

New complexity results and algorithms for min-max-min robust combinatorial optimization.

Jannis Kurtz^{*1}

¹Management Science, University of Siegen, Germany

Abstract

In this work we investigate the min-max-min robust optimization problem applied to combinatorial problems with uncertain cost-vectors which are contained in a convex uncertainty set. The idea of the approach is to calculate a set of k feasible solutions which are worst-case optimal if in each possible scenario the best of the k solutions would be implemented. It is known that the min-max-min robust problem can be solved efficiently if k is at least the dimension of the problem, while it is theoretically and computationally hard if k is small. While both cases are well studied in the literature nothing is known about the intermediate case, namely if k is smaller than but close to the dimension of the problem. We approach this open question and show that for a selection of combinatorial problems the min-max-min problem can be solved exactly and approximately in polynomial time if some problem specific values are fixed. Furthermore we approach a second open question and present the first implementable algorithm with pseudopolynomial runtime for the case that k is at least the dimension of the problem. The algorithm is based on a projected subgradient method where the projection problem is solved by the classical Frank-Wolfe algorithm. Additionally we derive a branch & bound method to solve the min-max-min problem for arbitrary values of k and perform tests on knapsack and shortest path instances. The experiments show that despite its theoretical impact the projected subgradient method cannot compete with an already existing method. On the other hand the performance of the branch & bound method scales very well with the number of solutions. Thus we are able to solve instances where k is above some small threshold very efficiently.

Robust Optimization, Min-max-min, Combinatorial Optimization, Complexity, Branch & Bound Algorithm, Projected Subgradient Method

1 Introduction

Combinatorial optimization problems nowadays emerge in many industries as production, health care, disaster management or transportation, just to name a few. Hence these problems are highly relevant for society and have to be solved by companies, non-profit organizations or governmental institutions. Usually solving optimization problems in practice involves uncertainties, e.g. uncertain traffic situations, uncertain demands or uncertain failures of a network, which have to be considered in the selected model. The optimization literature provides several ways to model uncertainties, e.g. stochastic optimization [13], robust optimization [5] or distributionally robust optimization [46, 28]. Focusing on robust optimization we assume that all possible realizations

*jannis.kurtz@uni-siegen.de

of the uncertain parameters are contained in a given uncertainty set and the aim is to find a solution which is optimal in the worst-case and feasible for all possible scenarios in the uncertainty set. Besides the robustness of the solutions also the absence of probability information makes the robust optimization approach convenient for applications with rare uncertain events which often cannot be modeled by a probability distribution due to missing data. Hence among other applications the robust optimization approach is tailored for safety-critical applications related to disaster management or critical infrastructures [38, 37, 41].

The robust optimization approach was intensively studied for convex and discrete uncertainty sets; see e.g. [8, 9, 12, 11, 35, 2, 19]. Despite its success it is often criticized to be too conservative since the calculated robust solution has to hedge against all scenarios in the given uncertainty set, which can lead to bad performances in the mean scenario. To overcome this problem several new robust models have been introduced; see e.g. [6, 26, 36, 1].

In this work we study the *min-max-min robust optimization problem* for combinatorial linear problems, which was first introduced in [17] to overcome the conservativeness of the classical robust approach. We consider deterministic optimization problems

$$\min_{x \in X} c^\top x \tag{P}$$

where $X \subseteq \{0,1\}^n$ is the set of incidence vectors of all feasible solutions and $c \in \mathbb{R}^n$ is an uncertain cost-vector which we assume is contained in a given convex uncertainty set $U \subseteq \mathbb{R}^n$. Based on the idea of k -adaptability ([10, 31, 45]) the min-max-min robust optimization problem is defined as

$$\min_{x^{(1)}, \dots, x^{(k)} \in X} \max_{c \in U} \min_{i=1, \dots, k} c^\top x^{(i)} \tag{M^3(k)}$$

where $k \in \mathbb{N}$ is a given parameter controlling the conservativeness of the problem. The main idea of the approach is to hedge against the uncertainty in the cost-vector in a robust manner, i.e. considering the worst-case costs over all scenarios in U , while providing more flexibility compared to the classical robust approach since multiple solutions are calculated and can be used to react to the emerging uncertain scenarios. In contrast to the k -adaptability problem the min-max-min robust approach has no two-stage structure and is therefore tailored for combinatorial problems where no second-stage decisions exist, i.e. where a set of complete solutions has to be prepared in advance. This can be inevitable in many applications regarding the construction of transportation plans, e.g. evacuation plans for buildings, airports or cities [20, 43, 39, 40] or route plans for robot systems [44]. To exemplify the benefit of the min-max-min approach consider a network problem where any component of the network may fail in the future, i.e. any prepared classical robust solution is unusable if a component required by the solution will fail. In contrast to this the min-max-min problem hedges against all possible network failures by providing a set of k solutions such that (if possible) in each failure scenario at least one of the k solutions is still usable.

Since 2015 the min-max-min robust approach was studied for several uncertainty sets and combinatorial problems. It was studied for convex uncertainty sets in [17, 16] and for discrete uncertainty sets in [18]. Regarding its complexity and solvability Problem (M³(k)) is a very interesting problem due to the unusual connection between its problem parameters and its complexity. A reasonable assumption would be that Problem (M³(k)) gets harder to solve with increasing k . However this is not true from a theoretical point of view as well as from a computational point of view. While for discrete uncertainty sets Problem (M³(k)) is weakly or strongly NP-hard for most of the classical combinatorial problems, in the case of convex uncertainty the problem can be solved in polynomial time if $k \geq n$ and if (P) can be solved in polynomial time. On the other hand it is NP-hard for each fixed $k \in \mathbb{N}$ even if U is a polyhedron. The authors in [17] present an

efficient algorithm for the case $k \geq n$ and a fast heuristic for each fixed $k \in \mathbb{N}$. Later in [21] it was shown that the problem can be solved in polynomial time for several combinatorial problems if U is a convex budgeted uncertainty set and $k = 2$. In [22] faster exact and heuristic algorithms for the same uncertainty set were presented. For the discrete budgeted uncertainty set the authors in [27] derive exact and fast heuristic algorithms and show that the problem is weakly or strongly NP-hard for most of the classical combinatorial problems. For binary uncertainty sets defined by linear constraints it was shown in [23] that Problem $(M^3(k))$ is Σ_2^P -complete. Recently an efficient exact algorithm based on smart enumeration was derived in [4] for problems where X does not contain too many good solutions. In [25] the min-max-min robust problem was applied to the vehicle routing problem where a set of k possible routes has to be prepared in advance which are robust to uncertain traffic scenarios. The idea of the min-max-min robust approach was also applied to the regret robust approach in [24].

Our Contributions

- While for small k (even $k = 2$) Problem $(M^3(k))$ is NP-hard and at least as hard as the classical robust optimization problem and very hard to solve algorithmically, for larger k the problem can be solved efficiently, more precisely in oracle-polynomial time for $k \geq n$. Nevertheless nothing was known about the problem between these two cases, especially if k is smaller than but close to n . In Section 3 we approach this question by showing that for a selection of combinatorial problems, Problem $(M^3(k))$ can be solved in time polynomial in n if the difference $n - k$ and some problem specific values are fixed. Additionally we show for a selection of combinatorial problems that the problem admits a PTAS under similar assumptions.
- While in [17] the authors can prove that Problem $(M^3(k))$ can be solved in oracle-polynomial time if $k \geq n$ by using results from [29], they are not able to provide a specific implementable polynomial-time algorithm. In Section 4 we approach this gap and provide an easy implementable algorithm which has runtime polynomial in n and some problem specific values. The algorithm is derived by using a projected subgradient method where the projection problem is solved by the classical Frank-Wolfe algorithm. Incidentally the derived method can also be used to solve any classical robust optimization problem with convex feasible region.
- In Section 5 we derive a lower bound for Problem $(M^3(k))$ and present an exact branch & bound algorithm for the problem which scales very well with the number of solutions k .
- In Section 6 we study the polynomial-time algorithm derived in Section 4 and the exact branch & bound method derived in Section 5 on classical knapsack and shortest path instances which were used in the min-max-min literature. We can show that despite its theoretical impact the projected subgradient method cannot compete with the already existing method from [17]. On the other hand we can show that our branch & bound method is struggling with instances for small k , while performing very well with increasing number of solutions. Thus we are able to solve instances for larger values of k (where $k < n$) than in the existing works on min-max-min robust optimization.

1.1 Notation

We define $[k] := \{1, \dots, k\}$ for $k \in \mathbb{N}$ and $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$. We denote by $\|x\| := \sqrt{\sum_{i \in [n]} x_i^2}$ the euclidean norm and by $\|x\|_\infty := \max_{i \in [n]} |x_i|$ the maximum norm. The convex

hull of a finite set $S = \{s^1, \dots, s^m\}$ is denoted by

$$\text{conv}(S) = \left\{ s = \sum_{i \in [m]} \lambda_i s^i : \lambda_i \geq 0 \forall i \in [m], \sum_{i \in [m]} \lambda_i = 1 \right\}.$$

The vector of all ones is denoted by $\mathbf{1}$ and the i -th unit vector by e_i .

2 Preliminaries

In [17] the authors study Problem $(M^3(k))$ for convex uncertainty sets U and show, by using lagrangian relaxation, that Problem $(M^3(k))$ for any $k \in \mathbb{N}$ is equivalent to problem

$$\min_{x \in X(k)} \max_{c \in U} c^\top x \quad (1)$$

where $X(k)$ is the set of all convex combinations derived by at most k solutions in X , i.e.

$$X(k) := \left\{ x \in \mathbb{R}^n : x = \sum_{i \in [k]} \lambda_i x^{(i)}, x^{(i)} \in X, \lambda \in \mathbb{R}_+^k, \sum_{i \in [k]} \lambda_i = 1 \right\}.$$

By the theorem of Carathéodory it follows that each point in $\text{conv}(X)$ can be described by a convex combination of at most $n + 1$ points in X , therefore it holds $X(k) = \text{conv}(X)$ for all $k \geq n + 1$. Since for any given point $x \in \text{conv}(X)$ and $\mu \in \mathbb{R}_+$ we have

$$\max_{c \in U} c^\top (\mu x) = \mu \max_{c \in U} c^\top x$$

an optimal solution is always attained on the boundary of $\text{conv}(X)$ if $k \geq n + 1$, i.e. can be described by a convex combination of at most n solutions in X . It follows that for each $k \geq n$ Problem $(M^3(k))$ is equivalent to the problem

$$\min_{x \in \text{conv}(X)} \max_{c \in U} c^\top x. \quad (2)$$

For an optimal solution x^* of Problem (2) the corresponding optimal solution $x^{(1)}, \dots, x^{(n)}$ of Problem $(M^3(k))$ can be calculated in polynomial time, if we can linearly optimize over X in polynomial time; see [29, 17] for more details.

Problem (2) is a convex problem, since the objective function $f(x) := \max_{c \in U} c^\top x$ is convex and $\text{conv}(X)$ is a convex set. Unfortunately for many classical combinatorial problems no outer-description of polynomial size for $\text{conv}(X)$ is known. Nevertheless the authors in [17] prove that Problem (2) and therefore the min-max-min robust problem can be solved in polynomial time if the underlying deterministic problem (P) can be solved in polynomial time which is stated in the theorem below. We say in the following that we have a *linear minimization oracle* for a set $S \subseteq \mathbb{R}^n$, if we have a procedure which returns an optimal solution of

$$\min_{x \in S} c^\top x$$

for any cost vector $c \in \mathbb{Q}^n$ in constant runtime. We assume that the returned solution always has polynomial size in the size of the input cost vector c . The equivalent definition holds for maximization problems.

Theorem 1 ([17]). *Let $\varepsilon \in (0, 1) \cap \mathbb{Q}$ and $k \geq n$. Assume that U is a non-empty convex set for which we have a linear maximization oracle and $\|c\| \leq M$ holds for all $c \in U$. Given a linear minimization oracle for X , we can solve Problem $(M^3(k))$ up to an additive error of at most ε in time polynomial in n and the encoding lengths of M and ε .*

The latter theorem states that, if we can linearly maximize over U in polynomial time and if we can linearly minimize over X in polynomial time, then we can solve the min-max-min robust problem in polynomial time. However the proof in [17] is not constructive, i.e. no implementable algorithm with a polynomial runtime guarantee is presented. Instead the authors present a column-generation algorithm to solve Problem $(M^3(k))$ for $k \geq n$ where iteratively the deterministic problem (P) and an adversary problem over U is solved. They show that this algorithm is very efficient on random instances of the knapsack problem and the shortest path problem. Furthermore the same algorithm was used successfully in [25] for the min-max-min version of the capacitated vehicle routing problem. We will adapt this algorithm to calculate strong lower bounds for Problem $(M^3(k))$ for any k which can be used in our branch & bound method presented in Section 5. Unfortunately no runtime guarantee for the column-generation algorithm is known and therefore no implementable polynomial time algorithm for $M^3(n)$ is known. We approach this gap and present an explicit algorithm in Section 4 where the number of oracle calls is polynomial in n , M and $\frac{1}{\varepsilon}$.

On the other hand it is shown in [17] that Problem $(M^3(k))$ with an uncertain constant is NP-hard for any fixed $k \in \mathbb{N}$, even if U is a polyhedron given by an inner description and $X = \{0, 1\}^n$. This results fits to the computational experience which was made in other publications, where it turns out that Problem $(M^3(k))$ is very hard to solve for small $k \in \mathbb{N}$; see [4, 22]. Nevertheless to tackle the problem even for small k in [17] the authors present an heuristic algorithm which is based on the column-generation algorithm mentioned above. The idea is to solve Problem $(M^3(k))$ for $k = n$ with this algorithm and afterwards select k of the calculated solutions with largest induced weights, given by the optimal convex combination of Problem (2). It is shown that this heuristic calculates solutions which are very close to the optimal value of Problem $(M^3(k))$. We will use this heuristic to calculate feasible solutions in our branch & bound method presented in Section 5.

3 Complexity results for an intermediate number of solutions

The min-max-min robust problem is known to be easy to solve if $k \geq n$, both theoretically (see Theorem 1) and practically (see [17]), while it is NP-hard for any fixed $k \in \mathbb{N}$ even if U is a polyhedron and $X = \{0, 1\}^n$. Moreover recent results indicate that it is computationally very hard to solve $(M^3(k))$ exactly for general convex uncertainty sets U even if k is fixed and small, e.g. $k \in \{2, 3, 4\}$; see [17, 22, 4]. However nothing is known for the intermediate setting when $k = n - l$ for a fixed number $l \in [n - 1]$. Furthermore no algorithms with approximation guarantees are known for the problem. In this section we approach these gaps by showing that for several combinatorial problems surprisingly Problem $(M^3(k))$ can be solved up to an additive error of $\varepsilon > 0$ in time polynomial in n if ε and the largest maximum-norm over all scenarios in U are fixed. Furthermore we show that under similar assumptions the min-max-min problem has a polynomial-time approximation scheme (PTAS) for several combinatorial problems.

In the following we study Problem $(M^3(k))$ where $k = n - l$ for a given parameter $l \in [n - 1]$. We denote the exact optimal value of Problem $(M^3(k))$ by $\text{opt}(k)$. Furthermore we assume that $\|c\|_\infty \leq M_\infty$ for all $c \in U$. Note that considering a maximum-norm bound for U for any $n \in \mathbb{N}$ is

less restrictive than bounding the euclidean-norm since the latter grows with increasing n even if the entries of the scenarios remain of the same size. In the following we denote the number of non-zero entries of $x \in X$ by $|x|$, i.e.

$$|x| := |\{i \in [n] : x_i = 1\}|.$$

We assume in the following that we have a linear minimization oracle for X and a linear maximization oracle for U .

The main idea of this section is to bound the difference $\text{opt}(n-l) - \text{opt}(n)$ and to show that for large n , which is still polynomial in the input, $\text{opt}(n-l)$ is close to $\text{opt}(n)$. To this end we first prove the following lemma which generalizes the result used in the proof of Theorem 6 in [17].

Lemma 2. *Assume that $|x| \leq p(n)$ for all $x \in X$ for a given function $p : \mathbb{N} \rightarrow [0, n]$. Let $s, k \in [n-1]$ where $s < k$, then it holds*

$$\text{opt}(s) - \text{opt}(k) \leq 2M_\infty p(n) \frac{k-s}{s+1}.$$

Proof. Let $x^*(k)$ be an optimal solution of Problem (1) with parameter k . Then by the results in [17] we have

$$\text{opt}(k) = \max_{c \in U} c^\top x^*(k)$$

and there exists a convex combination $x^*(k) = \sum_{i \in [k]} \lambda_i x^i$ where $x^i \in X$ and $\lambda \in \mathbb{R}_+^k$ with $\sum_{i \in [k]} \lambda_i = 1$. We may assume without loss of generality that $\lambda_1 \geq \dots \geq \lambda_k$. Define a solution $x(s)$ by

$$x(s) := \sum_{i \in [s-1]} \lambda_i x^i + \left(\sum_{i=s}^k \lambda_i \right) x^s,$$

then $x(s) \in X(s)$ and therefore $\text{opt}(s) \leq \max_{c \in U} c^\top x(s)$. Furthermore let $c^*(s) \in \arg \max_{c \in U} c^\top x(s)$. It follows

$$\begin{aligned} \text{opt}(s) - \text{opt}(k) &\leq \max_{c \in U} c^\top x(s) - \max_{c \in U} c^\top x^*(k) \\ &\leq c^*(s)^\top (x(s) - x^*(k)) \\ &= \sum_{i=s+1}^k \lambda_i c^*(s)^\top (x^s - x^i) \\ &\leq M_\infty 2p(n) \left(\sum_{i=s+1}^k \lambda_i \right) \end{aligned}$$

where the second inequality holds since $c^*(s)$ is a subgradient of the function $g(x) = \max_{c \in U} c^\top x$ in $x(s)$. For the first equality we used the definition of $x(s)$ and $x^*(k)$ and for the last inequality we used the assumption $|x| \leq p(n)$ and $|c^*(s)_j| \leq M_\infty$ for all $j \in [n]$. Due to the sorting $\lambda_1 \geq \dots \geq \lambda_k$ and since the sum over all λ_i is one, we have $\lambda_i \leq \frac{1}{i}$ and hence

$$\begin{aligned} M_\infty 2p(n) \left(\sum_{i=s+1}^k \lambda_i \right) &\leq M_\infty 2p(n) \left(\sum_{i=s+1}^k \frac{1}{i} \right) \\ &\leq M_\infty 2p(n) \frac{k-s}{s+1} \end{aligned}$$

which proves the result. \square

Note that the bound derived in the latter lemma is small if s is large compared to the difference $k - s$. Particularly if the difference $k - s$ is fixed and if $p(n)$ grows slower than s , then the bound tends to zero if s goes to infinity. This is the basic idea behind the following lemma.

Lemma 3. *Let $k = n - l$, $\varepsilon > 0$, $\delta \in (0, 1]$, $C > 0$ and $|x| \leq Cn^{1-\delta}$ for all $x \in X$. If*

$$n \geq l^{\frac{1}{\delta}} \left(\frac{2CM_{\infty}}{\varepsilon} + 1 \right)^{\frac{1}{\delta}} \quad (3)$$

then $\text{opt}(n) \leq \text{opt}(n - l) \leq \text{opt}(n) + \varepsilon$.

Proof. The inequality $\text{opt}(n) \leq \text{opt}(n - l)$ follows since $(M^3(k))$ attains smaller optimal values if more solutions are allowed. To prove the inequality $\text{opt}(n - l) \leq \text{opt}(n) + \varepsilon$ we apply Lemma 2 with $p(n) = Cn^{1-\delta}$, $s = n - l$ and $k = n$ and we obtain

$$\begin{aligned} \text{opt}(n - l) - \text{opt}(n) &\leq 2Cn^{1-\delta} M_{\infty} \frac{l}{n - l + 1} \\ &= 2CM_{\infty} \frac{l}{n^{\delta} - \frac{l}{n^{1-\delta}} + \frac{1}{n^{1-\delta}}} \\ &\leq 2CM_{\infty} \frac{l}{n^{\delta} - l} \\ &\leq \varepsilon \end{aligned}$$

where in the second inequality we used $n \geq 1$ and in the third inequality we used inequality (3). \square

The result of the latter lemma essentially says that, if we want to solve $(M^3(k))$ with $k = n - l$, we do not have to worry about instances where n is large, since we can solve these instances efficiently by solving $M^3(n)$. This is a pretty surprising result since we only have to care about the instances where n is bounded. If we fix all parameters of the bound in (3), then we can also solve $(M^3(k))$ for small n by enumerating all solutions, which leads to the following theorem.

Theorem 4. *Let $k = n - l$, $\varepsilon > 0$, $\delta \in (0, 1]$, $C > 0$ and $|x| \leq Cn^{1-\delta}$ for all $x \in X$. Then we can calculate an optimal solution of $(M^3(k))$ (up to an additive error of ε) in polynomial time in n , if all parameters M_{∞} , δ , C , ε and l are fixed.*

Proof. Given an instance of $(M^3(k))$ we check if condition (3) is true or not, which can be done in polynomial time, since the right hand side is a constant.

Case 1: If the condition is true, we solve $M^3(n)$ up to an additive error of ε , which can be done in polynomial time in n by Theorem 1 since M_{∞} and ε are constants. Given the solution of $M^3(n)$, which is of the form $x^* = \sum_{i \in [n]} \lambda_i x^i$ with $\lambda_1 \geq \dots \geq \lambda_n$, we construct the solution $x(k) = \sum_{i \in [k-1]} \lambda_i x^i + (\sum_{i=k}^n \lambda_i) x^k$ as in the proof of Lemma 2. By Lemma 3 it follows that

$$\max_{c \in U} c^{\top} x(k) \leq \text{opt}(n) + \varepsilon \leq \text{opt}(k) + \varepsilon$$

and for the objective value of the corresponding solution $x^{(1)}, \dots, x^{(k)}$ we have

$$\max_{c \in U} \min_{i=1, \dots, k} c^{\top} x^{(i)} = \min_{x \in \text{conv}(x^{(1)}, \dots, x^{(k)})} \max_{c \in U} c^{\top} x \leq \max_{c \in U} c^{\top} x(k) \leq \text{opt}(k) + \varepsilon$$

where the first inequality follows since $x(k) \in \text{conv}(x^{(1)}, \dots, x^{(k)})$.

Case 2: If condition (3) is not true, then n is bounded from above by the constant

$$\tau := l^{\frac{1}{\delta}} \left(\frac{2CM_{\infty}}{\varepsilon} + 1 \right)^{\frac{1}{\delta}}.$$

Therefore the number of possible solutions in X is in $\mathcal{O}(2^{\tau})$ and hence the number of solutions of $(M^3(k))$ is in $\mathcal{O}(2^{\tau k}) = \mathcal{O}(2^{\tau^2})$ since $k \leq n \leq \tau$. We can calculate an optimal solution in this case by enumerating all possible solutions of $(M^3(k))$ and comparing the objective values. Note that we can calculate the objective value of a given solution $x^{(1)}, \dots, x^{(k)}$ by solving the problem

$$\min_{x \in \text{conv}(x^{(1)}, \dots, x^{(k)})} \max_{c \in U} c^{\top} x$$

which can again be done in polynomial time in n by Theorem 1 using the oracles for X and U . \square

If the solutions of the underlying problem (P) fulfill the condition $|x| \leq Cn^{1-\delta}$ for all $x \in X$ for fixed $\delta \in (0, 1]$ and $C > 0$, then from the latter theorem it follows that we can solve $M^3(n-l)$ in polynomial time if l , M_{∞} and ε are fixed.

Corollary 5. *Let $k = n - l$ and $\varepsilon > 0$ and assume that M_{∞} , ε and l are fixed. Then we can calculate an optimal solution of $(M^3(k))$ (up to an additive error of ε) in polynomial time in n , if the underlying problem (P) is the spanning-tree problem on complete graphs, the matching problem on complete graphs or any polynomially solvable cardinality constrained combinatorial problem where $X \subseteq \{x \in \{0, 1\}^n : |x| = p\}$ for a fixed parameter $p \in [n]$.*

Proof. We have to show that for all of the mentioned problems $\delta \in (0, 1]$ and $C > 0$ exist, such that

$$|x| \leq Cn^{1-\delta} \tag{4}$$

for all $x \in X$. For the spanning-tree problem on complete graphs $G = (V, E)$ we have $|x| = |V| - 1$ for all solutions $x \in X$ and $n = |E| = |V|(|V| - 1)$. Hence $|x| \in \mathcal{O}(\sqrt{n})$ and condition (4) holds with $C = 1$ and $\delta = \frac{1}{2}$. For the matching problem on complete graphs $G = (V, E)$ we have $|x| \leq \frac{|V|}{2}$ for all solutions $x \in X$ and $n = |E| = |V|(|V| - 1)$. Hence $|x| \in \mathcal{O}(\sqrt{n})$ and condition (4) holds again with $C = 1$ and $\delta = \frac{1}{2}$. For all cardinality constrained problems condition (4) holds with $C = p$ and $\delta = 1$. All of the mentioned problems can be solved in polynomial time. \square

Note that besides the p -selection problem several combinatorial problems with cardinality constraint $|x| = p$ were studied in the literature; see [14] for an overview.

3.1 Polynomial-time approximation scheme

In this subsection we follow similar ideas as in the latter section and derive a multiplicative approximation factor.

Lemma 6. *Assume that $\underline{p} \leq |x| \leq p(n)$ for all $x \in X$ for a given function $p : \mathbb{N} \rightarrow [0, n]$ and $\underline{p} \in [n]$. Furthermore assume that $m_{\infty} > 0$ exists s.t. $c \geq m_{\infty} \mathbf{1}$ for all $c \in U$. Let $s, k \in [n - 1]$ where $s < k$, then it holds*

$$\text{opt}(s) \leq \left(1 + \frac{k - s}{s + 1} \frac{M_{\infty}}{m_{\infty}} \frac{2p(n)}{\underline{p}} \right) \text{opt}(k).$$

Furthermore if for all solutions in X it holds $|x| = p(n)$, then

$$\text{opt}(s) \leq \left(1 + \frac{k - s}{s + 1} \frac{2M_{\infty}}{m_{\infty}} \right) \text{opt}(k).$$

Proof. First note that due to the assumptions $\underline{p} \leq |x|$ and $c \geq m_\infty \mathbf{1} > 0$ we have $\text{opt}(k) > 0$ for all $k \in [n]$ and therefore the multiplicative approximation guarantee stated in the lemma is well defined.

By Lemma 2 we have

$$\text{opt}(s) \leq 2M_\infty p(n) \frac{k-s}{s+1} + \text{opt}(k) = \left(1 + 2M_\infty p(n) \frac{k-s}{s+1} \frac{1}{\text{opt}(k)}\right) \text{opt}(k). \quad (5)$$

Let $x^*(k)$ be an optimal solution of Problem (1) with parameter k . Then by the results in [17] we have

$$\text{opt}(k) = \max_{c \in U} c^\top x^*(k)$$

and there exists a convex combination $x^*(k) = \sum_{i \in [k]} \lambda_i x^i$ where $x^i \in X$ and $\lambda \in \mathbb{R}_+^k$ with $\sum_{i \in [k]} \lambda_i = 1$. Then it holds

$$\text{opt}(k) = \max_{c \in U} c^\top x^*(k) \geq m_\infty \mathbf{1}^\top x^*(k) \quad (6)$$

where the last inequality holds due to the assumption $c \geq m_\infty \mathbf{1}$ and since $x^*(k) \geq 0$. We can now reformulate

$$m_\infty \mathbf{1}^\top x^*(k) = \sum_{i \in [k]} \lambda_i m_\infty \mathbf{1}^\top x^i \geq \sum_{i \in [k]} \lambda_i m_\infty \underline{p} = m_\infty \underline{p}$$

where in the first inequality we used the assumption $\underline{p} \leq |x|$. Together with (6) we obtain $\text{opt}(k) \geq m_\infty \underline{p}$ and substituting this into (5) proves the result.

The second result follows since $p(n) = \underline{p}$ holds in this case. □

By analogue steps as in the proof of Lemma 3 we can derive the following lemma.

Lemma 7. *Let $k = n - l$, $\varepsilon > 0$, $\delta \in (0, 1]$, $C > 0$, $|x| \leq Cn^{1-\delta}$ for all $x \in X$ and U meets the assumption of Lemma 6. If*

$$n \geq l^{\frac{1}{\delta}} \left(\frac{2CM_\infty}{m_\infty \underline{p} \varepsilon} + 1 \right)^{\frac{1}{\delta}} \quad (7)$$

then $\text{opt}(n-l) \leq (1 + \varepsilon) \text{opt}(n)$. Furthermore if $|x| = p(n)$ for all $x \in X$ for any function p , then the same result holds already for

$$n \geq l \left(\frac{2CM_\infty}{m_\infty \varepsilon} + 1 \right).$$

Note that the assumption $|x| = p(n)$ for the second result is more restrictive since it says that for a given $n \in \mathbb{N}$ all solutions need to have the same number of 1-entries (which is the case for the spanning-tree problem or the matching problem). However we can use any function $p(n)$ here, which is more general than using the function $p(n) = Cn^{1-\delta}$ as in the first result. We will see that due to the second case we do not need to restrict the spanning-tree problem and the matching problem to complete graphs as in Corollary 5.

By the same argumentation as in the proof of Theorem 1 we obtain the following theorem.

Theorem 8. *Let $k = n - l$, $\delta \in (0, 1]$, $C > 0$, $|x| \leq Cn^{1-\delta}$ for all $x \in X$ and U meets the assumption of Lemma 6. Then $(M^3(k))$ admits a PTAS if all parameters M_∞ , m_∞ , δ , C and l are fixed.*

Note that a PTAS is not required to be polynomial in $\frac{1}{\varepsilon}$, therefore the same proof as in Theorem 4 can be used to prove the latter theorem. As a direct consequence we obtain the following corollary.

Corollary 9. *Let $k = n - l$, assume that U meets the assumption of Lemma 6 and M_∞, m_∞ and l are fixed. Then $(M^3(k))$ admits a PTAS if the underlying problem (P) is the spanning-tree problem, the matching problem or any polynomially solvable cardinality constrained combinatorial problem where $X \subseteq \{x \in \{0, 1\}^n : |x| = p\}$ for a fixed parameter $p \in [n]$.*

The proof of the latter corollary is equivalent to the proof of Corollary 5 but since in the spanning-tree problem and the matching problem for a given graph, each solution uses the same number of edges, we can use the second result in Lemma 7 where no restriction on the function p has to be made. Therefore the result holds for the spanning-tree problem and the matching problem on general graphs (in contrast to complete graphs in Corollary 5).

4 An oracle-based algorithm for convex robust optimization problems

In this section we derive an algorithm to solve Problem (2) and hence to solve the min-max-min problem $(M^3(k))$ for $k \geq n$, which only requires linear optimization oracles for the underlying deterministic problem and the uncertainty set. Since all results also hold for general convex robust optimization problems, we state all results for the more general problem

$$\min_{x \in S} \max_{c \in U} c^\top x \quad (8)$$

where $S \subset \mathbb{R}^n$ is a convex set with $\max_{x, x' \in S} \|x - x'\| \leq D$ and $U \subset \mathbb{R}^n$ is a convex set with $\max_{c \in U} \|c\| \leq M$. Note that the latter Problem can also be used to calculate lower bounds for classical discrete robust optimization problems.

We define the projection of $y \in \mathbb{R}^n$ onto the set S by

$$P(y) = \arg \min_{x \in S} \|x - y\|^2 \quad (9)$$

and the approximate projection with additive accuracy $\delta > 0$ by $P_\delta(y)$, i.e. it holds $P_\delta(y) \in S$ and

$$\|P_\delta(y) - y\|^2 \leq \|x - y\|^2 + \delta \quad \forall x \in S.$$

Note that $f(x) := \max_{c \in U} c^\top x$ is a convex function which we have to minimize over the convex set S . Furthermore for each $x_0 \in S$ and $c^* \in \arg \max_{c \in U} c^\top x_0$ it holds

$$f(x_0) + (c^*)^\top (x - x_0) = (c^*)^\top x_0 + (c^*)^\top (x - x_0) = (c^*)^\top x \leq f(x)$$

and therefore c^* is a subgradient for f in x_0 . The latter observations indicate that a subgradient method can be used to solve Problem (8).

The main algorithm in this section is based on a projected subgradient method (presented in Algorithm 1) where the projection problem is approximately solved (up to an additive error) by the classical Frank-Wolfe algorithm (presented in Algorithm 2). We show that the algorithm has a runtime guarantee which is a polynomial in n, M and $\frac{1}{\varepsilon}$ if we have linear optimization oracles for S and U . Note that the main difficulty is to incorporate the additive error of the approximate projection problem into the convergence analysis of the projected subgradient method. Therefore

we will present a detailed convergence analysis and derive a bound on the total number of iterations and hence oracle-calls.

Gradient-based methods were already used for convex robust optimization problems [7, 32, 42]. In [7] a gradient-based online learning method is presented which iteratively calls an oracle for the deterministic problem and an oracle for the projection problem onto U . Furthermore a binary search over the objective value has to be conducted to find an optimal solution. In [42] first-order algorithms are presented which are applied to a Lagrangian saddle-point reformulation of the robust problem which have the same or better convergence rate but rely on a Slater-type condition. Note that all of the above methods assume an oracle for the projection problem on either U or S (or both) and it is not clear if the convergence results hold if this oracle is replaced by an approximate projection given e.g. by a Frank-Wolfe type algorithm. Furthermore our method does not require a description of S or any Slater-type conditions.

Frank-Wolfe type algorithms were already used for robust optimization problems under ellipsoidal uncertainty in [15, 3]. Nevertheless directly applying a Frank-Wolfe type method to Problem (8) with more general uncertainty sets is not possible since the convergence analysis of FW type methods crucially relies on the curvature constant of the objective function, which can not be bounded for the objective function $\max_{c \in U} c^\top x$ in (8) with arbitrary convex uncertainty sets U .

Algorithm 1 Projected Subgradient Method

Input: $n \in \mathbb{N}$, $T \in \mathbb{N}$, $\alpha_t \in (0, 1] \forall t \in [T]$, linear maximization oracle for U (LMO)

- 1: $x^0 = 0$, $c^0 = 0$, $f_{\text{best}} = \infty$
 - 2: **for** $t = 1, \dots, T$ **do**
 - 3: calculate $x^t = P_{\alpha_{t-1}^4} (x^{t-1} - \alpha_{t-1} c^{t-1})$ by Algorithm 2
 - 4: receive $c^t \in \arg \max_{c \in U} c^\top x^t$ from the LMO
 - 5: **if** $(c^t)^\top x^t < f_{\text{best}}$ **then**
 - 6: $x^* = x^t$
 - 7: $f_{\text{best}} = (c^t)^\top x^t$
 - 8: **end if**
 - 9: **end for**
 - 10: **Return:** x^*
-

Algorithm 2 Frank-Wolfe Projection Algorithm

Input: $n \in \mathbb{N}$, $T \in \mathbb{N}$, $y \in \mathbb{R}^n$, linear minimization oracle for X (LMO)

- 1: choose $x^0 \in S$
 - 2: **for** $t = 1, \dots, T$ **do**
 - 3: receive $s \in \arg \min_{x \in S} (x^{t-1} - y)^\top x$ by the LMO
 - 4: set $x^t = x^{t-1} + \frac{2}{t+1}(s - x^{t-1})$
 - 5: **end for**
 - 6: **Return:** x^T
-

The main result we prove in this section is given in the following theorem.

Theorem 10. *For any $\varepsilon > 0$ Algorithm 1 returns a solution $x^* \in S$ with*

$$\max_{c \in U} c^\top x^* \leq \min_{x \in S} \max_{c \in U} c^\top x + \varepsilon$$

(i) after $\mathcal{O}(\frac{D^2+DM^2}{\varepsilon^2})$ subgradient iterations and $\mathcal{O}(\frac{D^3(D+M^2)^5}{\varepsilon^6})$ Frank-Wolfe iterations if we use a constant step size $\alpha_t = \frac{\varepsilon}{1+2D+M^2}$,

(ii) after $\mathcal{O}(\frac{(D+M^2)^4}{\varepsilon^4})$ subgradient iterations and $\mathcal{O}(\frac{D(D+M^2)^{16}}{\varepsilon^{16}})$ Frank-Wolfe iterations if we use step size $\alpha_t = t^{-\frac{3}{4}}$.

Note that in each subgradient iteration the linear maximization oracle for U is called once and in each Frank-Wolfe iteration the linear minimization oracle for S is called once. Since $D \leq \sqrt{n}$ for $S = \text{conv}(X)$ we directly attain the following corollary.

Corollary 11. *Problem $(M^3(k))$ with $k \geq n$ can be solved to optimality by Algorithm 1 up to an additive error of $\varepsilon > 0$*

(i) using at most $\mathcal{O}(\frac{n+\sqrt{n}M^2}{\varepsilon^2})$ LMO calls for U and $\mathcal{O}(\frac{n^2(\sqrt{n}+M^2)^5}{\varepsilon^6})$ LMO calls for X if we use a constant step size $\alpha_t = \frac{\varepsilon}{1+2\sqrt{n}+M^2}$,

(ii) using at most $\mathcal{O}(\frac{(\sqrt{n}+M^2)^4}{\varepsilon^4})$ LMO calls for U and $\mathcal{O}(\frac{\sqrt{n}(\sqrt{n}+M^2)^{16}}{\varepsilon^{16}})$ LMO calls for X if we use step size $\alpha_t = t^{-\frac{3}{4}}$.

To prove Theorem 10 we follow the classical convergence analysis for projected subgradient methods incorporating approximate projections with the appropriately chosen additive approximation guarantee α_{t-1}^4 . To this end we first prove the following lemma.

Lemma 12. *For all $x \in \mathbb{R}^n$ and $\delta > 0$ we have $\|P_\delta(y) - P(y)\|^2 \leq \delta$.*

Proof. Assume $y \in \mathbb{R}^n$ and define $x_p = P(y)$ and $x_a = P_\delta(y)$. Since x_p is the projection of y onto S , and since S is convex, it holds that

$$(y - x_p)^\top (x - x_p) \leq 0 \quad \forall x \in S. \quad (10)$$

We can now bound

$$\begin{aligned} \|x_p - y\|^2 + \delta &\geq \|y - x_a\|^2 \\ &= \|y - x_p + x_p - x_a\|^2 \\ &= \|y - x_p\|^2 + \|x_p - x_a\|^2 - 2(y - x_p)^\top (x_a - x_p) \end{aligned}$$

where the first inequality follows from the definition of $P_\delta(y)$. By resorting the terms from the last inequality we obtain

$$\|x_p - x_a\|^2 \leq \delta + 2(y - x_p)^\top (x_a - x_p) \leq \delta$$

where the second inequality follows from (10) and since $x_a \in S$. \square

Using the result of Lemma 12 we can prove the convergence of Algorithm 1.

Theorem 13. *For every $\varepsilon > 0$ Algorithm 1 returns a solution $x^* \in S$ with*

$$\max_{c \in U} c^\top x^* \leq \min_{x \in S} \max_{c \in U} c^\top x + \varepsilon$$

(i) after $T \geq \frac{D(M^2+1+2D)}{\varepsilon^2}$ iterations if we use constant step size $\alpha_t = \frac{\varepsilon}{1+2D+M^2}$,

(ii) after $T \geq \frac{(D+(M^2+1+2D)3)^4}{\varepsilon^4}$ iterations if we use step size $\alpha_t = t^{-\frac{3}{4}}$

Proof. Let x^* be the optimal solution of Problem (8), $t \in [T-1]$ and define $y = x^t - \alpha_t c^t$. Then it holds

$$\begin{aligned}
& \|x^{t+1} - x^*\|^2 \\
&= \|P_{\alpha_t^4}(y) - x^*\|^2 \\
&= \|P_{\alpha_t^4}(y) - P(y) + P(y) - x^*\|^2 \\
&= \|P_{\alpha_t^4}(y) - P(y)\|^2 + \|P(y) - x^*\|^2 + 2 \left(P_{\alpha_t^4}(y) - P(y) \right)^\top (P(y) - x^*) \\
&\leq \alpha_t^4 + \|P(y) - x^*\|^2 + 2 \|P_{\alpha_t^4}(y) - P(y)\| \|P(y) - x^*\| \\
&\leq \alpha_t^4 + \|P(y) - x^*\|^2 + 2\alpha_t^2 D \\
&\leq \|P(y) - x^*\|^2 + (1 + 2D)\alpha_t^2,
\end{aligned}$$

where in the first equality we applied Step 3 in Algorithm 1, the first inequality is given by the Cauchy-Schwarz inequality and Lemma 12, in the second inequality we applied Lemma 12 and the assumption that the diameter of S is bounded by D and the third inequality is true since $\alpha_t \in (0, 1]$.

Since $P(y)$ is the projection of y onto S and S is convex, we have $\|P(y) - x^*\|^2 \leq \|y - x^*\|^2$ for all $x \in S$ and therefore, substituting $y = x^t - \alpha_t c^t$ we can continue

$$\begin{aligned}
& \|P(y) - x^*\|^2 + (1 + 2D)\alpha_t^2 \\
&\leq \|x^t - \alpha_t c^t - x^*\|^2 + (1 + 2D)\alpha_t^2 \\
&= \|x^t - x^*\|^2 + \alpha_t^2 (\|c^t\|^2 + 1 + 2D) - 2\alpha_t (c^t)^\top (x^t - x^*).
\end{aligned}$$

Since c^t is a subgradient of $f(x) = \max_{c \in U} c^\top x$ in x^t and since $\|c^t\| \leq M$ we have

$$\begin{aligned}
& \|x^t - x^*\|^2 + \alpha_t^2 (\|c^t\|^2 + 1 + 2D) - 2\alpha_t (c^t)^\top (x^t - x^*) \\
&\leq \|x^t - x^*\|^2 + \alpha_t^2 (M^2 + 1 + 2D) - 2\alpha_t (f(x^t) - f(x^*))
\end{aligned}$$

and in summary we have the inequality

$$\|x^{t+1} - x^*\|^2 \leq \|x^t - x^*\|^2 + \alpha_t^2 (M^2 + 1 + 2D) - 2\alpha_t (f(x^t) - f(x^*)).$$

Applying the latter inequality iteratively we obtain

$$\|x^{t+1} - x^*\|^2 \leq \|x^1 - x^*\|^2 + \sum_{j \in [t]} \alpha_j^2 (M^2 + 1 + 2D) - 2 \sum_{j \in [t]} \alpha_j (f(x^j) - f(x^*))$$

and since $\|x^{t+1} - x^*\|^2 \geq 0$ we obtain

$$2 \sum_{j \in [t]} \alpha_j (f(x^j) - f(x^*)) \leq \|x^1 - x^*\|^2 + \sum_{j \in [t]} \alpha_j^2 (M^2 + 1 + 2D).$$

We can bound $\left(2 \sum_{j \in [t]} \alpha_j \right) (f_{\text{best}} - f(x^*))$ from above by the left-hand side of the latter inequality and using $\|x^1 - x^*\|^2 \leq D$ we get the final upper bound

$$f_{\text{best}} - f(x^*) \leq \frac{D + (M^2 + 1 + 2D) \sum_{j \in [t]} \alpha_j^2}{\left(2 \sum_{j \in [t]} \alpha_j \right)}. \quad (11)$$

Case (i): Substituting $\alpha_j = \frac{\varepsilon}{1+2D+M^2}$ in (11) we have

$$f_{\text{best}} - f(x^*) \leq \frac{D(M^2 + 1 + 2D) + t\varepsilon^2}{2t\varepsilon} = \frac{D(M^2 + 1 + 2D)}{2t\varepsilon} + \frac{\varepsilon}{2}.$$

Now substituting $t = T \geq \frac{D(M^2+1+2D)}{\varepsilon^2}$ we have $\frac{D(M^2+1+2D)}{2t\varepsilon} \leq \frac{\varepsilon}{2}$ and therefore it holds

$$f_{\text{best}} - f(x^*) \leq \varepsilon$$

after at least T iterations which proves result (i).

Case (ii): For step size $\alpha_j = j^{-\frac{3}{4}}$ we have

$$\sum_{j \in [t]} \alpha_j^2 = \sum_{j \in [t]} \frac{1}{j^{\frac{3}{2}}} \leq \zeta\left(\frac{3}{2}\right) < 3$$

where $\zeta\left(\frac{3}{2}\right)$ denotes the value of the Riemann zeta function at $\frac{3}{2}$. On the other hand we have

$$\sum_{j \in [t]} \alpha_j = \sum_{j \in [t]} \frac{1}{j^{\frac{3}{4}}} \geq \int_1^{t+1} x^{-\frac{3}{4}} dx = 4(t+1)^{\frac{1}{4}} - 4 \geq (t+1)^{\frac{1}{4}} \geq t^{\frac{1}{4}}$$

where the second inequality holds for all $t \geq 3$ which is a valid assumption e.g. if we assume $D, M \geq 1$ and $\varepsilon \leq 1$. Using the latter two results together with (11) we obtain

$$f_{\text{best}} - f(x^*) \leq \frac{D + (M^2 + 1 + 2D)3}{t^{\frac{1}{4}}}$$

and hence for $t = T \geq \frac{(D+(M^2+1+2D)3)^4}{\varepsilon^4}$ we have $f_{\text{best}} - f(x^*) \leq \varepsilon$ which proves result (ii). \square

Next we have to bound the number of iterations of the Frank-Wolfe algorithm which is used to calculate the approximate projection $P_{\alpha_t^4}$ in Step 3.

Lemma 14. *For every $y \in \mathbb{R}^n$ and $\delta > 0$ Algorithm 2 returns a solution $x_p \in S$ with*

$$\|x_p - y\|^2 \leq \min_{x \in S} \|x - y\|^2 + \delta$$

after $T \geq \frac{4D^2}{\delta}$ iterations.

Proof. First note that the gradient of $g(x) := \|x - y\|^2$ in $x_0 \in S$ is given by $\nabla g(x_0) = 2(x_0 - y)$ and therefore Step 3 in Algorithm 2 is equivalent to minimizing $\nabla g(x^{t-1})^\top x$ over S . Hence Algorithm 2 is the classical Frank-Wolfe algorithm applied to the problem

$$\min_{x \in S} \|x - y\|^2. \tag{12}$$

Furthermore the gradient of g is Lipschitz continuous with Lipschitz constant 2. In this case it follows from classical convergence results (see e.g. [33])

$$g(x^t) - g(x^*) \leq \frac{4D^2}{t+2} \leq \frac{4D^2}{t}$$

where x^* is the optimal solution of (12). If we choose $t = T \geq \frac{4D^2}{\delta}$ we obtain $g(x^t) - g(x^*) \leq \delta$ which proves the result. \square

Using all the latter results we finally can prove Theorem 10.

Proof of Theorem 10. Since Theorem 13 provides a bound on the number of subgradient iterations, we can now bound the step size α_t and therefore the number of Frank-Wolfe iterations. Combining both, the number of subgradient iterations and Frank-Wolfe iterations, we can then prove the result.

Case (i): First note that the number of required subgradient iterations by Theorem 13 is in $\mathcal{O}(\frac{D(D+M^2)}{\varepsilon^2})$. Substituting the accuracy $\delta = \alpha_t^4$ for the constant step size $\alpha_t = \frac{\varepsilon}{1+2D+M^2}$ into the bound on the number of Frank-Wolfe iterations we obtain that

$$T = \frac{4D^2}{\alpha_t^4} = \frac{4D^2(1+2D+M^2)^4}{\varepsilon^4} \in \mathcal{O}\left(\frac{D^2(D+M^2)^4}{\varepsilon^4}\right)$$

iterations of the Frank-Wolfe algorithm are enough to ensure the demanded accuracy of α_t^4 in Step 3 of Algorithm 1. Multiplying this bound with the bound on the number of subgradient iterations yields a bound on the number of total Frank-Wolfe iterations which is in $\mathcal{O}(\frac{D^3(D+M^2)^5}{\varepsilon^6})$.

Case (ii): First note that the number of required subgradient iterations by Theorem 13 is in $\mathcal{O}(\frac{(D+M^2)^4}{\varepsilon^4})$. Therefore we can bound the number of Frank-Wolfe iterations in the j -th subgradient iteration by

$$T = \frac{4D}{\alpha_j^4} = 4Dj^3 \in \mathcal{O}\left(\frac{D(D+M^2)^{12}}{\varepsilon^{12}}\right)$$

where the right bound follows from $j \in \mathcal{O}(\frac{(D+M^2)^4}{\varepsilon^4})$. Multiplying the bound on T with the bound on the number of subgradient iterations yields a bound on the number of total Frank-Wolfe iterations which is in $\mathcal{O}(\frac{D(D+M^2)^{16}}{\varepsilon^{16}})$. \square

Note that for practical implementations, instead of using a maximal number of steps T in Algorithm 2, the well-known duality gap can be used as a stopping criterion for the Frank-Wolfe algorithm, i.e. we can stop the algorithm if in the current iteration

$$\nabla f(x^{t-1})^\top (x^{t-1} - s) = 2(x^{t-1} - y)^\top (x^{t-1} - s) \leq \delta$$

is true for the given accuracy $\delta > 0$.

5 An exact branch & bound algorithm

In this section we adapt the idea in [34] to derive an exact branch & bound algorithm for Problem $(M^3(k))$ where the deterministic problem (P) can be accessed by an oracle, i.e. any algorithm for the deterministic problem can be used and we do not require an integer programming formulation of the problem. The main idea of the algorithm is to use a classical branch & bound method on the solution space which is given by the concatenation of k feasible solutions, i.e. on

$$X_k := \left\{ \left(x^{(1)}, \dots, x^{(k)} \right) \in \{0, 1\}^{nk} : x^{(1)}, \dots, x^{(k)} \in X \right\}.$$

The main task is, given fixations on some of the variables in X_k , to efficiently calculate a lower bound for Problem $(M^3(k))$. In the following we denote the set of indices which are fixed to one in solution x^i by $J_1^i \subseteq [n]$ and the set of indices which are fixed to zero by $J_0^i \subseteq [n]$, where $J_0^i \cap J_1^i = \emptyset$. Furthermore we denote the set of solutions in X complying the given fixations by

$$X(J_0^i, J_1^i) := \{x \in X : x_j = 0 \forall j \in J_0^i, x_j = 1 \forall j \in J_1^i\},$$

and the set of feasible solutions in X_k in the branch & bound node by

$$X(J_0, J_1) := X(J_0^1, J_1^1) \times \cdots \times X(J_0^k, J_1^k).$$

We can now derive the following lower bound. Given fixations $J_0^i, J_1^i \subseteq [n]$ for all $i \in [k]$ s.t. $X(J_0, J_1)$ is non-empty, by interchanging the first min and the max operator we have

$$\min_{(x^1, \dots, x^k) \in X(J_0, J_1)} \max_{c \in U} \min_{i=1, \dots, k} c^\top x^i \quad (13)$$

$$\geq \max_{c \in U} \min_{(x^1, \dots, x^k) \in X(J_0, J_1)} \min_{i=1, \dots, k} c^\top x^i \quad (14)$$

$$= \max_{c \in U, z \in \mathbb{R}} \{z : z \leq c^\top x^i \quad \forall x^i \in X(J_0^i, J_1^i), i \in [k]\} \quad (15)$$

where the last equality was derived by the classical level set transformation. Problem (15) can be solved using the idea behind the efficient algorithm derived in [17] which was used by the authors to solve Problem (2). The main idea of the algorithm is to iteratively generate the constraints $z \leq c^\top x^i$ by solving the deterministic problem (P) for the current optimal solution c^* of a relaxation of (15). Since all sets $X(J_0^i, J_1^i)$ contain a finite number of solutions, we have a finite number of constraint to be generated. In [17] the authors observe that this algorithm is very efficient and only needs a small number of iterations compared to the dimension n of the problem. In our case we have to adapt the algorithm such that the different fixations for the solutions are considered, therefore instead of using one oracle call in each iteration we have to perform k oracle calls. The adapted algorithm is presented in Algorithm 3.

Algorithm 3 Lower bound calculation

Input: $n \in \mathbb{N}$, convex set $U \subset \mathbb{R}^n$, linear minimization oracle for X (LMO), fixations J_0^i, J_1^i for all $i \in [k]$

- 1: choose any $(x^1, \dots, x^k) \in X(J_0, J_1)$ and set $X' := \{(x^1, \dots, x^k)\}$
- 2: **repeat**
- 3: calculate an optimal solution z^*, c^* of

$$\max_{c \in U, z \in \mathbb{R}} \{z : z \leq c^\top x^1, \dots, z \leq c^\top x^k \quad \forall (x^1, \dots, x^k) \in X'\}$$

- 4: for each $i \in [k]$, calculate an optimal solution \bar{x}^i of

$$\min_{x^i \in X(J_0^i, J_1^i)} (c^*)^\top x^i$$

- 5: add $(\bar{x}^1, \dots, \bar{x}^k)$ to X'
 - 6: **until** $(c^*)^\top \bar{x}^i \geq z^*$ for all $i \in [k]$
 - 7: **Return:** z^*
-

Since the solutions \bar{x}^i are derived by minimizing over $X(J_0^i, J_1^i)$ the stopping criterion is equivalent to

$$z^* \leq c^\top x^i \quad \forall x^i \in X(J_0^i, J_1^i) \quad \forall i \in [k]$$

and hence (z^*, c^*) is a feasible solution of (15). Therefore z^* is smaller or equal to the optimal value of (15). On the other hand z^* was calculated in Step 3 as the optimal value of a relaxation of Problem (15) and therefore z^* is at least as large as the optimal value of (15). Combining both

the returned z^* must be optimal for Problem (15) and is therefore a lower bound for $(M^3(k))$ under the given fixations.

Note that in Algorithm 3 in Step 3 we have to solve an adversarial problem over U , which can be done very efficiently for classical uncertainty sets as polyhedra or ellipsoids. In Step 4 we have to solve the underlying deterministic problem k times for given sets of fixations. Fixations can be handled easily for most of the classical combinatorial problems by either adding positive and negative big- M values to the costs of the fixed indices or by incorporating fixation-constraints if an integer programming formulation is used.

We can now implement a classical branch & bound procedure over the solution space X_k by calculating lower bounds in each branch & bound node using Algorithm 3. In the following we present several strategies to improve the procedure.

Calculate feasible solutions Algorithm 3 calculates a large set of feasible solutions in $X(J_0, J_1)$ which can be used in the branch & bound algorithm. Even all possible combinations of k solutions (regarding the fixations) are feasible. Nevertheless checking the objective value for all of them is costly since for each feasible solution $x^{(1)}, \dots, x^{(k)}$ we have to solve the problem

$$\max_{c \in U, z \in \mathbb{R}} \{z : z \leq c^\top x^i \forall i \in [k]\}.$$

Instead we use the following heuristic to generate feasible solutions. During the calculations of Algorithm 3 we collect all calculated solutions $(\bar{x}^1, \dots, \bar{x}^k)$ and all worst-case scenarios c^* . We denote the set of all these scenarios by C and the set of all candidates for solution $i \in [k]$ by X^i . Since we have to consider the given fixations in the current node for each $i \in [k]$ we can choose only one solution from each X^i . All these solutions combined give a feasible min-max-min solution for the current branch & bound node. The selection of the solutions is done as follows: for each $x \in \bigcup_{i \in [k]} X^i$ we calculate the average performance on all scenarios in C , i.e.

$$\text{avg}(x) := \frac{1}{|C|} \sum_{c \in C} c^\top x.$$

We then pick the solution with the best average performance and delete all solutions in the same set X^i . We repeat this process k times but in the following instead of calculating the average performance we calculate the average over all scenarios of the difference to the best value over all already selected solutions, i.e.

$$\text{avg-diff}(x) := \frac{1}{|C|} \sum_{c \in C} \left(c^\top x - \min_{x' \text{ already selected}} c^\top x' \right).$$

The idea is to check which solution improves the current objective value the most in average. After repeating this calculation k times we obtain a feasible solution for the current branch & bound node.

Furthermore in the root node of the branch & bound tree we use the heuristic in [17] to calculate a feasible solution for Problem $(M^3(k))$. The idea of the heuristic is to solve Problem (2) efficiently and pick k of the calculated solutions sorted by the induced weights of the optimal convex combination. It was shown in [17] that this heuristic often calculates solutions which are very close to the optimal value.

Branching decision To find a branching variable we consider all solutions $x \in X$ which were calculated in Algorithm 3 for which $(c^*)^\top x = z^*$ in the last iteration of the algorithm. Then we

try to find an index in one of these solutions, which is not fixed yet, and for which $c_j^* > 0$ and $x_j = 1$. If we fix $x_j = 0$ for this index, then $(c^*)^\top x < z^*$ and therefore branching on this index will have an effect on the lower bound in one of the child nodes. Note that we have to add the index to the set of fixations corresponding to the chosen solution. If we do not find any index as described, we just branch on the first non-fixed index.

Avoid symmetries Since for a given solution (x^1, \dots, x^k) each permutation is also a feasible solution with the same objective value, the number of possible branch & bound nodes can be reduced significantly if we avoid symmetric fixations. To this end we record all already performed fixations and prune the branch & bound tree if a fixation appears which is a permutation of an already performed fixation. To check if a fixation is a permutation of another fixation, first for each solution x^i we sort the fixation indices and afterwards we sort the fixation vectors for all x^1, \dots, x^k lexicographically. The lexicographically sorted array of vectors is unique for all permutations of the same fixations.

Furthermore in each branch & bound node, if the fixations for two solutions x^i and x^j are exactly the same, then the corresponding oracle calls in Step 4 of Algorithm 3 will always return the same optimal solution. Therefore at the beginning of Algorithm 3 we sort out all indices in $[k]$ whose fixations are duplicates of already existing fixations and then run the algorithm with the reduced number of solutions $k' \leq k$.

Warm-starts One idea to improve the runtime of the branch & bound method could be to perform warm-starts in each node by inherit all feasible solutions from the parent node to the child node which satisfy the child node fixations. Nevertheless this strategy is omitted in the computations since in the tests it turned out that this does not improve the runtime significantly while at the same time consuming more memory which is because for each unprocessed node a large list of solutions has to be stored. This effect is possibly because the oracle calls for our underlying problems do not consume much runtime and therefore recalculating good solutions in the child node is not more costly than the calculations regarding the warm-starts. Nevertheless when applying the algorithm to very hard deterministic problems the warm-start strategy could become useful.

6 Computations

In this section we test Algorithm 1 and the branch & bound method derived in Section 5 on random instances which were already used in the min-max-min literature. We test the branch & bound method for the shortest path problem and the knapsack problem and show that in contrast to already existing methods the branch & bound method struggles on instances with small k while the efficiency increases with increasing k . Additionally we compare the subgradient method presented in Algorithm 1 with several step sizes to the column-generation method from [17] on the knapsack instances. Our experiments show that the latter method clearly outperforms the subgradient method. In the following we denote the column-generation method by CG, the subgradient method by SG, the Frank-Wolfe algorithm by FW and the branch & bound method by BB.

All algorithms were implemented in Python 3.6.8 and were executed on two AMD EPYC 7452 CPUs with 32 cores, 2.35-3.35 GHz, 128 MB Cache and a RAM of 256 GB DDR4 and 3200 MHz. All optimization problems appearing in Algorithm 1, 2 and 3 were solved by Gurobi 9.1.1 [30]. For all calculations we set a timelimit of 7200 seconds.

6.1 Instances

We test all algorithms on the minimum knapsack problem which is given by

$$\min_{\substack{a^\top x \geq b \\ x \in \{0,1\}^n}} \hat{c}^\top x$$

where the instances were generated as in [22, 27]. For each dimension $x \in \{50, 100, 200, 400\}$ we generate 10 random knapsack instances where the costs \hat{c}_i and the weights a_i were drawn from a uniform distribution on $\{1, \dots, 100\}$ and the knapsack capacity b was set to 35% of the sum of all weights. For each knapsack instance we generated a budgeted uncertainty set

$$U = \left\{ c \in \mathbb{R}^n : c_i = \hat{c}_i + \delta_i d_i, \sum_{i \in [n]} \delta_i \leq \Gamma, \delta \in [0, 1]^n \right\}$$

where each d_i was drawn uniformly from $\{1, \dots, \hat{c}_i\}$ and the budget parameter was set to $\Gamma = \lfloor \frac{n}{20} \rfloor$. For the number of solutions we consider the cases $k \in \{2, 4, \dots, 20\}$.

For the shortest path problem we use the original instances from [31] which were also used in several other publications of the min-max-min literature. We consider random graphs $G = (V, E)$ with $|V| \in \{20, 30, 40, 50\}$ nodes corresponding to points in the Euclidean plane with random coordinates in $[0, 10]$. For each dimension the authors generated 100 graphs. We consider budgeted uncertainty sets described as above where the mean values \hat{c}_{ij} on edge (i, j) are set to the euclidean distance of node i and j and the deviation values are set to $d_{ij} = \frac{\hat{c}_{ij}}{2}$. The budget parameter Γ was chosen in $\in \{3, 6\}$ and for the number of solutions we again consider the cases $k \in \{2, 4, \dots, 20\}$.

6.2 Subgradient Method

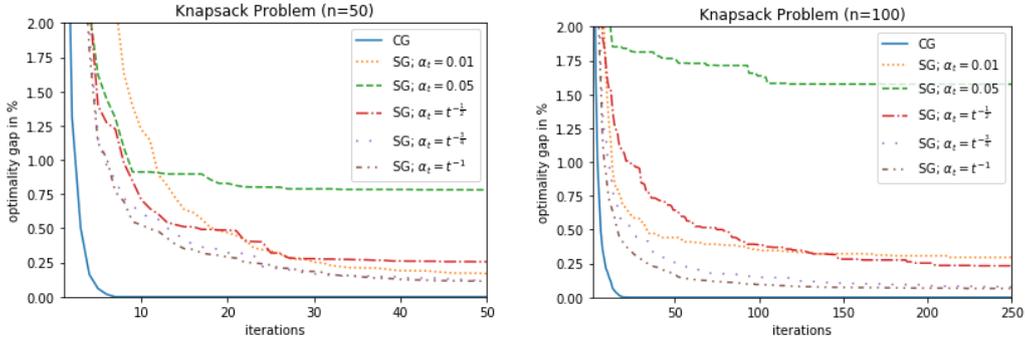


Figure 1: Optimality gap for CG and SG dependent on the number of processed iterations.

The subgradient method was tested on the random knapsack instances described in the latter section. Note that we omit experiments for the shortest path problem for the following reasons: in each of the FW iterations a shortest path problem with potentially negative costs and hence potentially negative cycles has to be solved. Therefore the classical shortest path formulation can not be used and we have to solve a much harder problem in each of the iterations. Furthermore the experiments for the knapsack problem already show that SG cannot compete with CG even if the underlying problem is computationally tractable. Since in the CG no negative costs can

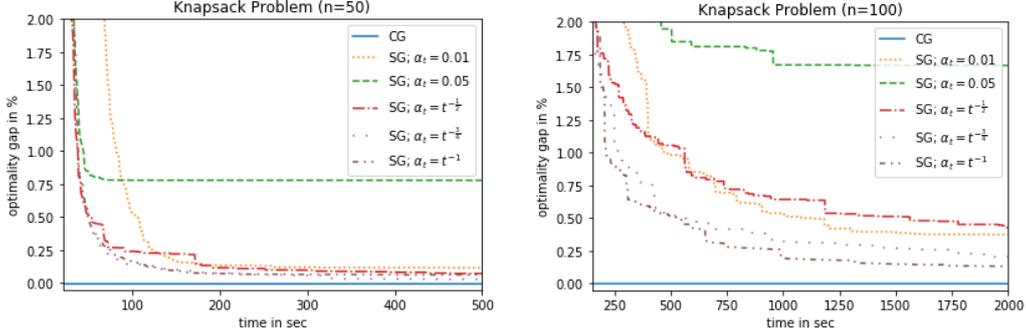


Figure 2: Optimalty gap for CG and SG dependent on the processed time.

appear for the chosen uncertainty sets and we can therefore solve the classical shortest path problem in each of the iterations, there is no hope that the SG can compete with CG on the shortest path instances.

We implemented FW with step size $\frac{2}{t+1}$ and set the maximum number of iterations to $T = 10000$. Furthermore after each iteration we check if the Frank-Wolfe duality gap is below 0.0001 in which case we already stop the algorithm. For the LMO in Step 3 we implemented the knapsack formulation in Gurobi.

The SG was also stopped after a maximum of $T = 10000$ iterations or if there was no update of the objective value in the last 200 iterations. Note that the case that the subgradient is close to 0 can not appear since the subgradient in our algorithm is always given by a scenario in U which does not contain the zero vector. We varied the step sizes of the SG in $\{0.01, 0.05, 0.1t^{-\frac{1}{2}}, 0.1t^{-\frac{3}{4}}, 0.1t^{-1}\}$. The LMO in Step 4 was implemented in Gurobi.

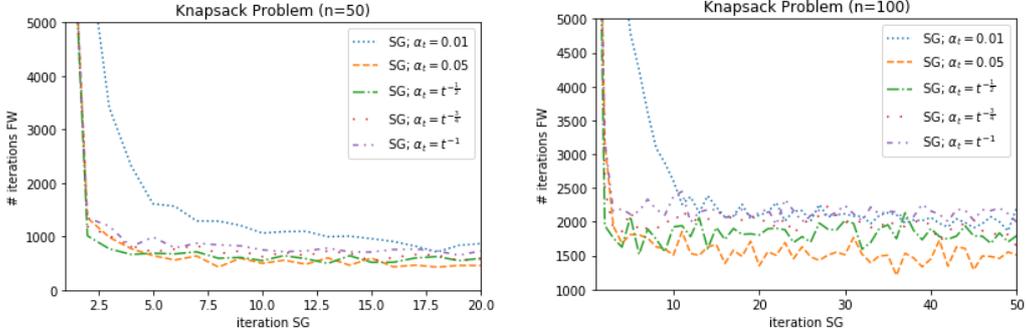


Figure 3: Average number of FW iterations dependent on the processed SG iterations.

In Figure 1 we show the optimality gap in % over the number of iterations of the subgradient method for the instances with $n \in \{50, 100\}$. The detailed results can be found in the Appendix in Table 1. The results show that the CG clearly outperforms the SG for all step sizes. The first advantage of the CG is that it always finds an optimal solution while the SG often achieves a small optimality gap but does not find the optimal solution. Furthermore the CG finds the exact optimal solution after very few iterations while the SG takes a multiple of iterations to achieve a gap close to zero. While the CG in average takes only 4 iterations for the instances with $n = 50$ and 16 iterations for the instances with $n = 100$ the SG takes more than 10 times

of the iterations. Regarding the step size the best choice is $\alpha_t = t^{-1}$ or $\alpha_t = t^{-\frac{3}{4}}$ while for a constant step size of $\alpha_t = 0.05$ the SG cannot close the optimality gap. For the instances with $n = 100$ even the step sizes $\alpha_t = 0.01$ and $\alpha_t = t^{-\frac{1}{2}}$ do not have a gap below 0.25% after 250 iterations.

In Figure 2 we show the optimality gap in % over the processed runtime. As expected the picture is very similar to Figure 1. While the CG finds the optimal solutions in seconds the SG for all step sizes takes a multiple of time to achieve an optimality gap close to zero.

Figure 3 shows the average number of FW iterations which have to be performed in each of the subgradient iterations. The results show that the number of FW iterations is very large in the first SG iterations, possibly since in the beginning the SG often produces solutions which are not contained in the feasible set and therefore the projection problem is harder to solve. Interestingly the step sizes which perform well regarding the overall runtime, namely $\alpha_t = t^{-1}$ and $\alpha_t = t^{-\frac{3}{4}}$, have the largest number of FW iterations. This indicates that the number of iterations of the SG has a larger impact on the runtime than the number of iterations of the FW. Nevertheless this behaviour can change if the deterministic problem is much harder to solve, since then the LMO calls of the FW are more expensive.

6.3 Branch & Bound Method

We tested the branch & bound method on the instances of the knapsack problem and of the shortest path problem. The branch & bound method was implemented as described in Section 5 and all subproblems in Algorithm 3 were solved by Gurobi.

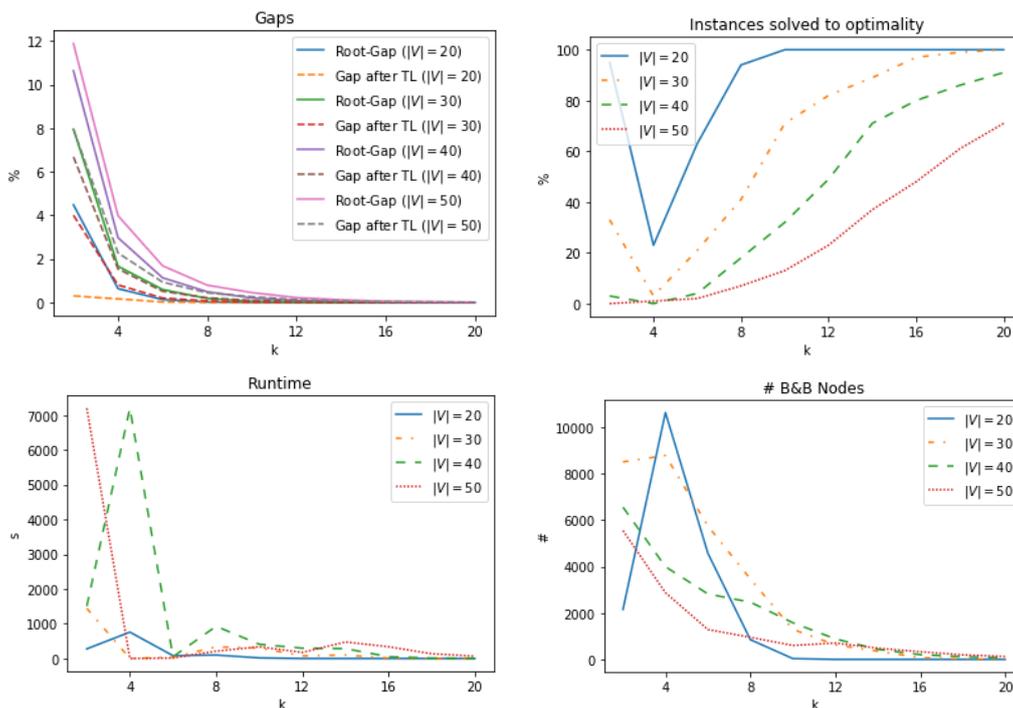


Figure 4: Results for the shortest path problem with budgeted uncertainty ($\Gamma = 3$).

6.3.1 Shortest Path Problem

In Figure 4 we show results of BB for the shortest path instances. The plots show the variation of the following metrics (averaged over all problem instances) over k : the root gap in %, i.e. the percental difference between the objective value of the calculated solution and the lower bound in the root node of the BB tree; the optimality gap in %, i.e. the percental difference between the objective value of the best known solution and the lower bound after the timelimit; the percental number of instances which were solved to optimality during the timelimit; the runtime in seconds averaged only over the instances which could be solved during timelimit; the number of processed BB nodes.

The results indicate that both the root gap and the optimality gap decrease rapidly with increasing k and are larger for a larger problem dimension. The reason for the decrease in k is that the heuristic we use in the root node calculates the optimal solution for Problem $(M^3(k))$ with $k \geq n$. Therefore for larger k , the root gap gets closer to zero and hence the performance of the BB gets better. Note that the root gap can already be zero for $k < n$. Figure 4 shows that already for $k = 8$ both gaps are smaller than 1% for all dimensions. The plot in the upper right shows that the number of instances, which could be solved to optimality during the timelimit, is small if k is small and increases with larger k . For $k = 20$ nearly all instances could be solved for the smaller graphs and more than 60% for the graphs with $|V| = 50$. Interestingly it seems that the hardest instances are the ones for $k = 4$. Furthermore we can see that the number of optimally solved instances decreases if the dimension of the problem increases. The runtime plot (in the lower left) indicates that often the instances with small k are hard to solve. Note that for smaller dimensions the runtime is always small since we only show the runtime for the instances which were solved during the timelimit. This indicates that either the instances could be solved very fast or they could not be solved during the timelimit. As the plot in the lower right indicates, the large runtime can primarily be explained by the number of processed BB nodes which is very large for small k and decreases rapidly, which is again due to the decreasing root gap. The detailed results can be found in the Appendix in Table 2 and 3.

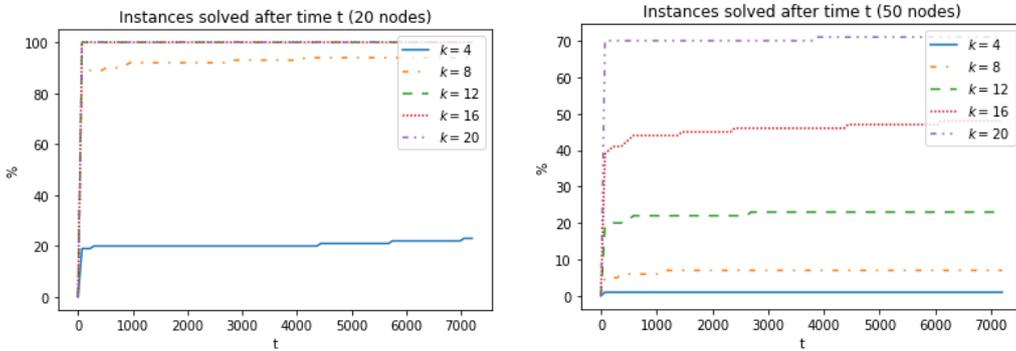


Figure 5: Results for the shortest path problem with budgeted uncertainty ($\Gamma = 3$).

In Figure 5 we show the percental number of solved instances after a processed time t for the instances with 20 nodes on the left and with 50 nodes on the right. The plots show that most of the instances could be solved after a few seconds and all other instances could not be solved during timelimit. Furthermore we can again see the effect that the number of solved instances increases with k and decreases with $|V|$.

In summary the results show that the BB method is able to solve instances even in larger dimensions if at the same time k is not too small. Already for $k = 8$ the optimality gap is below

1% and for $k = 20$ nearly all instances could be solved for $|V| \in \{20, 30, 40\}$. The trend indicates that for all dimensions with increasing k more and more instances can be solved and finally after some threshold all instances can be solved to optimality by the heuristic we use in the root node. At the same time the BB method struggles to solve instances for small k (especially $k = 4$), in which case the already existing methods from the min-max-min literature should be preferred; see e.g. [4]. On the other hand the performance of the existing methods drops with increasing k which motivates the use of our BB method.

6.3.2 Knapsack Problem

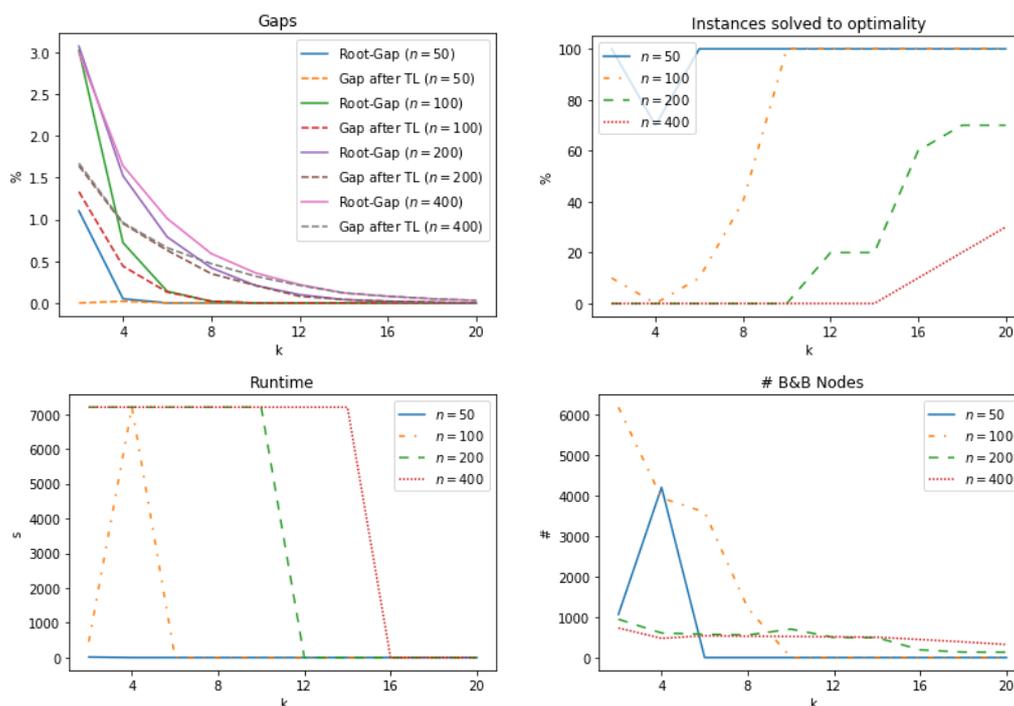


Figure 6: Results for the knapsack problem with budgeted uncertainty.

In Figure 6 we show the results of BB for the knapsack instances. The plots show the same metrics as described in the latter subsection.

The results show very similar effects as for the shortest path problem. Figure 6 indicates that, by the same reason as for the shortest path problem, both the root gap and the optimality gap decrease rapidly with increasing k and are larger for larger problem dimensions. Again already for $k = 8$ both gaps are below 1% for all problem dimensions. The plot in the upper right shows, that the number of instances which could be solved to optimality during the timelimit is small, if k is small and increases with larger k . Here for the larger dimensions the growth of the curves starts late, namely after $k = 10$. Also the number of solved instances for $k = 20$ is smaller than for the shortest path problem which may be due to the different problem dimensions but also since the number of infeasible variable fixations and therefore the number of infeasible nodes in the BB tree is larger for the shortest path problem than for the knapsack problem. Interestingly again it seems that the hardest instances are the ones for $k = 4$. The runtime plot (in the lower

left) indicates that the instances with $k \leq 16$ are hard to solve if $n > 50$. As the plot in the lower right indicates, the large runtime can primarily be explained by the number of processed BB nodes which is very large for small k and decreases rapidly. The detailed results can be found in the Appendix in Table 4.

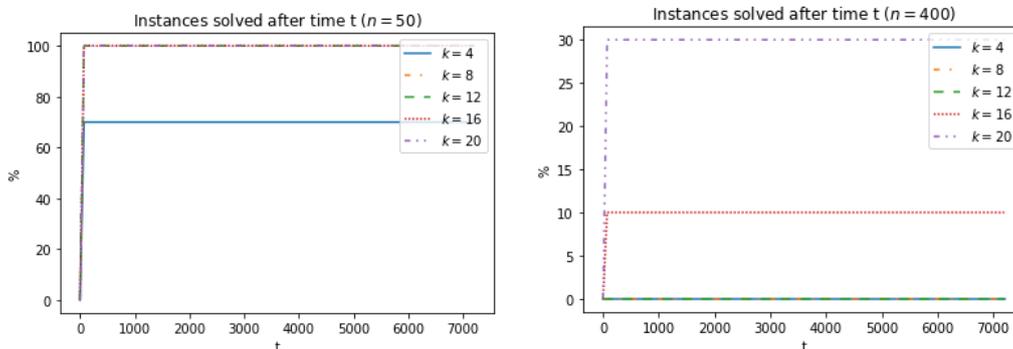


Figure 7: Results for the knapsack problem with budgeted uncertainty.

In Figure 7 we show the percental number of solved instances over the runtime t for the instances with $n = 50$ on the left and with $n = 400$ on the right. The plots show that most of the instances could be solved after a few seconds and all other instances could not be solved during timelimit. Furthermore we can again see the effect that the number of solved instances increases with k and decreases with n .

In summary the results show that, similar to the shortest path instances, the BB method is able to solve instances even in larger dimensions if at the same time k is not too small. Already for $k = 8$ the optimality gap is below 1% and for $k = 20$ more than 60% of the instances could be solved for $n \in \{50, 100, 200\}$. The trend indicates that with increasing k more and more instances can be solved and finally after some threshold all instances can be solved to optimality by the heuristic in the root node. At the same time the BB method struggles to solve instances for small k (especially $k = 4$), in which case again the already existing methods from the min-max-min literature should be preferred. On the other hand the performance of the existing methods drops with increasing k which motivates the use of our BB method.

7 Conclusion

In this work we study the min-max-min robust problem for combinatorial optimization problems with uncertain costs. We focus on the case when the number of calculated solutions k is smaller than but close to the dimension n of the problem. We show that surprisingly if k is close to n , the min-max-min problem remains theoretically tractable and can be solved exactly and approximately in polynomial time for a large set of combinatorial problems if some problem parameters are fixed. Furthermore this theoretical result is supported by the derived branch & bound method which is able to solve instances very efficiently if k gets closer to n while struggling on instances with very small k . Nevertheless the generality of the method, scaling well for increasing k , motivates its use. Additionally in this work the first explicit pseudopolynomial algorithm for the case $k \geq n$ is derived. Despite its theoretical impact we show that the algorithm cannot compete with the already known column-generation algorithm.

This work demonstrates and extends the knowledge about the difficulty of the min-max-min

problem algorithmically and theoretically. While the cases where k is small or close to n are now well understood a future direction could be to study the problem for the case where the number of solutions lies in between 1 and n , namely when $k = \beta n$ for a fixed parameter $\beta \in (0, 1)$. Furthermore it would be interesting if results similar to this work can be derived for discrete uncertainty sets.

References

- [1] D. Adjiashvili, S. Stiller, and R. Zenklusen. Bulk-robust combinatorial optimization. *Mathematical Programming*, 149(1):361–390, 2015.
- [2] H. Aissi, C. Bazgan, and D. Vanderpooten. Min-max and min-max regret versions of combinatorial optimization problems: A survey. *European Journal of Operational Research*, 197(2):427–438, 2009.
- [3] C. Al Dahik, Z. Al Masry, S. Chrétien, J.-M. Nicod, and L. Rabehasaina. A frank-wolfe based algorithm for robust discrete optimization under uncertainty. In *2020 Prognostics and Health Management Conference (PHM-Besançon)*, pages 247–252. IEEE, 2020.
- [4] A. Arslan, M. Poss, and M. Silva. Min-max-min robust combinatorial optimization with few recourse solutions. 2020.
- [5] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust optimization*. Princeton university press, 2009.
- [6] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.
- [7] A. Ben-Tal, E. Hazan, T. Koren, and S. Mannor. Oracle-based robust optimization via online learning. *Operations Research*, 63(3):628–638, 2015.
- [8] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.
- [9] A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1):1–13, 1999.
- [10] D. Bertsimas and C. Caramanis. Finite adaptability in multistage linear optimization. *IEEE Transactions on Automatic Control*, 55(12):2751–2766, 2010.
- [11] D. Bertsimas and M. Sim. Robust discrete optimization and network flows. *Math. Program.*, 98(1-3):49–71, 2003.
- [12] D. Bertsimas and M. Sim. The price of robustness. *Operations Research*, 52(1):35–53, 2004.
- [13] J. R. Birge and F. Louveaux. *Introduction to stochastic programming*. Springer Science & Business Media, 2011.
- [14] M. Bruglieri, M. Ehrgott, H. W. Hamacher, and F. Maffioli. An annotated bibliography of combinatorial optimization problems with fixed cardinality constraints. *Discrete Applied Mathematics*, 154(9):1344–1357, 2006.
- [15] C. Buchheim, M. De Santis, F. Rinaldi, and L. Trieu. A frank-wolfe based branch-and-bound algorithm for mean-risk optimization. *Journal of Global Optimization*, 70(3):625–644, 2018.

- [16] C. Buchheim and J. Kurtz. Min-max-min robustness: a new approach to combinatorial optimization under uncertainty based on multiple solutions. *Electronic Notes in Discrete Mathematics*, 52:45–52, 2016.
- [17] C. Buchheim and J. Kurtz. Min–max–min robust combinatorial optimization. *Mathematical Programming*, 163(1):1–23, 2017.
- [18] C. Buchheim and J. Kurtz. Complexity of min–max–min robustness for combinatorial optimization under discrete uncertainty. *Discrete Optimization*, 28:1–15, 2018.
- [19] C. Buchheim and J. Kurtz. Robust combinatorial optimization under convex and discrete cost uncertainty. *EURO Journal on Computational Optimization*, 6(3):211–238, 2018.
- [20] E. M. Cepolina. A methodology for defining building evacuation routes. *Civil engineering and environmental systems*, 22(1):29–47, 2005.
- [21] A. Chassein and M. Goerigk. On the complexity of min–max–min robustness with two alternatives and budgeted uncertainty. *Discrete Applied Mathematics*, 2020.
- [22] A. B. Chassein, M. Goerigk, J. Kurtz, and M. Poss. Faster algorithms for min-max-min robustness for combinatorial problems with budgeted uncertainty. *European Journal of Operational Research*, 279(2):308–319, 2019.
- [23] M. Claus and M. Simmoteit. A note on $\sigma 2p$ -completeness of a robust binary linear program with binary uncertainty set. *Operations Research Letters*, 48(5):594–598, 2020.
- [24] A. Crema. Min max min robust (relative) regret combinatorial optimization. *Mathematical Methods of Operations Research*, 92(2):249–283, 2020.
- [25] L. Eufinger, J. Kurtz, C. Buchheim, and U. Clausen. A robust approach to the capacitated vehicle routing problem with uncertain costs. *INFORMS Journal on Optimization*, 2(2):79–95, 2020.
- [26] M. Fischetti and M. Monaci. Light robustness. In *Robust and online large-scale optimization*, pages 61–84. Springer, 2009.
- [27] M. Goerigk, J. Kurtz, and M. Poss. Min–max–min robustness for combinatorial problems with discrete budgeted uncertainty. *Discrete Applied Mathematics*, 285:707–725, 2020.
- [28] J. Goh and M. Sim. Distributionally robust optimization and its tractable approximations. *Operations research*, 58(4-part-1):902–917, 2010.
- [29] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012.
- [30] Gurobi Optimization, LLC. Gurobi optimizer reference manual, 2021.
- [31] G. A. Hanasusanto, D. Kuhn, and W. Wiesemann. K-adaptability in two-stage robust binary programming. *Operations Research*, 63(4):877–891, 2015.
- [32] N. Ho-Nguyen and F. Kılınç-Karzan. Exploiting problem structure in optimization under uncertainty via online convex optimization. *Mathematical Programming*, 177(1):113–147, 2019.
- [33] M. Jaggi. Revisiting frank-wolfe: Projection-free sparse convex optimization. In *International Conference on Machine Learning*, pages 427–435. PMLR, 2013.

- [34] N. Kämmerling and J. Kurtz. Oracle-based algorithms for binary two-stage robust optimization. *Computational Optimization and Applications*, 77(2):539–569, 2020.
- [35] P. Kouvelis and G. Yu. *Robust discrete optimization and its applications*, volume 14. Springer Science & Business Media, 2013.
- [36] C. Liebchen, M. Lübbecke, R. H. Möhring, and S. Stiller. Recoverable robustness. *ARRIVAL-Project-Technical Report*, 66, 2007.
- [37] Y. Liu, H. Lei, D. Zhang, and Z. Wu. Robust optimization for relief logistics planning under uncertainties in demand and transportation time. *Applied Mathematical Modelling*, 55:262–280, 2018.
- [38] J. Marques, M. C. Cunha, J. Sousa, and D. Savić. Robust optimization methodologies for water supply systems design. *Drinking Water Engineering and Science*, 5(1):31–37, 2012.
- [39] G. Mishra, S. Mazumdar, and A. Pal. Improved algorithms for the evacuation route planning problem. In *Combinatorial Optimization and Applications*, pages 3–19. Springer, 2015.
- [40] V. Pillac, P. Van Hentenryck, and C. Even. A conflict-based path-generation heuristic for evacuation planning. *Transportation Research Part B: Methodological*, 83:136–150, 2016.
- [41] K. Postek, D. Den Hertog, J. Kind, and C. Pustjens. Adjustable robust strategies for flood protection. *Omega*, 82:142–154, 2019.
- [42] K. Postek and S. Shtern. First-order algorithms for robust optimization problems via convex-concave saddle-point lagrangian reformulation. *arXiv preprint arXiv:2101.02669*, 2021.
- [43] M. Rungta, G. J. Lim, and M. Baharnemati. Optimal egress time calculation and path generation for large evacuation networks. *Annals of Operations Research*, 201(1):403–421, 2012.
- [44] M. Skutella and W. Welz. Route planning for robot systems. In *Operations research proceedings 2010*, pages 307–312. Springer, 2011.
- [45] A. Subramanyam, C. E. Gounaris, and W. Wiesemann. K-adaptability in two-stage mixed-integer robust optimization. *Mathematical Programming Computation*, pages 193–224, 2020.
- [46] W. Wiesemann, D. Kuhn, and M. Sim. Distributionally robust convex optimization. *Operations Research*, 62(6):1358–1376, 2014.

8 Appendix

Table 1 shows the following values (averaged over all problem instances) from left to right: the dimension of the knapsack problem, the budget parameter Γ , the applied method, the applied step size, the runtime in seconds, the number of performed iterations, the average number of iterations performed by the FW algorithm. All values were rounded to two decimal places.

Tables 2, 3 and 4 show the following values (averaged over all problem instances) from left to right: the number of nodes in the graph for the shortest path problem or the dimension of the knapsack problem; the budget parameter Γ ; the number of solutions k ; the number of instances which could be solved during timelimit; the runtime in seconds averaged only over all instances which could be solved during timelimit; the number of processed branch & bound nodes; the root gap in %, i.e. the percental difference between the objective value of the calculated solution and the lower bound in the root node of the BB tree; the optimality gap in %, i.e. the percental difference between the objective value of the best known solution and the lower bound after the timelimit. All values were rounded to two decimal places.

n	Γ	method	step size	t	# iterations	# iterations FW
50	2	CG	-	0.02	4.0	-
50	2	SG	0.01	357.79	107.8	708.45
50	2	SG	0.05	177.92	34.1	571.62
50	2	SG	$t^{-\frac{1}{2}}$	460.86	177.9	600.35
50	2	SG	$t^{-\frac{3}{4}}$	358.79	128.8	620.43
50	2	SG	t^{-1}	351.43	83.3	742.04
100	5	CG	-	0.48	15.9	-
100	5	SG	0.01	5652.09	120.2	2059.33
100	5	SG	0.05	4408.31	66.3	1574.73
100	5	SG	$t^{-\frac{1}{2}}$	8329.38	217.4	1841.7
100	5	SG	$t^{-\frac{3}{4}}$	9621.69	300.8	1910.66
100	5	SG	t^{-1}	7142.89	160.3	2268.21

Table 1: Computational results of SG and CG for the knapsack problem.

$ V $	Γ	k	# solved	t	# nodes	root-gap (%)	opt-gap (%)
20	3	2	95	278.1	2152.91	4.48	0.3
20	3	4	23	759.36	10636.98	0.63	0.16
20	3	6	63	82	4572.87	0.13	0.02
20	3	8	94	99.97	851.45	0.02	0
20	3	10	100	19.07	34.74	0	0
20	3	12	100	0.47	1	0	0
20	3	14	100	0.53	1	0	0
20	3	16	100	0.6	1	0	0
20	3	18	100	0.68	1	0	0
20	3	20	100	0.74	1	0	0
20	6	2	75	993.4	6101.55	9.84	2.07
20	6	4	2	0.16	12494.21	1.96	0.84
20	6	6	15	5.86	9772.18	0.54	0.16
20	6	8	54	235.62	4743.42	0.14	0.04
20	6	10	82	68.61	1670.33	0.04	0.01
20	6	12	94	116.25	582.12	0.01	0
20	6	14	98	59.82	177.38	0	0
20	6	16	100	9.87	8.8	0	0
20	6	18	100	0.97	1	0	0
20	6	20	100	1.06	1	0	0
30	3	2	33	1443.06	8513.32	7.95	4
30	3	4	3	1.81	8788.83	1.66	0.8
30	3	6	21	39.94	5750.57	0.59	0.2
30	3	8	41	336.62	3466.22	0.18	0.07
30	3	10	71	301.41	1295.98	0.05	0.02
30	3	12	82	76.98	627.69	0.02	0.01
30	3	14	89	100.44	352.55	0	0
30	3	16	97	13.26	76.86	0	0
30	3	18	99	2.65	20.43	0	0
30	3	20	100	3.01	1	0	0
30	6	2	4	1211.29	10411.49	15.47	10.32
30	6	4	0	7200	7277.34	4.85	2.66
30	6	6	4	4.48	5452.29	1.64	0.81
30	6	8	7	6	4622.01	0.75	0.35
30	6	10	20	643.32	3161.97	0.35	0.17
30	6	12	36	557.38	2125.72	0.16	0.08
30	6	14	58	240.64	1129.94	0.05	0.03
30	6	16	76	100.83	526.39	0.02	0.01
30	6	18	87	9.12	240.26	0.01	0
30	6	20	94	42.04	105.15	0	0

Table 2: Computational results of the branch & bound method for the shortest path problem.

$ V $	Γ	k	# solved	t	# nodes	root-gap (%)	opt-gap (%)
40	3	2	3	1505.9	6562.02	10.63	6.68
40	3	4	0	7200	3995.19	2.97	1.54
40	3	6	4	50.36	2820.19	1.13	0.51
40	3	8	18	926.98	2466.98	0.49	0.21
40	3	10	32	410.75	1580.77	0.2	0.09
40	3	12	49	297.64	883.71	0.09	0.05
40	3	14	71	284.46	434.6	0.04	0.02
40	3	16	80	51.79	204.27	0.02	0.01
40	3	18	86	14.06	117.25	0.01	0.01
40	3	20	91	7.42	64.84	0	0
40	6	2	0	7200	5419.42	19.71	13.72
40	6	4	0	7200	3118.55	7.19	4.58
40	6	6	0	7200	2244.44	3.1	1.9
40	6	8	0	7200	1468.32	1.61	1.02
40	6	10	1	2.75	1844.09	0.86	0.54
40	6	12	13	754.45	1286.65	0.49	0.32
40	6	14	20	440.87	962.1	0.27	0.19
40	6	16	33	324.57	666.71	0.15	0.11
40	6	18	45	82.13	455.86	0.08	0.06
40	6	20	58	61.57	295.99	0.04	0.03
50	3	2	0	7200	5553.62	11.88	7.94
50	3	4	1	0.92	2874.36	3.98	2.27
50	3	6	2	14.53	1286.76	1.68	0.93
50	3	8	7	212.42	952.06	0.79	0.44
50	3	10	13	338.12	602.68	0.45	0.26
50	3	12	23	174.77	704.89	0.22	0.13
50	3	14	37	476.45	481.27	0.12	0.07
50	3	16	48	339.01	329.72	0.05	0.03
50	3	18	61	137.17	196.5	0.03	0.02
50	3	20	71	68.2	125.02	0.01	0.01
50	6	2	0	7200	2748.7	22.73	15.83
50	6	4	0	7200	1321.67	9.27	6.44
50	6	6	0	7200	1019.67	4.49	3.01
50	6	8	1	186.22	749.36	2.48	1.71
50	6	10	2	664.75	596.81	1.49	1.03
50	6	12	2	15.93	423.85	0.93	0.69
50	6	14	5	16.85	429.04	0.61	0.47
50	6	16	11	197.97	431.34	0.38	0.29
50	6	18	13	433.38	365.68	0.24	0.18
50	6	20	21	346.38	292.2	0.14	0.12

Table 3: Computational results of the branch & bound method for the shortest path problem.

n	Γ	k	# solved	t	# nodes	root-gap (%)	opt-gap (%)
50	2	2	100	15.29	1068.6	1.1	0
50	2	4	70	0.01	4210.6	0.05	0.02
50	2	6	100	0.01	1	0	0
50	2	8	100	0.01	1	0	0
50	2	10	100	0.01	1	0	0
50	2	12	100	0.01	1	0	0
50	2	14	100	0.01	1	0	0
50	2	16	100	0.01	1	0	0
50	2	18	100	0.01	1	0	0
50	2	20	100	0.01	1	0	0
100	5	2	10	455.02	6190.8	3.02	1.33
100	5	4	0	7200	3948.7	0.72	0.44
100	5	6	10	0.36	3596	0.14	0.13
100	5	8	40	0.34	1225.1	0.02	0.02
100	5	10	100	0.12	1	0	0
100	5	12	100	0.12	1	0	0
100	5	14	100	0.12	1	0	0
100	5	16	100	0.13	1	0	0
100	5	18	100	0.13	1	0	0
100	5	20	100	0.13	1	0	0
200	10	2	0	7200	948.7	3.07	1.64
200	10	4	0	7200	608.6	1.52	0.95
200	10	6	0	7200	576.9	0.79	0.63
200	10	8	0	7200	558.6	0.42	0.35
200	10	10	0	7200	704.7	0.21	0.21
200	10	12	20	0.45	498.5	0.1	0.08
200	10	14	20	0.49	493.5	0.04	0.04
200	10	16	60	1.18	192.3	0.02	0.02
200	10	18	70	1.34	135.8	0.01	0.01
200	10	20	70	1.4	134.6	0	0
400	20	2	0	7200	732.1	2.99	1.67
400	20	4	0	7200	476.3	1.64	0.96
400	20	6	0	7200	541.8	1.01	0.66
400	20	8	0	7200	528.9	0.59	0.47
400	20	10	0	7200	521.5	0.36	0.32
400	20	12	0	7200	513.9	0.22	0.21
400	20	14	0	7200	504.3	0.12	0.12
400	20	16	10	1.12	446.9	0.08	0.08
400	20	18	20	1.02	389.3	0.05	0.05
400	20	20	30	1.06	323	0.03	0.03

Table 4: Computational results of the branch & bound method for the knapsack problem.