What is the optimal cutoff surface for ore bodies with more than one mineral?

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Abstract

In mine planning problems, cutoff grade optimization defines a threshold at every time period such that material above this value is processed, and the rest is considered waste. In orebodies with multiple minerals, which occur in practice, the natural extension is to consider a cutoff surface. We show that in two dimensions the optimal solution is a line, and in three dimensions it is a hyperplane.
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1 Introduction

The standard practice in mine planning is to first define the contour of a mine by solving the ultimate pit problem, and then to perform a cutoff grade optimization to define at each moment in time which material should be mined and processed; the rest is considered waste. The seminal work of K. Lane [1] established a unified framework to perform cutoff grade optimization, taking into account economic factors, production capacities, and the time value of money. The algorithm proposed in [1] is widely used in commercial software for the mining industry, and its optimality has been characterized by [2].

Several extensions of Lane's algorithm have been proposed. In [3], the authors propose a dynamic optimization formulation for underground mines, while in [4] the author extends Lane's framework by taking into account the stockpiling option. The work in [5] includes supply uncertainty into the cutoff grade optimization problem, showing a 14% difference between minimum and maximum production rates.

In this work we study deposits with multiple minerals, focusing on the two-dimensional case. Mines that contain more than one economic mineral are common, see [6] for a discussion focused on host rock affinity and ore-genetic models. Examples of such deposits include copper-gold [7], lead-zinc [8] and copper-lead-zinc [9].

In [10], an extension of Lane's original paper is proposed for the case of two minerals. The authors state that in principle any kind of *curve* may be considered to separate ore from waste. However, they restrict their search to lines, and apply a grid-search method to find the optimal coefficients at each iteration. A series of publications focuses on the multiple minerals case, using genetic algorithms, golden section search method, and refinements of the grid search method. For an excellent and comprehensive review of those extensions we refer the reader to [11].

In two dimensions, we have that each axis represents a mineral, and each point in the positive orthant is a *grade-pair* that corresponds to the grade of each mineral. In principle, any two-dimensional *set* is a valid region separating ore from waste. For example, for a given function $f(\cdot)$ we could have a set defined by grade pairs (x, y) such that $y \geq f(x)$. One could also have a region defined by the union—or intersection—of two or more different sets.

Our main contribution is to show that in two dimensions it is optimal to consider grade-pairs that can be defined as the region above a line. Indeed, we will start by considering *any set*, and then show that a necessary condition for optimality is that the set is precisely the region above a line. We will show results that relate different optimal lines, depending on which of the processes is the limiting one. We also show that in three dimensions the optimal cutoff surface is a hyperplane.

The importance of our theoretical result is that algorithms, heuristics and numerical schemes designed to find the optimal cutoff grade in a mine with multiple minerals can restrict themselves to the search of lines, or hyperplanes.

2 The mining scheduling problem

Following [1] and [10], we consider the mining operation as a succession of three stages: mining, concentrating and refining. The production scheduling problem consist in determining, for each time period, the amount of material that should be processed in each one such stages[1]. The exposition will be done for the two-mineral case, and we will later extend it to three dimensions.

2.1 Notation

The relevant decision variables in this case are as follows:

[1]In our paper we consider an homogeneous orebody, hence, the optimization scheduling problem is aspatial.

- $Q_{m,t}$ is the material to be extracted from the deposit,
- $Q_{c,t}$ is the extracted material to be sent to the concentrator, and
- $Q^1_{r,t}$ and $Q^2_{r,t}$ are the amount of minerals 1 and 2 to be refined, respectively.

In the transition from the mining stage to the concentrator, the decision maker must define a criterion to distinguish valuable ore, that will be sent to the concentrator and processed for mineral recovery, from waste that will be sent to a dump. In the single mineral case, at each time period the decision maker must determine a cutoff grade g_t; all material with a grade above g_t is ore, the rest is waste. In the two minerals case, the extension of the concept of cutoff grade is more involved. The grade distribution is now two-dimensional, and the decision maker must identify a set of grade-pairs such that all material with a pair of grade belonging to a certain region is ore.

Each one of the stages has its own costs and constraints, and as it is standard in the literature and in applications, we will assume them constant over time. In the following, we introduce the notation to represent the economic and operational aspects of the mining scheduling problem:

- M (tons): Maximum amount of material that can be extracted (mine capacity) per period.
- C (tons): Maximum amount of extracted material that can be sent to the concentrator (concentrating/mill capacity) per period.
- R_i (lbs): Maximum amount of mineral i that can be refined per period.
- z_i (lbs/ton): Maximum proportion of mineral i that can be recovered from ore.
- $s_i(\$/lbs)$: Sale price per unit or mineral i.
- m (\$/ton): Unit cost of extracted material.
- c (\$/ton): Unit cost of concentrated material.
- r_i (\$/lbs): Unit cost of refined mineral i.
- f : Fixed cost of operating the mine throughout one time period.
- d : Discount rate.
- $\delta = \frac{1}{1+d}$: Discount factor.
- T: Last time period that the mine will be operative. The value of T is also a variable to be determined.

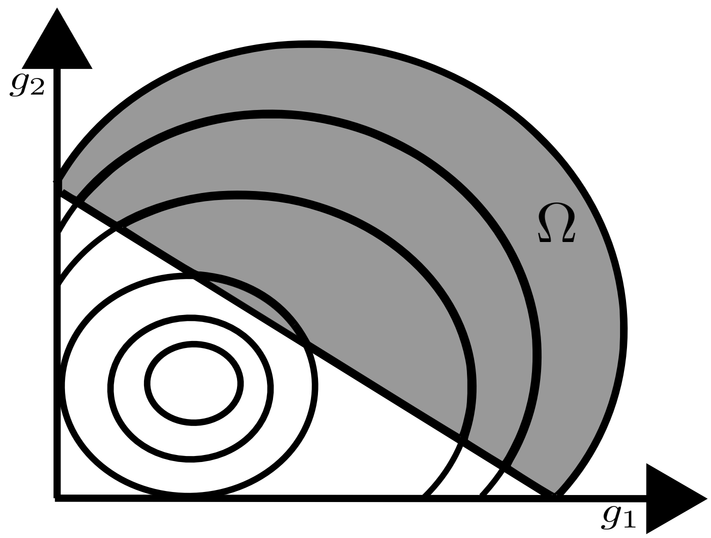

Figure 1: A cutoff curve represented by a line, with admissible grade-pairs in gray.

Before stating the scheduling problem, we need to introduce what we call a *grade density function*. We say that a function $\lambda : [0, \bar{g}_1] \times [0, \bar{g}_2] \to [0, 1]$ is a grade density function describing an homogeneous orebody if it is Lebesgue-integrable with $\int_0^{\bar{g}_1} \int_0^{\bar{g}_2} \lambda(g_1, g_2) dg_1 dg_2 = 1$, where $\bar{g}_i$ is the highest grade of mineral i present in the mine. Intuitively, for a given pair of grades (g_1, g_2), the quantity $\lambda(g_1, g_2) dg_1 dg_2$ represents the percentage of the ore that has grade g_1 of mineral 1 and g_2 of mineral 2. Analogously, the integral $\int_{g_2}^{\bar{g}_1} \int_{g_1}^{\bar{g}_2} \lambda(g_1, g_2) dg_1 dg_2$ represents the percentage of the mine that has, simultaneously, grade *at least* g_1 of mineral 1, and *at least* g_2 of mineral 2.

We still need one additional definition in order to relate the quantities Q_m, Q_c and Q_r^i. We define *set of admissible grade-pairs*, denoted by $\Omega \subseteq [0, \bar{g}_1] \times [0, \bar{g}_2]$, as the set of grade-pairs of the mineral that is selected to be concentrated, that is, all material with grades $(g_1, g_2) \in \Omega$ is sent to the concentrator; the rest is waste. Figure 1 (an adaptation of a figure included in [10]) shows some of the level sets of the grade density function λ together with the set Ω in gray, which is defined by the line in this case and the shaded area.

2.2 Problem statement

Using the definitions of Section 2.1 and letting the *set of admissible grade-pairs* (g_1, g_2) *in time period* t be denoted by $\Omega_t \subset \Omega$ we have

$$Q_{c,t} = Q_{m,t} \iint_{\Omega_t} \lambda(g_1, g_2) dg_1 dg_2, \tag{1}$$

$$Q^i_{r,t} = Q_{m,t} z_i \iint_{\Omega_t} g_i \lambda(g_1, g_2) dg_1 dg_2, \quad i = 1, 2. \tag{2}$$

If the initial amount of material in the mine is U_o, the optimization problem we want to solve is:

$$V(U_o) = \begin{cases} \max & \sum_{t=0}^{T} \delta^t [b_1 Q^1_{r,t} + b_2 Q^2_{r,t} - cQ_{c,t} - mQ_{m,t} - f] \\ s.a. & Q^i_{r,t} \leq R_i, \qquad \text{for } i = 1, 2, \\ & Q_{c,t} \leq C, \\ & Q_{m,t} \leq M, \\ & \sum_{t=1}^{T} Q_{m,t} \leq U_o, \end{cases} \tag{3}$$

where $Q_{c,t}$ and $Q^i_{r,t}$ are given by (1) and (2), and $b_i = z_i(s_i - r_i)$ for $i = 1, 2$.

Let Ω be a set of admissible pairs of grades corresponding to the solution of the optimization problem (3) in time period t. Based on the hypothesis that we send first the material with the best grade to the concentrator, we easily see that:

$$(g_1, g_2) \in \Omega \Rightarrow \{(x_1, x_2) \mid x_1 \geq g_1 \text{ and } x_2 \geq g_2\} \subset \Omega. \tag{4}$$

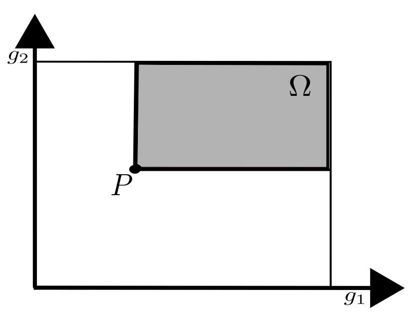

Figure 2: Region Ω for P.

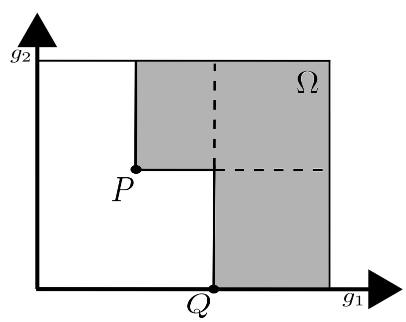

Figure 3: Region Ω for P and Q.

In figures 2 and 3 we illustrate the region implied by Remark 1 given (a) one and (b) two admissible grade-pairs.

Remark 1. *Property* (4) *is very powerful in the characterization of* Ω*. Indeed, it implies that the set* Ω *is connected. More importantly,* Ω *can be defined as the region located above the graph of a non-increasing function* $h_\Omega : [0, \bar{g}_1] \to [0, \bar{g}_2]$*. There is no loss of generality in considering only the sets defined by continuous* *functions* [2]*. We denote this graph as* α *(*$(g_1, g_2) \in \alpha \Leftrightarrow g_2 = h_\Omega(g_1)$*) and name it* cutoff curve [3].

3 Dynamic programming formulation

The first step is to observe that the objective function of problem (3) can be expressed in terms of Q_m and Ω only. Indeed, using definitions (1) and (2), we have

$$\begin{aligned} v(Q_m, \Omega) &:= b_1 Q_m \iint_\Omega g_1 \lambda(g_1, g_2) dg_1 dg_2 + \\ b_2 Q_m &\iint_\Omega g_2 \lambda(g_1, g_2) dg_1 dg_2 \\ &- c Q_m \iint_\Omega \lambda(g_1, g_2) dg_1 dg_2 - m Q_m - f \\ &= Q_m \iint_\Omega (b_1 g_1 + b_2 g_2 - c) \lambda(g_1, g_2) dg_1 dg_2 - m Q_m - f. \end{aligned} \tag{5}$$

Using (5), the optimization problem (3) can be expressed as

$$\max_{\{Q_{m,t}, \Omega_t\}_{t=0}^T} \sum_{t=0}^T \delta^t v(Q_{m,t}, \Omega_t)$$

$$s.t. \;\; z_1 Q_{m,t} \iint_{\Omega_t} g_1 \lambda(g_1, g_2) dg_1 dg_2 \leq R_1, \tag{6}$$

$$z_2 Q_{m,t} \iint_{\Omega_t} g_2 \lambda(g_1, g_2) dg_1 dg_2 \leq R_2, \tag{7}$$

$$Q_{m,t} \iint_{\Omega_t} \lambda(g_1, g_2) dg_1 dg_2 \leq C, \tag{8}$$

$$Q_{m,t} \leq M, \tag{9}$$

$$\sum_{t=1}^T Q_{m,t} \leq U_o. \tag{10}$$

The optimization problem assumes that the exploitation of the mine has not started yet. However, by simply replacing U_o by the amount left in the mine U in the right hand side of constraint (10), we can solve the problem for any configuration of the mine. Moreover, if we have U tons left in the mine and we

[2]Proof available from the authors upon request.

[3]With this definition it may be the case that Ω comprises regions of $[0, \bar{g}_1] \times [0, \bar{g}_2]$ where λ is zero. We may feel tempted to exclude those regions from the set of admissible grade-pairs, motivating a more complicated definition of Ω. However, including these regions or not, does not change the values of Q_c, Q_r^1 and Q_r^2. Hence, there is no loss of generality in considering only sets defined as the regions to the right and above a cutoff curve.

extract Q_m, then the amount of material left in the mine in the next period will be $U - Q_m$. Hence, the value function $V(U)$ must satisfy the Bellman equation:

$$V(U) = \max_{Q_m,\Omega}\{v(Q_m,\Omega) + \delta V(U - Q_m)\} \tag{11}$$

$$s.t. \quad Q_m \le U \tag{12}$$

$$(6), (7), (8) \text{ and } (9). \tag{13}$$

We note that in problem (11)-(13) Ω is a decision variable. Since the function V depends on Ω only through the function $v(\cdot)$, we can write the problem as a bilevel optimization problem:

$$V(U) = \max_{Q_m \in [0,\min\{M,U\}]}\{v(Q_m) + \delta V(U - Q_m)\}, \tag{14}$$

where

$$v(Q_m) = Q_m \max_{\Omega}\left\{\iint_\Omega (b_1 g_1 + b_2 g_2 - c)\lambda(g_1, g_2) dg_1 dg_2\right\} - mQ_m - f \tag{15}$$

$$s.t. \; z_i \iint_\Omega g_i \lambda(g_1, g_2) dg_1 dg_2 \le R_i/Q_m, \qquad \text{for } i = 1, 2, \tag{16}$$

$$\iint_\Omega \lambda(g_1, g_2) dg_1 dg_2 \le C/Q_m. \tag{17}$$

Observe that constraints (16) and (17) are a slight reformulation of (6) and (7) while the feasible interval for Q_m is equivalent to (9) and (12). By separating the decision Q_m and Ω, we will be able to characterize the optimal solution of problem (15)–(17). Our aim is to find necessary conditions that Ω must satisfy, we will not compute the value of $v(\cdot)$. Based on that, we disregard the term $(-mQ_m - f)$ of (15) from now on.

4 Solving the lower level problem

In this section we attempt to characterize the optimal set Ω that solves problem (15)–(17). Thanks to Remark 1 we know that Ω is the region located above the graph of a function h_Ω.

We denote by $\hat{\alpha}$ the closed curve formed by the border of Ω, which is the concatenation of α with segments belonging to the border of $[0, \bar{g}_1] \times [0, \bar{g}_2]$ [4]. In the following, we will represent Ω via a characterization of $\hat{\alpha}$. We know thanks to Remark 1 that $\hat{\alpha}$ must be simple and continuous. However, when solving the problem, we will restrict our attention to curves that are, in addition, piecewise smooth.

(A_1) We assume that Ω has a piecewise smooth border.

[4] In such a way that the point $(\bar{g}_1, \bar{g}_2)$ belongs to $\hat{\alpha}$.

We make this assumption in order to comply with Green's Theorem, which will allow us to transform the double integrals in (15)-(17) into tractable line integrals. Additionally, to apply Green's Theorem we need to extend the grade distribution function λ to an open region containing $[0, \bar{g}_1] \times [0, \bar{g}_2]$ (we denote this extension by $\hat{\lambda}$) and we also need an additional assumption on $\hat{\lambda}$

(A_2) There exists a twice differentiable function $\mathcal{M}$ with

$$\mathcal{M}_{g_1,g_2}(g_1, g_2) = \hat{\lambda}(g_1, g_2).$$

We are now ready to apply Green's theorem, which we restate here for convenience.

Green's Theorem Let $\hat{\alpha}$ be a positively oriented, piecewise smooth, simple closed curve in a plane, and let Ω be the region bounded by $\hat{\alpha}$. If $\mathbf{F}(g_1, g_2) = (F_1(g_1, g_2), F_2(g_1, g_2))$ is defined on an open region containing Ω and has continuous partial derivatives there, then

$$\iint_\Omega \left(\frac{\partial F_1}{\partial g_1} + \frac{\partial F_2}{\partial g_2} \right) dg_1\, dg_2 = \oint_{\hat{\alpha}} (-F_2\, dg_1 + F_1\, dg_2),$$

where the path of integration along $\hat{\alpha}$ is anticlockwise.

In order to apply Green's Theorem to $v(Q_m)$, defined in (15), we set $F_1 = b_2 g_2 \mathcal{M}_{g_2}(g_1, g_2)$ and $F_2 = (b_1 g_1 - c)\mathcal{M}_{g_1}(g_1, g_2)$. We obtain

$$\begin{aligned} &\iint_\Omega [b_1 g_1 + b_2 g_2 - c] \lambda(g_1, g_2) dg_1 dg_2 \\ &= \oint_{\hat{\alpha}} -(b_1 g_1 - c)\mathcal{M}_{g_1}(g_1, g_2) dg_1 + b_2 g_2 \mathcal{M}_{g_2}(g_1, g_2) dg_2 \end{aligned}$$

Applying Green's Theorem to (16) and (17), we obtain an equivalent reformulation of the problem as follows:

$$\begin{aligned} v(Q_m) = \max_{r(t)} \oint_{\hat{\alpha}} &-(b_1 g_1 - c)\mathcal{M}_{g_1}(g_1, g_2) dg_1 + \\ &b_2 g_2 \mathcal{M}_{g_2}(g_1, g_2) dg_2 \qquad (18) \\ \text{s.t.} \oint_{\hat{\alpha}} &-g_1 \mathcal{M}_{g_1}(g_1, g_2) dg_1 \leq \frac{R_1}{z_1 Q_m}, \qquad (19) \\ \oint_{\hat{\alpha}} &g_2 \mathcal{M}_{g_2}(g_1, g_2) dg_2 \leq \frac{R_2}{z_2 Q_m}, \qquad (20) \\ \oint_{\hat{\alpha}} &-\mathcal{M}_{g_1}(g_1, g_2) dg_1 \leq \frac{C}{Q_m}, \qquad (21) \\ &\hat{\alpha} \subset [0, \bar{g}_1] \times [0, \bar{g}_2]. \qquad (22) \end{aligned}$$

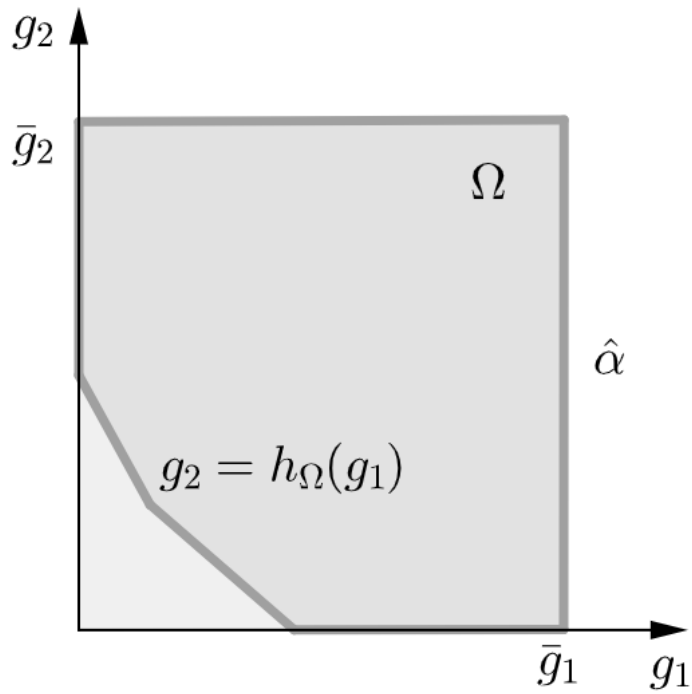

Figure 4: Curve $\hat{\alpha}$.

4.1 Euler-Lagrange equations

We are now in a position to derive necessary conditions that the curve $\hat{\alpha}$ must satisfy to be an optimal solution of problem (18)-(22). We focus on the characterization of the part of $\hat{\alpha}$ that belongs to the interior of $[0, \bar{g}_1] \times [0, \bar{g}_2]$, i.e., when constraint (22) is inactive (what we defined previously as α). Thanks to Remark 1, we know that the curve α is the graph of a continuous function h_Ω, hence it can be parametrized by $r(g_1) = (g_1, h_\Omega(g_1))$. With this parametrization we can write $dg_2 = h'_\Omega dg_1 = g'_2 dg_1$ in the computation of the integrals in (18) and (20)
We define

$$\begin{aligned} F(g_1, g_2, g'_2) &= b_2 g_2 \mathcal{M}_{g_2}(g_1, g_2) g'_2 - (b_1 g_1 - c)\mathcal{M}_{g_1}(g_1, g_2), \\ G^1_r(g_1, g_2, g'_2) &= -g_1 \mathcal{M}_{g_1}(g_1, g_2), \\ G^2_r(g_1, g_2, g'_2) &= g_2 \mathcal{M}_{g_2}(g_1, g_2) g'_2, \\ G_c(g_1, g_2, g'_2) &= -\mathcal{M}_{g_1}(g_1, g_2). \end{aligned}$$

For the optimization problem defining $v(Q_m)$, we define the Lagrangian operator as

$$\begin{aligned}\mathcal{L}(g_1, g_2, g_2') &= F - \mu_1 G^1 r - \mu_2 G_r^2 - \mu_c G_c \\ &= \Big[(-b_1 + \mu_1) g_1 + c + \mu_c\Big] \mathcal{M}_{g_1}(g_1, g_2) \\ &+ \Big[(b_2 - \mu_2) g_2\Big] \mathcal{M}_{g_2}(g_1, g_2) g_2'\end{aligned}$$

where μ_1, μ_2 and μ_c are non-negative Lagrange multipliers associated with constraints (19) to (21). Given that we are interested in solutions where constraint (22) is inactive, we do not consider it in the Lagrangian.

The following is the necessary Euler condition for $g_2 = h_\Omega(g_1)$ to be optimal:

$$\frac{\partial \mathcal{L}(g_1, g_2, g_2')}{\partial g_2} = \frac{d}{dg_1} \frac{\partial \mathcal{L}(g_1, g_2, g_2')}{\partial g_2'}. \tag{23}$$

In the following computations we omit the dependence of $\mathcal{M}$'s partial derivatives on (g_1, g_2) to ease the notation.

$$\begin{aligned}\frac{\partial}{\partial g_2} \mathcal{L}(g_1, g_2, g_2') &= (b_2 - \mu_2)[\mathcal{M}_{g_2} + g_2 \mathcal{M}_{g_2 g_2}] g_2' \\ &\quad + [-(b_1 - \mu_1) g_1 + c + \mu_c] \lambda, \\ \frac{\partial}{\partial g_2'} \mathcal{L}(g_1, g_2, g_2') &= (b_2 - \mu_2) g_2 \mathcal{M}_{g_2}, \\ \frac{d}{dg_1} \frac{\partial}{\partial g_2'} \mathcal{L}(g_1, g_2, g_2') &= (b_2 - \mu_2)[g_2' \mathcal{M}_{g_2} + g_2(\lambda + \mathcal{M}_{g_2 g_2} g_2')].\end{aligned}$$

With these computations, (23) is equivalent to $[(b_1 - \mu_1) g_1 + (b_2 - \mu_2) g_2 - c - \mu_c)] \lambda(g_1, g_2) = 0$. Hence, to satisfy the Euler necessary conditions we need:

$$(b_1 - \mu_1) g_1 + (b_2 - \mu_2) g_2 - c - \mu_c = 0 \tag{24}$$

for all $(g_1, g_2) \in \alpha$ in the region where $\lambda(g_1, g_2) \neq 0$. Hence, the curve α is a line.

We know that when a constraint is not active, its associated Lagrange multiplier is zero. We consider first the case where none of the constraints (19) to (21) are active, which induces the simplest expression of the cutoff line. Observe that constraints (19) and (20) are a reformulation of constraint (6), while (21) is a reformulation of (7). If all of them are inactive, either (8) or (12) must be active in the original formulation, implying that it is the mining capacity that is limiting the exploitation of the mine.

Case 1: when only the mining constraint is active If (19), (20) and (21) are not active then $\mu_c = \mu_1 = \mu_2 = 0$ and the equation of the cutoff line is reduced to

$$b_1 g_1 + b_2 g_2 - c = 0. \tag{25}$$

In the following we characterize the position of the cutoff line relative to (25) when only one of constraints (19) to (21) is active.

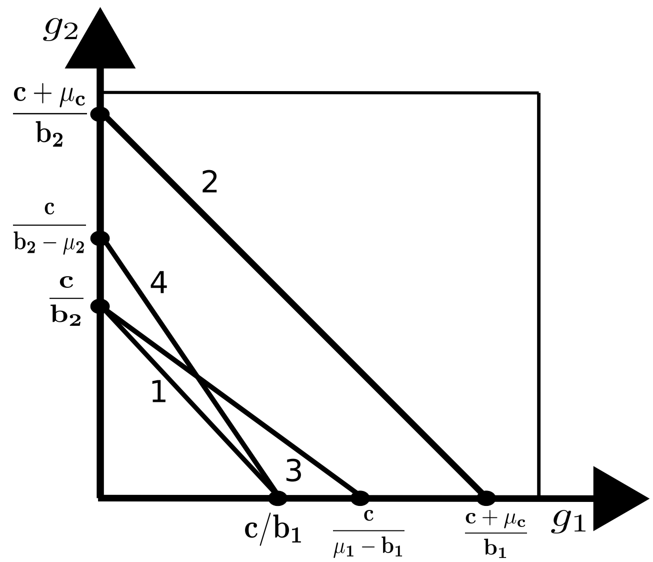

Figure 5: Optimal lines for each of the four cases.

Case 2: when only the concentrator constraint is active If only (21) is active, then $\mu_1 = \mu_2 = 0$ and the equation of the cutoff line is reduced to

$$b_1 g_1 + b_2 g_2 - c - \mu_c = 0.$$

Observe that the cutoff line when the concentrator constraint is active is strictly above (25). This is intuitively correct. Indeed, when the capacity of the concentrator is limiting, we send a smaller proportion of the extracted material to the concentrator with respect to what would be sent in Case 1.

Case 3: when only refinery 2 constraint active If only (19) is active, we have $\mu_c = \mu_2 = 0$ and the equation of the cutoff line is reduced to

$$(b_1 - \mu_1)g_1 + b_2 g_2 - c = 0.$$

Compared to Case 1, the slope of the cutoff line is less steep, implying that we tend to process ore with a higher grade of mineral 1.

Case 4: when only refinery 2 constraint is active If only (20) is active, we have $\mu_c = \mu_1 = 0$ and the equation of the cutoff line is reduced to

$$b_1 g_1 + (b_2 - \mu_2)g_2 - c = 0.$$

Compared to Case 1, the slope of the cutoff line is steeper, implying that we tend to process ore with a higher grade of mineral 2.

Figure 5 illustrates the relative position of the optimal line for each of the four cases. If more than one of the constraints (19) to (21) is active, we cannot give a precise statement on the relative position of the corresponding cutoff line with respect to Case 1 as we have more than one multiplier different from zero. However, the optimal curve is also a line in each of those cases according to the Euler conditions derived in (24).

5 Extension to three-minerals mines

When we consider a three mineral orebody, we represent the grades of each mineral as a point in a three-dimensional space. We refer to this point as a *grade-vector*. The set of admissible grade-vectors Ω is now a region of $[0, \bar{g}_1] \times [0, \bar{g}_2] \times [0, \bar{g}_3]$.

Following exactly the same steps as before, we obtain that the value function $V(U)$ is characterized by (14) where function $v(Q_m)$ is now defined by

$$v(Q_m) = Q_m \max_{\Omega} \left\{ \sum_{i=1,2,3} b_i \iiint_{\Omega} x_i \lambda(x_1, x_2, x_3) dx_1 dx_2 dx_3 - c \iiint_{\Omega} \lambda(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right\} - mQ_m - f \tag{26}$$

$$\text{s.t.} \quad z_i \iiint_{\Omega} x_i \lambda(x_1, x_2, x_3) dx_1 dx_2 dx_3 \le R_i/Q_m, i = 1, 2, 3, \tag{27}$$

$$\iiint_{\Omega} \lambda(x_1, x_2, x_3) dx_1 dx_2 dx_3 \le C/Q_m, \tag{28}$$

$$\Omega \subseteq \prod_{i=1}^{3} [0, \bar{g}_i].$$

We will use the Divergence Theorem, which we now state for convenience

Divergence Theorem Suppose Ω is a subset of $\mathbf{R}^3$ which is compact and has a piecewise smooth boundary S. If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of Ω, then

$$\iiint_{\Omega} (\nabla \cdot \mathbf{F}) \, d\Omega = \oiint_{S} (\mathbf{F} \cdot \hat{n}) \, dS,$$

where $\hat{n}$ is the outward pointing unit normal at each point of S.

Analogously to §4 we make the following assumptions: there exists a function $\hat{\lambda}$ that is an extension of λ to an open region U containing $\prod [0, \bar{g}_i]$ and there exists a twice differentiable function, $\mathcal{M} : U \to \mathbf{R}$, such that $\mathcal{M}_{x_1, x_3} = \hat{\lambda}$.

Let $\mathbf{F} : U \to \mathbf{R}^3$ be defined by

$$\begin{aligned} F_1(x_1, x_2, x_3) &= b_3 x_3 \mathcal{M}_{x_3}, \\ F_2(x_1, x_2, x_3) &= 0, \\ F_3(x_1, x_2, x_3) &= (b_1 x_1 + b_2 x_2 - c)\mathcal{M}_{x_1}. \end{aligned}$$

It is easy to verify that $\nabla \cdot \mathbf{F} = \left(\sum_{i=1}^3 b_i x_i - c\right) \lambda(x_1, x_2, x_3)$, which is exactly the function to be integrated in (26).

The Divergence Theorem yields

$$\iiint_\Omega \left(\sum_{i=1}^3 b_i x_i - c\right) \lambda(x_1, x_2, x_3)\, d\Omega =$$
$$\oiint_S (b_3 x_3 \mathcal{M}_{x_3}, 0, (b_1 x_1 + b_2 x_2 - c)\mathcal{M}_{x_1}) \cdot \hat{n}\, dS.$$

Repeating the process with the integrands of the constraints, we can express $v(Q_m)$ as the optimal value of the optimization problem (26)-(28) as follows:

$$Q_m \max_\Omega \left\{ \oiint_S (b_3 x_3 \mathcal{M}_{x_3}, 0, (b_1 x_1 + b_2 x_2 - c)\mathcal{M}_{x_1}) \cdot \hat{n}\, dS \right\} \tag{29}$$

$$\text{s.t.} \quad \oiint_S (0, 0, x_i \mathcal{M}_{x_1}) \cdot \hat{n}\, dS \leq R_i/(z_i Q_m), \qquad \text{for } i = 1, 2, \tag{30}$$

$$\oiint_S (x_3 \mathcal{M}_{x_3}, 0, 0) \cdot \hat{n}\, dS \leq R_3/(z_3 Q_m), \tag{31}$$

$$\oiint_S (0, 0, \mathcal{M}_{x_1}) \cdot \hat{n}\, dS \leq C/Q_m, \tag{32}$$

$$S \subset [0, \bar{g}_1] \times [0, \bar{g}_2] \times [0, \bar{g}_3]. \tag{33}$$

We have disregarded the term $(-mQ_m - f)$ of (26) because it does not depend on Ω. Our goal is to find necessary conditions that Ω must satisfy, not to compute the value of $v(\cdot)$. To evaluate the integrals in (29)–(33) we need a parametrization of S, $r : \mathbf{R}^2 \to \mathbf{R}^3$, $r(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$. Associated to the optimization problem defining $v(Q_m)$ we have the following Lagrangian operator

$$\mathcal{L} = (\mathbf{F} - \mu_1 \mathbf{G}_r^1 - \mu_2 \mathbf{G}_r^2 - \mu_3 \mathbf{G}_r^3 - \mu_c \mathbf{G}_c) \cdot \hat{n},$$

where

$$\begin{aligned} \mathbf{F} &= (b_3 x_3 \mathcal{M}_{x_3}, 0, (b_1 x_1 + b_2 x_2 - c)\mathcal{M}_{x_1}), \\ \mathbf{G}_r^i &= (0, 0, x_i \mathcal{M}_{x_1}), \qquad i = 1, 2, \\ \mathbf{G}_r^3 &= (x_3 \mathcal{M}_{x_3}, 0, 0), \\ \mathbf{G}_c &= (0, 0, \mathcal{M}_{x_1}), \end{aligned}$$

and μ_i with $i = 1, 2, 3$ and μ_c are non-negative Lagrange multipliers associated with constraints (30) to (32). We are only interested on the characterization S in the interior of $\prod_{i=1}^{3}[0, \bar{g}_i]$, hence we do not consider (33) in the Lagrangian operator.

In the following, we consider a parametrization of S of the form $r(u, v) = (u, v, h_\Omega(u, v))$. Remark 1 can be generalized to three dimensions and assure the existence of such parametrization with a continuous h_Ω. When solving the problem we will restrict our attention to surfaces that are in addition, piecewise smooth.

> (A_3) Ω has a piecewise smooth boundary S. Hence, we assume that h_Ω is continuously differentiable almost everywhere.

It is easy to see that $\hat{n} = (h_u, h_v, -1)$. We then have

$$\begin{aligned}\mathcal{L} &= \big((b_3 - \mu_3)x_3\mathcal{M}_{x_3}, 0, [(b_1 - \mu_1)x_1 \\ &\quad + (b_2 - \mu_2)x_2 - c - \mu_c]\mathcal{M}_{x_1}\big) \cdot \hat{n} \\ &= (b_3 - \mu_3)h(u, v)\mathcal{M}_{x_3}(u, v, h(u, v))h_u(u, v) \\ &\quad - [(b_1 - \mu_1)u + (b_2 - \mu_2)v - c - \mu_c]\mathcal{M}_{x_1}(u, v, h(u, v)),\end{aligned}$$

where we sometimes omit the dependence of the partial derivatives of $\mathcal{M}$ on $(u, v, h(u, v))$, and we omit the subindex Ω of the function h_Ω.

Observe that $\mathcal{L}$ can be expressed as $\mathcal{L}(u, v, z, (\xi_1, \xi_2))$, where $z = h(u, v)$ and $\xi = \nabla h(u, v)$. Following [12], we can express the necessary Euler condition that any critical point of $\mathcal{L}$ must satisfy as

$$\begin{aligned}\frac{\partial}{\partial z}\mathcal{L}(u, v, h(u, v), \nabla h(u, v)) &= \frac{d}{du}\frac{\partial}{\partial \xi_1}\mathcal{L}(u, v, h(u, v), \nabla h(u, v)) \\ &\quad + \frac{d}{dv}\underbrace{\frac{\partial}{\partial \xi_2}\mathcal{L}(u, v, h(u, v), \nabla h(u, v))}_{=0}.\end{aligned} \tag{34}$$

We then have

$$\begin{aligned}&\frac{\partial}{\partial z}\mathcal{L}(u, v, h(u, v), \nabla h(u, v)) = (b_3 - \mu_3)h(u, v)h_u(u, v)\mathcal{M}_{x_3x_3} \\ &\quad + (b_3 - \mu_3)\mathcal{M}_{x_3}h_u(u, v) - [(b_1 - \mu_1)u + (b_2 - \mu_2)v - c - \mu_c]\lambda, \\ &\frac{\partial}{\partial \xi_1}\mathcal{L}(u, v, h(u, v), \nabla h(u, v)) = (b_3 - \mu_3)h(u, v)\mathcal{M}_{x_3}(u, v, h(u, v)), \\ &\frac{d}{du}\frac{\partial}{\partial \xi_1}\mathcal{L}(u, v, h(u, v), \nabla h(u, v)) = (b_3 - \mu_3)[h_u(u, v)\mathcal{M}_{x_3} \\ &\quad + h(u, v)(\lambda + \mathcal{M}_{x_3x_3}h_u(u, v))].\end{aligned}$$

With these computations, (34) is equivalent to $[(b_1 - \mu_1)u + (b_2 - \mu_2)v + (b_3 - \mu_3)h(u, v) - c - \mu_c)]\lambda(u, v, h(u, v)) = 0$, implying that

$$(b_1 - \mu_1)x + (b_2 - \mu_2)y + (b_3 - \mu_3)z - c - \mu_c) = 0. \tag{35}$$

for all $(x, y, z) \in S \subset (0, \bar{g}_1) \times (0, \bar{g}_2) \times (0, \bar{g}_3)$ in the region where $\lambda(x, y, z) \neq 0$.

We note that equation (35) is very similar to equation (24), which characterizes the curve $\hat{\alpha}$ for the two-minerals case. Evidently, the optimal surfaces for the three-minerals case are hyperplanes. Following the exact same steps as described at the end of Section 4, that is, isolating which constraints are active in each case, we can characterize the relative positions of the optimal hyperplanes when only one constraint is active.

6 Conclusions

Cutoff grade is a classical criterion that differentiates between ore and waste in open pit mining operations. K. Lane's classical work [1] proposed an algorithm for deposits with a single mineral, and in [2] the authors showed that the algorithm works both in theory and in practice. The two-minerals case is more involved, since ore and waste are separated by a cutoff curve, instead of a number. It was suggested in a follow up paper by K. Lane and coauthors [10] that the optimal curve in this case would be a line, but no proof is provided.

In this paper we start by considering the two-minerals case. We write an explicit optimization formulation for the problem, and using dynamic programming we obtained a bilevel formulation whose lower-level decision variable is the cutoff curve. Applying Green's Theorem, and considering all possible cases in which the constraints are binding, we showed that the optimal curve must be a line. We also characterized the relative position of the optimal lines, depending on which constraints are active. We discuss the intuition behind the result; for instance when the refinery constraint is active for the first mineral, then the optimal line will be such that material with higher grades of the first mineral will be processed. Using the Divergence Theorem, we extended our result for the three-minerals case, and we showed that the optimal cutoff surface is a hyperplane.

Our results offer a clear guideline for the construction of algorithms to solve mine planning problems with multiple minerals. In two and three dimensions, the search for the optimal cutoff region should be restricted to the ones defined by lines and hyperplanes, respectively. For the less common case of more than three minerals, similar results can be derived by using the generalized Stokes Theorem. One possible avenue of future research is to consider randomness in some of the parameters, such as the year capacities and the prices of each mineral, and aim at characterizing the optimal cutoff surface in this case.

References

[1] K. F. Lane, Choosing the optimum cut-off grade, Mines Quarterly 59 (1964) 811–829.

[2] M. Goycoolea, P. Lamas, B. K. Pagnoncelli, A. Piazza, Lane's algorithm revisited, Management Science (2020).

[3] X.-w. Gu, Q. Wang, D.-z. Chu, B. Zhang, Dynamic optimization of cutoff grade in underground metal mining, Journal of Central South University of Technology 17 (3) (2010) 492–497.

[4] M. Asad, Cutoff grade optimization algorithm with stockpiling option for open pit mining operations of two economic minerals, International Journal of Surface Mining, Reclamation and Environment 19 (3) (2005) 176–187.

[5] M. W. A. Asad, R. Dimitrakopoulos, A heuristic approach to stochastic cutoff grade optimization for open pit mining complexes with multiple processing streams, Resources Policy 38 (4) (2013) 591–597.

[6] S. K. Haldar, Introduction to mineralogy and petrology, Elsevier, 2020.

[7] A. Kumar, R. Dimitrakopoulos, Application of simultaneous stochastic optimization with geometallurgical decisions at a copper–gold mining complex, Mining Technology (2019).

[8] S. Mohr, D. Giurco, M. Retamal, L. Mason, G. Mudd, Global projection of lead-zinc supply from known resources, Resources 7 (1) (2018) 17.

[9] Y.-H. Fan, H. Wang, Application of remote sensing to identify copper–lead–zinc deposits in the heiqia area of the west kunlun mountains, chinas, Scientific Reports 10 (1) (2020) 1–13.

[10] K. Lane, D. Hamilton, J. Parker, Cutoff grades for two minerals, in: Application of Computers and Mathematics in the Minerals Industries. International symposium. 18, 1984, pp. 485–491.

[11] M. W. A. Asad, M. A. Qureshi, H. Jang, A review of cut-off grade policy models for open pit mining operations, Resources Policy 49 (2016) 142–152.

[12] I. Ekeland, R. Temam, Convex analysis and variational problems, SIAM, 1999.