A Fixed Point Approach with a New Solution Concept for Set-valued Optimization [1]

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Abstract: We present a fixed point approach to find the whole solution set of a set-valued optimization problem though a parametric problem, in which the height of the level set of the objective function is regarded as the parameter. First, the solution concept based on the vector approach is considered in this method. Then, we propose another solution concept which additionally takes the maximal part of the set into consideration and compare it with the solution concept based on the set approach. The fixed point approach is also extended to set-valued optimization with respect to this solution concept. Finally, a special case of this theory is investigated particularly, for which the new solution concept actually provides a vectorization.

Keywords: Set-valued Optimization; Min-max Minimizer; Fixed Point Approach; Set Order Relation

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1 Introduction

Set-valued optimization is a versatile branch of applied mathematics. It stands for the optimization problems where the objective map or the constraint maps are set-valued, which offers a vital unification and extension of both scalar and vector optimization problems. There is a large number of problems related to various areas that can be formulated and studied within the realm of set-valued optimization. Fields like duality for vector optimization problems, vector variational inequalities, robust optimization, control theory, artificial intelligence, inverse problems in image processing, economics and management all benefit a lot from the improvement of the theories of set-valued optimization. The investigations as well as applications of set-valued optimization have been of tremendous interest these years, see [3, 4, 7, 10, 14]. Meanwhile, the development of theories, like set-valued analysis [2], variational analysis [17], [19], [20], provide abundant concepts and results for set-valued maps, considerably enriching the tools to cope with set-valued optimization problems.

In applications, there are mainly two approaches for defining the solution concept for a set-valued optimization problem. One is the vector approach, in which the optimality is understood as the Pareto minimality of the image of the objective map and the solution is usually identified by a pair in the graph. The other one is based on comparisons among different sets, with respect to a certain set relation, for which a variety of set relations have been introduced and studied, for instance [6, 9, 11–13]. These two approaches represent different perspectives in the understanding of minimization and they both have numerous applications in proper areas. Obviously, the solution for an optimization problem is usually not a single point, even for the scalar problems. In some situations, it is needed to find out the whole solution set, like in bi-level programming, where the final decision is made on the solutions of another optimization problem.

In general, it is not an easy task to generate the whole solution set. Scalarization might provide an option for finding the entire solution set to a vector or set-valued optimization problem. Regarding the scalarized problem as a parametric optimization problem, we can try to obtain all the solutions if the involved parameter ranges through the whole parameter set. However, this is only doable when the problem has a special structure. For instance, if we are dealing with a scalarization of a multiobjective optimization problem by linear continuous functions, every weakly efficient solution in Pareto sense can be reached if the problem is convex. But it may fail when the problem lacks convexity or what we try to get is the efficient solution set.

In this paper, we focus on set-valued problems where the solution concept is based on the vector approach. In [8], the authors proposed a way to obtain the entire efficient solution set of a constraint vector optimization problem. In this approach, a characterization of solutions is given through the fixed points of a set-valued map, defined by a parametric problem. This method has been applied in [1] to discuss the ways of finding global solution of some bi-level vector optimizations. In this paper, we try to exploit the idea in [8] to investigate the set-valued case. As the method in [8] is presented for vector optimization problems, the extension of the fixed point approach to set-valued problems with the solution concept based on the vector approach is more conceivable. Hence, we shall consider the solution concept based on the vector approach, which is more difficult to handle when finding the whole solution set. Associated with a scalarization technique, a parametric problem is constructed. Although a scalar map is involved here, it is fixed, while the height of the level set of the objective function is treated as the parameter. We will show that all the solutions can be reached as the parameter ranges through the entire set of parameters. It is worth to mention that this works for general set-valued optimization problems. It does not require any convexity to guarantee the results.

If we consider the solution concept based on the vector approach, a solution depends on a special point in the value set, while the other points are ignored. In some situations, we may need to take the whole value set into consideration to make the decision, for which we propose another solution concept. It is still defined though the idea of vector approach but involves the maximal part of the value set. We give a comparison between it and the solutions based on set relations. It turns out that they are completely different solution concepts in general. Then, we extend the fixed point result to a set-valued optimization problem with respect to the new solution concept, cooperating with some scalarization methods, which leads a way to detect all the solutions. A particular case that the minimal and maximal parts of every value set of the objective map are both single points is considered. In such a situation, the new solution concept virtually provides a vectorization for the set-valued problem. Relationships between assertions concerning maximal fixed points and solutions of set optimization problems are studied by Li and Tammer in [15], [16].

The paper is organized as follows. In Section 2, we recall some notions of a set-valued map and give the concepts as well as properties of some basic set relations. In Section 3, considering the solution concept based on the vector approach, we establish a fixed point approach to find the entire solution set for a constraint set-valued optimization problem. In Section 4, a new solution concept is proposed, and a comparison of it with the solution concept based on set relations is also given there. In Section 5, we extend the fixed point approach to the set-valued problem equipped with this new solution concept. The situation where the minimal and maximal parts of every value of the objective map are single points are investigated particularly.

2 Preliminaries

Let X, Y be real Banach spaces with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, while K_X and K_Y are both closed, convex and pointed cones with nonempty interior, which induce a partial order $\leq_{K_X}$ in X and $\leq_{K_Y}$ in Y, respectively. K_Y (K_X, respectively) is called an ordering cone. For a subset $A \subset Y$, denote the topological interior of A by $\operatorname{int} A$, and the closed ball with the center x and the radius ε by $B(x, \varepsilon)$.

Definition 2.1. *A set $A \subset Y$ is said to be bounded in order if A has an upper bound and a lower bound w.r.t. $\leq_{K_Y}$, i.e., there exist $\bar{y}, \hat{y} \in Y$ such that $\hat{y} \leq_{K_Y} a \leq_{K_Y} \bar{y}$ for all $a \in A$.*

Definition 2.2. *The ordering cone K_Y is called normal if there is a number $m > 0$ such that, for all $y_1, y_2 \in Y$:*

$$0_Y \leq_{K_Y} y_1 \leq_{K_Y} y_2 \quad \textit{implies} \quad \|y_1\|_Y \leq m\|y_2\|_Y.$$

Definition 2.3. *The ordering cone K_Y is called regular if every increasing sequence which is bounded in order is convergent.*

Proposition 2.1. *[5, Proposition 19.2.] If the ordering cone K_Y is regular, then it is also normal.*

For a set-valued map $F : X \rightrightarrows Y$, the domain and image of F are

$$\text{dom } F{:=}\{x \in X : F(x) \neq \emptyset\} \text{ and } \text{Im } F{:=}\{y \in Y : y \in F(x) \text{ for some } x \in X\},$$

while, the image by F of the set $S \subset X$ is $F(S) := \underset{x \in S}{\cup} F(x)$. The graph of the set-valued map F is defined as

$$\text{graph } F{:=}\{(x, y) \in X \times Y : y \in F(x)\}.$$

In the case where $X = Y$, an element $x \in X$ is said to be an fixed point of F if $x \in F(x)$.

Next, we recall some convexity notions for set-valued maps.

Definition 2.4. *Let $F : X \rightrightarrows Y$. We say that F is convex if* graph F *is a convex set.*

Definition 2.5. *Let $F : X \rightrightarrows Y$. We say that F is K_Y-convex if the set-valued map $F_{K_Y} : X \rightrightarrows Y$, $F_{K_Y}(\cdot) := F(\cdot) + K_Y$, is convex.*

It is obvious that if F is convex, then so are $F(x)$ for every $x \in X$, dom F, and Im F, while, F is K_Y-convex if and only if

$$\forall\, \lambda \in [0, 1],\ \forall\, x_1,\ x_2 \in X : \quad \lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2).$$

For a nonempty subset A of Y and $\bar{y} \in A$, we say that $\bar{y}$ is a minimal element of A if $(\{\bar{y}\} - K_Y) \cap A = \{\bar{y}\}$, or equivalently, $(\{\bar{y}\} - K_Y \setminus \{0_Y\}) \cap A = \emptyset$. Denote by Min A the set of all the minimal elements of A. The maximal element of A can be defined similar, and the set of all the maximal elements of A is denoted by Max A. This approach leads to a solution concept based on the vector approach for a general set-valued optimization problem:

$$\min_{x \in S} F(x), \tag{SOP}$$

where $F : X \rightrightarrows Y$ and $S \subseteq X$.

In several assertions of this paper, we consider set optimization problems using the solution concept of minimizers introduced in the following Definition 2.6.

Definition 2.6. *Let $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in$ graph F. The pair $(\bar{x}, \bar{y}) \in$ graph F is called a minimizer(maximizer) of the problem* (SOP) *if $\bar{y} \in$ Min $F(S)$ ($\bar{y} \in$ Max $F(S)$).*

Besides this, another way to give a solution concept for (SOP) is comparing the values of F by some set relations (for an overview see [10, Section 2.6]). We first introduce two basic set preorders that can be used to construct some other ones.

Definition 2.7. ([12], [13]) *Let $A, B \in 2^Y$ be arbitrary nonempty sets. Then, the lower set less relation $\preceq^l_{K_Y}$ is defined by*

$$A \preceq^l_{K_Y} B :\Longleftrightarrow A + K_Y \supseteq B$$

and the upper set less relation $\preceq^u_{K_Y}$ is defined by

$$A \preceq^u_{K_Y} B :\Longleftrightarrow A \subseteq B - K_Y.$$

With our assumption that K_Y is a pointed, closed, convex cone, it is shown in [10, Remark 2.6.10]:

$$A \preceq^l_{K_Y} B \text{ and } B \preceq^l_{K_Y} A \implies \text{Min } A = \text{Min } B.$$

Furthermore, under the additional assumptions

$$A \subseteq \text{Min } A + K_Y \text{ and } B \subseteq \text{Min } B + K_Y, \tag{1}$$

we obtain

$$A \preceq^l_{K_Y} B \text{ and } B \preceq^l_{K_Y} A \iff \text{Min } A = \text{Min } B.$$

Similarly, it holds that

$$A \preceq^u_{K_Y} B \text{ and } B \preceq^u_{K_Y} A \implies \text{Max } A = \text{Max } B.$$

Moreover, under the additional assumptions

$$A \subseteq \text{Max } A - K_Y \text{ and } B \subseteq \text{Max } B - K_Y, \tag{2}$$

we have

$$A \preceq^u_{K_Y} B \text{ and } B \preceq^u_{K_Y} A \iff \text{Max } A = \text{Max } B.$$

We can see that the set relations work better for those sets satisfying (1) and (2). For the sake of convenience, here we give the following assumption for F:

$$\forall x \in S: \quad F(x) \subseteq \text{Min } F(x) + K_Y \text{ and } F(x) \subseteq \text{Max } F(x) - K_Y. \tag{Assumption $\mathscr{A}$}$$

3 Fixed Point Approach Based on Minimizers

We consider the following set-valued optimization (SOP):

$$\min_{x \in S} F(x),$$

where $F : X \rightrightarrows Y$ and S is a subset of X. In this section, we study the solution concept based on the vector approach, which is introduced in Definition 2.6, trying to characterize the entire solution set by a fixed point approach.

For each $y \in F(S)$, the sublevel set of F of height y w.r.t. K_Y, as it is called in [10], is defined by

$$\text{Lev}_y(F) := \{x \in X : F(x) \cap (\{y\} - K_Y) \neq \emptyset\}.$$

It can be seen as an extension of the level set of a real value function. When F is single-valued and $(Y, K_Y) = (\mathbb{R}, \mathbb{R}_+)$, $\text{Lev}_y(F)$ just degenerates to $\{x \in X : F(x) \leq y\}$. In our fixed point approach for characterizing the whole set of minimizers of (SOP), $\text{Lev}_y(F)$ is a very crucial part. Here, y will play the role of a parameter.

For some $y \in F(S)$, setting

$$S(y) := \{x \in S : F(x) \cap (\{y\} - K_Y) \neq \emptyset\}$$

and

$$F_y(x) := F(x) \cap (\{y\} - K_Y),$$

we introduce the following parametric set optimization problem using the solution concept of minimizers introduced in Definition 2.6:

$$\min_{x \in S(y)} F_y(x). \tag{SOP)-(y}$$

Taking into account the definition of the sublevel set $\mathrm{Lev}_y(F)$, the feasible set of (SOP)-(y) is actually $S \cap \mathrm{Lev}_y(F)$. For $\mathrm{Lev}_y(F)$, we have the following result.

Proposition 3.1. *The sublevel set* $\mathrm{Lev}_y(F)$ *is convex for all* $y \in \mathrm{Im}\ F$ *if and only if*

$$\forall \lambda \in [0,1],\ x_1,\ x_2 \in S\ :\ (F(x_1) + K_Y) \cap (F(x_2) + K_Y) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + K_Y. \tag{3}$$

Consequently, if S *is a convex set and* F *satisfies (3), then* $S(y)$ *is convex for all* $y \in \mathrm{Im}\ F$.

Proof.

(a) Suppose that (3) holds. Picking arbitrary $x_1,\ x_2 \in \mathrm{Lev}_y(F)$, there exist $y_i \in F(x_i)$ and $q_i \in K_Y$, $i = 1,2$ such that $y = y_1 + q_1 = y_2 + q_2$. According to (3), for any $\lambda \in [0,1]$,

$$y \in (F(x_1) + K_Y) \cap (F(x_2) + K_Y) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + K_Y,$$

which implies $\lambda x_1 + (1-\lambda)x_2 \in \mathrm{Lev}_y(F)$. This means that $\mathrm{Lev}_y(F)$ is convex.

(b) Assume that $\mathrm{Lev}_y(F)$ is convex. Consider $y \in (F(x_1) + K_Y) \cap (F(x_2) + K_Y)$, then $x_1,\ x_2 \in \mathrm{Lev}_y(F)$. As $\mathrm{Lev}_y(F)$ is convex, $\lambda x_1 + (1-\lambda)x_2 \in \mathrm{Lev}_y(F)$ for all $\lambda \in [0,1]$. Hence, there exists some $y' \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y' \leq_{K_Y} y$, implying $y \in F(\lambda x_1 + (1-\lambda)x_2) + K_Y$, i.e., (3) holds. Then, the convexity of $S(y)$ is just a result of $S(y) = S \cap \mathrm{Lev}_y(F)$.

□

It is not hard to verify that condition (3) is fulfilled when F is a K_Y-convex map. Hence, the convexity of $S(y)$ can also be guaranteed if F is K_Y-convex and S is convex. As the feasible set of every parametric problem (SOP)-(y) is a subset of S, while for every $x \in X$, there is also $F(x) \cap (\{y\} - K_Y) \subseteq F(x)$, we can obtain the following proposition concerning the relationship between minimizers of the set optimization problem (SOP) and minimizers of the parametric set optimization problem (SOP)-(y) (see Definition 2.6).

Proposition 3.2. *Supposing that* $(\bar{x}, \bar{y}) \in \mathrm{graph}\ F$ *is a minimizer of* (SOP)*, then for every* $y \in F(S)$*,* $(\bar{x}, \bar{y})$ *is also a minimizer of* (SOP)-(y) *whenever* $\bar{x} \in S(y)$.

As for the inverse statement, it is provided as follows.

Proposition 3.3. *For every $y \in F(S)$, the minimizers $(\bar{x}, \bar{y}) \in \text{graph } F_y$ of the parametric problem* (SOP)-(y) *are also minimizers of* (SOP).

Proof. Picking $y \in F(S)$, assume that $(\bar{x}, \bar{y}) \in \text{graph } F_y$ is a minimizer of (SOP)-(y) but not of (SOP). Since $(\bar{x}, \bar{y}) \in \text{graph } F_y$ implies $(\bar{x}, \bar{y}) \in \text{graph } F$ and $S(y) \subseteq S$, there exists some $(\hat{x}, \hat{y}) \in \text{graph } F$ such that $\hat{x} \in S$ and $\hat{y} \preceq_{K_Y \setminus \{0_Y\}} \bar{y}$. Together with $(\bar{x}, \bar{y}) \in \text{graph } F_y$, it follows that $\hat{y} \in \{y\} - K_Y$. This indicates $\hat{x} \in S(y)$ and $(\hat{x}, \hat{y}) \in \text{graph } F_y$, which contradicts that $(\bar{x}, \bar{y})$ is a minimizer of (SOP)-(y). □

Next, we consider a special case of the parametric problem. Supposing that for some $y \in F(S)$, a minimizer $(\bar{x}, \bar{y})$ of (SOP)-(y) has already been found, now we try to investigate the parametric problem with $y = \bar{y}$.

Proposition 3.4. *Consider $y \in F(S)$. If $(\bar{x}, \bar{y}) \in \text{graph } F$ is a minimizer of* (SOP)-(y)*, then $(\bar{x}, \bar{y})$ is also a minimizer of* (SOP)-($\bar{y}$).

Proof. Let us consider $y \in F(S)$ and a minimizer $(\bar{x}, \bar{y}) \in \text{graph } F$ of (SOP)-(y). This yields $\bar{x} \in S$ and $\bar{y} \in F(\bar{x}) \cap (\{y\} - K_Y)$. From $\bar{y} \in F(\bar{x}) \cap (\{\bar{y}\} - K_Y)$, it follows that $\bar{x} \in S(\bar{y})$ and $(\bar{x}, \bar{y}) \in \text{graph } F_{\bar{y}}$. Supposing that $(\bar{x}, \bar{y})$ is not a minimizer of (SOP)-($\bar{y}$), there is some $(\hat{x}, \hat{y}) \in \text{graph } F_{\bar{y}}$ with $\hat{x} \in S(\bar{y})$ such that $\hat{y} \preceq_{K_Y \setminus \{0_Y\}} \bar{y}$. This implies, together with $\bar{y} \in \{y\} - K_Y$, that $\hat{y} \in F(\hat{x}) \cap \{y\} - K_Y$. Hence, $\hat{x}$ is also feasible for (SOP)-(y) and $(\hat{x}, \hat{y}) \in \text{graph } F_y$, contradicting that $(\bar{x}, \bar{y})$ is a minimizer of (SOP)-(y). □

Combining the above propositions, we can observe that $(\bar{x}, \bar{y}) \in \text{graph } F$ is a minimizer of (SOP) if and only if $(\bar{x}, \bar{y})$ is a minimizer of the parametric problem at $y = \bar{y}$, which is summarized as the following fixed point result. In order to formulate the fixed point theorem, we denote the set of minimizers to the parametric set optimization problem (SOP)-(y) by Sol{(SOP) −(y)}.

Theorem 3.1. *The pair $(\bar{x}, \bar{y}) \in \text{graph } F$ is a minimizer of* (SOP) *if and only if $(\bar{x}, \bar{y})$ is a fixed point of the set-valued map:*

$$\begin{aligned} \Phi : \text{graph } F &\rightrightarrows \text{graph } F \\ (x, y) &\longmapsto \text{Sol}\{(\text{SOP}) - (y)\}. \end{aligned}$$

According to this theorem, taking into account that (SOP)-(y) is a parametric optimization problem w.r.t. the parameter y, all minimizers of (SOP) should be reached. However, the parametric problem is still a set-valued optimization problem. The objective function F_y of (SOP)-(y) takes a single value only when y is a minimal element of $F(S)$. Solving the (SOP)-(y) may not be easy in the general case.

If $F(x)$ can be written in some analytic form, for instance, if $F(x)$ is a segment in a vector space that can be expressed as convex combinations of the endpoints, some scalarization method can be applied, cooperating with the fixed point approach.

For a given scalarization function $l : X \to \mathbb{R}$ that is strongly K_Y-monotone (i.e., for all $y_1, y_2 \in Y$, $y_1 \preceq_{K_Y \setminus \{0_Y\}} y_2$ implies $l(y_1) < l(y_2)$) and $y \in F(S)$, we consider the following scalar problem:

$$\min\{l(y') : y' \in F(x) \cap (\{y\} - K_Y),\ x \in S(y)\}. \tag{(SP$_l$)-(y)}$$

A pair $(\bar{x}, \bar{y})$ is said to be a solution of (SP_l)-(y) (with $y \in F(S)$) if $\bar{y} \in F(\bar{x}) \cap (\{y\} - K_Y)$, $\bar{x} \in S$ and $l(\bar{y}) \leq l(y')$ for all (x', y') with $y' \in F(x') \cap (\{y\} - K_Y)$, $x' \in S$. We denote the set of solutions to $(\mathrm{SP}_l) - (y)$ by $\mathrm{Sol}\{(\mathrm{SP}_l) - (y)\}$.

Here, the scalarization map l is fixed, while we still see y as the parameter. For a minimizer $(\bar{x}, \bar{y})$ of (SOP), it might not be a solution of the scalar problem for every y. However, it is a solution when the parameter $y = \bar{y}$. Therefore, we are able to deduce the following fixed point result for $(\mathrm{SP}_l) - (y)$.

Theorem 3.2. *The pair* $(\bar{x}, \bar{y}) \in \text{graph } F$ *is a minimizer of* (SOP) *if and only if* $(\bar{x}, \bar{y})$ *is a fixed point of the set-valued map*

$$\begin{aligned} \Phi_l : \text{graph } F &\rightrightarrows \text{graph } F \\ (x, y) &\longmapsto \mathrm{Sol}\{(\mathrm{SP}_l) - (y)\}. \end{aligned}$$

Proof.

(a) Let $(\bar{x}, \bar{y}) \in \text{graph } F$ be a minimizer of (SOP). Then, there are $\bar{y} \in F(\bar{x}) \cap (\{\bar{y}\} - K_Y)$ and $\bar{x} \in S$, such that $(\bar{x}, \bar{y})$ is feasible for $(\mathrm{SP}_l) - (\bar{y})$. Hence, it is a minimizer for $(\mathrm{SP}_l) - (\bar{y})$. Otherwise, there exists some (x', y') with $y' \in F(x') \cap (\{\bar{y}\} - K_Y)$, $x' \in S$ such that $l(y') < l(\bar{y})$, which means $y' \leq_{K_Y} \bar{y}$ and $y' \neq \bar{y}$, contradicting that $(\bar{x}, \bar{y})$ is a minimizer of (SOP). This means that $(\bar{x}, \bar{y})$ is a fixed point of Φ_l.

(b) Conversely, suppose that $(\bar{x}, \bar{y}) \in \text{graph } F$ is a fixed point of Φ_l, i.e., it is a minimizer of $(\mathrm{SP}_l) - (\bar{y})$, but not a minimizer of (SOP). Then, we can find some $(\hat{x}, \hat{y}) \in \text{graph } F$ with $\hat{x} \in S$ and $\hat{y} \leq_{K_Y \setminus \{0_Y\}} \bar{y}$. Hence, $(\hat{x}, \hat{y})$ is feasible for $(\mathrm{SP}_l) - (\bar{y})$. Meanwhile, $l(\hat{y}) < l(\bar{y})$ since l is strongly K_Y-monotone. This is a contradiction to $(\bar{x}, \bar{y}) \in \mathrm{Sol}\{(\mathrm{SP}_l) - (\bar{y})\}$.

□

As we mentioned above, in our approach, we consider $y \in F(S)$ as the parameter to find the entire set of minimizers to (SOP). It can be applied together with scalarization methods in some cases. Unlike some classical ways, where the scalarization function ranges as parameter, the map l is fixed here. In order to stress this difference and to illustrate our fixed point approach, we give an example.

Example 3.1. *Consider* $(X, K_X) = (\mathbb{R}, \mathbb{R}_+)$, $(Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$ *and define the set-valued objective map by*

$$F(x) := [F^L(x), F^R(x)] = \{(1-t)F^L(x) + tF^R(x) : \ t \in [0, 1]\},$$

where $F^L(x) : X \to Y$ *and* $F^R(x) : X \to Y$ *are given by*

$$F^L(x) := (2x - x^2, 1 - x^2), \ F^R(x) := (4x - x^2, 1 - x^2 + 2x).$$

The feasible set S *is given by*

$$S := \{x \in \mathbb{R} \ : \ g_1(x) \geq 0, \ g_2(x) \geq 0\},$$

where $g_1(x) := x$, $g_2(x) := 1 - x$ *for* $x \in \mathbb{R}$.

For any $y \in F(x)$ *with* $x \in S$*,*

$$y = (y_1, y_2) = (-x^2 + (2 + 2t)x,\ -x^2 + 2tx + 1)$$

for some $t \in [0,1]$*. Let* $l = (\lambda_1, \lambda_2) \in \text{int } \mathbb{R}^2_+$ *be fixed. The corresponding parametric problem* $(\text{SP}_l) - (y)$ *(with* $y \in F(x)$*,* $x \in S$*) is*

$$\begin{aligned}\min\{\lambda_1(y_1') + \lambda_2(y_2') : \quad & y_1' = -x'^2 + (2 + 2t')x' \leq y_1, y_2' = -x'^2 + 2t'x' + 1 \leq y_2,\\ & x' \geq 0,\ 1 - x' \geq 0, t' \in [0,1]\}.\end{aligned}$$

If $t > 0$ *and* $x > 0$*, we can always find some* $t' \in [0, t)$ *such that*

$$y' = (-x^2 + (2 + 2t')x,\ -x^2 + 2t'x + 1) \leq_{\text{int } \mathbb{R}^2_+} y,$$

resulting in $l(y') < l(y)$*. This means* (x, y) *is not a solution of* $(\text{SP}_l) - (y)$*. If* $t = 0$ *or* $x = 0$*, then* $S(y) = \{x\}$ *and* $F(x) \cap (\{y\} - \mathbb{R}^2_+) = \{y\}$*, showing that the solution of* $(\text{SP}_l) - (y)$ *is* (x, y)*. Varying* y*, all the solutions of* $(\text{SP}_l) - (y)$ *are given by* $\{(x, y) : x \in [0,1],\ y = (2x - x^2,\ 1 - x^2)\}$*, which is the set of minimizers of* (SOP)*, as it obviously is to check.*

Next, we try to handle the set optimization problem (SOP) *by fixing* $l = (\lambda_1, \lambda_2) \in \text{int } \mathbb{R}^2_+$ *as parameter y. Consider the problem* (SP_l)*-*(λ_1, λ_2)*:*

$$\begin{aligned}\min\{\lambda_1(y_1') + \lambda_2(y_2') : \quad & y_1' = -x'^2 + (2 + 2t')x', y_2' = -x'^2 + 2t'x' + 1\\ & x' \geq 0,\ 1 - x' \geq 0, t' \in [0,1]\}.\end{aligned}$$

It is not hard to observe that the solutions of (SP_l)*-*(λ_1, λ_2) *are only given by*

$$\begin{cases} x = 1 \\ t = 0, \end{cases} \text{or} \begin{cases} x = 0 \\ t = 0 \end{cases} \text{and} \begin{cases} x = 1 \\ t = 0, \end{cases} \text{or} \begin{cases} x = 0 \\ t = 0, \end{cases}$$

depending on respectively $\lambda_1 < \lambda_2$ *or* $\lambda_1 = \lambda_2$ *or* $\lambda_1 > \lambda_2$*. Therefore,* (SP_l)*-*(λ_1, λ_2) *fails to detect all the minimizers of* (SOP) *even if* (λ_1, λ_2) *ranges through all the values in* $\text{int } \mathbb{R}^2_+$*.*

4 Min-max Minimizers of Set-valued Optimization Problems

In the last section, the set optimization problem (SOP) has been investigated with a solution concept based on the vector approach (see Definition 2.6). This concept only involves the information of the minimal part of $F(x)$. However, there might be cases where the minimal elements of $F(\bar{x})$ are small, while the maximal points of $F(\bar{x})$ are all very large. Meanwhile, there is another set, say $F(\hat{x})$, whose minimal points are just a little bit bigger than the ones of $F(\bar{x})$, while the maximal points are much more smaller than the ones of $F(\bar{x})$ (see Example 4.1). In some situations, we need to consider the whole set of $F(x)$. For instance, $F(x) = \{f(x, \xi) : \xi \in \Omega\}$. Here, $f(x, \xi) : X \times \Omega \to Y$ is a function involving a random variable ξ, where Ω is finite and every ξ is taken with the same probability. In this case, we might find that $\hat{x}$ is also a good choice.

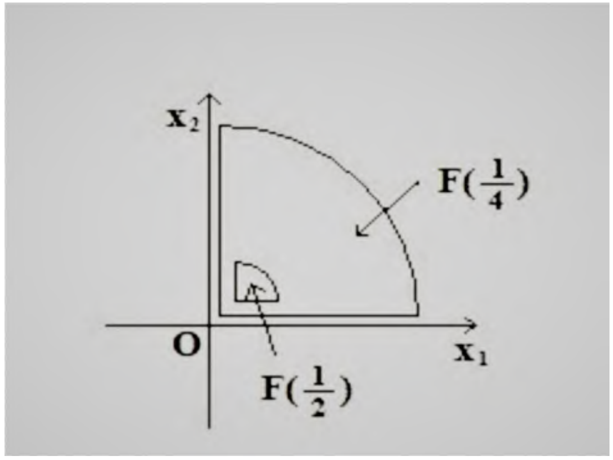

Figure 1: Example 4.1

Example 4.1. *Let* $(X, K_X) = (\mathbb{R}, \mathbb{R}_+)$ *and* $(Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$*, where the linear operations are defined pointwisely. Define* $F : X \rightrightarrows Y$ *as*

$$F(x):=f_1(x)k_0 + (B(0_{\mathbb{R}^2}, f_2(x)) \cap \mathbb{R}^2_+),$$

where

$$f_1(x):=\frac{1}{10}x,\ f_2(x):=\frac{1}{x^3},\ \text{and } k_0 = (1, 1).$$

That indicates (see (4), (5))

$$F_{min}(x) = \frac{1}{10}x(1, 1),$$
$$F_{max}(x) = \frac{1}{10}x(1, 1) + \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1^2 + y_2^2 = \frac{1}{x^3}\}.$$

Set $S = \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$*, then*

$$F(\frac{1}{4}) = (\frac{1}{40}, \frac{1}{40}) + (B(0_{\mathbb{R}^2}, 64) \cap \mathbb{R}^2_+),$$
$$F(\frac{1}{3}) = (\frac{1}{30}, \frac{1}{30}) + (B(0_{\mathbb{R}^2}, 27) \cap \mathbb{R}^2_+),$$
$$F(\frac{1}{2}) = (\frac{1}{20}, \frac{1}{20}) + (B(0_{\mathbb{R}^2}, 8) \cap \mathbb{R}^2_+),$$

see Figure 1.1.

According to Definition 2.6, $(\frac{1}{4}, (\frac{1}{40}, \frac{1}{40}))$ *will be taken as the only minimizer of (SOP). As for* $x = \frac{1}{2}$*, it will not be considered as a good decision, i.e., any pair* $(\frac{1}{2}, y) \in \text{graph } F$ *is not a minimizer of (SOP), even though the maximal points of* $F(\frac{1}{2})$ *are much smaller than the ones of* $F(\frac{1}{4})$.

Here, we give another solution concept for set-valued optimization problems. This solution concept is still based on the concept of efficient solutions to vector optimization problems, but it also takes the maximal part of the set into consideration.

For $F : X \rightrightarrows Y$, define

$$F_{min}(x):=\{y \in F(x) : \{y\} - K_Y\backslash\{0_Y\} \cap F(x) = \emptyset\} = \text{Min } F(x), \tag{4}$$

$$F_{max}(x):=\{y \in F(x) : \{y\} + K_Y\backslash\{0_Y\} \cap F(x) = \emptyset\} = \text{Max } F(x). \tag{5}$$

Definition 4.1. *An element* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max}) \in \text{graph } (F_{min}, F_{max})$ *is said to be a min-max minimizer of* (SOP) *if* $\bar{x} \in S$ *and there does not exist* $(x, y_{min}, y_{max}) \in \text{graph } (F_{min}, F_{max})$ *such that* $x \in S$ *and*

$$(y_{min}, y_{max}) \preceq_{K_Y \times K_Y \backslash\{(0_Y, 0_Y)\}} (\bar{y}_{min}, \bar{y}_{max}).$$

If we apply Definition 4.1 to Example 4.1, any point $(\frac{1}{2}, y_{max})$ with $y_{max} \in (\frac{1}{20}, \frac{1}{20}) + \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1^2 + y_2^2 = 8\}$ is also a minimizer.

When dealing with set-valued optimization problems, another widely used way to define the solution concept is using the set approach. Different kinds of set relations (see Definition 2.7) have been introduced for the comparison of sets. For a set relation $\preceq^*_{K_Y}$, we consider the set-valued optimization problem (SOP).

Definition 4.2. *An element* $\bar{x} \in S$ *is called a minimal solution of problem* (SOP) *w.r.t. the preorder* $\preceq^*_{K_Y}$ *if*

$$F(x) \preceq^*_{K_Y} F(\bar{x}) \textit{ for some } x \in S \implies F(\bar{x}) \preceq^*_{K_Y} F(x).$$

This is also a solution concept concerning the whole information of the set $F(x)$. It is necessary to make a comparison between the solution concept based on set relations and the min-max minimizers. We shall mainly focus on the set less relation $\preceq^s_{K_Y}$ (see Definition 4.3) and the min-max set less relation $\preceq^m_{K_Y}$ (see Definition 4.4).

The set less relation $\preceq^s_{K_Y}$ is introduced by [18] and [21] independently.

Definition 4.3. *Let* $K_Y \subseteq Y$ *be a proper, closed, convex and pointed cone. Furthermore, let* $A, B \in 2^Y$ *be arbitrarily chosen nonempty sets. Then, the set less relation* $\preceq^s_{K_Y}$ *is defined by*

$$A \preceq^s_{K_Y} B :\iff A \preceq^l_{K_Y} B \textit{ and } A \preceq^u_{K_Y} B \iff A + K_Y \supseteq B \textit{ and } A \subseteq B - K_Y.$$

For the preorder $\preceq^s_{K_Y}$, we have the following facts.

Fact 4.1. $\bar{x} \in S$ *is a minimal solution of problem* (SOP) *w.r.t. the preorder* $\preceq^s_{K_Y}$
does not imply:
There exist some $\bar{y}_{min} \in \text{Min } F(\bar{x})$ *and* $\bar{y}_{max} \in \text{Max } F(\bar{x})$ *such that* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ *is a min-max minimizer of* (SOP)*.*

Even if Assumption $\mathscr{A}$ *holds, this implication is still not true.*

We present two counter examples to illustrate this fact.

Example 4.2. *Let* $(X, K_X) = (Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$*. Suppose* $S = \{\bar{x}, \hat{x}\}$*, while* $F(\bar{x})$ *and* $F(\hat{x})$ *are presented in Figure 2.*

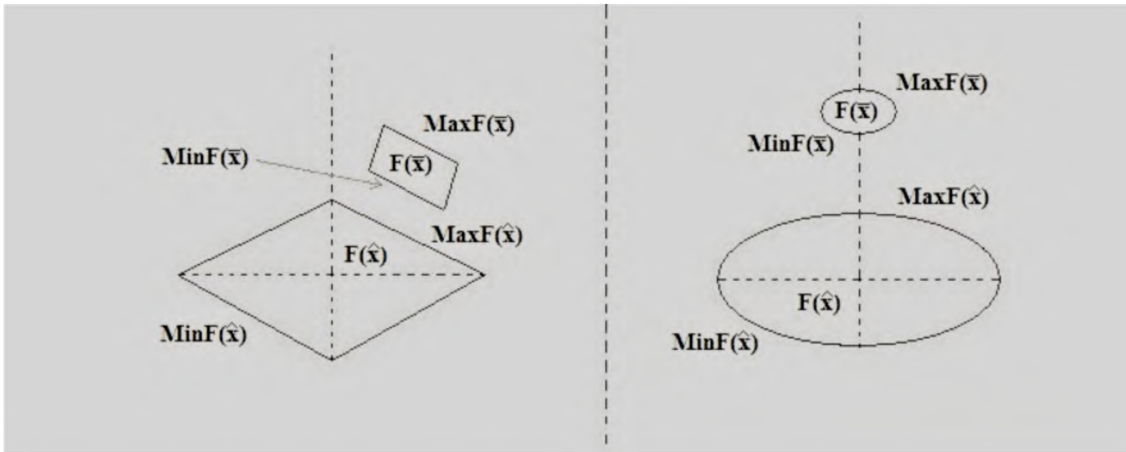

Figure 2: Example 4.2

We can see that Assumption $\mathscr{A}$ holds in both examples. As $F(\hat{x}) \not\subseteq F(\bar{x}) - K_Y$, which means $F(\hat{x}) \not\preceq^s_{K_Y} F(\bar{x})$, there does not exist $x \in S$ such that $F(x) \preceq^s_{K_Y} F(\bar{x})$. Therefore, $\bar{x}$ is a minimal solution of problem (SOP) *w.r.t. the preorder $\preceq^s_{K_Y}$.*

However, for any $(\bar{y}_{min}, \bar{y}_{max}) \in \text{Min } F(\bar{x}) \times \text{Max } F(\bar{x})$, we can always find some $(\hat{y}_{min}, \hat{y}_{max}) \in \text{Min } F(\hat{x}) \times \text{Max } F(\hat{x})$ such that

$$(\hat{y}_{min}, \hat{y}_{max}) \preceq_{(K_Y \times K_Y)\setminus\{(0_Y,0_Y)\}} (\bar{y}_{min}, \bar{y}_{max}).$$

This indicates that for any $(\bar{y}_{min}, \bar{y}_{max}) \in \text{Min } F(\bar{x}) \times \text{Max } F(\bar{x})$, $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is not a min-max minimizer of (SOP) *in the sense of Definition 4.1.*

Fact 4.2. *$(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of* (SOP) *in the sense of Definition 4.1 for any $\bar{y}_{min} \in \text{Min } F(\bar{x})$ and $\bar{y}_{max} \in \text{Max } F(\bar{x})$*
does not imply
$\bar{x} \in S$ is a minimal solution of problem (SOP) *w.r.t. the preorder $\preceq^s_{K_Y}$ in the sense of Definition 4.2. Even if Assumption $\mathscr{A}$ holds, this implication is still not true.*

We also present two counter examples here to illustrate this fact, one, where Assumption $\mathscr{A}$ is satisfied, while in the other one not.

Example 4.3. *Let $(X, K_X) = (Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$. Suppose $S = \{\bar{x}, \hat{x}\}$, while $F(\bar{x})$ and $F(\hat{x})$ are presented in Figure 3.*

$F(\hat{x})$ consists of the half open segment $(y^o, y^1]$ and the point y^2. $\text{Min } F(\hat{x})$ and $\text{Max } F(\hat{x})$ are both single point sets, i.e., $\text{Min } F(\hat{x}) = \{y^2\}$ and $\text{Max } F(\hat{x}) = \{y^1\}$.

We can see that Assumption $\mathscr{A}$ is not satisfied as $y^o \notin F(\hat{x})$ (consequently, $F(\hat{x}) \not\subseteq \text{Min } F(\hat{x}) + K_Y$). For every $\bar{y}_{min} \in F(\bar{x})$, $y^2 \not\preceq_{K_Y} \bar{y}_{min}$. This implies there does not exist any $(\hat{y}_{min}, \hat{y}_{max}) \in \text{Min } F(\hat{x}) \times \text{Max } F(\hat{x})$ such that

$$(\hat{y}_{min}, \hat{y}_{max}) \preceq_{(K_Y \times K_Y)} (\bar{y}_{min}, \bar{y}_{max}),$$

for any $\bar{y}_{min} \in \text{Min } F(\bar{x})$ and $\bar{y}_{max} \in \text{Max } F(\bar{x})$. Therefore, $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of (SOP) *in the sense of Definition 4.1 for any $\bar{y}_{min} \in \text{Min } F(\bar{x})$ and $\bar{y}_{max} \in \text{Max } F(\bar{x})$.*

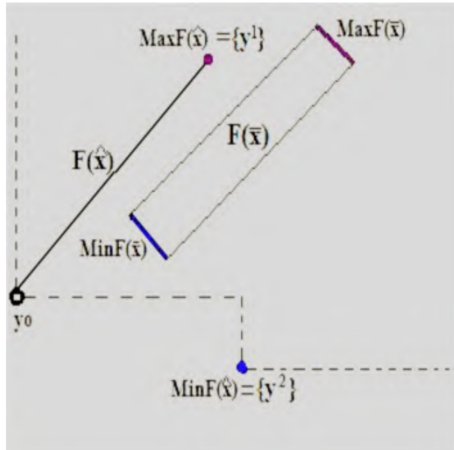

Figure 3: Example 4.3

However, $F(\bar{x}) \subseteq F(\hat{x}) + K_Y$ and $F(\hat{x}) \subseteq F(\bar{x}) - K_Y$, which means $F(\hat{x}) \preceq^s_{K_Y} F(\bar{x})$. Also, $F(\bar{x}) \npreceq^s_{K_Y} F(\hat{x})$. Hence, $\bar{x}$ is not a minimal solution of problem (SOP) *w.r.t. the preorder $\preceq^s_{K_Y}$ in the sense of Definition 4.2.*

Example 4.4. *Let $(X, K_X) = (Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$. Suppose $S = \{\bar{x}, \hat{x}\}$, while $F(\bar{x})$ and $F(\hat{x})$ are presented in Figure 4.*

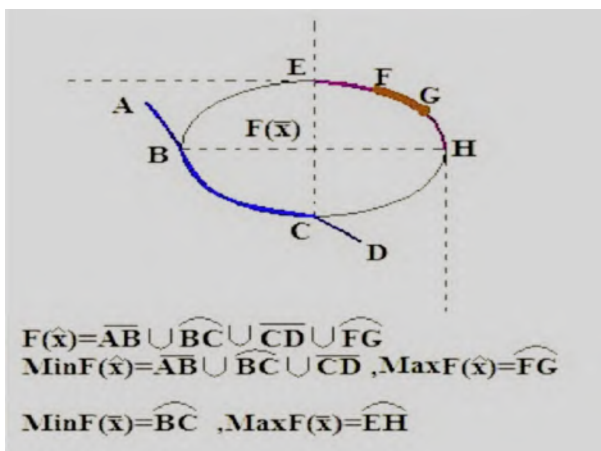

Figure 4: Example 4.4

We can see that Assumption $\mathscr{A}$ holds here. For every $(\bar{y}_{min}, \bar{y}_{max}) \in \text{Min } F(\bar{x}) \times \text{Max } F(\bar{x})$, there does not exist any $(\hat{y}_{min}, \hat{y}_{max}) \in \text{Min } F(\hat{x}) \times \text{Max } F(\hat{x})$ such that

$$(\hat{y}_{min}, \hat{y}_{max}) \preceq_{(K_Y \times K_Y)\setminus\{(0_Y, 0_Y)\}} (\bar{y}_{min}, \bar{y}_{max}),$$

Therefore, $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of (SOP) *for every $(\bar{y}_{min}, \bar{y}_{max}) \in \text{Min } F(\bar{x}) \times \text{Max } F(\bar{x})$.*

However, as

$$\text{Min } F(\bar{x}) \subseteq \text{Min } F(\hat{x}),$$
$$\text{Max } F(\hat{x}) \subseteq \text{Max } F(\bar{x}),$$

with Assumption $\mathscr{A}$, we can still deduce $F(\bar{x}) \subseteq F(\hat{x}) + K_Y$ and $F(\hat{x}) \subseteq F(\bar{x}) - K_Y$. This means $F(\hat{x}) \preceq^s_{K_Y} F(\bar{x})$. Also, $F(\bar{x}) \npreceq^s_{K_Y} F(\hat{x})$ as $F(\hat{x}) \not\subseteq F(\bar{x}) + K_Y$. Hence, $\bar{x}$ is not a minimal solution of problem (SOP) *w.r.t. the preorder $\preceq^s_{K_Y}$ in the sense of Definition 4.2.*

Besides the set less relation, another set relation we shall consider here is the minmax set less relation.

Set

$$\mathscr{F} := \{A \in 2^Y : \text{Min } A \neq \emptyset \text{ and } \text{Max } A \neq \emptyset\}.$$

Definition 4.4. *Let A, B be sets belonging to $\mathscr{F}$. Then, the minmax set less relation $\preceq^m_{K_Y}$ is defined by*

$$A \preceq^m_{K_Y} B :\Longleftrightarrow \text{Min } A \preceq^s_{K_Y} \text{Min } B \textit{ and } \text{Max } A \preceq^s_{K_Y} \text{Max } B.$$

When Assumption $\mathscr{A}$ holds, we have the following result.

Lemma 4.1. *Suppose Assumption $\mathscr{A}$ is fulfilled. If $F(\bar{x}) \preceq^m_{K_Y} F(\hat{x})$, then $F(\bar{x}) \preceq^s_{K_Y} F(\hat{x})$.*

Proof. Set $A := F(\bar{x})$ and $B := F(\hat{x})$. Then, it follows from $F(\bar{x}) \preceq^m_{K_Y} F(\hat{x})$ that

$$\text{Min } A \preceq^s_{K_Y} \text{Min } B \text{ and } \text{Max } A \preceq^s_{K_Y} \text{Max } B,$$

i.e.,

$$\text{Min } B \subseteq \text{Min } A + K_Y, \tag{6}$$

$$\text{Min } A \subseteq \text{Min } B - K_Y, \tag{7}$$

$$\text{Max } B \subseteq \text{Max } A + K_Y, \tag{8}$$

$$\text{Max } A \subseteq \text{Max } B - K_Y. \tag{9}$$

Applying (6), together with Assumption $\mathscr{A}$, we get

$$B \subseteq \text{Min } B + K_Y \subseteq \text{Min } A + K_Y + K_Y \subseteq A + K_Y. \tag{10}$$

Applying (9), together with Assumption $\mathscr{A}$, we get

$$A \subseteq \text{Max } A - K_Y \subseteq \text{Max } B - K_Y - K_Y \subseteq B - K_Y. \tag{11}$$

Then, (10) and (11) indicate $A \preceq^s_{K_Y} B$. □

Assumption $\mathscr{A}$ is necessary for the assertion above. Without it, the conclusion can not be guaranteed. We give an example to demonstrate that.

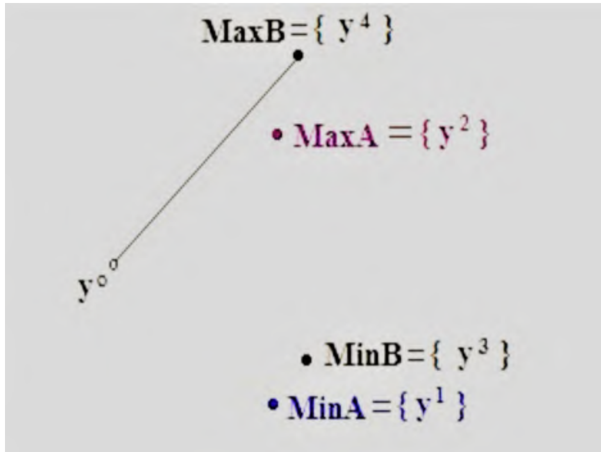

Figure 5: Example 4.5

Example 4.5. *Let* $(X, K_X) = (Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$*. Suppose* $A = \{y^1, y^2\}$*, while* B *consists of point* y^3 *and half open segment* $(y^o, y^4]$*, as presented in Figure 5.*

Assumption $\mathscr{A}$ *does not hold here. Also, there is* $A \preceq^m_{K_Y} B$*. However,* $A \npreceq^s_{K_Y} B$ *as* $B \not\subseteq A + K_Y$*.*

Considering minimal solutions of (SOP) w.r.t. preorder $\preceq^m_{K_Y}$ in the sense of Definition 4.2, in comparison with the concept of min-max minimizers in the sense of Definition 4.1, we obtain the following results.

Fact 4.3. $\bar{x} \in S$ *is a minimal solution of problem* (SOP) *w.r.t. the preorder* $\preceq^m_{K_Y}$ *in the sense of Definition 4.2*
does not imply:
There exist some $\bar{y}_{min} \in \text{Min } F(\bar{x})$ *and* $\bar{y}_{max} \in \text{Max } F(\bar{x})$ *such that* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ *is a min-max minimizer of* (SOP)*. Even if Assumption* $\mathscr{A}$ *holds, this implication is still not true.*

An example is presented here to demonstrate this fact.

Example 4.6. *Let* $(X, K_X) = (Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$*. Suppose* $S = \{\bar{x}, \hat{x}\}$*, while* $F(\bar{x})$ *and* $F(\hat{x})$ *are presented in Figure 6.*

It is clear that Assumption $\mathscr{A}$ *holds here. As* $\text{Max } F(\hat{x}) \not\subseteq \text{Max } F(\bar{x}) - K_Y$*, which means* $F(\hat{x}) \npreceq^m_{K_Y} F(\bar{x})$*, there does not exist* $x \in S$ *with* $F(x) \preceq^m_{K_Y} F(\bar{x})$*. Therefore,* $\bar{x}$ *is a minimal solution of problem* (SOP) *w.r.t. the preorder* $\preceq^m_{K_Y}$*.*

However, for any $(\bar{y}_{min}, \bar{y}_{max}) \in \text{Min } F(\bar{x}) \times \text{Max } F(\bar{x})$*, we can always find some* $(\hat{y}_{min}, \hat{y}_{max}) \in \text{Min } F(\hat{x}) \times \text{Max } F(\hat{x})$ *such that*

$$(\hat{y}_{min}, \hat{y}_{max}) \preceq_{(K_Y \times K_Y) \setminus \{(0_Y, 0_Y)\}} (\bar{y}_{min}, \bar{y}_{max}).$$

This indicates that for any $(\bar{y}_{min}, \bar{y}_{max}) \in \text{Min } F(\bar{x}) \times \text{Max } F(\bar{x})$*,* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ *is not a min-max minimizer of* (SOP) *in the sense of Definition 4.1.*

Fact 4.4. *The fact that there exist some* $\bar{y}_{min} \in \text{Min } F(\bar{x})$ *and* $\bar{y}_{max} \in \text{Max } F(\bar{x})$ *such that* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ *is a min-max minimizer of* (SOP) *in the sense of Definition 4.1*

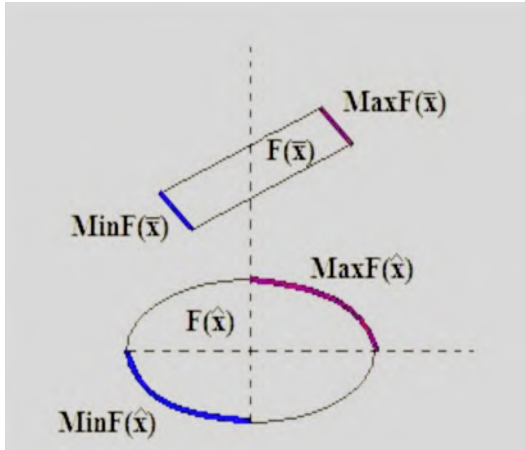

Figure 6: Example 4.6

does not imply:
$\bar{x} \in S$ *is a minimal solution of problem* (SOP) *w.r.t. the preorder* $\preceq^m_{K_Y}$ *in the sense of Definition 4.2. Even if Assumption* $\mathscr{A}$ *holds, this implication is still not true.*

We also present a counter example here to illustrate this fact.

Example 4.7. *Let* $(X, K_X) = (Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$. *Suppose* $S = \{\bar{x}, \hat{x}\}$, $F(\hat{x})$ *is the closed elliptical area, while* $F(\bar{x})$ *consists of segments* $[A, \bar{y}]$, $[\bar{y}, C]$ *and the curve* $\widehat{BD}$, *as presented in Figure 7.*

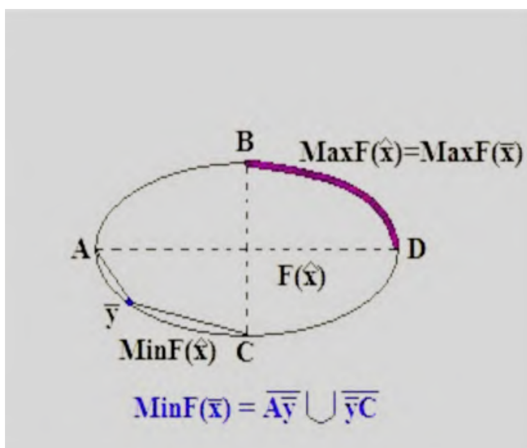

Figure 7: Example 4.7

Assumption $\mathscr{A}$ *holds for* F *here.*

For $\bar{x}$, *there is* $\bar{y} \in \operatorname{Min} F(\bar{x})$. *As* $\operatorname{Max} F(\bar{x}) = \operatorname{Max} F(\hat{x})$, *and there does not exist any* $y \in F(\bar{x}) \cup F(\hat{x})$ *such that* $y \preceq_{K_Y \setminus \{0_Y\}} \bar{y}$, $(\bar{x}, \bar{y}, \cdot)$, *where '·' can be any point in* $\operatorname{Max} F(\bar{x})$, *is a min-max minimizer of* (SOP).

However, $F(\hat{x}) \preceq^m_{K_Y} F(\bar{x})$, *while* $F(\bar{x}) \npreceq^m_{K_Y} F(\hat{x})$*(as* $\operatorname{Min} F(\bar{x}) \npreceq^s_{K_Y} \operatorname{Min} F(\hat{x})$*). Hence,* $\bar{x}$ *is not a minimal solution of problem* (SOP) *w.r.t. the preorder* $\preceq^m_{K_Y}$.

According to the fact above, the existence of min-max minimizers can not guarantee a solution w.r.t. the set relation $\preceq^m_{K_Y}$. However, it will provide a minimal point w.r.t. $\preceq^m_{K_Y}$ if all $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ are min-max minimizers, and Assumption $\mathscr{A}$ is not necessary for this assertion.

Proposition 4.1. *Supposing that for any $\bar{y}_{min} \in \text{Min } F(\bar{x})$ and $\bar{y}_{max} \in \text{Max } F(\bar{x})$, $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of* (SOP)*, then $\bar{x} \in S$ is a minimal solution of problem* (SOP) *w.r.t. the preorder $\preceq^m_{K_Y}$.*

Proof. Suppose there exists some $\hat{x}$ such that $F(\hat{x}) \preceq^m_{K_Y} F(\bar{x})$, which means

$$\text{Min } F(\hat{x}) \preceq^s_{K_Y} \text{Min } F(\bar{x}) \text{ and } \text{Max } F(\hat{x}) \preceq^s_{K_Y} \text{Max } F(\bar{x}), \tag{12}$$

i.e.

$$\text{Min } F(\bar{x}) \subseteq \text{Min } F(\hat{x}) + K_Y, \tag{13}$$

$$\text{Min } F(\hat{x}) \subseteq \text{Min } F(\bar{x}) - K_Y, \tag{14}$$

$$\text{Max } F(\bar{x}) \subseteq \text{Max } F(\hat{x}) + K_Y, \tag{15}$$

$$\text{Max } F(\hat{x}) \subseteq \text{Max } F(\bar{x}) - K_Y. \tag{16}$$

With (13) and (15), we can deduce that, for any $(\bar{y}_{min}, \bar{y}_{max}) \in \text{Min } F(\bar{x}) \times \text{Max } F(\bar{x})$, there is some $(\hat{y}_{min}, \hat{y}_{max}) \in \text{Min } F(\hat{x}) \times \text{Max } F(\hat{x})$ such that

$$(\hat{y}_{min}, \hat{y}_{max}) \preceq_{K_Y \times K_Y} (\bar{y}_{min}, \bar{y}_{max}).$$

If $\hat{y}_{min} \preceq_{K_Y \setminus \{0_Y\}} \bar{y}_{min}$ or $\hat{y}_{max} \preceq_{K_Y \setminus \{0_Y\}} \bar{y}_{max}$, then it contradicts that every $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of (SOP). Hence, it has to be $(\hat{y}_{min}, \hat{y}_{max}) = (\bar{y}_{min}, \bar{y}_{max})$. Then, the arbitrariness of $(\bar{y}_{min}, \bar{y}_{max})$ indicates

$$\text{Min } F(\bar{x}) \subseteq \text{Min } F(\hat{x}) \text{ and } \text{Max } F(\bar{x}) \subseteq \text{Max } F(\hat{x}). \tag{17}$$

Together with (14) and (16), we can get

$$\text{Min } F(\hat{x}) \subseteq \text{Min } F(\bar{x}) - K_Y \subseteq \text{Min } F(\hat{x}) - K_Y,$$

and

$$\text{Max } F(\hat{x}) \subseteq \text{Max } F(\bar{x}) - K_Y \subseteq \text{Max } F(\hat{x}) - K_Y.$$

This implies

$$\text{Min } F(\hat{x}) \subseteq \text{Min } F(\bar{x}) \text{ and } \text{Max } F(\hat{x}) \subseteq \text{Max } F(\bar{x}). \tag{18}$$

Hence, together with (17), we have

$$\text{Min } F(\hat{x}) = \text{Min } F(\bar{x}) \text{ and } \text{Max } F(\hat{x}) = \text{Max } F(\bar{x}),$$

which implies $F(\bar{x}) \preceq^m_{K_Y} F(\hat{x})$. Therefore, $\bar{x} \in S$ is a minimal solution of problem (SOP) w.r.t. the preorder $\preceq^m_{K_Y}$. □

When coping with the set-valued optimization problem (SOP), the min-max minimizer in the sense of Definition 4.1 and the minimal solution w.r.t. a certain set relation in the sense of Definition 4.2 both use more information of $F(x)$, rather than just the minimal points, to compare the values. However, they are different solution concepts in general, according to these facts and examples presented above.

5 Fixed Point Approach Based on Min-max Minimizers

Although Definition 4.1 involves the maximal part of $F(x)$, it is still a concept based on the vector approach. This makes it easier to extend our fixed point approach to the concept of min-max minimizers, which we will investigate in this section.

Consider the general set-valued optimization problem (SOP):

$$\min_{x \in S} F(x),$$

where $F : X \rightrightarrows Y$ and $S \subseteq X$. To apply the concept of min-max minimizers (Definition 4.1), we introduce the notion $\text{Dom } F := \{x \in X : F_{min}(x) \neq \emptyset, \ F_{max}(x) \neq \emptyset\}$, and assume $S \subseteq \text{Dom } F$.

For $x' \in S$, picking $y'_{min} \in F_{min}(x')$ and $y'_{max} \in F_{max}(x')$, define the set

$$S(y'_{min}, y'_{max}) := \{x \in S : F_{min}(x) \cap (\{y'_{min}\} - K_Y) \neq \emptyset, \ F_{max}(x) \cap (\{y'_{max}\} - K_Y) \neq \emptyset\}.$$

With a similar argument as in the proof of Proposition 3.1, it is not hard to observe that $S(y'_{min}, y'_{max})$ is convex if S is convex and

$$\begin{aligned} \forall \lambda \in [0,1], \ x_1, \ x_2 \in S : \quad & (F_{min}(x_1) + K_Y) \cap (F_{min}(x_2) + K_Y) \subseteq F_{min}(\lambda x_1 + (1-\lambda)x_2) + K_Y, \\ & (F_{max}(x_1) + K_Y) \cap (F_{max}(x_2) + K_Y) \subseteq F_{max}(\lambda x_1 + (1-\lambda)x_2) + K_Y. \end{aligned}$$

Now, we introduce the parametric problem

$$\min_{x \in S(y'_{min}, y'_{max})} F(x). \qquad \text{(SOP)-}(y'_{min}, y'_{max})$$

Proposition 5.1. *Suppose* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max}) \in \text{graph}(F_{min}, F_{max})$ *is a min-max minimizer of* (SOP)*, then it is also a min-max minimizer of* (SOP)-(y'_{min}, y'_{max}) *for any* (y'_{min}, y'_{max}) *such that* $\bar{x} \in S(y'_{min}, y'_{max})$.

Proof. Assume $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ fails to be a min-max minimizer of (SOP)-(y'_{min}, y'_{max}) for some (y'_{min}, y'_{max}) with $\bar{x} \in S(y'_{min}, y'_{max})$. Then, there exist $\hat{x} \in S$ and $(\hat{y}_{min}, \hat{y}_{max}) \in F_{min}(\hat{x}) \times F_{max}(\hat{x})$, satisfying $(\hat{y}_{min}, \hat{y}_{max}) \preceq_{K_Y \times K_Y \setminus \{(0_Y, 0_Y)\}} (\bar{y}_{min}, \bar{y}_{max})$. This contradicts that $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of (SOP).

□

Proposition 5.2. *Given* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max}) \in \text{graph}\,(F_{min}, F_{max})$*. If* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ *(with* $\bar{x} \in S$*) is a min-max minimizer of* (SOP)-$(\bar{y}_{min}, \bar{y}_{max})$*, then* $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ *is also a min-max minimizer of* (SOP).

Proof. If $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is not a min-max minimizer of (SOP), then we can find $\hat{x} \in S$ and $(\hat{y}_{min}, \hat{y}_{max}) \in F_{min}(\hat{x}) \times F_{max}(\hat{x})$ with

$$(\hat{y}_{min}, \hat{y}_{max}) \preceq_{K_Y \times K_Y \setminus \{(0_Y, 0_Y)\}} (\bar{y}_{min}, \bar{y}_{max}). \tag{19}$$

This indicates

$$\hat{y}_{min} \in F_{min}(\hat{x}) \cap (\{\bar{y}_{min}\} - K_Y) \text{ and } \hat{y}_{max} \in F_{max}(\hat{x}) \cap (\{\bar{y}_{max}\} - K_Y),$$

showing that $\hat{x}$ is also feasible for the parametric problem (SOP)-$(\bar{y}_{min}, \bar{y}_{max})$. Then, (19) gives a contradiction to the fact that $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of (SOP)-$(\bar{y}_{min}, \bar{y}_{max})$. □

Noting that $\bar{x} \in S$ is naturally feasible for (SOP)-$(\bar{y}_{min}, \bar{y}_{max})$ whenever $(\bar{y}_{min}, \bar{y}_{max}) \in F_{min}(\bar{x}) \times F_{max}(\bar{x})$, as $\bar{y}_{min} \in F_{min}(\bar{x}) \cap (\{\bar{y}_{min}\} - K_Y)$ and $\bar{y}_{max} \in F_{max}(\bar{x}) \cap (\{\bar{y}_{max}\} - K_Y)$. Hence, combining Propositions 5.1 and 5.2, we are able to provide a fixed point result for min-max minimizers in the sense of Definition 4.1. We denote the set of min-max minimizers to (SOP)-(y'_{min}, y'_{max}) by $(\min, \max)$-Sol$\{(\text{SOP}) - (y'_{min}, y'_{max})\}$.

Theorem 5.1. *Let $(\bar{x}, \bar{y}_{min}, \bar{y}_{max}) \in$ graph (F_{min}, F_{max}) be given. Then, $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a min-max minimizer of* (SOP) *if and only if $(\bar{x}, \bar{y}_{min}, \bar{y}_{max})$ is a fixed point of the following set-valued map:*

$$\begin{aligned} \Psi : \text{graph } (F_{min}, F_{max}) &\rightrightarrows \text{graph } (F_{min}, F_{max}) \\ (x', y'_{min}, y'_{max}) &\longmapsto (\min, \max)\text{-Sol}\{(\text{SOP}) - (y'_{min}, y'_{max})\}. \end{aligned}$$

Remark 5.1. *The notion of min-max minimizer depends on the information of the minimal as well as the maximal points of $F(x)$. Tackling the problem* (SOP) *with this idea, we need to figure out what are $F_{min}(x)$ and $F_{max}(x)$, which is not an easy task in the general case. For this problem, the fixed point approach provided in Section 3 might be used as an auxiliary step. Considering the solution concept in Definition 2.6, solving the problem $\min_{x \in S_x} F(x)$, where $S_x = \{x\}$, should give us the set $F_{min}(x)$, while, $F_{max}(x)$ can also be obtained in a similar way.*

Due to the special structure of min-max minimizer concept, it is worth paying special attentions to a case in which both F_{min} and F_{max} are single value maps. Throughout the rest of this section, we assume that F_{min}, F_{max} are both vector-valued functions. In such a situation, the min-max minimizer idea will perform better as F_{min} and F_{max} are easier not only to obtain but also to cope with. More importantly, F_{min} and F_{max} can give a better description for the entire set $F(x)$, especially when Assumption $\mathscr{A}$ is fulfilled, which means for every $x \in S$

$$\forall y \in F(x): \quad F_{min}(x) \preceq_{K_Y} y \preceq_{K_Y} F_{max}(x) \tag{20}$$

in this case. For instance, if every $F(x)$ is a segment with endpoints $F_{min}(x)$ and $F_{max}(x)$, then F is completely characterized by F_{min} and F_{max}.

Since F_{min} and F_{max} are single value maps, for every $x \in S$, the point $(x, y_{min}, y_{max}) \in$ graph(F_{min}, F_{max}) can be specified by $(x, F_{min}(x), F_{max}(x))$. In view of this, the definition of min-max minimizers can be reformulated as follows.

Definition 5.1. *For the set-valued optimization problem* (SOP)*, an element $\bar{x} \in S$ is said to be a min-max solution of* (SOP) *if $\bar{x} \in S$ and there does not exist any $x \in S$ such that*

$$(F_{min}(x), F_{max}(x)) \preceq_{K_Y \times K_Y \setminus \{(0_Y, 0_Y)\}} (F_{min}(\bar{x}), F_{max}(\bar{x})).$$

Remark 5.2. (i) In this particular case, it is a straightforward consequence of the definition of '$\preceq^m_{K_Y}$' that

$\bar{x} \in S$ is a min-max solution of (SOP) if and only if $\bar{x}$ is a minimal solution of (SOP) w.r.t. $\preceq^m_{K_Y}$.

(ii) If in addition, Assumption $\mathscr{A}$, i.e., (20) holds, then

$\bar{x} \in S$ is a min-max solution of (SOP) if and only if $\bar{x}$ is a minimal solution of (SOP) w.r.t. $\preceq^s_{K_Y}$.

Note that (20) is not always true even if $F_{min}(x)$ and $F_{max}(x)$ are both vectors. Some other conditions are required to guarantee it.

Proposition 5.3. *Suppose K_Y is regular. If $F(x)$ is closed and bounded in order for some $x \in X$, then one has*

$$\forall y \in F(x): \quad F_{min}(x) \preceq_{K_Y} y \preceq_{K_Y} F_{max}(x).$$

Proof. Picking an arbitrary $y \in F(x)$, we suppose $y \not\succeq_{K_Y} F_{min}(x)$. As $F_{min}(x)$ is the only minimal point of $F(x)$, there must be some $\tilde{q}_1 \in K_Y \setminus \{0_Y\}$ such that $y - \tilde{q}_1 \in F(x)$. Set

$$E_1 := \{\tilde{q}_1 \in K_Y \setminus \{0_Y\} : y - \tilde{q}_1 \in F(x)\} \text{ and } \varepsilon_1 := \sup\{\|\tilde{q}_1\|_Y : \tilde{q}_1 \in E_1\}.$$

It is not hard to verify that $\varepsilon_1 \neq +\infty$, since $F(x)$ is bounded in order and K_Y is regular. Then, for $\delta_1 = 1$, there exists some $q_1 \in E_1$ such that $\|q_1\|_Y > \varepsilon_1 - \delta_1 = \varepsilon_1 - 1$. Setting $y_1 = y - q_1$, we have $y_1 \not\succeq_{K_Y} F_{min}(x)$. Hence, there is some $\tilde{q}_2 \in K_Y \setminus \{0_Y\}$ such that $y_1 - \tilde{q}_2 \in F(x)$. Continue this procedure, we can get $y_n = y - \sum_{i=1}^{n} q_i \in F(x)$ with $y_n \not\succeq_{K_Y} F_{min}(x)$. As $F_{min}(x)$ is the only minimal point of $F(x)$, setting

$$E_{n+1} = \{\tilde{q}_{n+1} \in K_Y \setminus \{0_Y\} : y_n - \tilde{q}_{n+1} \in F(x)\},$$

then, $E_{n+1} \neq \emptyset$. Let $\varepsilon_{n+1} = \sup\{\|\tilde{q}_{n+1}\|_Y : \tilde{q}_{n+1} \in E_{n+1}\}$. For $\delta_{n+1} = \frac{1}{n+1}$, we can find some $q_{n+1} \in E_{n+1}$ such that $\|q_{n+1}\|_Y > \varepsilon_{n+1} - \delta_{n+1} = \varepsilon_{n+1} - \frac{1}{n+1}$. Set $y_{n+1} = y_n - q_{n+1}$. As $y_{n+1} \not\succeq_{K_Y} F_{min}(x)$ for every $n \in \mathbb{N}$, we can obtain a sequence $\{y_n\}_{n=1}^{+\infty} = \{y - \sum_{i=1}^{n} q_i\}_{n=1}^{+\infty} \subseteq F(x)$, which is decreasing and bounded in order.

Since K_Y is regular, there exists some $\bar{y} \in Y$ such that $y_n \to \bar{y}$, as $n \to +\infty$. This gives

$$\|q_n\|_Y \to 0 \quad \text{as} \quad n \to +\infty. \tag{21}$$

Meanwhile, as $F(x)$ is closed, we have $\bar{y} \in F(x)$. Therefore, we can find some $\bar{q} \in K_Y \setminus \{0_Y\}$ with $\bar{y} - \bar{q} \in F(x)$. According to Proposition 2.1, there is a number $m > 0$ such that, for all $y_1, y_2 \in Y$:

$$0_Y \preceq_{K_Y} y_1 \preceq_{K_Y} y_2 \quad \Longrightarrow \quad \|y_1\|_Y \leq m\|y_2\|_Y.$$

Since (21), there is some $N > 0$ such that $\|q_N\|_Y + \frac{1}{N+1} < \frac{1}{m}\|\bar{q}\|_Y$. Consider $y_{N-1} = y - \sum_{i=1}^{N-1} q_i$. Setting $\hat{q} = y_{N-1} - (\bar{y} - \bar{q})$, then $\hat{q} \in E_N$ and

$$\hat{q} = y_{N-1} - (\bar{y} - \bar{q}) = \sum_{i=N}^{+\infty} q_i + \bar{q} \succeq_{K_Y} \bar{q}.$$

Applying Proposition 2.1, we can deduce that $m\|\hat{q}\|_Y \geq \|\bar{q}\|_Y$. Hence, $\|\hat{q}\|_Y \geq \frac{1}{m}\|\bar{q}\|_Y > \|q_N\|_Y + \frac{1}{N+1} > \varepsilon_N$, which contradicts $\hat{q} \in E_N$. Hence, we obtain $y \succeq_{K_Y} F_{min}(x)$.

For $y \preceq_{K_Y} F_{max}(x)$, it can be proved similarly. □

Next, we give some examples here to show that all the conditions in Proposition 5.3 are necessary for the conclusion.

Example 5.1. *Let* $(X, K_X) = (\mathbb{R}, \mathbb{R}_+)$*,* $(Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$ *and* $S = (0, 1)$*. Define* $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ *by*

$$F(x){:=}\{(x, -x)\} \cup \{y^{(n)} = (2 - \sum_{i=1}^{n} x^n, -n)\}_{n=1}^{+\infty}.$$

Then, every $F(x)$ *is closed and* K_Y *is regular. Taking* $x = \frac{1}{2}$ *for instance, there is*

$$F(\frac{1}{2}) = \{(\frac{1}{2}, -\frac{1}{2})\} \cup \{y^{(n)} = (2 - \sum_{i=1}^{n} \frac{1}{2^n}, -n)\}_{n=1}^{+\infty}.$$

It is not bounded in order as $-n \to -\infty$*.* $(\frac{1}{2}, -\frac{1}{2})$ *is the only minimal point of* $F(\frac{1}{2})$*, while* $y^{(1)}$ *is the only maximal point of* $F(\frac{1}{2})$*. However,* $y^{(n)} \not\geq (\frac{1}{2}, -\frac{1}{2})$ *for all* $n \in \mathbb{N}$*.*

Example 5.2. *Let* $(X, K_X) = (\mathbb{R}, \mathbb{R}_+)$*,* $(Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$ *and* $S = [0, 1]$*. Define* $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ *by*

$$\begin{aligned} F(x) &:= \{(x + 1, x)\} \cup ((x, x + 1), (x + 1, x + 2)] \\ &= \{(x + 1, x)\} \cup \{t(x, x + 1) + (1 - t)(x + 1, x + 2) : t \in [0, 1)\}. \end{aligned}$$

Then, every $F(x)$ *is bounded in order and* K_Y *is regular. Taking* $x = 1$ *for instance, we have that* $F(1) = \{(2, 1)\} \cup ((1, 2), (2, 3)]$*. As* $((1, 2), (2, 3)]$ *is a half open segment,* $F(1)$ *is not closed.* $(2, 1)$ *is the only minimal point of* $F(1)$*, while* $(2, 3)$ *is the only maximal point of* $F(1)$*. However,* $y_t \not\geq_{K_Y} (2, 1)$ *for all* $t \in (0, 1)$*, where* $y_t = t(1, 2) + (1 - t)(2, 3) = (2 - t, 3 - t)$*. Hence, the conclusion* $y \geq_{K_Y} F_{min}(x)$ *fails due to that* $F(x)$ *is not closed.*

Example 5.3. *Let* $(X, K_X) = (\mathbb{R}, \mathbb{R}_+)$*,* $S = [0, 1]$*, while*

$$Y := \{h : [0, 1] \to \mathbb{R} : h \text{ is continuous on } [0, 1]\},$$

and

$$K_Y := \{h \in Y \ : \ \forall\, t \in [0, 1] : \ h(t) \geq 0\},$$

$$\forall h \in Y : \ \|h\|_Y = \sup_{t \in [0,1]} |h(t)|.$$

Define $F : \mathbb{R} \rightrightarrows Y^2$ *by*

$$F(x) \ := \ \{h_0(t) + x, \ h'_0(t) + x\} \cup \{h_n(t) + x\}_{n=1}^{+\infty},$$

where

$$\forall\, t \in [0, 1] : \ h_0(t) := 4t - 2, \ h'_0(t) \equiv 2, \ h_n(t) := t^n (n \in \mathbb{N}).$$

Then, every $F(x)$ *is closed and bounded in order. Observe that* h_n *is decreasing and bounded in order, but not uniformly convergent, which shows that* K_Y *is not regular. Taking* $F(0) = \{h_0(t), \ h'_0(t)\} \cup \{h_n(t)\}_{n=1}^{+\infty}$ *for instance, then* h_0 *and* h'_0 *are the only minimal point and the only maximal point of* $F(0)$*, respectively. However, as* $h_0(t) > h_1(t) \geq h_n(t)$ *for all* $n \in \mathbb{N}$ *when* $t \in (\frac{2}{3}, 1]$*,* $h_n \not\geq_{K_Y} h_0$ *for all* $n \in \mathbb{N}$*.*

As F_{min} and F_{max} are both vector-valued maps, given $(y'_{min}, y'_{max}) \in (F_{min}, F_{max})(S) := \bigcup_{x\in S}(F_{min}(x), F_{max}(x))$, the level set $S(y'_{min}, y'_{max})$ can be simplified as

$$S(y'_{min}, y'_{max}) = \{x \in S : F_{min}(x) \preceq_{K_Y} y'_{min},\ F_{max}(x) \preceq_{K_Y} y'_{max}\}.$$

It is convex if

$$\begin{aligned}\forall\lambda \in [0,1],\ x_1, x_2 \in S,\ y \in Y: \quad & F_{min}(x_i) \preceq_{K_Y} y,\ i = 1,2 \ \Rightarrow\ F_{min}(\lambda x_1 + (1-\lambda)x_2) \preceq_{K_Y} y,\\ & F_{max}(x_i) \preceq_{K_Y} y,\ i = 1,2 \ \Rightarrow\ F_{max}(\lambda x_1 + (1-\lambda)x_2) \preceq_{K_Y} y.\end{aligned}$$

For the parametric problem

$$\min_{x\in S(y'_{min}, y'_{max})} F(x), \qquad \text{(SOP)-}(y'_{min}, y'_{max})$$

as a corollary of Theorem 5.1, we obtain the following fixed point result for min-max solutions of (SOP) in the sense of Definition 5.1. We denote the set of min-max solutions to (SOP)-$(F_{min}(x'), F_{max}(x'))$ by (min, max)-Sol$\{(\text{SOP}) - (F_{min}(x'), F_{max}(x'))\}$.

Corollary 5.1. *Let $\bar{x} \in S$ be given. Then, $\bar{x}$ is a min-max solution of* (SOP) *if and only if $\bar{x}$ is a fixed point of the following set-valued map:*

$$\begin{aligned}\Psi:\ & X \rightrightarrows X\\ & x' \longmapsto (\min, \max)\text{-Sol}\{(\text{SOP}) - (F_{min}(x'), F_{max}(x'))\}.\end{aligned}$$

An example is presented here to illustrate that we can use this fixed point approach to handle the set optimization problem (SOP) equipped with the min-max solution concept.

Example 5.4. *Let $(X, K_X) = (\mathbb{R}, \mathbb{R}_+)$ and $(Y, K_Y) = (\mathbb{R}^2, \mathbb{R}^2_+)$. Define $S{:=}[0,2]$ and $F : X \rightrightarrows Y$ as*

$$F(x){:=}\{\lambda F_1(x) + (1-\lambda)F_2(x) : \lambda \in [0,1]\},$$

where $F_1(x){:=}(x^2, x^2)$, $F_2(x){:=}(|x-1|+3, |x-1|+3)$. Then, it is not hard to observe that $F_{min} = F_1$ and $F_{max} = F_2$. Picking some $t \in [0,2]$, the corresponding parametric problem (SOP)-$(F_{min}(t), F_{max}(t))$ *is*

$$\min_{x\in S(F_{min}(t), F_{max}(t))} F(x),$$

where

$$\begin{aligned}& S(F_{min}(t), F_{max}(t))\\ =\ & \{x \in [0,2] : F_{min}(x) \preceq_{\mathbb{R}^2_+} F_{min}(t),\ F_{max}(x) \preceq_{\mathbb{R}^2_+} F_{max}(t)\}\\ =\ & \{x \in [0,2] : x \le t,\ |x-1| \le |t-1|\}.\end{aligned}$$

We divide it into two cases.

Case 1. If $t \in (1,2]$, then

$$S(F_{min}(t), F_{max}(t)) = \{x \in [0,1] : 2 - x \le t\} \cup \{x \in (1,2] : x \le t\}.$$

Since $t \in (1,2]$, for any t, there is some $x \in (1,2]$ with $x < t$, which implies $F_{min}(x) \preceq_{\text{int } \mathbb{R}^2_+} F_{min}(t)$ and $F_{max}(x) \preceq_{\text{int } \mathbb{R}^2_+} F_{max}(t)$. Hence, t is not a min-max solution of (SOP)-$(F_{min}(t), F_{max}(t))$*, i.e., t is not a fixed point of Ψ.*

Case 2. If $t \in [0,1]$, then

$$S(F_{min}(t), F_{max}(t)) = \{x \in [0,1] : x = t\}.$$

Therefore, every t with $t \in [0,1]$ is a fixed point of Ψ, consequently a min-max solution of (SOP)*. That gives the whole set of min-max solutions of* (SOP)*, as is easy to observe.*

The idea of min-max minimizers is making use of the minimal and the maximal parts of a set to do the comparison. This concept actually provides a vectorization for the (SOP) when Min $F(x)$ and Max $F(x)$ are single point sets for all $x \in S$. Therefore, it is more convenient to associate our fixed point approach with some scalarization method. Fixing a scalarization function $l : Y \to \mathbb{R}$ that is strongly K_Y-monotone, we introduce the scalar parametric problem (SP$_l$)-(y'_{min}, y'_{max}):

$$\min_{x \in S(y'_{min}, y'_{max})} l(F_{min}(x)) + l(F_{max}(x)), \qquad \text{(SP}_l\text{)-}(y'_{min}, y'_{max})$$

where $(y'_{min}, y'_{max}) \in (F_{min}, F_{max})(S)$ is the parameter. We denote the set of solutions to (SP$_l$)-($F_{min}(x'), F_{max}(x')$) by Sol{(SP$_l$)-($F_{min}(x'), F_{max}(x')$)}.

We show the following fixed point theorem for min-max solutions of (SOP) in the sense of Definition 5.1.

Theorem 5.2. *Suppose $\bar{x} \in S$. Then, $\bar{x}$ is a min-max solution of* (SOP) *if and only if $\bar{x}$ is a fixed point of the following set-valued map:*

$$\begin{aligned} \Psi_l : \quad X &\rightrightarrows X \\ x' &\longmapsto \mathrm{Sol}\{(\mathrm{SP}_l)\text{-}(F_{min}(x'), F_{max}(x'))\}. \end{aligned}$$

Proof.

(a) Assume that $\bar{x} \in S$ is a min-max solution for (SOP), but not a solution of (SP$_l$)-($F_{min}(\bar{x}), F_{max}(\bar{x})$). That means there exists some $\hat{x} \in S(F_{min}(\bar{x}), F_{max}(\bar{x}))$ such that

$$l(F_{min}(\hat{x})) + l(F_{max}(\hat{x})) < l(F_{min}(\bar{x})) + l(F_{max}(\bar{x})). \tag{22}$$

Note that $\hat{x} \in S(F_{min}(\bar{x}), F_{max}(\bar{x}))$ indicates $F_{min}(\hat{x}) \preceq_{K_Y} F_{min}(\bar{x})$ and $F_{max}(\hat{x}) \preceq_{K_Y} F_{max}(\bar{x})$. Hence, together with (22), it follows that

$$(F_{min}(\hat{x}), F_{max}(\hat{x})) \preceq_{K_Y \times K_Y \setminus \{(0_Y, 0_Y)\}} (F_{min}(\bar{x}), F_{max}(\bar{x})),$$

contradicting that $\bar{x}$ is a min-max solution of (SOP).

(b) Suppose that there is some fixed point of Ψ_l, say $\bar{x}$, which is not a min-max solution of (SOP). Then, we can find some $\hat{x} \in S$ with

$$(F_{min}(\hat{x}), F_{max}(\hat{x})) \preceq_{K_Y \times K_Y \setminus \{(0_Y, 0_Y)\}} (F_{min}(\bar{x}), F_{max}(\bar{x})).$$

That implies $\hat{x}$ is feasible for (SP$_l$)-($F_{min}(\bar{x}), F_{max}(\bar{x})$), while at least one of $l(F_{min}(\hat{x})) < l(F_{min}(\bar{x}))$ and $l(F_{max}(\hat{x})) < l(F_{max}(\bar{x}))$ holds as l is strongly K_Y-monotone. This is a contradiction to the assumption.

□

If we need to figure out the entire min-max solution set of (SOP), we should keep l fixed and regard (SP) as a parametric problem w.r.t. (y'_{min}, y'_{max}). Then, according to Theorem 5.2, all the min-max solutions will be found.

6 Conclusion

In this paper, we investigate a constrained set-valued optimization problem associated with the solution concept based on the vector approach. Then, a new solution concept involving the maximal points is introduced. This concept differs from the one defined through set less relations in general. Then, we establish fixed point approaches for the set-valued optimization problems with respect to these two solution concepts based on the vector approach, respectively. In addition, we pay special attentions to the case in which both the minimal and maximal parts of every value set of the objective function are single points. In such a situation, our method associated with the new solution concepts can lead a better performance.

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