

On the Formulation Dependence of Convex Hull Pricing

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Convexhull pricing provides a potential solution for reducing out-of-market payments in wholesale electricity markets. This paper revisits the theoretical construct of convexhull pricing and explores its important but underappreciated formulation-dependence property. Namely, convexhull prices may change for different formulations of the same unit commitment problem. After a conceptual exposition of the property, its practical significance is illustrated with two reformulations commonly observed in the market clearing process. Sufficient conditions under which convexhull prices will be preserved by a reformulation are also explored. These findings contribute to a better understanding of convexhull pricing and demonstrate the importance of continued theoretical research into the method.

Key words: convexhull price, electricity markets, formulation dependence, unit commitment, uplift payment

1. Introduction

Non-convexity is an inherent feature of wholesale electricity markets due to the presence of binary unit commitment decisions. While modern optimization techniques allow for the solution of practical Unit Commitment (UC) problems, price formation remains challenging. Unlike a convex market clearing problem where constraint shadow prices *support*¹ the cleared quantities and therefore form market clearing prices, a non-convex problem typically does not have a price that supports the cleared quantities. As a result, Independent System Operators (ISOs) have traditionally relied on a combination of *fixed-commitment pricing*, where Locational Marginal Prices (LMPs) are derived from a convex economic dispatch problem with commitment decisions fixed at their optimal values, and out-of-market *uplift payments* such as Make-Whole Payments (MWPs) and Lost Opportunity Costs (LOCs)² to support the

cleared quantities. These out-of-market payments are nontransparent, obscure economic signals, and raise stakeholder concerns.

As an alternative to fixed-commitment pricing, the convex hull pricing concept emerged more than a decade ago to address the out-of-market payments (Gribik et al. 2007, MISO 2011). Under convex hull pricing, the market price is the change of the *convex hull* value of the optimal UC cost with respect to a load perturbation. The mathematical foundation for convex hull pricing can be traced to duality theory developed in the 1960s and 1970s (Rockafellar 1970, Shapiro 1971, Fisher and Shapiro 1974, Geoffrion 1974), with convex hull prices being the optimal dual variables of a UC problem. However, the economic foundation of convex hull pricing is relatively new. Early investigations into the economic properties of the dual variables of integer programs in the 1960s met with limited success (Gomory and Baumol 1960). More recently, a minimum uplift pricing idea was explored for electricity markets (Ring 1995, Madrigal 2000), but it was not until Gribik et al. (2007) that a rigorous relationship between the duality gap and the minimum uplift payment was established under certain conditions.

The minimum uplift interpretation of the convex hull price triggered interest from both academia and industry, with the primary research focus being the efficient computation of convex hull prices or, equivalently, the solution of the Lagrangian dual problem. Despite the concaveness of the dual function, conventional algorithms such as the subgradient method often exhibit slow convergence for the dual of large UC problems. Therefore, different methods such as a surrogate simplex cutting-plane method (Wang et al. 2013), an extreme-point subdifferential method (Wang et al. 2013a, b), and a Dantzig-Wolfe decomposition method (Andrianesis et al. 2020) have been proposed for the UC dual problem. Different formulations of the convex hull pricing problem have also been explored. Based on theoretical results in Falk (1969), a primal formulation of convex hull pricing was obtained in Schiro et al. (2015) and Hua and Baldick (2016), with a modified Benders decomposition being proposed as a solution method in Knueven et al. (2019). A network flow-derived formulation and Bienstock–Zuckerberg-Based solution algorithm were proposed in Álvarez et al. (2020). The idea of relaxing binary commitment variables (i.e., integer

relaxation) and conditions under which the integer-relaxed UC problem will result in the convex hull price were explored in Chao (2019). A linear program convex hull pricing formulation obtained from the integer relaxation of a dynamic-programming-based UC formulation was presented in Yu et al. (2019).

Instead of focusing on computational issues, this paper revisits the theoretical foundation of convex hull pricing and explores its formulation-dependence property. This property has not been rigorously studied but has important implications. To this end, it is concluded that well-intended UC problem reformulations for improving computational efficiency may inadvertently affect convex hull prices and thus reduce the price transparency. This formulation-dependence property is illustrated by two small but practical examples. Furthermore, sufficient conditions for preserving the convex hull price under different reformulations are also derived.

The remainder of the paper is organized as follows. Section 2 summarizes the mathematical foundation and the economic interpretation of convex hull pricing. Section 3 demonstrates the dependence of convex hull prices on UC formulations through two examples and discusses the practical implications of this property. Section 4 then presents sufficient conditions under which different UC formulations will preserve the convex hull price. Section 5 tests the formulation dependence property on ISO-scale problems, and Section 6 concludes the paper.

2. The basics of convex hull pricing

Consider a general UC problem formulation:

$$\min_{x,y} f(x) \tag{1}$$

$$\text{s. t. } H(x,y) = 0 \tag{2}$$

$$G(y) \leq 0 \tag{3}$$

$$x \in X, \tag{4}$$

where vector x represents unit³ variables composed of continuous dispatch variables p and binary commitment variables u (i.e., $x = [p, u]^T$), set X represents the feasible region defined by *unit constraints* (e.g., dispatch range, minimum up/down times, and ramp constraints), vector y represents network variables (e.g., voltage phase angles and power flows), $f(x)$ represents the cost associated with the unit variables x (e.g., startup/no-load and dispatch costs), $H(x, y)$ represents nodal power balance equations that couple unit variables x and network variables y , and $G(y)$ represents *network constraints* (e.g., flow limits and N-1 security constraints).

In this paper, nodal power balance equations $H(x, y)$ are assumed linear and separable in x and y , i.e.,

$$H(x, y) = H_x(x) + H_y(y), \quad (5)$$

where $H_x(x)$ represents the nodal aggregation of unit powers⁴ and $H_y(y)$ represents the nodal aggregation of network flows. Moreover, X is assumed compact and regularity conditions are assumed where necessary. The above assumptions are reasonable for a practical UC problem. For simplicity, ancillary service products are not considered but the paper's results hold with their presence.⁵

Denote a vector of perturbations to the n nodal balance constraints by $d \in R^n$, i.e.,

$$H(x, y) = d, \quad (6)$$

and denote the optimal UC objective cost (1) for perturbation d by $v(d)$, i.e.,

$$v(d) = \min_x f(x), \text{ s.t. (3) -- (4) and (6)}. \quad (7)$$

With binary commitment variables, $v(d)$ may be nonconvex and/or discontinuous. The basic idea of convex hull pricing is to define the market price from the convex hull⁶ of $v(d)$, denoted by $v^c(d) \equiv \text{conv}(v(d))$. A convex hull $v^c(d)$ is illustrated in Figure 1. Note that the convex hull concept has a “global” feature: $v^c(0)$ at point D is related to points A and B rather than $v(0)$ at point P and its immediate neighborhood.

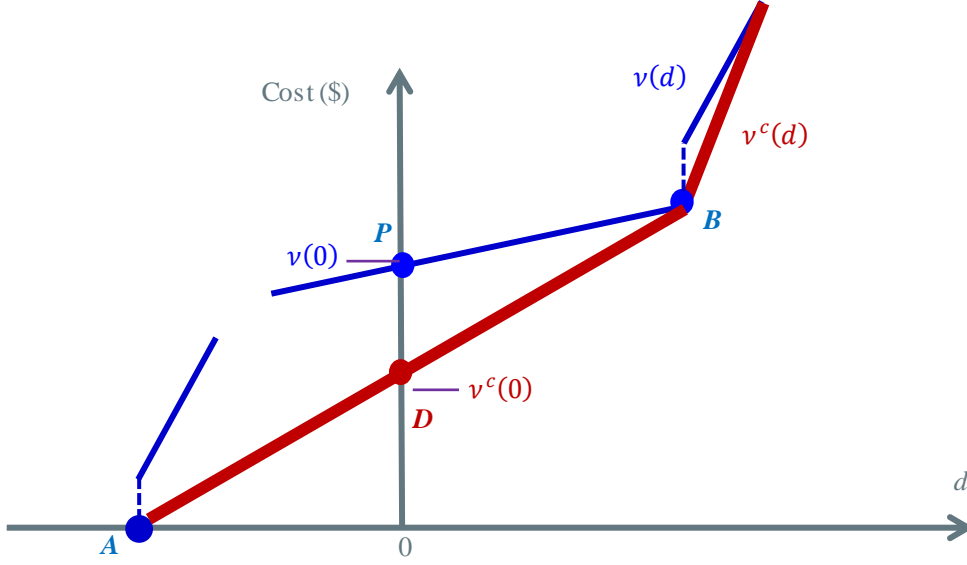


Figure 1. Illustration of a convex hull.

The subgradient of $v^c(d)$ defines the convex hull price. Specifically, the convex hull price denoted by LMP^c for the original problem (1)-(4) is defined as

$$LMP^c \equiv \partial v^c(d)|_{d=0}. \quad (8)$$

Note that LMP^c may depend on $v(d)$ at perturbation levels other than $d = 0$.

Mathematically, convex hull $v^c(d)$ can be constructed as the bi-conjugate of $v(d)$ (Rockafellar 1970):

$$v^c(d) = v^{**}(d) = \max_{\lambda} \left\{ d^T \lambda - \max_{d'} \{ \lambda^T d' - v(d') \} \right\}. \quad (9)$$

Substituting (7) into (9),

$$\begin{aligned} v^c(d) &= \max_{\lambda} \left\{ d^T \lambda - \max_{d'} \left\{ \lambda^T d' - \min_{\substack{x \in X, y \\ H(x, y) = d' \\ G(y) \leq 0}} f(x) \right\} \right\} \\ &= \max_{\lambda} \min_{d'} \min_{\substack{x \in X, y \\ H(x, y) = d' \\ G(y) \leq 0}} \{ \lambda^T (d - d') + f(x) \} \end{aligned}$$

$$= \max_{\lambda} \min_{\substack{x \in X, y \\ G(y) \leq 0}} \{f(x) - \lambda^T(H(x, y) - d)\}. \quad (10)$$

The max-min problem (10) is the Lagrangian dual problem of the perturbed UC problem (1), (3)-(4) and (6) with λ relaxing the perturbed nodal balance constraint (6). Based on (10), the subgradient of $v^c(d)$ is the optimal dual variable $\lambda^*(d)$, i.e.,

$$\partial v^c(d) = \lambda^*(d), \text{ with } \lambda^*(d) = \arg \max_{\lambda} \min_{\substack{x \in X, y \\ G(y) \leq 0}} \{f(x) - \lambda^T(H(x, y) - d)\}. \quad (11)$$

From (8) and (11), the *convex hull price* for the original UC problem (1)-(4) can be obtained as the optimal dual variable of the nodal balance constraint (2), i.e.,

$$LMP^c = \lambda^*(0) = \arg \max_{\lambda} \min_{\substack{x \in X, y \\ G(y) \leq 0}} \{f(x) - \lambda^T H(x, y)\}. \quad (12)$$

Without a transmission network, it was proven in Gribik et al. (2007) that the duality gap of a UC problem is also the minimum out-of-market “*uplift payment*” needed for the market price to support cleared quantities, where the uplift for price λ is defined as the difference between market participants’ maximum surplus⁷ from self-scheduling and the surplus from the optimal market schedule $x^*(0)$,⁸ i.e.,

$$U^M(\lambda) \equiv \max_{x \in X} \{\lambda^T H_x(x) - f(x)\} - (\lambda^T H_x(x^*(0)) - f(x^*(0))). \quad (13)$$

With network constraints, an additional *network uplift* needs to be introduced to connect the total uplift payment with the duality gap, i.e., the network uplift in response to the price λ is

$$U^N(\lambda) \equiv \max_{\substack{y \\ G(y) \leq 0}} \lambda^T H_y(y) - \lambda^T H_y(y^*(0)), \quad (14)$$

where the maximum term can be viewed as a hypothetical *for-profit* network operator’s maximum profit from self-scheduling, and the second term represents the profit from the optimal network schedule $y^*(0)$.

Based on (13) and (14), it can be derived that the total uplift payment

$$U^{Total}(\lambda) = U^M(\lambda) + U^N(\lambda) = v(0) - \min_{\substack{x \in X, y \\ G(y) \leq 0}} \{f(x) - \lambda^T H(x, y)\}. \quad (15)$$

Minimizing the total uplift in (15) yields

$$\begin{aligned} \min_{\lambda} U^{Total}(\lambda) &= \min_{\lambda} \left\{ v(0) - \min_{\substack{x \in X, y \\ G(y) \leq 0}} \{f(x) - \lambda^T H(x, y)\} \right\} \\ &= v(0) - \max_{\lambda} \min_{\substack{x \in X, y \\ G(y) \leq 0}} \{f(x) - \lambda^T H(x, y)\} \\ &= v(0) - v^c(0) = \text{Duality Gap}. \end{aligned} \quad (16)$$

Based on (16), the price that minimizes the total uplift is the convex hull price defined in (12), thus providing an economic interpretation for the convex hull price.

It is important to note that the for-profit network operator assumed for the network uplift definition does not exist in existing electricity markets. As a result, the *actual* total uplift may not equal the duality gap undermining the economic foundation of convex hull pricing. Although new pricing ideas such as minimizing the weighted sum of network and participant uplifts (Garcia 2019) were explored, more research on the economic foundation of convex hull pricing is needed.

3. Formulation dependence of convex hull pricing

In Section 2, the convex hull price for a general UC problem is defined as the optimal dual variables of the nodal balance constraints. This definition imposes few restrictions on the UC formulation. However, a UC problem can be formulated with different sets of variables and constraints. For instance, commitment logic can be represented by different mathematical models (Knueven et al. 2018) and only a subset of security constraints are included in the UC formulation in practice. Therefore, a natural question to ask is *whether the UC formulation affects the convex hull price*. The answer has important implications since

ISOs commonly use UC reformulations to improve computational performance, and a formulation-dependent convex hull price creates new market transparency concerns.

At their core, UC reformulations preserve the primal optimal solution. However, preserving the primal solution does not guarantee the preservation of the dual solution or the convex hull price. This result can be intuitively understood from Figure 1. For a given UC problem (1)-(4), the nodal perturbation d is 0. While different UC formulations should yield the same primal optimal cost $v(0)$ at point P , they may yield different values of $v(d)$ at $d \neq 0$. As discussed in Section 2, the values of $v(d)$ at $d \neq 0$ may affect the convex hull price at $d = 0$ (i.e., the “global” feature of convex hull). It follows that different UC formulations may result in different convex hull prices at $d = 0$.

The following subsections present two small examples to illustrate the dependence of convex hull pricing on the UC formulation. Example 1 was inspired by formulation tightening efforts meant to reduce computational time, and Example 2 was inspired by ISOs’ common security analysis practice.

3.1 Impact of UC formulation tightening

The UC problem is a Mixed Integer Program (MIP). Despite advances in MIP algorithms and computing power, solving large-scale UC problems remains a challenge. To reduce the MIP solution time, many studies have focused on tightening the UC formulation (Morales-Espana et al. 2013, 2014, 2015; Yan et al. 2020) by introducing additional constraints that better represent the problem feasibility region. Also, additional constraints that reflect *a priori* knowledge of the optimal solution may be added by the ISO. These additional constraints typically do not affect the fixed-commitment pricing but may change the convex hull of the optimal UC cost and thus affect the convex hull price. Example 1 shows that constraint tightening can alter the convex hull price of the UC problem.

Example 1. Consider a two-unit one-hour unit commitment problem. The unit parameters are listed in Table 1. The system demand is 210MW. Transmission is not considered.

Table 1. Unit parameters for Example 1

<i>Unit</i>	p_{\min} (MW)	p_{\max} (MW)	<i>Incremental Cost</i> (\$/MWh)	<i>Commitment Cost</i> (\$)
1	0	200	10	0
2	50	50	20	0

The unit commitment problem can be formulated as follows.

$$\begin{aligned}
& \min_{p_1, u_1, p_2, u_2} 10p_1 + 20p_2 \\
& \text{s. t.} \quad p_1 + p_2 = 210 \quad (\lambda) \quad \text{Power balance} \\
& \quad \quad 0 \cdot u_1 \leq p_1 \leq 200u_1, \quad u_1 \in \{0,1\}, \quad \text{Unit 1 constraints} \quad (17) \\
& \quad \quad 50u_2 \leq p_2 \leq 50u_2, \quad u_2 \in \{0,1\}, \quad \text{Unit 2 constraints}
\end{aligned}$$

where u_1 and u_2 , respectively, represent the commitment decisions of Unit 1 and Unit 2, and p_1 and p_2 represent the dispatch levels of the two units. Since no single unit can meet the demand, both units are committed. The block-loaded Unit 2 will be dispatched at 50MW, and Unit 1 will be dispatched to meet the remaining load of 160MW (= 210–50). The total generation cost is \$2,600 (= 160 × 10 + 50 × 20). The unit commitment solution is listed in Table 2.

Table 2. Unit commitment solution for Example 1

<i>Unit</i>	<i>Dispatch</i> (MW)	<i>Commitment Status</i>
1	160	On
2	50	On
Total Generation Cost: \$2,600		

The convex hull price is obtained from the dual problem of (17), i.e.,

$$\begin{aligned}
& \max_{\lambda} \min_{\substack{p_1, u_1, p_2, u_2 \\ 0u_1 \leq p_1 \leq 200u_1 \\ 50u_2 \leq p_2 \leq 50u_2 \\ u_1, u_2 \in \{0,1\}}} \{10p_1 + 20p_2 + \lambda(210 - p_1 - p_2)\}. \quad (18)
\end{aligned}$$

The dual problem (18) can be solved with the convex hull price $\lambda^* = \$20/\text{MWh}$ and an optimal dual cost of \$2200. The duality gap is \$400 ($= 2600 - 2200$), which reflects the LOC of the less expensive Unit 1 not being fully dispatched to 200MW. For comparison, the fixed-commitment price is \$10/MWh with an uplift of \$500, which reflects the MWP of the more expensive Unit 2. The pricing results are listed in Table 3. It can be seen that the convex hull price leads to a smaller uplift payment than the fixed-commitment price.

Table 3. Pricing results for Example 1 without tightening constraint

<i>Original UC formulation</i>	<i>Market Price (\$/MWh)</i>	<i>Uplift Payment (\$)</i>
Fixed-commitment pricing (<i>LMP</i>)	10	500
Convexhull pricing (<i>LMP^c</i>)	20	400

Now consider a reformulation that tightens the UC formulation (17). Suppose that a heuristic method identifies that Unit 2 must be committed and the constraint $u_2 \geq 1$ is added to the UC formulation. This additional constraint does not change the feasible region of the UC problem and thus results in the commitment and dispatch solution in Table 2. The convex hull price for the reformulated UC problem with the tightening constraint $u_2 \geq 1$ can be obtained from the following dual problem:

$$\max_{\lambda} \min_{\substack{p_1, u_1, p_2, u_2 \\ 0u_1 \leq p_1 \leq 200u_1 \\ 50u_2 \leq p_2 \leq 50u_2 \\ u_1, u_2 \in \{0,1\} \\ u_2 \geq 1}} \{10p_1 + 20p_2 + \lambda(210 - p_1 - p_2)\}. \quad (19)$$

Solving (19) yields the convex hull price $\lambda^* = \$10/\text{MWh}$ for the tightened UC formulation, with Unit 2 requiring a \$500 MWP to recover its bid-in cost. It can be verified that the fixed-commitment price for the reformulated UC problem is also \$10/MWh. These pricing results are listed in Table 4.

Table 4. Pricing results for Example 1 with tightening constraint

<i>Tightened UC reformulation</i>	<i>Market Price (\$/MWh)</i>	<i>Uplift Payment (\$)</i>
Fixed-commitment pricing (<i>LMP</i>)	10	500
Convexhull pricing (<i>LMP^c</i>)	10	500

Comparing Table 3 with Table 4, it can be seen that the tightened formulation did not change the fixed-commitment price. However, the convex hull prices are different. The fundamental reason for this difference is that the tightening constraint $u_2 \geq 1$, while not affecting the optimal solution of the UC problem (17) with a 210MW load, will affect the optimal solution when load falls below 200MW when the commitment of Unit 2 is unnecessary. Due to the global feature of the convex hull price illustrated in Figure 1, the additional constraint affects the optimal cost value at other load/perturbation levels and thus affects the convex hull price for (17).

3.2 Impact of non-binding security constraints

Practical UC problems often involve transmission security constraints and are known as Security Constrained Unit Commitment (SCUC) problems. Due to the large number of potential N-1 contingencies, a screening process is commonly used to limit the number of security constraints included in the SCUC problem. The process analyzes contingency power flows for a given SCUC solution and identifies the violated and near-violation constraints for the next SCUC iteration. The final solution from this iterative process satisfies all security constraints even though only a small subset of them was formally enforced. The identified security constraints are included in the ISO's pricing problem.

Obviously, security constraints that have not been identified in the screening process for $d = 0$ may be binding or violated at load perturbations of $d \neq 0$. Therefore, their presence or absence in the UC formulation can change the shape of $v(d)$ in Figure 1. As a result, the convex hull $v^c(d)$ may be altered and the convex hull price of the original SCUC problem at $d = 0$ may be different. A similar observation was briefly mentioned in Gribik et al (2007). The following Example 2 shows that the presence of a non-binding security constraint in a SCUC problem can lead to different convex hull prices.

Example 2. Consider a two-bus two-unit one-hour unit commitment problem. The system topology is depicted in Figure 2 with two equal-impedance lines having individual capacities of 100MW. Contingencies are defined on both lines. The load and unit parameters are listed in Table 5.

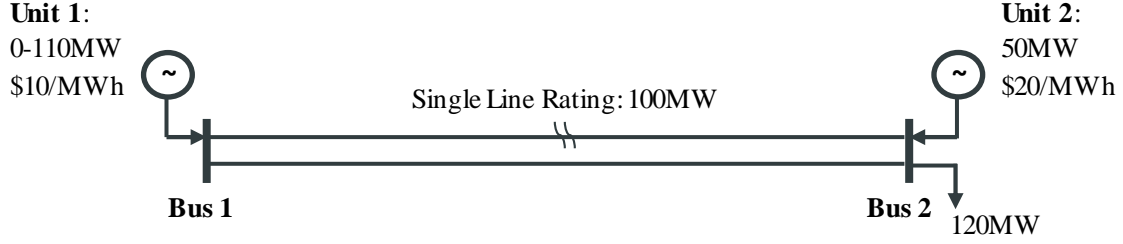


Figure 2. System topology for Example 2

Table 5. Load and unit parameters for Example 2

Unit	p_{\min} (MW)	p_{\max} (MW)	Incremental Cost (\$/MWh)	Commitment Cost (\$)
1	0	110	10	0
2	50	50	20	0
Bus 1 Load = 0 MW, Bus 2 Load = 120 MW				

Consider a typical iterative process for the SCUC problem. The initial solution without security constraints will commit both units since neither can satisfy the load alone. The block-loaded Unit 2 will produce 50MW and Unit 1 will produce the remaining load of 70MW ($= 120 - 50$). The flow on each transmission line from Bus 1 to Bus 2 will be 35MW ($= 70/2$), much less than the line capacity. N-1 contingency analysis will result in a post-contingency flow of 70MW on the remaining line, which is still below the line capacity. Thus, no security constraints will be identified for the final SCUC problem with flow from Node 1 to Node 2 being denoted by f_{12} :

$$\begin{aligned}
 & \min_{p_1, u_1, p_2, u_2, f_{12}} 10p_1 + 20p_2 \\
 & \text{s. t.} \quad p_1 - f_{12} = 0 \quad (\lambda_1) \quad \text{Node 1 balance} \\
 & \quad \quad f_{12} + p_2 - 120 = 0 \quad (\lambda_2) \quad \text{Node 2 balance} \\
 & \quad \quad 0u_1 \leq p_1 \leq 110u_1, u_1 \in \{0,1\} \quad \text{Unit 1 constraints} \\
 & \quad \quad 50u_2 \leq p_2 \leq 50u_2, u_2 \in \{0,1\}. \quad \text{Unit 2 constraints}
 \end{aligned} \tag{20}$$

The optimal solution of (20) is summarized in Table 6.

Table 6. SCUC solution for Example 2

<i>Unit</i>	<i>Dispatch p (MW)</i>	<i>Commitment Status</i>
1	70	“On”
2	50	“On”
Total Generation Cost: \$1,700		

The convex hull prices for the SCUC formulation (20) are the optimal dual variables associated with the power balance constraints in the following dual problem:

$$\begin{aligned} \max_{\lambda_1, \lambda_2} \quad & \min_{p_1, u_1, p_2, u_2, f_{12}} \{10p_1 + 20p_2 + \lambda_1(f_{12} - p_1) + \lambda_2(120 - f_{12} - p_2)\}. \\ & 0u_1 \leq p_1 \leq 110u_1 \\ & 50u_2 \leq p_2 \leq 50u_2 \\ & u_1, u_2 \in \{0, 1\} \end{aligned} \quad (21)$$

Solving (21) yields $\lambda_1^* = \lambda_2^* = \$20/\text{MWh}$ with an optimal dual cost of \$1,300. The duality gap is \$400 (= 1700–1300), which reflects the LOC of the less expensive Unit 1 that is not fully dispatched to 110MW. For comparison, the fixed-commitment price is \$10/MWh at both buses with the expensive Unit 2 requiring a MWP of \$500. These pricing results are listed in Table 7.

Table 7. Pricing results for Example 2 without non-binding security constraint

<i>SCUC without non-binding constraint</i>	<i>Price at Bus 1 (\$/MWh)</i>	<i>Price at Bus 2 (\$/MWh)</i>	<i>Uplift Payment (\$)</i>
Fixed-commitment pricing (LMP)	10	10	500
Convexhull pricing (LMP ^c)	20	20	400

Now consider adding the non-binding security constraint in the SCUC problem, i.e.,

$$\begin{aligned} \min_{p_1, u_1, p_2, u_2, f_{12}} \quad & 10p_1 + 20p_2 \\ \text{s. t.} \quad & p_1 - f_{12} = 0 \quad (\lambda_1) \quad \text{Node 1 balance} \\ & f_{12} + p_2 - 120 = 0 \quad (\lambda_2) \quad \text{Node 2 balance} \\ & f_{12} \leq 100 \quad (\mu) \quad \text{Security constraint} \\ & 0u_1 \leq p_1 \leq 110u_1, \quad u_1 \in \{0, 1\} \quad \text{Unit 1 constraint} \end{aligned} \quad (22)$$

$$50u_2 \leq p_2 \leq 50u_2, u_2 \in \{0,1\}. \quad \text{Unit 2 constraint}$$

The new security constraint $f_{12} \leq 100$ does not affect the primal SCUC solution since it is not binding at the optimal solution. As a result, the primal solution of SCUC formulation (22) is the same as Table 6.

The convex hull price for the SCUC formulation (22) can be obtained from the following dual problem:

$$\max_{\lambda_1, \lambda_2} \min_{\substack{p_1, u_1, p_2, u_2, f_{12} \\ 0u_1 \leq p_1 \leq 110u_1 \\ 50u_2 \leq p_2 \leq 50u_2 \\ u_1, u_2 \in \{0,1\} \\ f_{12} \leq 100}} \{10p_1 + 20p_2 + \lambda_1(f_{12} - p_1) + \lambda_2(120 - f_{12} - p_2)\}. \quad (23)$$

Solving (23) yields the convex hull prices of $\lambda_1^* = \$10/\text{MWh}$ and $\lambda_2^* = \$20/\text{MWh}$. The duality gap is \$300 ($= 1700 - 1400$), which reflects the network uplift from the post-contingency line flow being below the line capacity of 100MW. It can be calculated that the fixed-commitment prices for (22) are \$10/MWh at both buses. The pricing results for (22) are listed in Table 8.

Table 8. Pricing results for Example 2 with non-binding security constraint

<i>SCUC with non-binding constraint</i>	<i>Price at Bus 1 (\$/MWh)</i>	<i>Price at Bus 2 (\$/MWh)</i>	<i>Uplift Payment (\$)</i>
Fixed-commitment pricing (<i>LMP</i>)	10	10	500
Convexhull pricing (<i>LMP^c</i>)	10	20	300

Comparing the pricing results in Tables 7 and 8, it can be seen that the presence of a non-binding transmission security constraint led to different convex hull prices. This observation raises important questions about convex hull pricing for SCUC problems. Should a SCUC formulation include binding, near-binding, or all security constraints for convex hull pricing? Moreover, it should be noted that the contingency list used in an ISO's security analysis process is based on engineering judgment and does not consider every possible transmission element contingency. Does convex hull pricing need to include unlikely contingencies since they may affect the convex hull price? The theoretical foundation of convex hull pricing does not provide an answer, but intuition seems to suggest that every possible transmission security constraint must be considered for convex hull pricing.

4. Reformulations that preserve the convex hull price

It was shown in Section 3 that the convex hull price may depend on the UC formulation. However, certain reformulations will not affect the convex hull price. This section explores some of these reformulations and formalizes the findings in two subsections. Subsection 4.1 reveals that, under certain conditions, the reformulation of unit and network constraints will not affect the convex hull price, and Subsection 4.2 extends the equivalence of different LMP representations to convex hull pricing.

4.1. Reformulation of unit and network constraints

For convenience, let C be the vector of individual unit costs, so the cost function $f_i(x_i)$ of Unit i can be replaced by C_i with the constraint $C_i \geq f_i(x_i)$ added to the unit's feasible set X_i . The general UC problem can then be represented as

$$\min_{C,p,u,y} e^T C \quad (24)$$

$$\text{s. t.} \quad H_x(p) + H_y(y) = 0 \quad (25)$$

$$G(y) \leq 0 \quad (26)$$

$$(C, p, u) \in X. \quad (27)$$

Note that $H_x(p)$ references variable p instead of x to indicate that commitment variables do not affect the form of nodal power balance constraints. The feasible set X has also been modified to include the new cost variable vector C .

With (24)-(27), different UC formulations can be represented by the choices of commitment variables u , network variables y , nodal flow aggregation form H_y , network constraints G , and unit constraints X .

Without loss of generality, consider a different UC formulation with $(u', y', H'_{y'}, G', X')$ as

$$\min_{c,p,u',y'} e^T C \quad (28)$$

$$\text{s. t.} \quad H_x(p) + H'_{y'}(y') = 0 \quad (29)$$

$$G'(y') \leq 0 \quad (30)$$

$$(C, p, u') \in X', \quad (31)$$

Note that $(u', y', H'_{y'}, G', X')$ in the UC formulation (28)-(31) may have different dimensions from (u, y, H_y, G, X) in the UC formulation (24)-(27). Below, conditions for the two UC formulations to yield the same convex hull prices are explored.

It was shown in Section 3 that the convex hull price at $d = 0$ depends on the values of $v(d)$ at other perturbation levels and that reformulations altering those values may result in different convex hull prices. Therefore, an intuitive condition for the two formulations of a UC problem to have the same convex hull price is that they yield identical $v(d)$ values for *every* perturbation level d . This intuition leads to Theorem 1. For convenience, denote $proj_{v_j}(V)$ as the projection of a vector set V onto the vector space of its component v_j , i.e., $proj_{v_j}(V) \equiv \{v_j | (v_j, v_{-j}) \in V\}$ with v_{-j} denoting the components other than v_j .

Theorem 1. The UC formulations (24)-(27) and (28)-(31) have the same convex hull price if

$$\text{conv} \left(proj_{(c,p)}(X) \right) = \text{conv} \left(proj_{(c,p)}(X') \right) \text{ and } \text{conv} \left(proj_q(S) \right) = \text{conv} \left(proj_q(S') \right),^9$$

where $S = \{(q, y) | q = H_y(y) \text{ and } G(y) \leq 0\}$ and $S' = \{(q, y') | q = H'_{y'}(y') \text{ and } G'(y') \leq 0\}$.

[Proof]. With λ being the multiplier for (25), the dual problem of (24)-(27) is

$$\max_{\lambda} \min_{\substack{(c,p,u) \in X, y \\ G(y) \leq 0}} \left\{ e^T C + \lambda^T (H_x(p) + H_y(y)) \right\}$$

$$\begin{aligned}
&= \max_{\lambda} \left\{ \min_{(C,p,u) \in X} \{e^T C + \lambda^T H_x(p)\} + \min_{G(y) \leq 0} \lambda^T H_y(y) \right\} \\
&= \max_{\lambda} \left\{ \min_{(C,p,u) \in X} \{e^T C + \lambda^T H_x(p)\} + \min_{(q,y) \in S} \lambda^T q \right\} \\
&= \max_{\lambda} \left\{ \min_{(C,p) \in \text{proj}_{(C,p)}(X)} \{e^T C + \lambda^T H_x(p)\} + \min_{q \in \text{proj}_q(S)} \lambda^T q \right\}, \tag{32}
\end{aligned}$$

where the first equality is due to separability of unit and network variables, the second equality applies the definition of set S , and the third equality uses the fact that the objective cost involves only C , p , and q .

Moreover, based on the result of Schiro et al. (2015), the sets $\text{proj}_{(C,p)}(X)$ and $\text{proj}_q(S)$ can be replaced by their convex envelopes without affecting the dual solution. Namely, the dual problem of (24)-(27) is equivalent to

$$\max_{\lambda} \left\{ \min_{(C,p) \in \text{conv}(\text{proj}_{(C,p)}(X))} \{e^T C + \lambda^T H_x(p)\} + \min_{q \in \text{conv}(\text{proj}_q(S))} \lambda^T q \right\} \tag{33}$$

Similarly, with λ' being the multiplier for (29), the dual problem of (28)-(31) can be represented as

$$\max_{\lambda'} \left\{ \min_{(C,p) \in \text{conv}(\text{proj}_{(C,p)}(X'))} \{e^T C + \lambda'^T H_x(p)\} + \min_{q \in \text{conv}(\text{proj}_q(S'))} \lambda'^T q \right\} \tag{34}$$

The dual problems (33) and (34) of the two formulations and the resulting convex hull prices λ^* and λ'^* are the same if $\text{conv}(\text{proj}_{(C,p)}(X)) = \text{conv}(\text{proj}_{(C,p)}(X'))$ and $\text{conv}(\text{proj}_q(S)) = \text{conv}(\text{proj}_q(S'))$.

[End of Proof].

The condition described in Theorem 1 ensures that the two UC formulations yield the same convex characterizations for each unit and the network. Namely, the same convex hull $v^c(d)$ for the two formulations is obtained under this sufficient condition. A less general but more intuitive condition that ensures identical $v(d)$ values for the two formulations is described next.

Corollary 1. The UC formulations (24)-(27) and (28)-(31) have the same convex hull price if

$$proj_{(c,p)}(X) = proj_{(c,p)}(X') \text{ and } proj_q(S) = proj_q(S'). \quad (35)$$

Corollary 1 is directly implied by Theorem 1. Under its condition, unit cost-dispatch feasible regions are identical and network feasible regions are unchanged. For any perturbation d , it follows that optimization over these resources (both the units and the network) will yield the same optimal cost $v(d)$. Thus, the sufficient condition in Corollary 1 is consistent with the intuition that preserving $v(d)$ at every d preserves convex hull price. It can also be shown that the sufficient condition in Corollary 1 is not satisfied by the Section 3 examples: $proj_{(c,p)}(X) \neq proj_{(c,p)}(X')$ in Example 1 and $proj_q(S) \neq proj_q(S')$ in Example 2.

The following Example 3 illustrates a realistic application of Corollary 1.

Example 3. Consider extending the one-hour two-unit problem in Example 1 to a two-hour problem with the second hour load of 180MW. The unit parameters are summarized in Table 9 with the introduction of Minimum Run Time (MRT) and Minimum Down Time (MDT). No transmission network is considered.

Table 9. Unit parameters for Example 3

Unit	p_{\min} (MW)	p_{\max} (MW)	Incremental Cost (\$/MWh)	Commitment Cost (\$)	Initial Status	MRT (hour)	MDT (hour)
1	0	200	10	0	Off	1	1
2	50	50	20	0	Off	2	1
Hour 1 Load = 210 MW, Hour 2 Load = 180 MW							

Consider UC formulations based on the “1-bin” and “3-bin” commitment variable models that have been used in practice (Knueven et al. 2018).

In the 1-bin model, the binary variable $u_{i,t} \in \{0,1\}$ represents the “On” or “Off” status of Unit i at Hour t . Startup and shutdown actions are represented by the difference between the commitment statuses in consecutive hours. For this example, the 1-bin unit commitment formulation is

$$\min_{C,p,u} C_1 + C_2$$

$$\text{s. t.} \quad p_{11} + p_{21} = 210 \quad (\lambda_1) \quad \text{Hour 1 balance}$$

$$p_{12} + p_{22} = 180 \quad (\lambda_2) \quad \text{Hour 2 balance} \quad (36)$$

$$(C_1, p_{11}, p_{12}, u_{11}, u_{12}) \in X_1, \text{ with } X_1 = \left\{ (C_1, p_{11}, p_{12}, u_{11}, u_{12}) \in \mathcal{R}^5 \left| \begin{array}{l} C_1 = 10(p_{11} + p_{12}), \\ 0u_{11} \leq p_{11} \leq 200u_{11}, \\ 0u_{12} \leq p_{12} \leq 200u_{12}, \\ u_{11}, u_{12} \in \{0,1\} \end{array} \right. \right\}$$

$$(C_2, p_{21}, p_{22}, u_{21}, u_{22}) \in X_2, \text{ with } X_2 = \left\{ (C_2, p_{21}, p_{22}, u_{21}, u_{22}) \in \mathcal{R}^5 \left| \begin{array}{l} C_2 = 20(p_{21} + p_{22}), \\ 50u_{21} \leq p_{21} \leq 50u_{21}, \\ 50u_{22} \leq p_{22} \leq 50u_{22}, \\ u_{21} \leq u_{22}, \\ u_{21}, u_{22} \in \{0,1\} \end{array} \right. \right\}.$$

Both units need to be “On” for Hour 1 since neither can provide 210MW load alone, so the block-loaded Unit 2 will produce 50MW and Unit 1 will produce 170MW (= 210 – 50). Since Unit 2 has a 2-hour MRT, it has to be “On” for Hour 2 as well and will produce 50MW. Unit 1 will generate the remaining 130MW in Hour 2. The optimal cost is \$4,900. The primal optimal solution is shown in Table 10.

Table 10. UC solution for Example 3

Unit	Hour 1 Dispatch (MW)	Hour 1 Commitment	Hour 2 Dispatch (MW)	Hour 2 Commitment
1	160	“On”	130	“On”
2	50	“On”	50	“On”
Total Generation Cost: \$4,900				

The dual problem for the *I-bin* formulation (36) is:

$$\max_{\lambda_1, \lambda_2} \min_{\substack{(C_1, p_{11}, p_{12}, u_{11}, u_{12}) \in X_1 \\ (C_2, p_{21}, p_{22}, u_{21}, u_{22}) \in X_2}} \{C_1 + C_2 - \lambda_1(p_{11} + p_{21} - 210) - \lambda_2(p_{12} + p_{22} - 180)\}. \quad (37)$$

The optimal dual solution of (37) is $\lambda_1^* = \$30/MWh$ and $\lambda_2^* = \$10/MWh$, which are the hourly convex hull prices for the *I-bin* UC formulation. The optimal dual cost is \$4,100 with the \$800 duality representing the LOC of Unit 1 due to not producing at its 200MW capacity in the high-priced Hour 1.

The *3-bin* model introduces the additional startup variable $v_{i,t} \in \{0,1\}$ and shutdown variable $w_{i,t} \in \{0,1\}$ to represent startup and shutdown actions, respectively, of Unit i at Hour t . The *3-bin* UC formulation for this example is presented below.

$$\begin{aligned}
& \min_{C,p,u,v,w} C_1 + C_2 \\
& \text{s. t.} \quad p_{11} + p_{21} = 210 \quad (\lambda'_1) \quad \text{Hour 1 balance} \\
& \quad \quad p_{12} + p_{22} = 180 \quad (\lambda'_2) \quad \text{Hour 2 balance} \tag{38}
\end{aligned}$$

$$\begin{aligned}
\left(\begin{array}{c} C_1, p_{11}, p_{12}, u_{11}, u_{12}, \\ v_{11}, v_{12}, w_{11}, w_{12} \end{array} \right) \in X'_1, \text{ with } X'_1 = & \left\{ \left(\begin{array}{c} C_1, p_{11}, p_{12}, u_{11}, u_{12}, \\ v_{11}, v_{12}, w_{11}, w_{12} \end{array} \right) \in \mathcal{R}^9 \left| \begin{array}{l} C_1 = 10 \cdot (p_{11} + p_{12}), \\ 0u_{11} \leq p_{11} \leq 200u_{11}, \\ 0u_{12} \leq p_{12} \leq 200u_{12}, \\ u_{11} = v_{11} - w_{11} \\ u_{12} - u_{11} = v_{12} - w_{12} \\ v_{12} \leq u_{12} \\ w_{12} \leq 1 - u_{12} \\ u_{11}, u_{12} \in \{0,1\} \\ v_{11}, v_{12}, w_{11}, w_{12} \in \{0,1\} \end{array} \right. \right\} \\
\left(\begin{array}{c} C_2, p_{21}, p_{22}, u_{21}, u_{22}, \\ v_{21}, v_{22}, w_{21}, w_{22} \end{array} \right) \in X'_2, \text{ with } X'_2 = & \left\{ \left(\begin{array}{c} C_2, p_{21}, p_{22}, u_{21}, u_{22}, \\ v_{21}, v_{22}, w_{21}, w_{22} \end{array} \right) \in \mathcal{R}^9 \left| \begin{array}{l} C_2 = 20 \cdot (p_{21} + p_{22}), \\ 50u_{21} \leq p_{21} \leq 50u_{21}, \\ 50u_{22} \leq p_{22} \leq 50u_{22}, \\ u_{21} = v_{21} - w_{21} \\ u_{22} - u_{21} = v_{22} - w_{22} \\ v_{21} + v_{22} \leq u_{22} \\ w_{22} \leq 1 - u_{22} \\ u_{21}, u_{22} \in \{0,1\} \\ v_{21}, v_{22}, w_{21}, w_{22} \in \{0,1\} \end{array} \right. \right\}
\end{aligned}$$

The primal optimal solution of the UC formulation (39) is the same as the results in Table 10.

The dual problem for the *3-bin* formulation (38) is

$$\max_{\lambda'_1, \lambda'_2} \min_{\substack{(C_1, p_{11}, p_{12}, u_{11}, u_{12}, v_{11}, v_{12}, w_{11}, w_{12}) \in X'_1 \\ (C_2, p_{21}, p_{22}, u_{21}, u_{22}, v_{21}, v_{22}, w_{21}, w_{22}) \in X'_2}} \{C_1 + C_2 - \lambda'_1(p_{11} + p_{21} - 210) - \lambda'_2(p_{12} + p_{22} - 180)\}. \tag{39}$$

Solving (39) yields the optimal solutions $\lambda'_1^* = \$30/MWh$ and $\lambda'_2^* = \$10/MWh$, which are the hourly convex hull prices for the *3-bin* UC formulation. Furthermore, it can be verified that

$$proj_{(C_1, p_{11}, p_{12})}(X_1) = proj_{(C_1, p_{11}, p_{12})}(X'_1) = \{(C_1, p_{11}, p_{12}) | C_1 = 10(p_{11} + p_{12}), 0 \leq p_{11}, p_{12} \leq 200\},$$

$$proj_{(C_2, p_{21}, p_{22})}(X_2) = proj_{(C_2, p_{21}, p_{22})}(X'_2) = \{(0,0,0), (1000,0,50), (2000,50,50)\}.$$

Thus, the sufficient condition of Corollary 1 is satisfied and the *1-bin* and *3-bin* formulations indeed lead to the same convex hull prices: $\lambda_1'^* = \lambda_1^*$ and $\lambda_2'^* = \lambda_2^*$.

4.2. Alternative representations of the convex hull price

While convex hull prices are defined on perturbations to the nodal balance constraints, a UC problem may be formulated without these constraints. Consider such a UC formulation below.

$$\min_{C,p,u,y} e^T C \quad (40)$$

$$\text{s. t.} \quad M(H_x(p), y) \leq 0 \quad (41)$$

$$(C, p, u) \in X, \quad (42)$$

where (41) represents an alternative set of network constraints, e.g., a system balance constraint and shift-factor-based flow constraints, replacing the nodal balance constraints (25) and network constraints (26) in the UC formulation (24)-(27). Note that unit dispatch variables p appear in the form of nodal aggregations $H_x(p)$ in network constraints (41) since unit injections/withdrawals at the same node have the same network impact.

By introducing new network variables z with $H_x(p) = -z$ and replacing (41) with $M(-z, y) \leq 0$, the convex hull prices for (40)-(42) can be defined as the optimal dual variables for the new constraints $H_x(p) = -z$. The fixed-commitment prices, i.e., LMPs, can be defined in the same way but with commitment variables fixed at their optimal values. Moreover, it is well known that LMPs can be equivalently represented with shadow prices μ_{FC}^* of (41) (Litvinov 2008), i.e.,

$$LMP = \alpha_M^T \cdot \mu_{FC}^*, \quad (43)$$

where α_M is a vector of coefficients representing the impacts of nodal perturbations on the network constraints $M(\cdot)$. The natural question is: *Can the convex hull price be represented in a similar form?*

The below Theorem 2 provides the answer.

Theorem 2. With (43) under fixed-commitment pricing and linear network constraints (41), the convex hull prices, i.e., LMP^c , for the UC formulation (40)-(42) can be represented as

$$LMP^c = \alpha_M^T \cdot \mu^*, \quad (44)$$

where μ^* are the optimal dual variables of (41) for (40)-(42).

[*Proof*]. Based on Theorem 1, the convex hull prices LMP^c for (40)-(42) are identical to those of the following UC formulation:

$$\min_{C,p,y} e^T C \quad (45)$$

$$\text{s. t.} \quad M(H_x(p), y) \leq 0 \quad (46)$$

$$(C, p) \in \text{conv}(\text{proj}_{(C,p)} X). \quad (47)$$

The above (45)-(47) can be viewed as a convex economic dispatch problem with (47) defining each unit's cost as a function of its dispatch. Moreover, any economic dispatch problem can be constructed as a fixed-commitment problem with all units “On.” Therefore, the convex hull prices LMP^c for (45)-(47) can be obtained as LMPs for the below fixed-commitment problem:

$$\min_{C,p,y} e^T C, \quad \text{s. t. (46) – (47) and } u = 1. \quad (48)$$

With (43), LMP^c for (45)-(47), which is also the LMP for (48), can be represented as

$$LMP^c = \alpha_M^T \cdot \mu_{FC}^*. \quad (49)$$

Note that μ_{FC}^* are optimal dual variables or shadow prices of (46) in the fixed-commitment problem (48), i.e.,

$$\begin{aligned}
\mu_{FC}^* &= \arg \max_{\mu \geq 0} \min_{(C,p) \in \text{conv}(\text{proj}_{(C,p)} X)} \{e^T C + \mu \cdot M(H_x(p), y)\} \\
&= \arg \max_{\mu \geq 0} \min_{(C,p) \in \text{proj}_{(C,p)} X} \{e^T C + \mu \cdot M(H_x(p), y)\} \\
&= \arg \max_{\mu \geq 0} \min_{(C,p,u) \in X} \{e^T C + \mu \cdot M(H_x(p), y)\} = \mu^*
\end{aligned} \tag{50}$$

where the first equality applies the definition of shadow prices for (48), the second equality is due to the compactness of set X and the linearity of network constraints $M(\cdot)$, the third equality uses the fact that the dual objective does not involve u , and the last equality applies the definition of dual variables for (40)-(42). Then (44) follows from (49) and (50). *[End of proof].*

Theorem 2 indicates that alternative LMP representations under fixed-commitment pricing can be extended to convex hull pricing. The theorem expands the convex hull price preserving UC formulations to include those without nodal balance constraints. The below Example 4 illustrates Theorem 2.

Example 4. Consider a 3-bus 2-unit 1-hour UC problem with the network topology described in Figure 3. The unit parameters are listed in Table 11. The only load at Bus 2 is 120MW. All three transmission lines have the same impedance, and the capacity of each line is 60MW. Losses are not considered.

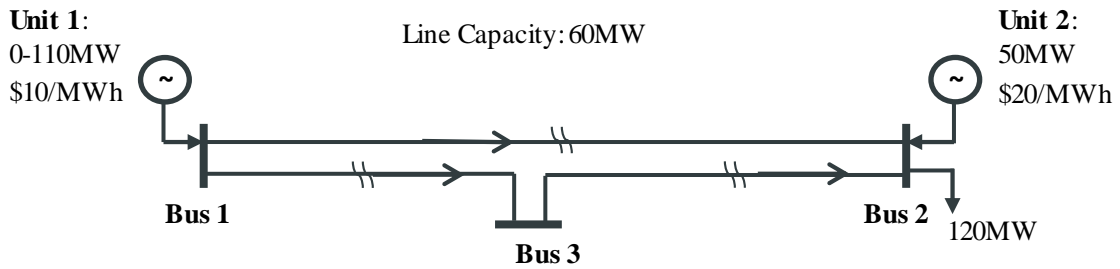


Figure 3. System topology for Example 4

Table 11. Unit parameters for Example 4

Unit	p_{\min} (MW)	p_{\max} (MW)	Incremental Cost (\$/MWh)	Commitment Cost (\$)
1	0	200	10	0
2	50	50	20	0

Consider the below UC formulation with the system power balance constraint instead of the nodal ones:

$$\begin{aligned}
& \min_{p_1, u_1, p_2, u_2} 10p_1 + 20p_2 \\
& \text{s. t.} \quad p_1 + p_2 = 120 \quad (\lambda_0) \quad \text{Load balance} \\
& \quad -60 \leq \frac{1}{3}p_1 - \frac{1}{3}(p_2 - 120) \leq 60 \quad (\mu'_{12}, \mu_{12}) \quad \text{Line-12 limits} \\
& \quad -60 \leq \frac{2}{3}p_1 + \frac{1}{3}(p_2 - 120) \leq 60 \quad (\mu'_{13}, \mu_{13}) \quad \text{Line-13 limits} \\
& \quad -60 \leq -\frac{1}{3}p_1 - \frac{2}{3}(p_2 - 120) \leq 60 \quad (\mu'_{32}, \mu_{32}) \quad \text{Line-32 limits} \quad (51) \\
& \quad 0u_1 \leq p_1 \leq 200u_1, \quad u_1 \in \{0,1\} \quad \text{Unit 1 constraints} \\
& \quad 50u_2 \leq p_2 \leq 50u_2, \quad u_2 \in \{0,1\}, \quad \text{Unit 2 constraints}
\end{aligned}$$

where the line constraints are modeled with the shift factors calculated by using Bus 3 as reference. The optimal dual of (51) can be represented as:

$$\max_{\substack{\lambda_0 \\ \mu'_{12}, \mu_{12} \geq 0 \\ \mu'_{13}, \mu_{13} \geq 0 \\ \mu'_{32}, \mu_{32} \geq 0}} \min_{\substack{p_1, u_1, p_2, u_2 \\ 0u_1 \leq p_1 \leq 200u_1 \\ 50u_2 \leq p_2 \leq 50u_2 \\ u_1, u_2 \in \{0,1\}}} \left\{ \begin{aligned} & 10p_1 + 20p_2 - \lambda_0 \cdot (p_1 + p_2 - 120) \\ & + \mu'_{12} \left(-60 - \frac{p_1}{3} + \frac{(p_2 - 120)}{3} \right) + \mu_{12} \left(\frac{p_1}{3} - \frac{(p_2 - 120)}{3} - 60 \right) \\ & + \mu'_{13} \left(-60 - \frac{2}{3}p_1 - \frac{(p_2 - 120)}{3} \right) + \mu_{13} \left(\frac{2}{3}p_1 + \frac{(p_2 - 120)}{3} - 60 \right) \\ & + \mu'_{32} \left(-60 + \frac{p_1}{3} + \frac{2(p_2 - 120)}{3} \right) + \mu_{32} \left(-\frac{p_1}{3} - \frac{2(p_2 - 120)}{3} - 60 \right) \end{aligned} \right\}. \quad (52)$$

Solving (52) yields the optimal dual variables: $[\lambda_0^*, \mu_{12}^*, \mu'_{12}^*, \mu_{13}^*, \mu'_{13}^*, \mu_{32}^*, \mu'_{32}^*] = [15, 15, 0, 0, 0, 0, 0]$.

Based on the LMP decomposition (Litvinov 2008) and Theorem 2, we have:

$$\begin{aligned}
LMP_1^c &= \lambda_0^* - \frac{1}{3}(\mu_{12}^* - \mu'_{12}^*) - \frac{2}{3}(\mu_{13}^* - \mu'_{13}^*) + \frac{1}{3}(\mu_{32}^* - \mu'_{32}^*) = 10, \\
LMP_2^c &= \lambda_0^* + \frac{1}{3}(\mu_{12}^* - \mu'_{12}^*) - \frac{1}{3}(\mu_{13}^* - \mu'_{13}^*) + \frac{2}{3}(\mu_{32}^* - \mu'_{32}^*) = 20, \quad (53) \\
LMP_3^c &= \lambda_0^* = 15.
\end{aligned}$$

Next we verify that the above LMP^c calculated using the optimal dual variables of UC formulation (51) are indeed the convex hull prices.

The convex hull prices for (51) can be defined via introducing new variables z for nodal power aggregations and substituting these variables z into (51), i.e.,

$$\begin{aligned}
& \min_{\substack{p_1, u_1, p_2, u_2, \\ z_1, z_2, z_3}} 10p_1 + 20p_2 \\
\text{s. t. } & p_1 = -z_1 & (\lambda_1) \\
& p_2 - 120 = -z_2 & (\lambda_2) \\
& 0 = -z_3 & (\lambda_3) \\
& (z_1, z_2, z_3) \in Z, \text{ with } Z = \left\{ (z_1, z_2, z_3) \in \mathcal{R}^3 \left| \begin{array}{l} -60 \leq -\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 60 \\ -60 \leq -\frac{2}{3}z_1 - \frac{1}{3}z_2 \leq 60 \\ -60 \leq \frac{1}{3}z_1 + \frac{2}{3}z_2 \leq 60 \end{array} \right. \right\} \\
& (p_1, u_1, p_2, u_2) \in X, \text{ with } X = \left\{ (p_1, u_1, p_2, u_2) \in \mathcal{R}^4 \left| \begin{array}{l} 0u_1 \leq p_1 \leq 200u_1 \\ 50u_2 \leq p_2 \leq 50u_2 \\ u_1, u_2 \in \{0,1\} \end{array} \right. \right\}.
\end{aligned} \tag{54}$$

The above formulation (54) includes the new nodal aggregation constraints that define z . The convex hull prices are then the optimal dual variables λ^* associated with these constraints, which are calculated from the below dual problem:

$$\max_{\lambda_1, \lambda_2, \lambda_3} \min_{\substack{(p_1, u_1, p_2, u_2) \in X, \\ (z_1, z_2, z_3) \in Z}} \{10p_1 + 20p_2 - \lambda_1(p_1 + z_1) - \lambda_2(p_2 - 120 + z_2) - \lambda_3 z_3\}. \tag{55}$$

Solving (55) yields the convex hull prices of $[\lambda_1^*, \lambda_2^*, \lambda_3^*] = [10, 15, 20]$ at the three buses, which are the same as those calculated from the alternative representation (53).

5. Numerical Testing

In this section, the formulation dependence of convex hull pricing is tested with ISO-scale problems. In particular, we focus on the impact of non-binding security constraints as described in Section 3.2 since it is a common ISO practice to exclude most of these constraints from the pricing problem.

Computing the convex hull price for large UC problems is generally challenging. However, it is known that under certain simplifying conditions, the convex hull price can be obtained through solving the integer-relaxed UC problem, which is a linear or quadratic program. One set of such conditions was presented in Chao (2019), i.e., for each unit,

- (i) its cost function is homogeneous of degree one in commitment and dispatch variables, and is convex in dispatch variables;
- (ii) the unit constraints involving both commitment and dispatch variables form a convex cone separable in commitment intervals; and
- (iii) the unit constraints involving only commitment variables form a unimodal network model.

The above conditions ensure that the integer-relaxation of binary commitment variables will lead to the convex hull for each individual unit, and thus the convex hull price is the dual solution of the integer relaxed problem. Since this paper is focused on the formulation dependence of convex hull price, we modified ISO New England (ISO-NE)'s Day-Ahead Market (DAM) formulation to satisfy the above conditions (i)-(iii). The “*modified UC*” formulation is then implemented in General Algebraic Modeling System (GAMS) with CPLEX being the optimization solver, and tested on actual ISO-NE DAM data involving 400+ generators, thousands of virtual bids, and a transmission network model of 3500 nodes, 5000 branches and 5000 contingencies. To avoid the complex contingency analysis, we consider two cases with different security constraint sets for each testing day:

- Case A with set S_A of *all* security constraints identified in the historical DAM run; and
- Case B with subset $S_B (\subseteq S_A)$ denoting *binding* security constraints of the modified UC problem.

Note that S_A includes non-binding constraints other than the binding ones of S_B . The convex hull prices for the two cases are obtained as the dual solutions of the modified UC problem with relaxed integer variables. Four days of year 2019 were selected for testing: the winter peak day of January 21, the

summer peak day of July 20, and two randomly selected days of March 14 and October 22. The hourly convex hull prices at the hub for the two cases are depicted in Figure 4.

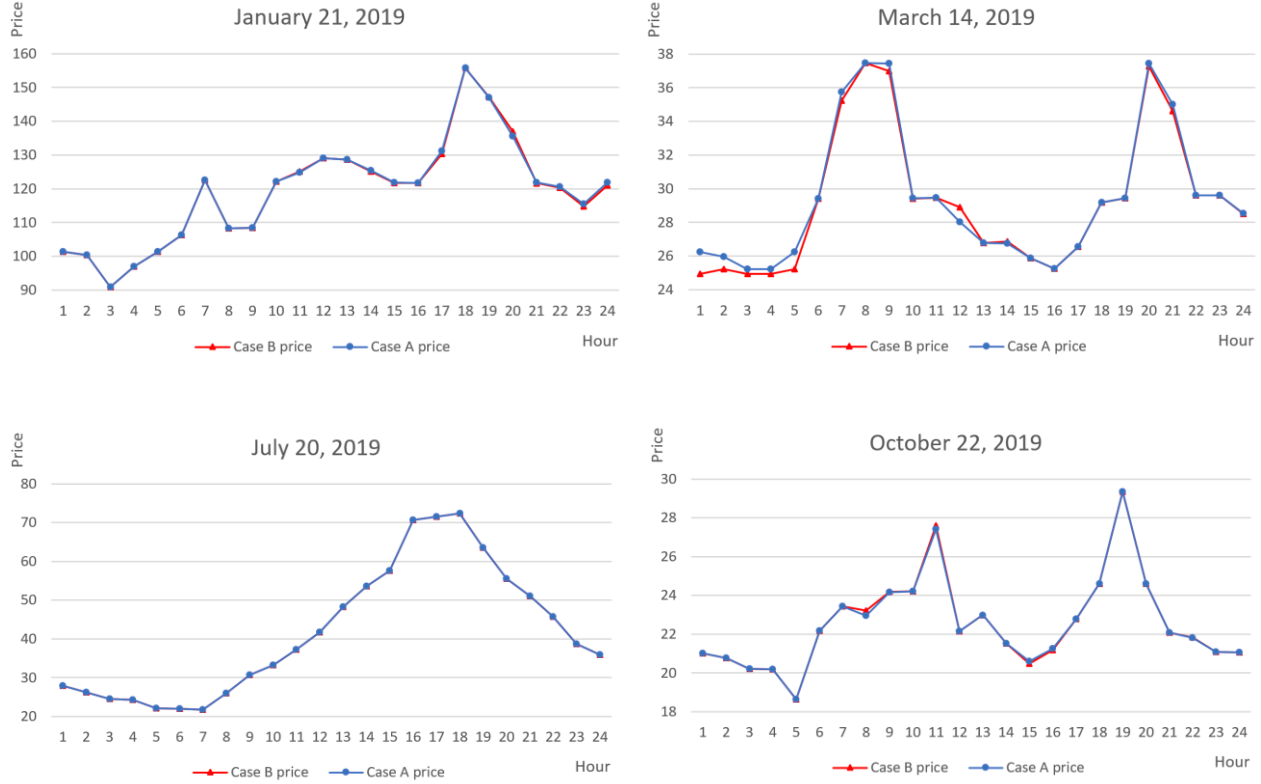


Figure 4. Comparison of convex hull prices in cases A and B for select days.

Among the four days tested, the summer peak day (July 20) yields the same hourly hub convex hull prices for security constraint sets S_A and S_B , while March 14 yields the largest price difference between the two cases. Both January 21 (winter peak) and October 22 yield small convex hull price differences. This suggests that the sensitivity of convex hull price to non-binding constraints is probably not correlated with the system load level. This is not surprising as the inclusion of non-binding constraints changes the $v(d)$ curve and thus its convex hull $v^c(d)$ in Figure 1, which can arbitrarily affect the convex hull price at any d (a representation of system loading) due to its global feature.

Also observe that the hourly price differences between Case A and Case B for the four test days are generally not large. This may attribute to the limited number of non-binding constraints considered in our

testing. Table 12 lists the numbers of security constraints modeled in cases A and B, i.e., $|S_A|$ and $|S_B|$, respectively. Note that $|S_A| - |S_B|$ indicates the number of non-binding constraints modeled in Case A.

Table 12. Numbers of security constraints and binding ones in cases A and B for the select days

<i>Date</i>	<i>Case A</i>		<i>Case B</i>	
	$ S_A $	<i>Binding in pricing (N_A)</i>	$ S_B $	<i>Binding in pricing (N_B)</i>
1/21/2019	195	46	42	42
3/14/2019	216	43	41	38
7/20/2019	244	3	1	1
10/22/2019	244	85	82	82

It can be seen from Table 12 that only a small portion of 5000 (contingencies) \times 5000 (branches) \times 24 (hours) security constraints are modeled for each case. Adding more non-binding security constraints into Case A in the sequence of their increasing distances from binding in the UC run will likely increase the price gap between the two cases, but such effect is expected to diminish since those constraints are also less likely to be binding in the pricing run. The table also lists numbers of the binding constraints from pricing runs in cases A and B, i.e., N_A and N_B , respectively. Note that $(N_A - N_B)$ is a good indicator of the convex hull price difference between the two cases, e.g., the largest $N_A - N_B$ occurs with the largest convex hull price gap in the March 14 example. Finally, it is observed that few constraints were modeled for the summer peak day of July 20. This is related to the increased DAM virtual biddings in anticipation of high system load for the peak day.

In sum, our testing on the ISO-scale problem shows that convex hull price is dependent on the non-binding constraints of the UC formulation, and the scale of these non-binding constraints' impact may vary with the testing data.

6. Conclusions

Convex hull pricing has attracted much interest from academia and industry as a potential solution for reducing out-of-market payments in electricity markets. This paper addresses a fundamental question about the formulation dependence of convex hull prices. With illustrative examples, it was shown that

convex hull prices can indeed be affected by the UC formulation. This formulation-dependence property was attributed to the global nature of convex hull pricing, and sufficient conditions that preserve convex hull prices were derived. These findings indicate that UC reformulations and contingency analysis processes must be carefully implemented if convex hull pricing is used. The formulation dependence property also implies that it may be advantageous for UC formulations to be public information. Finally, theoretical aspects of convex hull pricing such as its economic foundation need to be further explored.

Endnotes

1. A price *supports* cleared quantities if, given the price, each market participant obtains its maximal surplus at its cleared quantity. In other words, participants have no incentive to deviate from their cleared quantities.
2. A MWP is an out-of-market payment to ensure bid-in cost recovery for committed units, and a LOC is an out-of-market payment to ensure maximum profit. Currently, most ISOs compensate MWPs but little if any LOC.
3. The term “unit” in this paper represents both supply-side and demand-side resources. It also represents virtual bids in a day-ahead market and the forecasted load (as a fixed-consumption resource) in a real-time market.
4. The node- k aggregated power from units can be represented by $H_x^k(x) = \sum_i a_i x_i$. Without losses, coefficient a_i is 1 for supply resources at node- k , -1 for load resources at node- k , and 0 otherwise.
5. Market prices for ancillary services are defined on perturbations to their service requirements.
6. For a set S in an n -dimensional vector space, i.e., $S \in R^n$, its convex envelope/hull $\text{conv}(S)$ is the smallest convex set that contains S . For a function f defined on S , its convex hull $\text{conv}(f)$ is the function whose epigraph is the convex envelope of the epigraph of f .
7. The surplus for a supply-side participant is revenue less cost. The surplus for a demand-side participant is benefit less cost.

8. This uplift definition covers both MWP and LOC concepts.
9. Convexification of the network feasible region applies when nonlinear power flow equations or binary line switching decisions are modeled. With linear network constraints, the condition $\text{conv}(\text{proj}_q(S)) = \text{conv}(\text{proj}_q(S'))$ can simply be replaced with $\text{proj}_q(S) = \text{proj}_q(S')$.

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