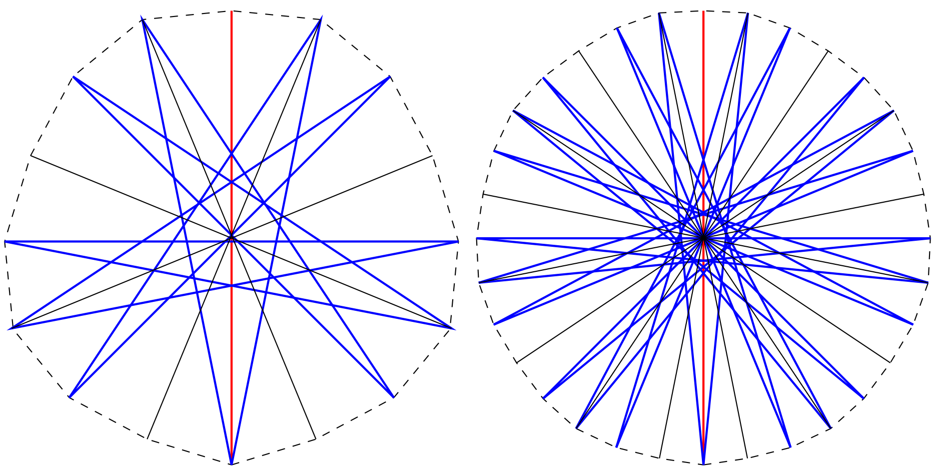
$$L(\mathtt{D}_n) = 2n\sin\frac{\pi}{2n}\cos\left(\frac{1}{2}\arctan\left(\tan\frac{2\pi}{n}\tan\frac{\pi}{n}\right) - \frac{1}{2}\arcsin\left(\frac{\sin(2\pi/n)\sin(\pi/n)}{\sqrt{4\sin^2(\pi/n)+\cos(4\pi/n)}}\right)\right),$$

$$W(\mathtt{D}_n) = \cos\left(\frac{\pi}{2n} + \frac{1}{2}\arctan\left(\tan\frac{2\pi}{n}\tan\frac{\pi}{n}\right) - \frac{1}{2}\arcsin\left(\frac{\sin(2\pi/n)\sin(\pi/n)}{\sqrt{4\sin^2(\pi/n)+\cos(4\pi/n)}}\right)\right),$$

and

$$\begin{aligned}\overline{L}_n - L(\mathtt{D}_n) &= \frac{\pi^9}{8n^8} + \frac{25\pi^{11}}{64n^{10}} + O\left(\frac{1}{n^{12}}\right),\\ \overline{W}_n - W(\mathtt{D}_n) &= \frac{\pi^5}{4n^5} + \frac{37\pi^7}{96n^7} + O\left(\frac{1}{n^8}\right).\end{aligned}$$

By construction, $\mathtt{D}_n$ is convex and small. We illustrate $\mathtt{D}_n$ for some n in Figure 6. □



(a) $(\mathtt{D}_{16}, 3.136548, 0.995107)$ (b) $(\mathtt{D}_{32}, 3.140331, 0.998793)$

Figure 6: Polygons $(\mathtt{D}_n, L(\mathtt{D}_n), W(\mathtt{D}_n))$ defined in Theorem 1: (a) Hexadecagon $\mathtt{D}_{16}$; (b) Triacontadigon $\mathtt{D}_{32}$

All polygons presented in this work were implemented as a MATLAB package: OPTIGON [12], which is freely available at `https://github.com/cbingane/optigon`. In OPTIGON, we provide MATLAB functions that give the coordinates of the vertices. One can also find an algorithm developed in [13] to find an estimate of the maximal area of a small n-gon when $n \geq 6$ is even.

Table 1 shows the perimeters of $\mathtt{D}_n$, along with the upper bounds $\overline{L}_n$, the perimeters of polygons $\mathtt{R}_n$ and $\mathtt{B}_n$. As suggested by Theorem 1, $\mathtt{D}_n$ provides a tighter lower bound on the maximal perimeter L_n^*

compared to the best prior convex small n-gon $\mathtt{B}_n$. For instance, we can note that

$$\begin{aligned} L_{64}^* - L(\mathtt{D}_{64}) &< \overline{L}_{64} - L(\mathtt{D}_{64}) &&= 1.33\ldots \times 10^{-11} &&< \overline{L}_{64} - L(\mathtt{B}_{64}) &&= 1.37\ldots \times 10^{-9}, \\ L_{128}^* - L(\mathtt{D}_{128}) &< \overline{L}_{128} - L(\mathtt{D}_{128}) &&= 5.19\ldots \times 10^{-14} &&< \overline{L}_{128} - L(\mathtt{B}_{128}) &&= 2.14\ldots \times 10^{-11}. \end{aligned}$$

The fraction $\frac{L(\mathtt{D}_n)-L(\mathtt{B}_n)}{\overline{L}_n - L(\mathtt{B}_n)}$ of the length of the interval $[L(\mathtt{B}_n), \overline{L}_n]$ containing $L(\mathtt{D}_n)$ shows that $L(\mathtt{D}_n)$ approaches $\overline{L}_n$ much faster than $L(\mathtt{B}_n)$ as n increases. Indeed, $L(\mathtt{D}_n) - L(\mathtt{B}_n) \sim \pi^7/(32n^6)$ for large n.

Table 2 displays the widths of $\mathtt{D}_n$, along with the upper bounds $\overline{W}_n$, the widths of $\mathtt{R}_n$ and $\mathtt{B}_n$. Again, when $n = 2^s$, $\mathtt{D}_n$ provides a tighter lower bound on the maximal width W_n^* compared to $\mathtt{B}_n$. We also remark that $W(\mathtt{D}_n)$ approaches $\overline{W}_n$ much faster than $W(\mathtt{B}_n)$ as n increases.

Table 1: Perimeters of $\mathtt{D}_n$

n	$L(\mathtt{R}_n)$	$L(\mathtt{B}_n)$	$L(\mathtt{D}_n)$	$\overline{L}_n$	$\frac{L(\mathtt{D}_n)-L(\mathtt{B}_n)}{\overline{L}_n-L(\mathtt{B}_n)}$
16	3.1214451523	3.1365427675	3.1365475080	3.1365484905	0.8283
32	3.1365484905	3.1403310687	3.1403311535	3.1403311570	0.9604
64	3.1403311570	3.1412772496	3.141277250919	3.141277250933	0.9903
128	3.1412772509	3.141513801123	3.14151380114425	3.14151380114430	0.9976

Table 2: Widths of $\mathtt{D}_n$

n	$W(\mathtt{R}_n)$	$W(\mathtt{B}_n)$	$W(\mathtt{D}_n)$	$\overline{W}_n$	$\frac{W(\mathtt{D}_n)-W(\mathtt{B}_n)}{\overline{W}_n-W(\mathtt{B}_n)}$
16	0.9807852804	0.9949956687	0.9951068324	0.9951847267	0.5880
32	0.9951847267	0.9987837929	0.9987931407	0.9987954562	0.8015
64	0.9987954562	0.9996980921	0.9996987472	0.9996988187	0.9016
128	0.9996988187	0.9999246565	0.9999246996	0.9999247018	0.9509
256	0.9999247018	0.9999811724	0.9999811752	0.9999811753	0.9755

4 Maximizing the perimeter

For $n = 2^s$ with $s \geq 4$, we can improve $L(\mathtt{D}_n)$ by adjusting the angles $\alpha_0, \alpha_1, \ldots, \alpha_{\frac{3n}{8}-1}$ from our parametrization of Section 3 to maximize the perimeter $L(\mathtt{P}_n)$ in (2a), creating a polygon $\mathtt{D}_n^*$ with larger perimeter. Thus, $L(\mathtt{D}_n^*)$ is the optimal value of the following optimization problem:

$$L(\mathtt{D}_n^*) = \max_{\boldsymbol{\alpha}} \quad \sum_{k=0}^{3n/8-1} 4c_k \sin\frac{\alpha_k}{2} \tag{7a}$$

$$\text{s.t.} \quad \sum_{k=1}^{3n/8-1} (-1)^{k-1} \sin\left(\sum_{i=0}^{k-1} c_i\alpha_i\right) = 1/2, \tag{7b}$$

$$\sum_{k=0}^{3n/8-1} c_k\alpha_k = \pi/2, \tag{7c}$$

$$0 \leq c_k\alpha_k \leq \pi/3 \qquad \forall k = 0, 1, \ldots, 3n/8-1, \tag{7d}$$

where c_k is 2 if $k = 3j-2$, and 1 otherwise.

This approach was already used by Mossinghoff [5] to obtain a convex small n-gon $\mathtt{B}_n^*$, for $n = 2^s$ with $s \geq 3$, with the same diameter graph as $\mathtt{B}_n$ but larger perimeter. We can show that

$$
\begin{aligned}
L(\mathtt{B}_n^*) = \max_{\boldsymbol{\alpha}} \quad & 4\sin\frac{\alpha_0}{2} + \sum_{k=1}^{n/4-1} 8\sin\frac{\alpha_k}{2} + 4\sin\frac{\alpha_{n/4}}{2} && \text{(8a)}\\
\text{s. t.} \quad & \sin\alpha_0 - \sum_{k=2}^{n/4}(-1)^k \sin\left(\alpha_0 + \sum_{i=1}^{k-1} 2\alpha_i\right) = -1/2, && \text{(8b)}\\
& \alpha_0 + \sum_{k=1}^{n/4-1} 2\alpha_k + \alpha_{n/4} = \pi/2, && \text{(8c)}\\
& 0 \leq \alpha_k \leq \pi/6 \qquad \forall k = 0, 1, \ldots, n/4-1, && \text{(8d)}\\
& 0 \leq \alpha_{n/4} \leq \pi/3. && \text{(8e)}
\end{aligned}
$$

Note that $L(\mathtt{B}_8^*) = L_8^*$. Then Mossinghoff asked if $L(\mathtt{B}_{16}^*) = L_{16}^*$ and if the maximal perimeter when $n = 2^s$ is always achieved by a polygon with the same diameter graph as $\mathtt{B}_n$. Numerical results in Table 3 show that both conjectures are not true. Indeed, for all $n = 2^s$ and $s \geq 4$, we have $L(\mathtt{B}_n^*) < L(\mathtt{D}_n) < L(\mathtt{D}_n^*)$.

Problems (7) and (8) were solved on the NEOS Server 6.0 using AMPL with Couenne 0.5.8. AMPL codes have been made available at https://github.com/cbingane/optigon. The solver Couenne [14] is a branch-and-bound algorithm that aims at finding global optima of nonconvex mixed-integer nonlinear optimization problems.

Table 3 shows the optimal values $L(\mathtt{D}_n^*)$ and $L(\mathtt{B}_n^*)$ for $n = 16, 32, 64$, along with the perimeters of $\mathtt{D}_n$, the upper bounds $\overline{L}_n$, and the fraction $\lambda_n^* := \frac{L(\mathtt{D}_n^*) - L(\mathtt{D}_n)}{\overline{L}_n - L(\mathtt{D}_n)}$ of the length of the interval $[L(\mathtt{D}_n), \overline{L}_n]$ where $L(\mathtt{D}_n^*)$ lies. The results support the following keypoints:

1. For each n, the optimal perimeter $L(\mathtt{B}_n^*)$ computed here agrees with the value obtained by Mossinghoff [5, 15].

2. For all $n = 2^s$ and $s \geq 4$, $L(\mathtt{B}_n^*) < L(\mathtt{D}_n) < L(\mathtt{D}_n^*)$, i.e., $\mathtt{B}_n^*$ is a suboptimal solution.

3. The fraction λ_n^* appears to approach a scalar $\lambda^* \in (0,1)$ as n increases, i.e., $\overline{L}_n - L(\mathtt{D}_n^*) = O(1/n^8)$.

The optimal angles α_k^* that produce $\mathtt{D}_n^*$ appear in Table 4. They exhibit a pattern of damped oscillation, converging in an alterning manner to a mean value around π/n. We remark that

$$W(\mathtt{D}_n^*) = \cos(\alpha_0^*/2) < W(\mathtt{D}_n)$$

for all n. We ask if $L(\mathtt{D}_{16}^*) = L_{16}^*$ and if $W(\mathtt{D}_{16}) = W_{16}^*$.

Table 3: Perimeters of $\mathtt{D}_n^*$

n	$L(\mathtt{B}_n^*)$	$L(\mathtt{D}_n)$	$L(\mathtt{D}_n^*)$	$\overline{L}_n$	λ_n^*
16	3.1365439563 [5]	3.1365475080	3.1365477165	3.1365484905	0.2122
32	3.1403310858 [5]	3.1403311535	3.1403311541	3.1403311570	0.1947
64	3.1412772498 [15]	3.141277250919	3.141277250922	3.141277250933	0.1908

Table 4: Angles α_k^* of $\mathtt{D}_n^*$

n	i	α_{6i}^*	α_{6i+1}^*	α_{6i+2}^*	α_{6i+3}^*	α_{6i+4}^*	α_{6i+5}^*
16	0	0.198316	0.194503	0.197746	0.194994	0.197164	0.196406
32	0	0.0982941	0.0980569	0.0982850	0.0980648	0.0982750	0.0980908
	1	0.0982593	0.0981082	0.0982205	0.0981293	0.0981985	0.0981752
64	0	0.0490948	0.0490800	0.0490947	0.0490801	0.0490945	0.0490806
	1	0.0490942	0.0490808	0.0490936	0.0490812	0.0490931	0.0490822
	2	0.0490926	0.0490827	0.0490915	0.0490833	0.0490909	0.0490846
	3	0.0490902	0.0490852	0.0490888	0.0490860	0.0490881	0.0490874

5 Conclusion

We provided tigther lower bounds on the maximal perimeter and the maximal width of convex small n-gons when n is a power of 2. For each $n = 2^s$ with integer $s \geq 4$, we constructed a convex small n-gon $\mathtt{D}_n$ whose perimeter and width are within $\pi^9/(8n^8) + O(1/n^{10})$ and $\pi^5/(4n^5) + O(1/n^7)$ of the maximal perimeter and the maximal width, respectively. We also showed that Mossinghoff's conjecture on the diameter graph of a convex small n-gon of maximal perimeter, when n is a power of 2, is not true, and proposed solutions $\mathtt{D}_n^*$ with the same diameter graph as $\mathtt{D}_n$ but larger perimeters.

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