The Graphical Traveling Salesperson Problem has no Integer Programming Formulation in the Original Space

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Abstract

The Graphical Traveling Salesperson Problem (GTSP) is the problem of assigning, for a given weighted graph, a nonnegative number x_e to each edge e such that the induced multi-subgraph is of minimum weight among those that are spanning, connected and Eulerian. Naturally, known mixed-integer programming formulations use integer variables x_e in addition to others. Denis Naddef posed the challenge of finding a (reasonably simple) mixed-integer programming formulation that has integrality constraints only on these edge variables. Recently, Carr and Simonetti (IPCO 2021) showed that such a formulation cannot consist of polynomial-time certifyiable inequality classes unless $\mathsf{NP} = \mathsf{coNP}$. In this note we establish a more rigorous result, namely that no such MIP formulation exists at all.

1 Introduction

Let $G = (V, E)$ be a graph and let $c \in \mathbb{R}^E$. The Graphical Traveling Salesperson problem is about finding c-minimum cost tour in G that visits each node at least once, where edges can be used multiple times. It can be formulated as the following constraint integer program due to Cornéjols, Fonlupt and Naddef [2].

$$\min \quad c^\mathsf{T} x \tag{1a}$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall \varnothing \neq S \subsetneq V \tag{1b}$$

$$\sum_{e \in \delta(v)} x_e \text{ is even} \quad \forall v \in V \tag{1c}$$

$$x_e \in \mathbb{Z}_{\geq 0} \quad \forall e \in E \tag{1d}$$

Here, $\delta(S) \coloneqq \{e \in E : |e \cap S| = 1\}$ and $\delta(v) \coloneqq \delta(\{v\})$ denote the cuts induced by node set $S \subseteq V$ and node $v \in V$, respectively. For each edge $e \in E$, the variable x_e indicates how often e is traversed in the tour.

The authors of [2] describe several classes of inequalities that are valid for the *GTSP polyhedron* $P_{gtsp}(G)$ defined as the convex hull of all feasible solutions, i.e.,

$$P_{gtsp}(G) := \text{conv}\{x \in \mathbb{Z}^E : x \text{ satisfies (1b), (1c) and (1d)}\}.$$

Among these were *path*, *wheelbarrow* and *bicycle* inequalities. In order to turn (1) into a mixed-integer programming model (MIP), constraint (1c) can be replaced by this pair of constraints:

$$\sum_{e \in \delta(v)} x_e = 2y_v \qquad\qquad \forall v \in V \qquad\qquad (2a)$$

$$y_v \in \mathbb{Z} \qquad\qquad \forall v \in V \qquad\qquad (2b)$$

These additional y-variables are artificial and their presence has no impact on the linear programming relaxation of (1). For this reason, Naddef posed the challenge of finding a simple mixed-integer programming formulation that involves, apart from the x-variables, only continuous variables [4]. According to [1], he had the formulation from [2] with *path*, *wheelbarrow* and *bicycle* inequalities in mind. There exist other (mixed-)integer programming formulations for the GTSP, see [1, 3], all of which requiring additional *integral* variables.

Recently, Carr and Simonetti showed that such a formulation cannot be nice in the sense that it cannot consist of inequality families for which one can certify membership in polynomial time, provided $\mathsf{NP} \neq \mathsf{coNP}$ (see Section 4.2 in [1]). The purpose of this paper is to show that the reason for the non-existence of a simple formulation does not lie in complexity theory. In fact, we show that no such formulation exists at all:

Theorem 1. *The GTSP has no mixed-integer programming formulation whose only integer variables are the x-variables from* (1)*.*
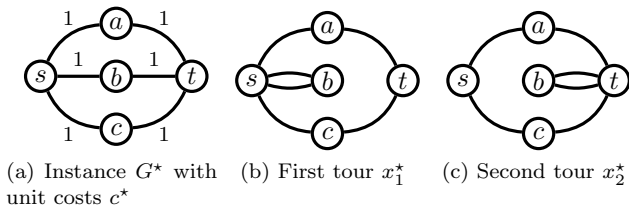
2 Nonexistence of the formulation

(a) Instance $G^\star$ with unit costs $c^\star$ (b) First tour $x_1^\star$ (c) Second tour $x_2^\star$

Figure 1: Example instance $G^\star = (V^\star, E^\star)$ with unit costs $c^\star = (1, 1, 1, 1, 1, 1)^\intercal$. To minimum-cost solutions are depicted in (1b) and (1c).

Proof of Theorem 1. We consider the graph $G^\star = (V^\star, E^\star)$ from Fig. 1 with unit edge costs $c^\star \in \mathbb{R}^{E^\star}$. It is easy to see that $G^\star$ has no Hamiltonian cycle, and therefore there is no solution of value $|V^\star| = 5$. Hence, the tours in Figure (1) (b) and (c), denoted by $x_1^\star, x_2^\star \in \mathbb{Z}^{E^\star}$ are optimal. Their midpoint is the point $x^\star = (1, 1, 1, 1, 1, 1)^\intercal$, which is integral but infeasible for (1) as it violates (1c). Moreover, since the tours are optimal, so is $x^\star$.

Assume, for the sake contradiction, that there exists a mixed-integer programming formulation $Q = \{(x, y) \in \mathbb{Z}^E \times \mathbb{R}^q : Ax + By \leq d\}$ that has integrality constraints only for the x-variables. Hence, the projection of Q onto the x-variables is the set of feasible solutions to (1), and hence

$$\min\{c^\intercal x : (x, y) \in Q\} \tag{3}$$

is equivalent to (1). In particular, feasibility of $x_1^\star$ and $x_2^\star$ for (1) implies that there exist $y_1^\star, y_2^\star \in \mathbb{R}^q$ such that $(x_i^\star, y_i^\star) \in Q$ for $i = 1, 2$. Now let $y^\star \in \mathbb{R}^q$ be the midpoint of $y_1^\star$ and $y_2^\star$. By convexity of the linear relaxation of Q and integrality of $x^\star$, also $(x^\star, y^\star)$ is an optimal solution to (3). This contradicts the fact that $x^\star$ is infeasible for (1). $\qquad\square$

References

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