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Optimization formulations for storage devices

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We consider a storage device, such as a pumped storage hydroelectric generator, that has a state-of-charge together with mutually exclusive charging and generating modes. We develop valid inequalities for a storage model that uses binary variables to represent the charging and generating modes. To investigate the model, we consider two contexts, standalone and large-scale. The standalone context involves the hydroelectric generator purchasing or selling electricity based on known or forecast prices. We consider properties of an optimization formulation with objective that evaluates the profit from sale of net generation and value of stored energy, present conditions for the optimum of the continuous relaxation of this optimization formulation to have binary values for the charging and generation commitment variables, and demonstrate the result numerically with a small example system. Analysis of the standalone context helps to explain why the combination of features in the storage model results in a difficult problem. The large-scale context embeds the model into a unit commitment and dispatch formulation for multiple generators. For several large-scale test cases, we numerically verify that the valid inequalities can improve the computation compared to the standard model in the literature.

Key words: Pumped storage hydroelectricity, state-of-charge constraints, unit commitment

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1. Introduction

We consider a storage device that has a state-of-charge (SOC) with a capability to both draw power, increasing its SOC, and generate power, decreasing its SOC. The notion of SOC applies to various storage technologies, including chemical batteries and pumped storage hydroelectric (PSH) systems [22].

Chemical batteries with appropriate power electronics that are capable of continuously varying from charging to discharging can be accommodated with a convex model under the assumption that it is never optimal to “throw away” energy; that is, under the assumption that electricity prices are strictly positive. A convex model can be incorporated into economic dispatch and unit commitment models, to be discussed in more detail in the next section, without additional binary variables under the assumption that prices are never negative. Even when electricity prices can be negative, the alternatives of charging or discharging in each time interval can be represented with a single binary variable for each interval if the storage device can continuously vary from charging to discharging.

In contrast, a larger challenge is presented by PSH, which typically has mutually exclusive and disjoint operating modes that are analogous to the configurations of a thermal generator and typically require at least two binary variables for each time interval. A key feature that distinguishes PSH operating modes from those of a typical generator is the opposing effects of pumping and generation modes on net production and on SOC. Moreover, the round trip efficiency of pumping and generating is strictly less than one. As will be discussed both theoretically and empirically, the combination of mutually exclusive charging and discharging modes together with SOC limits and round trip efficiency strictly less than one present a computational challenge.

The contributions of this paper are as follows. This paper formulates a model for a generic PSH generator with mutually exclusive operating modes. To derive some basic results about operation of the PSH, we consider a “standalone” optimization formulation for a single PSH generator with objective that evaluates the (operating) profit defined to be the sum of net sales of electricity plus the value of stored energy at the end of the time horizon. Again under the assumption that electricity prices are never negative, we present conditions for the optimum of the continuous relaxation of the formulation to have binary values for the pumping and generation mode commitment variables. We demonstrate this result numerically with a small example system.

The formulation provides valid inequalities that are also suitable for incorporating into a large-scale unit commitment model, such as solved by an electricity Independent System Operator (ISO). A contribution of this paper is an improved formulation of SOC constraints for PSH that is suitable for large-scale unit commitment formulations. We demonstrate that the tighter formulation can improve performance compared to existing models in the literature using a large-scale test system from the Midcontinent ISO. We also discuss the computational challenges and discuss how they arise from the generic form of the PSH model.

The remainder of the paper is organized as follows. Section 2 provides a literature survey of unit commitment and PSH modeling, and some related problems. Definitions and assumptions are presented in Section 3. Section 4 considers the vertex structure of the PSH model together with the vertex structure of simplified formulations to help elucidate why the full PSH model is challenging. Main results for the standalone profit maximization formulation are presented and discussed in Section 5 and numerical examples for both the standalone profit maximization formulation and the large-scale formulation are presented in Section 6. Section 7 concludes and describes several future extensions.

2. Literature survey

This section first briefly introduces economic dispatch and unit commitment and then discusses PSH models and related formulations. Then storage optimization problems are discussed more generally. Several connections to the development in the rest of the paper will be discussed.

2.1. Economic dispatch and unit commitment

Economic dispatch considers the choice of generation (or pumping) for multiple generators at a particular time or time interval and assumes that (discrete) decisions about whether or not to operate have already been taken for all of the generators. Economic dispatch is typically formulated with continuous variables and, in its basic form, is a convex problem assuming that operating costs are convex [43].

In contrast, the unit commitment problem involves a time horizon and includes decisions about whether each particular generator (or, in the case of PSH, pump) is operating in each of the time intervals in the time horizon [43]. The decision of whether or not to be in-service in each interval of the time horizon is represented by a binary commitment variable. Unit commitment is a non-convex problem because of discrete choices of whether or not to operate in each time interval together with start-up and no-load costs and non-zero minimum generation limits.

Modern formulations of unit commitment make use of the tremendous developments in mixed-integer linear programming (MIP) software over the last decades [16]. An important key to efficient solution of a MIP is a "tight" representation of the feasible set, meaning a characterization of the convex hull of the feasible set, or an approximation to it. Several researchers have contributed to finding compact representations of the convex hull of various features of unit commitment.

For example, typical generators, once started must stay on for a so-called minimum up-time, and once switched off must stay off for a so-called minimum down-time. A concise representation of the convex hull of the unit commitment feasible region when there are minimum up- and down-times was first reported in [32]. As another example, generators have limits on how rapidly their output can change over time. The convex hull of the unit commitment feasible region when there are ramp limitations was first reported in [8]. In combined-cycle units there can be specific constraints on production when a generator changes its operating mode. A tight formulation of ramping constraints during start-up and shut-down was developed in [26], consideration of transition power trajectories between different operating modes was developed in [19], and [45] provides a general characterization of the convex hull for thermal generators. It should be emphasized that none of these formulations consider SOC limits as required for representing PSH.

Although the convex hull of each of these issues has been characterized individually, the convex hull when multiple types of issues apply simultaneously is not simply the intersection of the constraints defining each convex hull. However, these constraints are all valid inequalities that can be included in a formulation that approximates the convex hull. Such an approach is discussed in [18].

This typically does not result in an exact characterization of the convex hull, even when the exact convex hull for each individual issue alone is known; however, a useful approximation to the convex hull can potentially be obtained [18]. The large-scale unit commitment cases in Section 6 make use of some of these valid inequalities.

2.2. Pumped storage hydroelectricity and related models

Turning to PSH models, it is important to note that the relevant timescales and issues depend on the amount of storage. For example, in large-scale seasonal storage, hydro-dominated systems such as Brazil, the relevant issues are long-term and relate to uncertain water inflows, water flow variability, and evaporation. In multi-year models, the value of energy stored for the future must be discounted appropriately. The discrete nature of PSH commitment modes may not be a significant issue in such long-term models.

In contrast, in other regions the amount of energy storage is much more limited and the decisions relate to medium-term issues such as weekly or even daily cycles of load and the complementary dispatch of thermal generation in the so-called hydro-thermal coordination problem [12, 27, **?**, 34, 35]. In this case, the discrete nature of PSH commitment is likely to be more significant. Moreover, optimization of weekly or daily PSH operation typically assumes negligible, or at least predictable, inflows of water to the system and negligible evaporation.[1] The formulation in the rest of this paper will focus on a daily or weekly context where PSH commitment must be explicitly considered, but where inflows and evaporation are ignored, and where discounting is ignored.

The feature of discrete choices that constrain a related continuous variable in unit commitment models is also common to a variety of other problems. Typically, a binary variable is used to represent either a fixed charge [1, 28] or enforce a non-zero lower limit on the continuous variable. For example, in capacitated facility location, the fixed charge is a cost for building any capacity at a particular location, and there are also costs that depend on the size of the capacity.

Both features of fixed charges and non-zero lower bounds apply to unit commitment problems, where the "fixed charge" corresponds to the start-up and no-load costs in operating modes and the lower limits are due to minimum generation limits when operating, as discussed in Section 2.1. These are represented through the binary commitment variables by terms in the objective and the constraints, respectively.

A key additional feature of the PSH formulation compared to these other fixed charge problems and compared to typical unit commitment problems is the combination of the mutual exclusivity of the pumping and generating operating modes of the PSH together with the opposing effects of the pumping and generation modes on the net generation and on the SOC and a roundtrip efficiency of less than one. The opposing effects of pumping and generating is akin to an "inventory" problem [41]; however, inventory problems typically do not involve binary fixed charges. Discussion in Section 4 will highlight the computational difficulties that are due to the combination of features in the PSH formulation.

An alternative to modeling the mutual exclusivity through constraints on the commitment binary variables is the "complementarity" formulation that constrains the product of the PSH pumping and generation to equal zero [25, 40, 44]. While the complementarity approach may have advantages in, for example, representation of chemical batteries that are capable of continuously varying from charging to discharging, the formulation in this paper for PSH will utilize commitment variables and mutual exclusivity will be represented with constraints on these binary variables.

North American ISOs have varying representations of PSH into their unit commitment and economic dispatch models. At one extreme, the PSH may simply offer to generate power and bid to buy power for pumping, analogous to the participation of thermal generators, but typically with some additional features to represent SOC limits. At the other extreme, PJM has partially integrated the representation of PSH characteristics into their day-ahead unit commitment model [14]; however, the currently used approach represents PSH as a refinement on an initial unit commitment and economic dispatch solution that does not include PSH optimization. The examples in Section 6 will report a fully integrated model with PSH and thermal generator characteristics considered in a single model.

2.3. General storage optimization models

In addition to the specific issues related to PSH as discussed in Section 2.2, there is a much broader literature on storage optimization in general. Although general storage models are not the focus of this paper, this section provides a very abbreviated introduction to this literature.

A first observation is that when uncertainties are significant, such as in longer-term storage models, the basic formulation involves sequential decision making under uncertainty, typically formulated as a Markov decision process [30, chapter 3]. See [31] for a comprehensive introduction to this literature. Second, the computational issues may dictate that exact solutions are not viable for realistic models of uncertainties, giving rise to various approximate methods [13].

A significant strand of literature considers the optimal decisions for a standalone storage device in response to a forecast, typically using expected profit as objective, but potentially considering risk using value-at-risk or conditional value-at-risk concepts [36]. See [**?**] for a survey of this literature. For example, [21, 46, 47] consider storage co-located with a wind farm, and formulate Markov decision processes to optimize the dispatch of the wind farm and storage against forecasts of stochastic wind production and prices.

This strand of work typically focuses on the calculation of an expected profit-maximizing policy for the standalone storage asset given an explicit characterization of the (assumed stationary) probability distribution of future prices and other uncertainties. From the perspective of a standalone asset owner, there are at least two sources of uncertainty in prices. The first source of uncertainty is due to underlying random events, including random production by intermittent renewables such as wind power due to weather variability.

The second source of uncertainty is due to uncertainty in knowledge about system conditions such as availability of other generation assets, transmission constraints, and other technical issues. This information, while known to the ISO, is typically not known in detail by the asset owner and this information can also change rapidly over the optimization horizon. The probability distributions of these uncertainties are typically non-stationary, so practical implementation requires that policies be updated very often or that the policies explicitly represent the non-stationarities.

Section 5 discusses a problem in this class under the further simplified conditions of a deterministic price forecast and no other uncertainties. That is, the analysis in Section 5 ignores both sources of uncertainty considered in the standalone asset optimization literature, resulting in a linear program [46] (if commitment variables are ignored) or a MIP (if commitment is included). These simplifications are made in order to investigate the computational complexity of generic PSH optimization problems.

In contrast to such standalone asset models, Section 6 will report results from embedding the characteristics of the PSH into a large-scale ISO unit commitment and dispatch model. Allowing the ISO to make the dispatch decisions for the PSH then eliminates the second source of uncertainty that involves knowledge of system conditions, by avoiding the need for the PSH asset owner to predict prices in the near-term that depend on these uncertainties.[2] North American ISO dispatch models typically handle the first source of uncertainty mostly through a combination of advance forecasting and preparation in the day-ahead market combined with recourse in real-time markets, the latter often featuring “lookahead dispatch,” which takes a model predictive control approach to dealing with uncertainties [33]. To summarize, incorporating a tractable model of PSH into ISO optimization can reduce uncertainties compared to standalone asset optimization; however, understanding the computational issues due to each asset alone as discussed in Section 5 is important in creating the tractable ISO model that will be discussed in Section 6.

In addition to problems that consider the operation of a given system, there is also a literature discussing the location and design of storage infrastructure. For example, [10, 11] consider planning the amount of compressed-air energy storage to be installed at a location.

3. Definitions and assumptions

This section defines various concepts in order to specify a generic PSH storage model with mutually exclusive pumping and generating modes and specify an electricity trade model. Several assumptions will then be established that will be helpful in proving results. The formulation abstracts from several sources, including [22, **?**, 29, 44, 46, 47, 39] for PSH and [8, 16, 32, 43] for standard thermal unit commitment formulations.

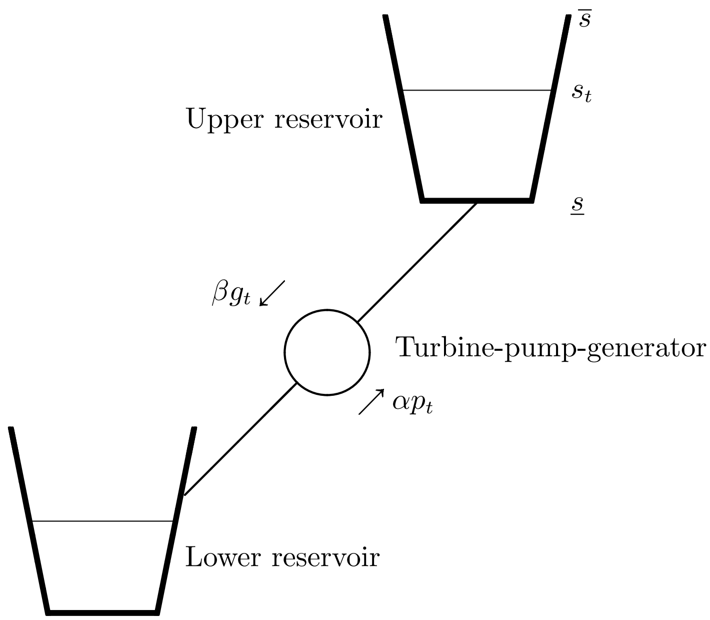

Figure 1 Schematic of pumped storage hydroelectric system.

Consider a time horizon divided into equal length time intervals, indexed by $t = 1, \ldots, T$, during which we are to schedule the operation of the storage device, such as the PSH illustrated schematically in Figure 1. As suggested in Figure 1, water can be pumped by the turbine-pump-generator from the lower reservoir to the upper reservoir, consuming electrical energy and storing it as gravitational potential energy, or water can be allowed to flow from the upper reservoir to the lower reservoir, generating electrical energy in the turbine-pump-generator from the gravitational potential energy. The volumes of water in the lower and upper reservoir, suggested by the thin horizontal lines showing the levels of water in the reservoirs, vary due to pumping and generating.

There are limits on pumping, generating, and the energy that is stored in the upper reservoir, as will be discussed in the next two sections. Subsequent sections present assumptions and further definitions and analysis.

3.1. Pumping and generating variables and limits

We define the electrical energy consumed from the grid for storage ("pumping" for PSH) during interval t to be $p_t \in \mathbb{R}$, while we define the electrical energy provided to the grid ("generating") during interval t to be $g_t \in \mathbb{R}$. When the PSH is operating, there are both upper and lower limits on both pumping and generation:

$$\underline{p} \leqslant p_t \leqslant \overline{p}, \ \forall t = 1, \dots, T, \tag{1}$$

$$\underline{g} \leqslant g_t \leqslant \overline{g}, \ \forall t = 1, \dots, T, \tag{2}$$

If the pump is not operating in interval t then $p_t = 0$, while if the generator is not operating in interval t then $g_t = 0$. Whereas an analogous model for a chemical battery would typically have $\underline{p} = \underline{g} = 0$, for a PSH it is usually the case that $\underline{p} > 0$ and $\underline{g} > 0$. That is, the pumping and generating regions result in the set of net electrical injections being non-convex and consisting of two disjoint intervals (together with the singleton corresponding to zero net injection). This non-convexity poses analogous problems to the commitment of thermal generators.

Moreover, many existing pumps can only be operated at a fixed power consumption level, in which case $\underline{p} = \overline{p} > 0$ [22, section 2.2].[3] Although storage resources can typically provide ancillary services, we do not consider this explicitly in the model. A future extension may include ancillary services building on, for example [5]. We collect the pumping and generation variables for the intervals $t = 1, \dots, T$ into vectors $p, g \in \mathbb{R}^T$.

Analogously to the representation of the configurations of thermal generators such as combined-cycle units [19], in order to represent the disjoint operating modes and the round trip efficiency of pumping and generating, we define two binary variables for each interval. In particular, for each $t = 1, \dots, T$, we define a binary commitment variable $u_t \in \{0, 1\}$ to represent whether the unit is pumping or not and a binary commitment variable $v_t \in \{0, 1\}$ to represent whether the unit is generating or not. This enables (1) and (2) to be generalized in the standard way to also represent the case of not pumping and not generating:

$$u_t \geqslant 0, \ \forall t = 1, \dots, T, \tag{3}$$

$$u_t \leqslant 1, \ \forall t = 1, \dots, T, \tag{4}$$

$$v_t \geqslant 0, \ \forall t = 1, \dots, T, \tag{5}$$

$$v_t \leqslant 1,\ \forall t = 1, \dots, T, \tag{6}$$

$$\underline{p}u_t \leqslant p_t \leqslant \overline{p}u_t,\ \forall t = 1, \dots, T, \tag{7}$$

$$\underline{g}v_t \leqslant g_t \leqslant \overline{g}v_t,\ \forall t = 1, \dots, T, \tag{8}$$

where the pumping and generation commitment variables, u_t and v_t, respectively, are also required to be integer-valued. We collect these commitment variables u_t and v_t for $t = 1, \dots, T$ together into vectors $u, v \in \mathbb{Z}^T$, respectively, where $\mathbb{Z}$ is the set of integers.

During any particular interval, pumping and generation modes are assumed to be mutually exclusive.[4] This is represented by:

$$u_t + v_t \leqslant 1, \forall t = 1, \dots, T. \tag{9}$$

The constraint (9), together with non-negativity requirements for u_t and v_t in (3) and (5), imply satisfaction of the upper bound constraints (4) and (6), so only (9) and the lower bound constraints (3) and (5) need to be enforced explicitly.

The resulting feasible region for choices of $(p_t, g_t) \in \mathbb{R}^2$ for a specific t is illustrated in Figure 2, for the case where $\underline{p} = 0.9, \overline{p} = 1.0, \underline{g} = 0.5$, and $\overline{g} = 1.0$. The feasible pumping and generating regions are shown as the two thick lines on the axes together with the bullet at the origin. The feasible region is non-convex, since it consists of two disjoint intervals and a point in $\mathbb{R}^2$. Moreover, as observed above, this results in the set of net electrical injections being non-convex, since it consists of two disjoint intervals together with a point in $\mathbb{R}$. For the example in Figure 2, the feasible net electrical generation is the set $[-1.0, -0.9] \cup [0.5, 1.0] \cup \{0\}$, reflecting the pumping mode, the generating mode, and not operating, respectively.

3.2. State-of-charge and limits

We define $s_t \in \mathbb{R}$ to be the SOC at the end of interval t. In Figure 1, the SOC is notionally determined by the volume of water in the upper reservoir. The SOC just prior to the beginning of the horizon is $s_0 \in \mathbb{R}$ and is assumed to be a known constant initial condition. There are conversion

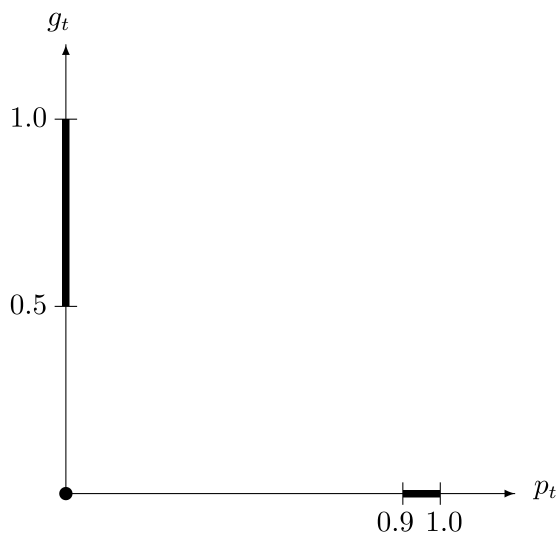

Figure 2 Example feasible region for pumping and generation for a particular t shown as thick lines and bullet.

losses both during pumping and generation and possibly also a units conversion between electrical energy represented by p_t and g_t (for example, measured in MWh) and the stored energy represented by s_t (for example, measured as volume of water in the upper reservoir of the PSH). To model this, the SOC evolves according to:

$$s_t = s_{t-1} + \alpha p_t - \beta g_t, \forall t = 1, \dots, T, \tag{10}$$

with $\beta > \alpha > 0$ implying that the round trip efficiency of pumping and subsequently generating is $\alpha/\beta < 1$.[5] The arrows in Figure 1 show the direction of flow of water when pumping and generating, respectively. We collect the SOC for $t = 1, \dots, T$ into a vector $s \in \mathbb{R}^T$; that is, s does not include the initial SOC, since s_0 is assumed to be known and constant. We ignore inflows of water into the system.

There is a maximum allowable SOC, $\overline{s}$ and a minimum allowable SOC, $\underline{s}$, which limit the values of $s_t, t = 0, \dots, T$:

$$\underline{s} \leqslant s_0 \leqslant \overline{s}, \tag{11}$$

$$\underline{s} \leqslant s_t \leqslant \overline{s}, \ \forall t = 1, \dots, T, \tag{12}$$

where, for later convenience, we have separated the conditions (11) for the initial SOC $t = 0$ from the conditions (12) for $t = 1, \dots, T$. In Figure 1, the maximum and minimum SOCs, $\overline{s}$ and $\underline{s}$, respectively, are illustrated by the conditions that the upper reservoir is literally full and empty, respectively. In practice the limits may be more restrictive due to other issues in operating the PSH and may vary with t. For example, the limits may also be due to capacity in the lower reservoir; however, such limits could still be expressed in the form of (11)–(12). In future work, we intend to consider the case of time varying limits in more detail.

The formulation (10)–(12) follows standard formulations such as, for example, [12, 46, 47]. However, due to the mutual exclusivity constraints, (12) can be reformulated to result in tighter valid inequalities and, as will be shown in Section 6, this reformulation can improve the computational performance compared to the standard formulation in the literature. In particular, for $t = 2, \ldots, T$, if s_{t-1} satisfies (12) for $t-1$, then we observe that the upper bound constraint in (12) cannot be violated for interval t if generating occurs in interval t. For $t = 1$, if s_0 satisfies (11) then the upper bound constraint in (12) cannot be violated for interval $t = 1$ if generating occurs in interval $t = 1$. Similarly, for $t = 2, \ldots, T$, if s_{t-1} satisfies (12) for $t-1$, then the lower bound constraint in (12) cannot be violated if pumping occurs in interval t. For $t = 1$, if s_0 satisfies (11) then the lower bound constraint in (12) cannot be violated for interval $t = 1$ if pumping occurs in interval $t = 1$. We can therefore rewrite (12) as the pair of constraints:

$$s_{t-1} + \alpha p_t \leqslant \overline{s}, \forall t = 1, \ldots, T, \tag{13}$$

$$s_{t-1} - \beta g_t \geqslant \underline{s}, \forall t = 1, \ldots, T. \tag{14}$$

The reformulation (13)–(14) reflects the observation that, if the SOC is feasible prior to the beginning of the time horizon (that is, if (11) is satisfed), then the SOC will stay feasible throughout the time horizon so long as:

- pumping in an interval $t = 1, \ldots, T$ does not result in the state variable s_t exceeding the maximum limit $\overline{s}$ at the end of the interval (that is, if (13) is satisfied), and
- generation in an interval $t = 1, \ldots, T$ does not result in the state variable s_t falling below the minimum limit $\underline{s}$ at the end of the interval (that is, if (14) is satisfied).

Figure 3 illustrates this observation, where the value s_{t-1} for interval $t-1$ is shown as being between the maximum and minimum SOCs, $\overline{s}$ and $\underline{s}$, respectively. It is only necessary to enforce the maximum SOC limit if pumping occurs in interval t, as illustrated by the value $s_{t-1}+\alpha p_t$ being below $\overline{s}$. Similarly, it is only necessary to enforce the minimum SOC limit if generation occurs in interval t, as illustrated by the value $s_{t-1}-\beta g_t$ being above $\underline{s}$.

Note that if (3), (5), (7), (8), and (9) are satisfied and the pumping and generation commitment variables, u_t and v_t, are required to be integer-valued then the standard formulation of the SOC limits (11)–(12) is equivalent to the reformulation (11) and (13)–(14). However, if the integrality requirement is relaxed, then the set of points satisfying (3), (5), (7), (8), (9), (11), and (13)–(14) is strictly contained in the set of points satisfying (3), (5), (7), (8), (9), (11), and (12) because:

$$\forall t=1,\dots,T, (s_{t-1}+\alpha p_t \leqslant \overline{s}) \Rightarrow (s_{t-1}+\alpha p_t-\beta g_t = s_t \leqslant \overline{s}),$$

$$\forall t=1,\dots,T, (s_{t-1}-\beta g_t \geqslant \underline{s}) \Rightarrow (s_{t-1}+\alpha p_t-\beta g_t = s_t \geqslant \underline{s}).$$

That is, (13) is tighter than the second inequality in (12) and (14) is tighter than the first inequality in (12) and so adopting (13)–(14) instead of (12) results in a closer approximation to the convex hull of the feasible region.[6]

The reformulation (13)–(14) of the SOC limits appears to be novel. It can also reflect the case of time-varying SOC limits, so long as the maximum SOC limit is non-decreasing with t and so long as the minimum SOC limit is non-increasing with t.

Because of the recursion (10), we can eliminate s from the formulation.[7] That is, applying (10) recursively, we have:

$$s_t = s_0 + \sum_{\tau=1}^{t}(\alpha p_\tau - \beta g_\tau), t=1,\dots,T. \tag{15}$$

To eliminate s from (13)–(14), first define the matrix $L \in \mathbb{R}^{T\times T}$ by

$$\forall i,j=1,\dots,T, L_{ij} = \begin{cases} 1, \text{ if } i \geqslant j, \\ 0, \text{ otherwise.} \end{cases}$$

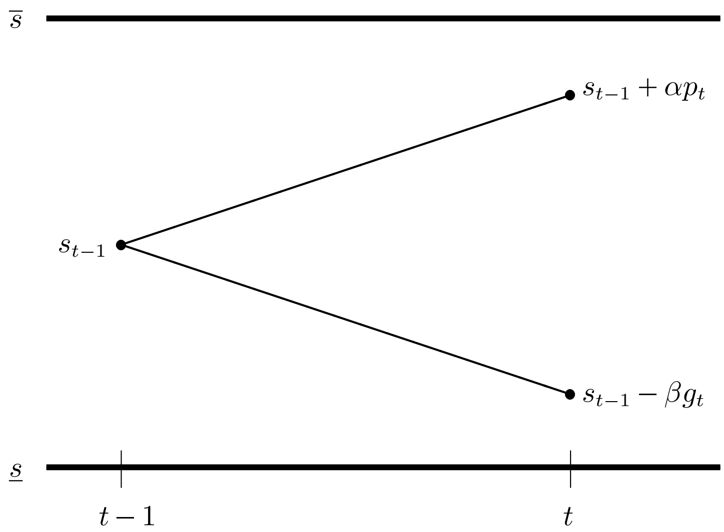

Figure 3 Illustration of SOC limits reformulation.

That is, L has ones on and under its diagonal and zeros elsewhere. Also define $\mathbf{I}\in\mathbb{R}^{T\times T}$ to be the identity matrix and define $\mathbf{1}\in\mathbb{R}^{T}$ to be the vector of all ones. Then we can use (10) to eliminate s and rewrite (13)–(14) in terms of s_0, p, and g. We have:

$$\alpha Lp+\beta(-L+\mathbf{I})g\leqslant \mathbf{1}(\overline{s}-s_0), \tag{16}$$

$$\alpha(L-\mathbf{I})p-\beta Lg\geqslant \mathbf{1}(\underline{s}-s_0), \tag{17}$$

3.3. Feasible pumping and generation region

Throughout the rest of the paper, we will assume that the storage must be operated to satisfy (3), (5), (7), (8), (9), (16), and (17) together with integrality of u and v. These inequalities and integrality requirements define the feasible operating region for a generic storage model. We define this feasible operating region to be the set:

$$\mathbb{S}=\{(p,g,u,v)\in\mathbb{R}^T\times\mathbb{R}^T\times\mathbb{Z}^T\times\mathbb{Z}^T|(3),\ (5),\ (7),\ (8),\ (9),\ (16),\ (17)\}.$$

Occasionally, we will collect (p,g,u,v) together into a vector x and, for convenience in discussing relaxations of the integrality constraints, we will occasionally consider x to be an element of $\mathbb{R}^T\times\mathbb{R}^T\times\mathbb{R}^T\times\mathbb{R}^T$ rather than of $\mathbb{R}^T\times\mathbb{R}^T\times\mathbb{Z}^T\times\mathbb{Z}^T$.

For example, we will need to consider relaxing the integrality constraints in $\mathbb{S}$ and so we define:

$$\mathbb{S}' = \{(p, g, u, v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T | (3), (5), (7), (8), (9), (16), (17)\},$$

so that $\mathbb{S}'$ is obtained from $\mathbb{S}$ by relaxing the integrality constraints. We will refer to $\mathbb{S}'$ as the "continuous relaxation" of $\mathbb{S}$. The next section will consider several basic assumptions that will be needed to characterize conditions under which the optimizer of a problem with feasible set $\mathbb{S}$ can be obtained by solving the continuous relaxation that has $\mathbb{S}'$ as its feasible set.

3.4. Basic assumptions

In addition to the generic storage model, we will need to make some mild assumptions in order to derive the results in Section 5. In particular, we make:

ASSUMPTION 1. We assume that (11) is satisfied, $\overline{p} > 0$, $\overline{g} > 0$, $\alpha\overline{p} + \beta\overline{g} < \overline{s} - \underline{s}$, and that $0 < \alpha < \beta$.

Requiring (11) to be satisfied merely means that the initial conditions are feasible; the non-negativity of the maximum limits $\overline{p}$ and $\overline{g}$ ensures that some pumping and some generating is possible; the restriction on $\overline{s} - \underline{s}$ means that the allowed range of SOC is larger than the energy that is involved in pumping and generating during one interval; while requiring $0 < \alpha < \beta$ establishes that the round-trip efficiency is less than one.

To derive some of the results in Section 5, however, we will also require additional, more restrictive, conditions including:

ASSUMPTION 2. We assume that $\alpha\overline{p} = \beta\overline{g}$.

Although Assumption 2 is quite restrictive, it can hold approximately in practice. Moreover, it should be emphasized that the representation of the SOC constraints as (16)–(17) is valid whether or not Assumption 2 is satisfied, so long as the mild condition (11) in Assumption 1 is satisfied.

For convenience, if Assumption 2 holds, we will write $\breve{s} = \alpha\overline{p} = \beta\overline{g}$. This condition means that the pump and generator are sized so that the increase in SOC due to pumping at the maximum pumping limit in an interval is the same as the decrease in the SOC due to generating at the maximum generating level in an interval. Assumption 2 means that, if p_t and g_t are either at their limits or off for each t, then the SOC will change by $0, \pm\check{s}$ in each interval.

In addition to Assumption 2, we will need to make an even stronger assumption to derive some of the results:

ASSUMPTION 3. We assume that $(\overline{s} - s_0)$ and $(s_0 - \underline{s})$ are both evenly divisible by $\alpha\overline{p} = \beta\overline{g}$.

Assumptions 2 and 3 imply that it is possible to scale the inequalities describing $\mathbb{S}$ and $\mathbb{S}'$ so that most of the entries of the coefficient matrix are $0, \pm 1$ and so that the right-hand side is integer-valued.

3.5. The inequalities that specify $\mathbb{S}$ and $\mathbb{S}'$

Define $\mathbf{0}$ to be either the zero matrix of size $T \times T$ or the zero row or column vector of length T, depending on context. Then we can represent the constraints: (3), (5), (7), (8), (9), (16), and (17) in the definitions of $\mathbb{S}$ and $\mathbb{S}'$ as:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\overline{p}\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \underline{p}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\overline{g}\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \underline{g}\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ \alpha L & \beta(-L+\mathbf{I}) & \mathbf{0} & \mathbf{0} \\ \alpha(-L+\mathbf{I}) & \beta L & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} p \\ g \\ u \\ v \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1}(\overline{s} - s_0) \\ \mathbf{1}(s_0 - \underline{s}) \end{bmatrix}, \tag{18}$$

where:

- the first block row of the coefficient matrix and the first sub-vector of the right-hand side of (18) represents (3),

- the second block row of the coefficient matrix and the second sub-vector of the right-hand side of (18) represents (5),
- the third and fourth block rows of the coefficient matrix and the third and fourth sub-vectors of the right-hand side of (18) represents (7),
- the fifth and sixth block row of the coefficient matrix and the fourth sub-vector of the right-hand side of (18) represents (8),
- the seventh block row of the coefficient matrix and the seventh sub-vector of the right-hand side of (18) represents (9),
- the eighth block row of the coefficient matrix and the eighth sub-vector of the right-hand side of (18) represents (16), and
- the ninth block row of the coefficient matrix and the ninth sub-vector of the right-hand side of (18) represents (17).

That is,

$$\begin{aligned} \mathbb{S} &= \{(p,g,u,v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{Z}^T \times \mathbb{Z}^T | (18)\}, \\ \mathbb{S}' &= \{(p,g,u,v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T | (18)\}. \end{aligned}$$

3.6. Profit from trading electricity

For a standalone PSH formulation in a market context, we assume that in each interval, $t = 1, \ldots, T$, there is a price, $c_t \in \mathbb{R}$, for trading electricity by the PSH, corresponding to the locational price at its bus. Pumping at level p_t in interval t requires a payment of $c_t p_t$ by the PSH, while generating electricity g_t in interval t results in a payment to the PSH of $c_t g_t$. Adding these payments across the time horizon, and defining $c \in \mathbb{R}^T$ to be a vector with entries $c_t, t = 1, \ldots, T$, results in a net payment to the PSH of:

$$\sum_{t=1}^{T} c_t(g_t - p_t) = c^\dagger(g - p), \tag{19}$$

where $\bullet^\dagger$ is the transpose.

In cases where the operation of the PSH is embedded into a larger optimization problem, we can interpret the vector c as being the opportunity costs from the larger formulation. We can consider (19) to be the contribution to social welfare due to pumping and generating, or equivalently, the profit to the PSH from pumping and generating when it is modeled as a "price taker" that cannot affect prices.[8] Note that we do not make any specific assumptions here about the sign of the entries of c; however, the results in Section 5 will explicitly consider the case where the entries of c are strictly positive and also consider alternatives.

3.7. Final state-of-charge

By (15), the SOC, s_T, at the end of the time horizon is:

$$s_T = s_0 + \sum_{\tau=1}^{T} (\alpha p_\tau - \beta g_\tau).$$

In some formulations, there may be a specified level for this final SOC. That is, for some given $\overline{\overline{s}}_T \in \mathbb{R}$, we impose the equality constraint:

$$s_0 + \sum_{\tau=1}^{T} (\alpha p_\tau - \beta g_\tau) = \overline{\overline{s}}_T. \tag{20}$$

For this case, we will modify Assumption 3 to:

ASSUMPTION 4. We assume that $(\overline{s} - s_0)$, $(s_0 - \underline{s})$, and $(\overline{\overline{s}}_T - s_0)$ are all evenly divisible by $\alpha\overline{p} = \beta\overline{g}$.

Instead of a specified level for the final SOC, there may be an economic value of the stored energy, $d \in \mathbb{R}$, at the end of the time horizon that is imputed from longer-term considerations. This then modifies the profit optimization objective from (19) to:

$$c^\dagger (g - p) + d \left(\sum_{\tau=1}^{T} (\alpha p_\tau - \beta g_\tau) \right) = C^\dagger g - D^\dagger p, \tag{21}$$

where we have ignored the constant term ds_0 and where the entries of $C, D \in \mathbb{R}^T$ are:

$$C_t = c_t - \beta d, t = 1, \ldots, T,$$

$$D_t = c_t - \alpha d, t = 1, \ldots, T.$$

Usually, we would expect that $d > 0$ reflecting the observation that the stored energy has useful value in the future.

3.8. Optimization formulations

We consider the problems of maximizing standalone PSH profit given the two alternative representations for final SOC from Section 3.7. First, suppose that there is a requirement on the level of the final SOC. Then the final SOC profit maximization problem is:

$$\max_{x\in\mathbb{S}}\{c^{\dagger}(g-p)|(20)\}. \tag{22}$$

We define the corresponding relaxed profit maximization problem to be:

$$\max_{x\in\mathbb{S}'}\{c^{\dagger}(g-p)|(20)\}. \tag{23}$$

This linear programming problem (23) is convex.

Second, with a specified economic value of stored energy, the final economic value profit maximization problem is:

$$\max_{x\in\mathbb{S}}\{C^{\dagger}g-D^{\dagger}p\}. \tag{24}$$

Similarly, we define the corresponding relaxed profit maximization problem to be:

$$\max_{x\in\mathbb{S}'}\{C^{\dagger}g-D^{\dagger}p\}, \tag{25}$$

which is again a linear programming problem.

4. Vertex structure

To aid in considering the properties of the optimizers of the formulations from Section 3.8, we will first consider the vertex structure of the relaxed feasible set $\mathbb{S}'$ and of several special cases. For $\begin{bmatrix} p \\ g \\ u \\ v \end{bmatrix} \in \mathbb{R}^{4T}$, the system (18) defines a (possibly empty) polytope $\mathbb{S}'$ whose extreme points are its vertices [17]. Each such vertex $\begin{bmatrix} \widehat{p} \\ \widehat{g} \\ \widehat{u} \\ \widehat{v} \end{bmatrix} \in \mathbb{R}^{4T}$ is a "basic feasible solution;" that is, there is a $4T$ element subset of the rows of (18) such that:

- the corresponding rows of the coefficient matrix are linearly independent and,
- if we treat these rows as equalities, then $\begin{bmatrix} \widehat{p} \\ \widehat{g} \\ \widehat{u} \\ \widehat{v} \end{bmatrix}$ is the unique solution of this set of linear equalities.

The development in Section 5 will seek conditions under which vertices of the system (18) have integer-valued entries for u and v at solutions of the optimization formulations from Section 3.8. Generically, the coefficient matrix and right-hand side of (18) has some non-integer values. To understand more clearly the values of u and v at the vertices of the system (18), it is convenient to first scale the coefficient matrix. Under Assumptions 2 and 3, the rows of the coefficient matrix can be scaled to have mostly integer values. In particular, re-arranging (18) and noting that, by

Assumption 2, $\breve{s} = \alpha\overline{p} = \beta\overline{g}$, we can obtain:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ L & -L+\mathbf{I} & \mathbf{0} & \mathbf{0} \\ -L+\mathbf{I} & L & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} p/\overline{p} \\ g/\overline{g} \\ u \\ v \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1}(\overline{s} - s_0)/\breve{s} \\ \mathbf{1}(s_0 - \underline{s})/\breve{s} \end{bmatrix}. \tag{26}$$

In the rest of this section, we will assume that Assumptions 2 and 3 hold and consider:

1. the vertices of $\mathbb{S}'$,
2. the vertices of a special case where SOC limits are not enforced, and
3. the vertices of a special case where mutual exclusivity constraints are not enforced.

We will see that the two special cases have more favorable vertex structures than $\mathbb{S}'$, highlighting that it is the combination of SOC limits and mutual exclusivity constraints in an integer linear programming formulation that present difficulties.

4.1. Vertices of $\mathbb{S}'$

Even under Assumptions 2 and 3, we will find that some of the vertices of $\mathbb{S}'$ do not have integer values for u and v. To see why this would be the case, we first observe that the result in [19, Appendix] regarding integrality is not applicable to $\mathbb{S}'$ because of the constraint structure. In particular, there are too many constraints involving the variables p and g in (26) to apply that result.

As a second observation, consider the property of total unimodularity [15, 23, 37, 38], which applies to matrices with entries that are $0, \pm 1$. In the special case that:

- $\underline{p} = 0$ or $\underline{p} = \overline{p}$, and
- $\underline{g} = 0$ or $\underline{g} = \overline{g}$,

then the coefficient matrix of (26) consists entirely of $0, \pm 1$ entries. Moreover, by Assumption 3, the right-hand side of (26) is integral. So, if the coefficient matrix *were* totally unimodular then the vertices of the system (26) would all have binary values for u and v. Unfortunately, even under these various assumptions, the coefficient matrix of (26) is not totally unimodular since some of the determinants of sub-matrices of the coefficient matrix of (26) are not equal to $0, \pm 1$. To see this, we will perform elementary row operations on the coefficient matrix, and identify a sub-matrix of the result that has determinant equal to 2.

Consider the coefficient matrix of (26) and subtract L times the third block row and subtract $(L - \mathbf{I})$ times the fifth row from the eighth row. This results in the following coefficient matrix:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & L & -L+\mathbf{I} \\ -L+\mathbf{I} & L & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Consider the following rows of this matrix:

1. the second row of the second block row,
2. the first row of the seventh block row,
3. the second row of the seventh block row, and
4. the first row of the eighth block row.

We re-order the coefficient matrix to make these the first four rows of the re-arranged matrix. Moreover, consider the following columns of this matrix:

1. the second column of the fourth block row (corresponding to v_2),
2. the second column of the third block row (corresponding to u_2),
3. the first column of the fourth block row (corresponding to v_1), and
4. the first column of the third block row (corresponding to u_1).

We re-order the coefficient matrix to make these the first four columns of the re-arranged matrix. This creates a lower block-triangular matrix of the form:

$$\begin{bmatrix} A & \mathbf{0} \\ B & C \end{bmatrix},$$

where $A \in \mathbb{R}^{4\times 4}$ is:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Direct calculation shows that the determinant of A is equal to -2 (and the determinant of sub-matrices of A are $0, \pm 1, \pm 2$). That is, A is not totally unimodular and *a fortiori* the full coefficient matrix of (26) is not totally unimodular. The structure of A is repeated throughout the coefficient matrix.

The existence of sub-matrices with determinant not equal to $0, \pm 1$ is due to the combination of mutual exclusivity constraints, binary variables to represent the pumping and generating operating regions, the opposite sign of the contribution of pumping and generating to the SOC, and the round trip efficiency being less than one. This is consistent with the observation in Section 2 that these features of the problem pose a challenge for efficient computation. To summarize, we are not able to use general results for totally unimodular coefficient matrices to analyze the vertices of the system (18).

Indeed, there are vertices of (18) that have non-integer-valued entries for u and v. For example, consider the parameter values $T = 2$, $\overline{s} = 2$, $s_0 = 1$, $\underline{s} = 0$, $\alpha\overline{p} = \beta\overline{g} = 1$, $\underline{p} = \underline{g} = 0$, and the specific values $\widehat{u}_1 = \widehat{v}_1 = 0.5, \widehat{p}_1 = 0.5/\alpha, \widehat{g}_1 = 0.5/\beta, \widehat{u}_2 = 1, \widehat{p}_2 = 1/\alpha$, and $\widehat{v}_2 = \widehat{g}_2 = 0$. There is a set of $4T = 8$ linearly independent constraints of the system (18) that are binding at the feasible point $\widehat{x} = \begin{bmatrix} \widehat{p} \\ \widehat{g} \\ \widehat{u} \\ \widehat{v} \end{bmatrix} \in \mathbb{R}^8$, namely:

$$
\begin{aligned}
\widehat{p}_1 &= 0.5/\alpha, \\
&= (1/\alpha) \times 0.5, \\
&= \overline{p}\,\widehat{u}_1, \\
\widehat{g}_1 &= 0.5/\beta, \\
&= (1/\beta) \times 0.5, \\
&= \overline{g}\,\widehat{v}_1, \\
\widehat{u}_1 + \widehat{v}_1 &= 0.5 + 0.5, \\
&= 1, \\
\widehat{v}_2 &= 0, \\
\widehat{p}_2 &= 1/\alpha, \\
&= (1/\alpha) \times 1, \\
&= \overline{p}\,\widehat{u}_2, \\
\widehat{g}_2 &= 0, \\
&= (1/\beta) \times 0, \\
&= \overline{g}\,\widehat{v}_2, \\
\widehat{u}_2 + \widehat{v}_2 &= 1 + 0,
\end{aligned}
$$

$$= 1,$$

$$s_0 + \alpha\widehat{p}_1 - \beta\widehat{g}_1 + \alpha\widehat{p}_2 = 1 + 0.5 - 0.5 + 1,$$

$$= \overline{s}.$$

These are the specific constraints considered in the construction above of the sub-matrix A. It is interesting to note that the pumping and generation lower limits in this example are both zero, so the existence of this vertex is not due to the non-convexity of non-zero lower limits on pumping and generation.

Consequently, conditions such that u and v are binary-valued must be based not only the structure of the coefficient matrix, but also on characteristics of the right-hand side and the form of the objective. Lemma 1 will show that if the value of electricity production is strictly positive, then this vertex $\widehat{x}$ cannot be a solution of a continuous relaxation of an electricity trading problem if Assumptions 1 and 2 hold.

Moreover, the form of (13)–(14) eliminates some of the non-binary-valued entries for u and v in the vertices of (18) compared to the standard formulation utilizing (12). For example, consider the same limits as in the example above but with $T = 1$, $s_0 = 2$, and the specific values: $\tilde{u}_1 = \tilde{v}_1 = \alpha\tilde{p}_1 = \beta\tilde{g}_1 = 0.5$. This point $\tilde{x} = \begin{bmatrix} \tilde{p} \\ \tilde{g} \\ \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathbb{R}^4$ is not feasible for (18), since:

$$s_0 + \alpha\tilde{p}_1 = 2 + 0.5,$$

$$\nleq 2,$$

$$= \overline{s}.$$

However, $\tilde{x}$ *is* a vertex of the modification of the system (18) where (13)–(14) are replaced by the standard formulation of SOC limits (12). In particular, there is a set of $4T = 4$ linearly independent constraints of the system (18) with (13)–(14) replaced by (12) that are binding at $\tilde{x}$, namely:

$$\tilde{p}_1 = 0.5/\alpha,$$

$$
\begin{aligned}
&= (1/\alpha) \times 0.5,\\
&= \overline{p}\,\tilde{u}_1,\\
\tilde{g}_1 &= 0.5/\beta,\\
&= (1/\beta) \times 0.5,\\
&= \overline{g}\,\tilde{v}_1,\\
\tilde{u}_1 + \tilde{v}_1 &= 0.5 + 0.5,\\
&= 1,\\
s_0 + \alpha\tilde{p}_1 - \beta\tilde{g}_1 &= 2 + 0.5 - 0.5,\\
&= \overline{s}.
\end{aligned}
$$

The specific form of (13)–(14) eliminates $\tilde{x}$ from being a vertex of the system (18), and so is tighter than the standard formulation in the literature. To summarize, although (18) does not characterize the convex hull, it is a tighter representation than the standard formulation using (12), consistent with the discussion in Section 3.2.

4.2. No SOC limits

Consider the special case of (18), or equivalently (26), where there are no SOC limits. In practice, this would correspond to a PSH having a very large range for $(\overline{s} - \underline{s})$ and with s_0 somewhere in the middle of this range. The inequalities describing the resulting feasable region can be obtained from (26) by deleting the eighth and ninth block rows to obtain:

$$
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}\\
\mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0}\\
-\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0}\\
\mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I}\\
\mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I}\\
\mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
p/\overline{p}\\
g/\overline{g}\\
u\\
v
\end{bmatrix}
\leqslant
\begin{bmatrix}
\mathbf{0}\\
\mathbf{0}\\
\mathbf{0}\\
\mathbf{0}\\
\mathbf{0}\\
\mathbf{0}\\
\mathbf{1}
\end{bmatrix}. \tag{27}
$$

The no SOC limits feasible operating region is the set:

$$\mathbb{M} = \{(p,g,u,v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{Z}^T \times \mathbb{Z}^T | (27)\}.$$

The corresponding continuous relaxation of $\mathbb{M}$ is:

$$\mathbb{M}' = \{(p,g,u,v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T | (27)\},$$

so that $\mathbb{M}'$ is obtained from $\mathbb{M}$ by relaxing the integrality constraints.

Now consider the first, second, and seventh block rows of (27). These constraints involve only the variables u and v. Moreover, if we temporarily ignore the variables p and g and temporarily ignore the constraints involving p and g, we note that the first, second, and seventh block rows characterize the convex hull of the region specified by:

- the requirement for u and v to be binary vectors, and
- the mutual exclusivity constraints.

Now consider all of the variables p, g, u, and v and all of the rows of the coefficient matrix of (27). By the result in [19, Appendix], which considers the convex hull of systems with both continuous and integer variables, we can show that all the vertices of this system have binary values for the entries of u and v. That is, a continuous optimization algorithm that seeks vertex solutions will automatically yield a solution that has binary values for u and v if we do not include the SOC limits. Under Assumptions 2 and 3, a formulation with mutual exclusivity constraints but without SOC limits has a more convenient structure for computation than the full formulation with both mutual exclusivity constraints and SOC limits. Consistent with this conclusion, we observe that the points $\widehat{x}$ and $\tilde{x}$ defined in Section 4.1 are not vertices of $\mathbb{M}'$ for the corresponding values of parameters because for both points there are, respectively, only 7 and 3 linearly independent constraints binding.

4.3. No mutual exclusivity constraints

In this section, we consider three special cases of the formulation where there are SOC limits but no mutual exclusivity constraints. We show that the cases with SOC limits but without mutual exclusivity constraints are also more convenient for computation than the full problem with both sets of constraints.

The three formulations we consider are:

1. where only pumping is possible, but not generation,
2. where only generation is possible, but not pumping,
3. where both pumping and generation is simultaneously possible.

In each case we will find that the convex hull of the feasible region is given by its continuous relaxation.

The case where only pumping is possible is a relevant model for energy-limited *consumption* of energy. This literally applies to the pumping of water for use in a gravity-fed water supply, but also applies, for example, to industrial loads that have some flexibility in the intensity of electricity consumption and also have a desired maximum amount of energy consumption over the scheduling horizon. In the latter example of an industrial load, we interpret "pumping" to also include consumption not necessarily literally associated with hydroelectric pumping. In this "pumping"-only case, we interpret $\overline{s} - s_0$ to be the maximum energy consumption over the scheduling horizon.

The case where only generation is possible is relevant to energy-limited reservoir hydroelectric generation, and to other energy-limited generation resources, if we interpret $s_0 - \underline{s}$ to be the maximum energy that can be used from the resource over the scheduling horizon.

The third case represents a so-called ternary PSH system [5][22, section 3.3], where there can be simultaneous charging and discharging associated with a single unit, resulting in a net injection or withdrawal of electric energy determined by the difference between pumping and generating in a single interval. That case will show that deleting the mutual exclusivity constraints in a pumping and generating model also results in a more tractable problem.

4.3.1. Pumping only Turning first to the case of pumping-only, the inequalities describing the feasible region can be obtained from (26) by deleting the second, fifth, sixth, seventh, and ninth block rows and deleting the second and fourth block columns to obtain:

$$\begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & (\underline{p}/\overline{p})\mathbf{I} \\ L & \mathbf{0} \end{bmatrix} \begin{bmatrix} p/\overline{p} \\ u \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1}(\overline{s}-s_0)/(\alpha\overline{p}) \end{bmatrix}. \tag{28}$$

The pumping-only feasible operating region is the set:

$$\mathbb{P} = \{(p,u) \in \mathbb{R}^T \times \mathbb{Z}^T | (28)\}.$$

The corresponding continuous relaxation of $\mathbb{P}$ is:

$$\mathbb{P}' = \{(p,u) \in \mathbb{R}^T \times \mathbb{R}^T | (28)\},$$

so that $\mathbb{P}'$ is obtained from $\mathbb{P}$ by relaxing the integrality constraints. We observe that if $\underline{p} = 0$ or $\underline{p} = \overline{p}$ then the coefficient matrix of (28) consists entirely of $0, \pm 1$ entries and is the concatenation of zero and identity matrices with L, which is a matrix having the "consecutive ones" property [15, 23, 37, 38], and so the coefficient matrix of (28) is totally unimodular. Therefore, since $\alpha\overline{p}$ divides evenly into $\overline{s} - s_0$ by Assumption 3, so that the right-hand side of the scaled system is integral, then the convex hull of $\mathbb{P}$ is its continuous relaxation $\mathbb{P}'$, and all vertices have integer-valued entries for $p/\overline{p}$ and u. To summarize, specialization to the case of pumping-only results in a formulation for which the convex hull can be straightforward to characterize compactly.[9] We observe that the points $(\widehat{p}, \widehat{u})$ and $(\tilde{p}, \tilde{u})$ defined in Section 4.1 are not vertices of $\mathbb{P}'$ for the corresponding values of parameters.

4.3.2. Generating only Similarly, in the case of generating-only, the inequalities describing the feasible region can be obtained from (26) by deleting the first, third, fourth, seventh, and eighth block rows and deleting the first and third block columns to obtain:

$$\begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & (\underline{g}/\overline{g})\mathbf{I} \\ L & \mathbf{0} \end{bmatrix} \begin{bmatrix} g/\overline{g} \\ v \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1}(s_0-\underline{s})/(\beta\overline{g}) \end{bmatrix}, \tag{29}$$

The generating-only feasible operating region is the set:

$$\mathbb{G} = \{(g, v) \in \mathbb{R}^T \times \mathbb{Z}^T | (29)\}.$$

The corresponding continuous relaxation of $\mathbb{G}$ is:

$$\mathbb{G}' = \{(g, v) \in \mathbb{R}^T \times \mathbb{R}^T | (29)\},$$

so that $\mathbb{G}'$ is obtained from $\mathbb{G}$ by relaxing the integrality constraints. Again, we observe that under the assumptions that $\underline{g} = 0$ or $\underline{g} = \overline{g}$, and since $\beta\overline{g}$ divides evenly into $s_0 - \underline{s}$ by Assumption 3, then we have that the convex hull of $\mathbb{G}$ is $\mathbb{G}'$, and all vertices have integer-valued entries for $g/\overline{g}$ and v. We observe that the points $(\widehat{g}, \widehat{v})$ and $(\tilde{g}, \tilde{v})$ defined in Section 4.1 are not vertices of $\mathbb{G}'$ for the corresponding values of parameters.

4.3.3. Ternary PSH Again, the inequalities describing the feasible region can be obtained by modifying (26). In this case, as well as deleting the mutual exclusivity constraints, it is necessary to add upper limits to the binary variables and change the SOC limits from (13)–(14) to the standard formulation (12), since with ternary PSH it *is* feasible to simultaneously pump and generate. This modifies (26) to:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I} \\ L & -L & \mathbf{0} & \mathbf{0} \\ -L & L & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} p/\overline{p} \\ g/\overline{g} \\ u \\ v \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1}(\overline{s} - s_0)/\breve{s} \\ \mathbf{1}(s_0 - \underline{s})/\breve{s} \end{bmatrix}. \qquad (30)$$

The ternary PSH feasible operating region is the set:

$$\mathbb{T} = \{(p, g, u, v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{Z}^T \times \mathbb{Z}^T | (30)\}.$$

The corresponding continuous relaxation of $\mathbb{T}$ is:

$$\mathbb{T}' = \{(p, g, u, v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T | (30)\},$$

so that $\mathbb{T}'$ is obtained from $\mathbb{T}$ by relaxing the integrality constraints.

Similarly to the previous cases of pumping-only and generation-only, if $\underline{p} = 0$ or $\underline{p} = \overline{p}$ and if $\underline{g} = 0$ or $\underline{g} = \overline{p}$ then Assumptions 2 and 3 result in the coefficient matrix being totally unimodular and the right-hand side being integral, so that all vertices have integer-valued entries for $p/\overline{p}, g/\overline{g}, u$, and v. To summarize, deleting the mutual exclusivity constraints results in a formulation for which the convex hull can be straightforward to characterize compactly. We observe that the points $\widehat{x}$ and $\tilde{x}$ defined in Section 4.1 are not vertices of $\mathbb{T}'$ for the corresponding values of parameters.

4.4. Contrast with general model and summary

Under Assumptions 2 and 3, if there are either no SOC limits or no mutual exclusivity constraints, then the convex hull of the feasible region has a compact representation. In contrast, and as discussed in Section 4, analogous results do not hold for the general PSH model with both pumping and generating, SOC limits, and mutual exclusivity constraints. That is, it appears that the convex hull of $\mathbb{S}$ does not in general have a compact representation. In particular, as demonstrated in Section 4.1, the analogous assumptions for PSH: Assumptions 2 and 3; that $\underline{p} = 0$ or $\underline{p} = \overline{p}$; and, that $\underline{g} = 0$ or $\underline{g} = \overline{g}$, do not result in a convenient characterization of the convex hull of $\mathbb{S}$.

To summarize, the analysis in this section further illustrates that the combination of mutual exclusivity constraints, round trip efficiency less than one, and the coupling over time due to the SOC limits underlies the computational challenge of PSH compared to the case of a generator without energy limits. We will see that this is consistent with the computational results in Section 6.

5. Standalone PSH operation

5.1. Properties of solutions to optimization formulations

In this section, we will prove results about the maximizers of the formulations (22) and (24) primarily under the assumption of strictly positive electricity prices; that is, if $c > \mathbf{0}$. It is well-known that electricity prices can occasionally be negative, particularly in the context of high renewable penetrations [2, 46, 47], so the assumption of $c > \mathbf{0}$ will not always be satisfied in practice. Nevertheless, $c > \mathbf{0}$ is a reasonable assumption on typical days at most locations.

To derive the results, we will characterize properties of a further relaxation of $\mathbb{S}$. Instead of enforcing (7) and (8), consider the following constraints:

$$p_t \geqslant 0, \forall t = 1, \ldots, T, \tag{31}$$

$$p_t \leqslant \overline{p} u_t, \forall t = 1, \ldots, T, \tag{32}$$

$$g_t \geqslant 0, \forall t = 1, \ldots, T, \tag{33}$$

$$g_t \leqslant \overline{g} v_t, \forall t = 1, \ldots, T. \tag{34}$$

Define

$$\mathbb{S}'' = \{(p, g, u, v) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T | (3), (5), (9), (31), (32), (33), (34), (16), (17)\}.$$

That is, instead of enforcing $\underline{p} u_t \leqslant p_t$ as in (7), we only enforce non-negativity of p_t in (31). Similarly, instead of enforcing $\underline{g} v_t \leqslant g_t$ as in (8), we only enforce non-negativity of g_t in (33).

Lemma 1. *Consider the optimization formulation (24) with a specified economic value d for the terminal SOC, and corresponding coefficients C and D. Suppose that Assumptions 1 and 2 are satisfied. Consider a solution of the further relaxed problem:*

$$\max_{x \in \mathbb{S}''} \{C^\dagger g - D^\dagger p\}. \tag{35}$$

We have the following:

1. *Suppose that $c_t > 0$ for some interval $t \in \{1, \ldots, T\}$. Then it is never optimal to simultaneously pump and generate in interval t.*

2. *Suppose that it is optimal to simultaneously pump and generate in some interval $t \in \{1, \ldots, T\}$. Then if $d > 0$ the maximum SOC limit is binding for some $\tau > t$.*

Proof We consider the optimality conditions of the convex linear problem (35) to derive relations to prove both items in the conclusion. For the given t, the constraints defining $\mathbb{S}''$ are as follows (with the corresponding non-negative Lagrange multiplier for optimality conditions of (35) in parenthesis after each inequality constraint):

$$u_t \geqslant 0, \qquad (\mu_{36t}) \tag{36}$$

$$v_t \geqslant 0, \qquad (\mu_{37t}) \tag{37}$$

$$u_t + v_t \leqslant 1, \qquad (\mu_{38t}) \tag{38}$$

$$p_t \geqslant 0, \qquad (\mu_{39t}) \tag{39}$$

$$p_t \leqslant \overline{p} u_t, \qquad (\mu_{40t}) \tag{40}$$

$$g_t \geqslant 0, \qquad (\mu_{41t}) \tag{41}$$

$$g_t \leqslant \overline{g} v_t, \qquad (\mu_{42t}) \tag{42}$$

$$s_0 + \sum_{\tau=1}^{t-1} (\alpha p_\tau - \beta g_\tau) + \alpha p_t \leqslant \overline{s}, \qquad (\mu_{43t}) \tag{43}$$

$$s_0 + \sum_{\tau=1}^{t-1} (\alpha p_\tau - \beta g_\tau) - \beta g_t \geqslant \underline{s}, \qquad (\mu_{44t}). \tag{44}$$

The optimality conditions of (35) include the following stationarity conditions for all $t = 1, \dots, T$ (with corresponding primal variables in parenthesis after each constraint):

$$-\mu_{36t} + \mu_{38t} \qquad - \overline{p}\mu_{40t} = 0, \ (u_t) \tag{45}$$

$$-\mu_{37t} + \mu_{38t} \qquad - \overline{g}\mu_{42t} = 0, \ (v_t) \tag{46}$$

$$-\mu_{39t} + \mu_{40t} + \alpha\mu_{43t} + \alpha \sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}) + D_t = 0, \ (p_t) \tag{47}$$

$$-\mu_{41t} + \mu_{42t} + \beta\mu_{44t} - \beta \sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}) - C_t = 0, \ (g_t) \tag{48}$$

where we note that (47) and (48) include terms in $\mu_{43\tau}$ and $\mu_{44\tau}$, which are defined to be the Lagrange multipliers on the constraints analogous to (43) and (44), respectively, for $\tau > t$, since the primal variables p_t and g_t appear in those constraints.

Summing equations (45), (46), $\overline{p}\times$(47), and $\overline{g}\times$(48) yields:

$$\begin{aligned}
0 &= -\mu_{36t} - \mu_{37t} + 2\mu_{38t} - \overline{p}\mu_{39t} - \overline{g}\mu_{41t} + \alpha\overline{p}\mu_{43t} + \beta\overline{g}\mu_{44t} \\
&\quad + (\alpha\overline{p} - \beta\overline{g}) \sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}) + \overline{p}D_t - \overline{g}C_t, \\
&= -\mu_{36t} - \mu_{37t} + 2\mu_{38t} - \overline{p}\mu_{39t} - \overline{g}\mu_{41t} + \alpha\overline{p}\mu_{43t} + \beta\overline{g}\mu_{44t} + \overline{p}D_t - \overline{g}C_t, \\
&\quad \text{since } \alpha\overline{p} = \beta\overline{g} \text{ by Assumption 2,} \\
&= -\mu_{36t} - \mu_{37t} + 2\mu_{38t} - \overline{p}\mu_{39t} - \overline{g}\mu_{41t} + \alpha\overline{p}\mu_{43t} + \beta\overline{g}\mu_{44t} + \overline{p}(c_t - \alpha d) - \overline{g}(c_t - \beta d), \\
&\quad \text{by definition of } C_t \text{ and } D_t, \\
&= -\mu_{36t} - \mu_{37t} + 2\mu_{38t} - \overline{p}\mu_{39t} - \overline{g}\mu_{41t} + \alpha\overline{p}\mu_{43t} + \beta\overline{g}\mu_{44t} + (\overline{p} - \overline{g})c_t, \qquad (49)
\end{aligned}$$

since $\alpha\overline{p} = \beta\overline{g}$ by Assumption 2. Furthermore:

$$\begin{aligned}
\overline{p} - \overline{g} &= (\alpha\overline{p} - \alpha\overline{g})/\alpha, \\
&> (\alpha\overline{p} - \beta\overline{g})/\alpha, \text{ since } \overline{g} > 0 \text{ and } 0 < \alpha < \beta \text{ by Assumption 1,} \\
&= 0, \qquad (50)
\end{aligned}$$

by Assumption 2, so the last term of (49) has the same sign as c_t.

We now turn to the specific items in the conclusion:

1. Suppose that $c_t > 0$. Then, on re-arranging (49),

$$\begin{aligned}
\mu_{36t} + \mu_{37t} + \overline{p}\mu_{39t} + \overline{g}\mu_{41t} &= 2\mu_{38t} + \alpha\overline{p}\mu_{43t} + \beta\overline{g}\mu_{44t} + (\overline{p} - \overline{g})c_t, \\
&\geqslant (\overline{p} - \overline{g})c_t, \\
&\quad \text{since the Lagrange multipliers are non-negative} \\
&\quad\quad \text{by complementary slackness,} \\
&> 0, \text{ by (50) and since } c_t > 0,
\end{aligned}$$

This means that for (49) to be satisfied at least one of $\mu_{36t}, \mu_{37t}, \overline{p}\mu_{39t}$, and $\overline{g}\mu_{41t}$ must be strictly greater than zero. We consider the alternatives for $\mu_{36t}, \mu_{37t}, \overline{p}\mu_{39t}$, and $\overline{g}\mu_{41t}$:

If $\mu_{36t} > 0$ then by complementary slackness $u_t = 0$ and so, by (39) and (40), $p_t = 0$;

If $\mu_{37t} > 0$ then by complementary slackness $v_t = 0$ and so, by (41) and (42), $g_t = 0$;

If $\overline{p}\mu_{39t} > 0$ then by complementary slackness $p_t = 0$; and,

If $\overline{g}\mu_{41t} > 0$ then by complementary slackness $g_t = 0$.

That is, in each case, at least one of p_t or g_t is equal to zero as claimed.

2. Suppose that it is optimal to simultaneously pump and generate in some interval $t \in \{1, \dots, T\}$. That is, by assumption, $p_t > 0$ and $g_t > 0$ so that $\mu_{39t} = \mu_{41t} = 0$. Substituting into (47) and (48) and noting that $C_t = c_t - \beta d$ and $D_t = c_t - \alpha d$, we obtain:

$$\mu_{40t} + \alpha\mu_{43t} + \alpha \sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}) + c_t - \alpha d = 0, \tag{51}$$

$$\mu_{42t} + \beta\mu_{44t} - \beta \sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}) - c_t + \beta d = 0. \tag{52}$$

Summing equations (51) and (52) yields:

$$\mu_{40t} + \mu_{42t} + \alpha\mu_{43t} + \beta\mu_{44t} - (\beta - \alpha) \sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}) + (\beta - \alpha)d = 0.$$

Since $\mu_{40t}, \mu_{42t}, \mu_{43t}$, and μ_{44t} are all non-negative by complementary slackness, and since $\beta - \alpha > 0$ by Assumption 1, we have that:

$$d \leqslant \sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}). \tag{53}$$

Therefore, if $d > 0$ then, by (53), $\sum_{\tau=t+1}^{T} (\mu_{43\tau} - \mu_{44\tau}) > 0$ and so $\mu_{43\tau} > 0$ for at least one interval $\tau > t$. That is, the maximum SOC limit must be binding for at least one interval $\tau > t$.

Q.E.D.

Item 1 of Lemma 1 allows us to consider the vertex $\widehat{x} = \begin{bmatrix} \widehat{p} \\ \widehat{g} \\ \widehat{u} \\ \widehat{v} \end{bmatrix} \in \mathbb{R}^8$ described in Section 4.1 in the context of optimization formulation (35). That vertex involves simultaneous pumping and generating in interval $t = 1$ and we also observe that the maximum SOC limit is binding in interval

$\tau = 2 > 1$. By item 1 of Lemma 1, this vertex would be ruled out as a solution of problem (35) if $c_1 > 0$.

To further interpret Lemma 1, note that simultaneous pumping and generation involves disposing of stored energy. That is, the change in SOC is strictly lower with simultaneous pumping and generation as compared to the change in SOC where only pumping or only generation was occurring with the same net electrical injection. Negative prices alone, without the maximum SOC limit being subsequently binding, are insufficient to make simultaneous pumping and generation desirable. This is because negative prices encourage maximum pumping and no generation, unless maximum SOC limits are binding at a subsequent interval. If prices are, however, negative and maximum SOC limits are also binding at a subsequent interval then it can improve the objective to both pump and dispose of stored energy. Similar results have been observed before in the context of zero lower limits on pumping and generation [39]; however, here we prove this result in the somewhat more general setting of non-zero lower limits, albeit under the somewhat restrictive Assumption 2 and in the absence of uncertainty.

Corollary 1. *Consider the optimization formulation (22) with $c > \mathbf{0}$ and a specified level $\overline{\overline{s}}_T$ for the final SOC. Suppose that Assumptions 1 and 2 are satisfied. Consider a solution of the further relaxed problem:*

$$\max_{x \in \mathbb{S}''} \{c^\dagger (g - p) | (20)\}. \tag{54}$$

For any given interval t, it is never optimal to simultaneously pump and generate.

Proof Note that in the proof of the first item of Lemma 1, the economic value d of the terminal SOC could be of either sign without affecting (49). Re-intepreting d as the value of the Lagrange multiplier on the equality constraint (20) in (54), the same conclusion follows. Q.E.D.

Theorem 1. *Consider the optimization formulation (24) with $c > \mathbf{0}$, an economic value d for the terminal SOC, and corresponding coefficients C and D. Suppose that Assumptions 1, 2, and 3 are* *satisfied. Then there is an optimizer* $x^\star = \begin{bmatrix} u^\star \\ v^\star \\ p^\star \\ g^\star \end{bmatrix}$ *of the relaxed problem (35) that is an element of* $\mathbb{S}$.

Proof Consider any optimizer $x^{\star\star} = \begin{bmatrix} u^{\star\star} \\ v^{\star\star} \\ p^{\star\star} \\ g^{\star\star} \end{bmatrix}$ of the relaxed problem (35) and corresponding vector of Lagrange multipliers $\mu^{\star\star}$ and vector of SOCs $s^{\star\star}$. We construct an optimizer $x^\star$ of problem (35) and corresponding vector of SOCs $s^\star$ with the required properties by successively specifying the entries of $x^\star$ and $s^\star$.

We divide the construction into several steps:

- Using Lemma 1 to specify some of the entries of $x^\star$ to be the same as the corresponding entries of $x^{\star\star}$,
- Considering the remaining "uneven" intervals of $x^{\star\star}$,
- Defining the notion of "epoch" and its properties,
- Considering the coefficients of the objective for the uneven intervals in an epoch,
- Evaluating the contribution of uneven intervals to the objective, and
- Specifying the remaining entries of $x^\star$.

The details are described in the Appendix. Q.E.D.

Corollary 2. *Consider the optimization formulation (24) with* $c > \mathbf{0}$*, an economic value* d *for the terminal SOC, and corresponding coefficients* C *and* D*. Suppose that Assumptions 1, 2, and 3* *are satisfied. Then there is an optimizer* $x^\star = \begin{bmatrix} u^\star \\ v^\star \\ p^\star \\ g^\star \end{bmatrix}$ *of the relaxed problem (25) that is an element of* $\mathbb{S}$.

Proof Note that the solution $x^\star$ constructed in the proof of Theorem 1 is an element of $\mathbb{S}$ and $\mathbb{S}'$. Since $x^\star$ optimizes the objective over the larger set $\mathbb{S}''$, and since $x^\star$ is also an element of $\mathbb{S}$ and $\mathbb{S}'$, it also optimizes the objective over both $\mathbb{S}'$ and $\mathbb{S}$. Q.E.D.

COROLLARY 3. *Consider the optimization formulation (22) with* $c > \mathbf{0}$ *and a specified level* $\overline{\overline{s}}_T$ *for the final SOC. Suppose that Assumptions 1, 2, and 4 are satisfied. Then there is an optimizer* $x^\star = \begin{bmatrix} u^\star \\ v^\star \\ p^\star \\ g^\star \end{bmatrix}$ *of the relaxed problem (23) that is an element of* $\mathbb{S}$.

Proof Note that in this case we have that, for the last epoch defined in the proof of Theorem 1, $s_{e_L} - s_{e_{L-1}} = \overline{\overline{s}} - s_{e_{L-1}}$ is evenly divisible by $\check{s}$ by assumption 4 and we can apply the construction from the proof of Theorem 1. Q.E.D.

5.2. Discussion

The results in this section are a manifestation of the observation for PSH that if prices are low enough and SOC constraints do not bind then it is optimal to pump at full capacity, while if prices are high enough and SOC constraints do not bind it is optimal to generate at full capacity. The ratio of the pumping to the generating interval prices is less than or equal to the round-trip efficiency. Similar observations have been made in various settings [46, 47, 39, 43]. Moreover, with strictly positive electricity prices, the mutual exclusivity constraints will be satisfied by the solution of the continuous relaxation of the problem. Similar observations and proofs also apply to the complementarity formulation of the mutual exclusivity constraints [25, 40, 44].[10]

Turning to the SOC limits, we observe that the somewhat unrealistic divisability requirements in Assumption 2 avoid a situation where optimal pumping and generation must be "uneven;" that is, neither zero nor at capacity. The uneven intervals considered in the proof of Theorem 1 correspond to primal degeneracy and will likely be a rare occurrence given the heterogeneity of prices (and therefore heterogeneity of objective coefficients) over time. That is, the adjustments described in the proof of Theorem 1 to uneven intervals occur only when there are repeated coefficients in the objective that occur during pumping and generating intervals. It is acknowledged that similar results could also be obtained for the standard formulation of the SOC limits (12).

Even if Assumption 2 is not satisfied it will typically be the case that the solution can be modified to have binary values for u and v. For example, suppose that the optimal generation g_t for interval t in the solution of a relaxed problem was between maximum and minimum and suppose that p_t was equal to zero. Then even if the continuous relaxation also yielded non-binary values for u_t and v_t, these could be set equal to 0 and 1, respectively, to create a feasible solution for the unrelaxed problem.

However, if the constraints and objective are embedded into a larger optimization problem as in the large-scale case studies to be presented in Section 6.2, it can easily be the case that uneven pumping and generation is required with either the standard or the tightened formulation, and that under these circumstances the solution of the relaxed problem may not satisfy the constraint of no simultaneous pumping and generating. Furthermore, if the entries of c are zero or negative or if Assumptions 2, 3, 4 do not hold then the solution of the relaxed problem may also exhibit the unphysical solution of simultaneous pumping and generating.

That is, it is generally necessary to explicitly enforce the integrality of u and v as will be exemplified in Section 6.1; however, the results in this section suggest that there may only be a few non-binary entries of u and v in the solution of relaxed problems and, moreover, the tightened formulation rules out some non-binary solutions as compared to the standard formulation. This facilitates solution in a large-scale problem by reducing the number of branch-and-bound or branch-and-cut steps. However, as mentioned in Endnote 7, a countervailing issue is that the tightened formulation involves more non-zeros. The next section will explore solution numerically to compare the standard and tightened formulations.

6. Numerical examples

We illustrate the main results with two sets of case studies. In Section 6.1, Corollary 2 is numerically verified with an example having a standalone PSH unit. In Section 6.2, large-scale Midcontinent ISO cases with over one thousand generators and three PSH units are investigated.

6.1. Standalone PSH case

Consider a PSH with parameters shown in Table 1. The feasible region for pumping and generation is illustrated in Figure 4. The pump is fixed power so that $\underline{p} = \overline{p}$. The example is similar to, but not exactly the same as, the example in Section 4.

We consider the relaxed problem (25). For the sake of a simple illustration, we assume that the economic value of stored energy at the end of the time horizon is zero, so that $d = 0$. Therefore, it is expected that the SOC will reach the minimum SOC limit $\underline{s} = 0.0$ at the end of the time horizon. We first consider an example with $T = 2$ in Section 6.1.1 and then an example with $T = 24$ in Section 6.1.2. The objective coefficients represent the electricity price for each interval. We consider several different scenarios for the electricity prices for each example and observe that in the solved cases:

- for non-negative electricity prices the formulation results in an optimizer that is an element of $\mathbb{S}$, but
- for negative electricity prices we can obtain solutions of the relaxed problem (25) with non-binary values for u and v.

That is, as discussed in Section 5.2, it is in general necessary to explicitly enforce the integrality constraints.

6.1.1. Two interval example We present three scenarios for the electricity prices $c_t, t = 1, 2$. The results of each scenario are presented in pairs of columns in Table 2. Time interval $t \in 1, 2$ is

Table 1 Parameters for examples Section 6.1.

$\underline{p}$	$\overline{p}$	$\underline{g}$	$\overline{g}$	$\underline{s}$	$\overline{s}$	s_0	α	β
1.0	1.0	0.0	0.81	0.0	0.9	0.0	0.9	1/0.9

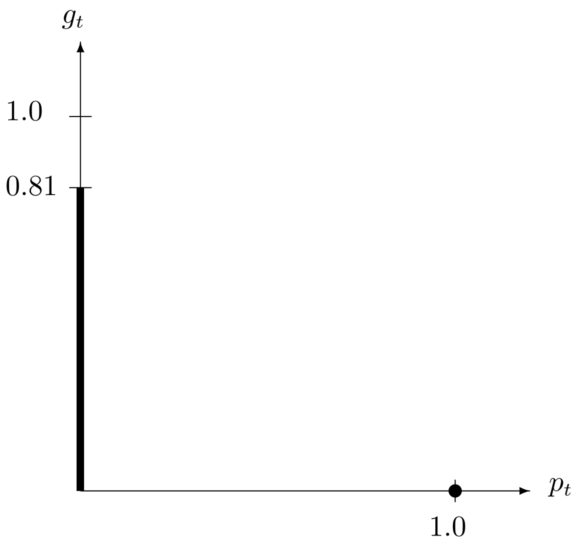

Figure 4 Feasible region for pumping and generation for examples in Section 6.1 for a particular t shown as thick line and bullet at $(1.0, 0.0)$.

Table 2 Results for two interval example in Section 6.1.1.

t	1	2	1	2	1	2
Prices	Positive		Zero		Negative	
c_t	20	30	0	0	−20	−30
u_t	1	0	0	0	0.5	1
p_t	1.0	0.0	0.0	0.0	0.5	1.0
v_t	0	1	1	0	0.5	0
g_t	0.0	0.81	0.0	0.0	0.405	0.0
s_t	0.9	0.0	0.0	0.0	0.0	0.9

indicated in the first row. The sign and the values of the price c_t for each scenario are listed in the second and third row, respectively. The pumping commitment variable u_t and pumping value p_t are shown in the fourth and fifth rows, respectively. The generation commitment variable v_t and

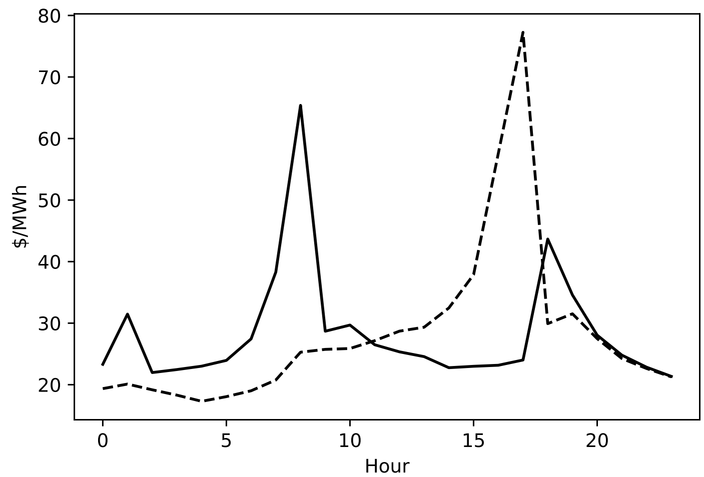

Figure 5 Prices for single PSH unit case study. The dashed line shows prices for a single peak day, while the solid line shows prices for a two peak day.

generation value g_t are shown in the sixth and seventh rows, respectively. The SOC s_t is shown in the eighth row. A similar arrangement will be used for subsequent tables.

For positive and zero prices, the results from the second to the fifth columns in Table 2 show that the optimizer of the relaxed problem (25) is an element of $\mathbb{S}$ and successfully avoids non-integer values for u_t and v_t. It should be noted that Corollary 2 does not guarantee this result for zero prices; however, the scenario with zero prices does satisfy the conclusion of Corollary 2. When the prices are negative, the results shown in the last two columns in Table 2 indicate that an optimizer of the relaxed problem (25) may indeed have non-integer values for u_t and v_t, as occurs for both $u_1 = 0.5$ and $v_1 = 0.5$.

6.1.2. Twenty-four interval example Next we solve the relaxed problem (25) for a twenty-four interval horizon; that is, $T = 24$, representing a day of 24 hours. Figure 5 shows the two price scenarios used for the twenty-four interval example. The dashed line indicates the hourly prices during a day with a single main peak, while the solid line indicates the prices during a day with two main peaks.

The solution of the relaxed problem (25) for the price scenario with a single main peak is presented in Table 3. The unit pumps in interval $t = 5$ at its maximum capacity $\overline{p} = 1.0$, generates in interval $t = 18$ at its maximum capacity $\overline{g} = 0.81$, and has zero pumping and generation in the other intervals. The intervals with non-zero pumping or generating are bolded in Table 3.

Table 3 Solution of relaxed problem (25) for prices with a single main peak.

t	1	2	3	4	**5**	6	7	8	9	10	11	12
u_t	0	0	0	0	**1**	0	0	0	0	0	0	0
p_t	0	0	0	0	**1**	0	0	0	0	0	0	0
v_t	1	1	1	1	0	1	1	1	1	1	1	1
g_t	0	0	0	0	0	0	0	0	0	0	0	0
s_t	0	0	0	0	**0.9**	0.9	0.9	0.9	0.9	0.9	0.9	0.9

t	13	14	15	16	17	**18**	19	20	21	22	23	24
u_t	0	0	0	0	0	0	0	0	0	0	0	0
p_t	0	0	0	0	0	0	0	0	0	0	0	0
v_t	1	1	1	1	1	**1**	1	1	1	1	1	1
g_t	0	0	0	0	0	**0.81**	0	0	0	0	0	0
s_t	0.9	0.9	0.9	0.9	0.9	**0**	0	0	0	0	0	0

The pumping commitment variable has value $u_t = 0$ except for $t = 5$. The generation commitment variable has value $v_t = 1$ except for $t = 5$; however, the generation is zero in the other intervals except for $t = 18$. Similarly to the scenario with positive prices in section 6.1.1, and consistent with Corollary 2, the solution of the relaxed problem (25) is an element of $\mathbb{S}$.

The solution of the relaxed problem (25) for the price scenario with two main peaks is presented in Table 4. The unit pumps in three intervals $t = 1, 3, 15$ at its maximum capacity $\overline{p} = 1.0$, generates in three intervals $t = 2, 9, 19$ at its maximum capacity $\overline{g} = 0.81$, and has zero pumping and generation in the other intervals. The intervals with non-zero pumping or generating are bolded in Table 4.

The pumping commitment variable has value $u_t = 0$ except for $t = 1, 3, 15$. The generation commitment variable has value $v_t = 1$ except for $t = 1, 3, 15$; however, the generation is again zero in these other intervals. Again consistent with Corollary 2, the solution of the relaxed problem (25) is an element of $\mathbb{S}$.

Table 4 Solution of relaxed problem (25) for prices with two main peaks.

t	**1**	**2**	**3**	4	5	6	7	8	**9**	10	11	12
u_t	**1**	0	**1**	0	0	0	0	0	0	0	0	0
p_t	**1**	0	**1**	0	0	0	0	0	0	0	0	0
v_t	0	**1**	0	1	1	1	1	1	**1**	1	1	1
g_t	0	**0.81**	0	0	0	0	0	0	**0.81**	0	0	0
s_t	**0.9**	**0**	**0.9**	0.9	0.9	0.9	0.9	0.9	**0**	0	0	0

t	13	14	**15**	16	17	18	**19**	20	21	22	23	24
u_t	0	0	**1**	0	0	0	0	0	0	0	0	0
p_t	0	0	**1**	0	0	0	0	0	0	0	0	0
v_t	1	1	0	1	1	1	**1**	1	1	1	1	1
g_t	0	0	0	0	0	0	**0.81**	0	0	0	0	0
g_t	0	0	**0.9**	0.9	0.9	0.9	**0**	0	0	0	0	0

6.2. Midcontinent ISO case study

In this section, a large-scale computational study is presented for seven Midcontinent ISO production cases. Standard and tightened SOC limits are compared. The cases each have over 1000 generators and represent approximately 220 transmission constraints.

Except for the "Without SOC" model, detailed models of both generators and PSH are included based on the configuration-based PSH model that was presented in [20]. That is, we assume that PSH operating characteristics are directly modeled in the production case. In contrast, the "Without SOC" model will represent the PSH implicitly through bids and offers. Both the standard formulation of the SOC limits (12) ([20, equation (11)]) and the tightened SOC formulation (13)–(14) have been implemented in the Midcontinent ISO prototype day-ahead security-constrained unit commitment tool HIPPO [7].

As observed in Endnote 7, to preserve sparsity in the formulation, the SOC s is explicitly represented. Moreover, although the tightened formulation of the SOC constraint provides a better approximation to the convex hull of $\mathbb{S}$, it also requires more non-zero elements in the constraint matrix than the standard formulation of SOC limits. These two issues are countervailing.

Because of their countervailing effects on computational effort, we test both the standard and the tightened SOC limit formulations using HIPPO with seven Midcontinent ISO cases. All the system reserve requirements and transmission security constraints are included. We perform all tests on a 2.5 GHz (32 processors) Intel Core Processor (Haswell, no TSX, IBRS) with 256GB RAM. All optimization problems are solved with Gurobi 8.1. The cases involve hourly operation over a 36-hour span and have different load and generator scenarios. The model generated from each case has over 6×10^5 rows, 6×10^5 columns, and 4×10^6 non-zeros.

Each case is solved with three models, labeled as follows in Table 5:

Without SOC: with no SOC constraints represented, and the PSH units dispatched on the basis of bids for pumping during a window of pumping intervals and offers for generation during a window of generation intervals,

Standard SOC: using the standard formulation of SOC constraints for the configuration-based PSH model, and

Tightened SOC: using the tightened formulation of SOC constraints for the configuration-based PSH model.

A cutoff computation time of 3600 seconds is imposed. The linear programming (LP) relaxations of the Standard and the Tightened SOC formulations correspond to the relaxed problem (25). Due to the intrinsic randomness built into a Mixed Integer Programming solver like Gurobi, we test each model for each case five times to have more robust results. A parameter named Random Seed Number in Gurobi is designed to introduce a perturbation that typically leads to different solution paths. Therefore the Random Seed Number is set to a different number (from one to five) every time a model is tested with the same case. The average wall clock MIP stopping times are listed in Table 5. The MIP stopping time for the Without SOC runs provides a gauge of the effect on problem difficulty of representing three units (out of over 1000 generators) with an SOC instead

Table 5 Numerical comparison for standard and tightened SOC constraints in Midcontinent ISO cases.

Case	Without SOC	Standard SOC		Tightened SOC		Stand.-Tight.
	Stopping Time [s] (STD)	LP Objective [$]	Stopping Time [s] (STD)	Tight.-Stand. LP Obj. [$]	Stopping Time [s] (STD)	Stopping Time [s] (STD)
1	333(16)	9.5×10^6	409(14)	11	425(27)	−16(21)
2	234(9)	17.5×10^6	262(9)	3,447	285(10)	−23(14)
3	353(23)	14.2×10^6	409(22)	572	382(13)	27(20)
4	302(16)	6.2×10^6	1835(1312)	952	1201(1070)	634(2100)
5	3709(24)	1.4×10^6	3420(569)	0	3663(11)	−243(565)
6	513(252)	16.4×10^6	761(275)	770	660(199)	101(303)
7	219(14)	10.9×10^6	224(45)	18	241(34)	3(57)

Table 6 Numerical comparison for standard and tightened SOC constraints in Midcontinent ISO case 5.

Case	Standard SOC		Tightened - Standard	
	Objective [$]	Lower Bound [$]	Objective [$]	Lower Bound [$]
5	1.4×10^6	1.4×10^6	22,992	1,841

of as generator offers and pump bids. The sample standard deviation (STD) of the MIP stopping time is calculated and listed in the parentheses in the same cell to the right of the value of the average Stopping Time.

The MIP optimality gap is set to 1% for cases 1 to 5. The MIP optimality gap is set to 0.1% for cases 6 and 7. The MISO production MIP gap target is 0.1%; however, cases 1 to 5 are very hard cases and the solver cannot get a solution close to the MIP gap of 0.1% within the cutoff time of 3600 seconds. Therefore, a 1% gap is set for those cases instead in order that the solver could get a solution in a reasonable time and the results can be used to compare between models.

A higher LP objective indicates a tighter model and a smaller stopping time indicates that the MIP solver could find a feasible solution within the optimality gap in less time. The LP objective and the time taken to get a feasible solution within the optimality gap for both models varies with different cases. The fifth column in Table 5 shows the differences between the Tightened SOC model optimal LP objective minus the Standard SOC model optimal LP objective. The results empirically verify that the LP objective of the model with the tightened SOC constraints is always equal to or higher than the LP objective of the model with the standard SOC constraints.

The MIP stopping times shown in Table 5 show that the tightened model has a moderate effect on computational burden, mostly either maintaining roughly the same or somewhat reducing the computational burden, with the exception of case 5. In four out of the seven cases (cases 1 to 3 and case 7), the average stopping time for the standard model and the tightened model are within 50 seconds of each other. For cases 4 and 6, the tightened model is solved faster than the standard model by more than 100 seconds on average. For these two cases, it is observed that the LP objective of the tightened model is significantly higher than the standard model indicating that the tightened model provides a better lower bound when the MIP starts. That is likely to contribute to the tightened model's better MIP stopping time performance for cases 4 and 6.

For case 5, the tightened model is solved slower than the standard model on average. Notice that the LP objectives are the same for both the standard model and the tightened model in case 5. That means the tightened model is not as helpful in case 5 as it is for the other cases, at least at the root node.

To assess the consistency between the result for each run, the average and the sample standard deviation (listed in the paranthesis in each cell) of the stopping time differences (the result of the standard model minus the result of the tightened model) are listed in the last column in Table 5. Except for case 4, the sample standard deviations are relatively low. That indicates that, except for case 4, the conclusion from the average stopping time does not come from a single or a few runs with extreme results. For case 4, the large STD comes from the fact that the tightened model is solved much slower (more than 2000 seconds) than the standard model in one particular run, while the tightened model is consistently solved faster than the standard model in the other four runs.

Case 5 is an extremely hard case and the solver could not get a solution close to even the 1% MIP gap for most of the runs for both the standard and the tightened model. Therefore, the average of the objective and the best lower bound for each model at the cutoff time, or at the time a solution with a less than 1% MIP gap was found before the cutoff time (only occurs in one run with the standard model) are listed in Table 6 for a further comparison. Notice that the differences on objective and lower bound between the two models (the result of the tightened SOC minus the result of the Standard SOC model) are listed in the last two columns of the table. The objective of this case is relatively small (about one tenth of most of the other cases). The lower objective is due to the presence of virtual bids/offers and due to violation penalties. In this case it is observed that although the tightened SOC model provides a better (higher) lower bound, it does not help the solver to find a better (lower) objective.

The MIP solver explores the solutions in an iterative process and apparently there are other factors affecting the computational time. To summarize, the numerical results of the tested cases in Table 5 show that, with one exception in the seven cases considered, the tightened SOC constraints typically have approximately neutral or improved impact on the computation time.

7. Conclusion and future work

In this paper we have presented tighter valid inequalities for a generic, but basic, storage device that has mutually exclusive pumping and generating modes and SOC limits. The constraint formulation is therefore suitable for incorporating into unit commitment models. We also used a standalone PSH model to explain why the combination of mutual exclusivity constraints with round trip efficiency less than one and SOC limits results in a challenging model. Several numerical examples demonstrate the results.

The basic storage model does not consider all of the features of storage systems. For example, in PSH systems there may be multiple turbine-pump-generators sharing a single reservoir and PSH systems may share several of the types of constraints that are common in thermal generators, such as ramp limitations [8], minimum up- and down-times [32], and transition power trajectories [19].

Moreover, PSH may also provide ancillary services. As presented in [18], we can also incorporate the characterizations of each additional issue as valid inequalities in the formulation developed in this paper, and a useful formulation including multiple issues can potentially be obtained [18].

Future work includes:

- investigating more numerical cases to understand where the tightened formulation leads to improved computational performance,
- representing other limits on operation such as ramp limitations,
- representing head and variable efficiency effects and cascaaded reservoirs,
- expanding the formulation to include ancillary services as well as energy,
- investigating the case where storage can continuously vary from charging to discharging as is typical in chemical battery storage models, and
- investigating whether it is possible to accommodate multi-turbine reservoirs, where a single SOC applies across multiple turbine-pump-generators.

Appendix. Proof of Theorem 1

Using Lemma 1 to specify some of the entries of $x^\star$ To begin the specification of the entries of $x^\star$, observe that by Lemma 1, for each $t = 1, \dots, T$, at least one of $p_t^{\star\star}$ or $g_t^{\star\star}$ is equal to zero. For each t such that $p_t^{\star\star} = 0$, define $u_t^\star = 0$ and $p_t^\star = 0$, and, if it is the case for this t that $g_t^{\star\star} = \overline{g}$, then also define $v_t^\star = 1$ and $g_t^\star = \overline{g}$. Similarly, for each t such that $g_t^{\star\star} = 0$, define $v_t^\star = 0$ and $g_t^\star = 0$ and, if it is the case for this t that $p_t^{\star\star} = \overline{p}$, then also define $u_t^\star = 1$ and $p_t^\star = \overline{p}$. Note that, for each t, either the entries $u_t^\star$, $p_t^\star$, $v_t^\star$, and $g_t^\star$ have all been specified and satisfy (36)–(42), or only $u_t^\star$ and $p_t^\star$ remain unspecified, or only $v_t^\star$ and $g_t^\star$ remain unspecified. In the remainder of the proof, we specify any remaining unspecified values in such a way that $x^\star \in \mathbb{S}$.

Considering the "uneven" intervals of $x^{\star\star}$ The remaining unspecified values correspond to intervals t having either:

1. $u_t^\star$ and $p_t^\star$ unspecified, so that $0 < p_t^{\star\star} < \overline{p}$, $u_t^{\star\star} > 0$, and $g_t^\star = g_t^{\star\star} = 0$, $v_t^\star = 0$, or
2. $v_t^\star$ and $g_t^\star$ unspecified, so that $0 < g_t^{\star\star} < \overline{g}$, $v_t^{\star\star} > 0$, and $p_t^\star = p_t^{\star\star} = 0$, $u_t^\star = 0$,

where we observe that only one of these two cases can occur in any given interval. We call these the "uneven" intervals for the solution $x^{\star\star}$ and refer to them, respectively, either as case 1 or 2 uneven intervals of $x^{\star\star}$, or simply case 1 or 2 uneven intervals when the context is clear.

We call the rest of the intervals "even" intervals for the solution $x^{\star\star}$ and note that the change of SOC in any even interval is either 0 or $\pm\breve{s} = \pm\alpha\overline{p} = \pm\beta\overline{g}$. Our specification of $x^\star$ will ensure that all of its intervals are even and that the SOC constraints remain satisfied; that is, it will ensure that $x^\star \in \mathbb{S}$.

Defining the notion of "epoch" and its properties To facilitate specifying the rest of the intervals, consider the intervals t such that the SOC s_t equals $\overline{s}$ or $\underline{s}$. Label these intervals as $e_1, e_2, \ldots$. Note that, in principle, s_T might or might not equal $\overline{s}$ or $\underline{s}$; however, for convenience we include $t = T$ as the last labeled interval irrespective of its value. Also define $t = 0$ as the 0-th labeled interval, $e_0 = 0$, so that the labeled intervals are $e_0 = 0, e_1, \ldots, e_L = T$. For $\ell = 1, \ldots, L$, define the ℓ-th "epoch" to be the collection of intervals $e_{\ell-1} + 1, \ldots, e_\ell$. That is: the first epoch includes intervals $1, \ldots, e_1$; the second epoch (if there is one) includes intervals $e_1 + 1, \ldots, e_2$; while the last epoch includes the intervals $e_{L-1} + 1, \ldots, e_L = T$.

Now consider a particular epoch $\ell \in \{1, \ldots, L\}$ and write t_k for the k-th uneven interval in the ℓ-th epoch, with $k \in \{1, \ldots, K\}$, and where there are K uneven intervals in total in the ℓ-th epoch. The set of all uneven intervals in epoch ℓ is $\{t_1, \ldots, t_k\}$, while the set of all even intervals in epoch ℓ is $\{e_{\ell-1} + 1, \ldots, e_\ell\} \setminus \{t_1, \ldots, t_k\}$. We define Δs to be the sum of the changes in SOC due to the *uneven* pumping and generation intervals in epoch ℓ. That is, $\Delta s = \sum_{k=1}^{K} (\alpha p^{\star\star}_{t_k} - \beta g^{\star\star}_{t_k})$ differs from $s^{\star\star}_{e_\ell} - s^{\star\star}_{e_{\ell-1}}$ by the sum of the changes in the SOC due to the *even* pumping and generation intervals in epoch ℓ. (We are slightly abusing notation by not including the index, ℓ, of the epoch in the notation for the uneven intervals and Δs in the ℓ-th epoch; however, the epoch index will always be clear from the context.)

We note three properties in each epoch that will be useful in the balance of the proof:

1. For each ℓ, except possibly for $\ell = L$, we have that $s^{\star\star}_{e_\ell} - s^{\star\star}_{e_{\ell-1}}$ is evenly divisible by $\breve{s} = \alpha\overline{p} = \beta\overline{g}$, where we define $s^{\star\star}_{e_0} = s_0$, and recall that by Assumption 3, $(\overline{s} - s^{\star\star}_{e_0}) = (\overline{s} - s_0)$ and $(s^{\star\star}_{e_0} - \underline{s}) = (s_0 - \underline{s})$ are both evenly divisible by $\breve{s} = \alpha\overline{p} = \beta\overline{g}$, so that:

$$
\begin{aligned}
(s^{\star\star}_{e_1} - s^{\star\star}_{e_0}) &= (s^{\star\star}_{e_1} - s_0), \\
&= \begin{cases} (\overline{s} - s_0), & \text{if } s^{\star\star}_{e_1} = \overline{s}, \\ (\underline{s} - s_0), & \text{if } s^{\star\star}_{e_1} = \underline{s}, \end{cases}
\end{aligned}
$$

is also evenly divisible by $\breve{s} = \alpha\overline{p} = \beta\overline{g}$.

2. For each ℓ, except possibly for $\ell = L$, Δs is evenly divisible by $\breve{s}$, since for these epochs we have that:

$$
\begin{aligned}
\Delta s &= \sum_{k=1}^{K} (\alpha p^{\star\star}_{t_k} - \beta g^{\star\star}_{t_k}), \text{ by definition}, \\
&= s^{\star\star}_{e_\ell} - s^{\star\star}_{e_{\ell-1}} - \sum_{t \in \{e_{\ell-1}+1, \ldots, e_\ell\} \setminus \{t_1, \ldots, t_k\}} (\alpha p^{\star\star}_t - \beta g^{\star\star}_t), \\
&= \text{(an integer multiple of } \breve{s}) - \text{ (an integer multiple of } \breve{s}),
\end{aligned}
$$

since $\alpha p_t^{\star\star} = \beta g_t^{\star\star} = \breve{s}$ for even intervals $t \in \{e_{\ell-1}+1, \ldots, e_\ell\} \backslash \{t_1, \ldots, t_k\}$.

3. For each ℓ, and for each interval $\tau \in \{e_{\ell-1}+1, \ldots, e_\ell - 1\}$ (that is, for all the intervals in the range, not just the uneven intervals), the SOC $s_\tau^{\star\star}$ does not equal $\overline{s}$ nor $\underline{s}$. That is, neither (43) nor (44) are binding for these intervals, so that by complementary slackness the corresponding Lagrange multipliers satisfy $\mu_{43\tau}^{\star\star} = \mu_{44\tau}^{\star\star} = 0$ for all $\tau \in \{e_{\ell-1}+1, \ldots, e_\ell - 1\}$.

Considering the coefficients of the objective for the uneven intervals in an epoch We characterize D_{t_k} for case 1 uneven intervals and characterize C_{t_k} for case 2 uneven intervals in the ℓ-th epoch. First consider case 1 uneven intervals t_k in the ℓ-th epoch. These intervals satisfy $0 < p_{t_k}^{\star\star} < \overline{p}$, $u_{t_k}^{\star\star} > 0$, and $g_{t_k}^{\star} = g_{t_k}^{\star\star} = 0$. Since $u_{t_k}^{\star\star} > 0$ and $p_{t_k}^{\star\star} > 0$, we have by complementary slackness that $\mu_{36t_k}^{\star\star} = 0$ and $\mu_{39t_k}^{\star\star} = 0$. Therefore, by (45), $\mu_{38t_k}^{\star\star} - \overline{p}\mu_{40t_k}^{\star\star} = 0$ and, so, either (38) and (40) are both binding with strictly positive values of Lagrange multipliers or $\mu_{38t_k}^{\star\star} = \mu_{40t_k}^{\star\star} = 0$.

We claim that (38) and (40) cannot be both binding with strictly positive values of Lagrange multipliers so that the only possibility is $\mu_{38t_k}^{\star\star} = \mu_{40t_k}^{\star\star} = 0$. To prove this, we consider each of the following two alternative cases for values of $v_{t_k}^{\star\star}$ and for both cases we prove by contradiction:

$v_{t_k}^{\star\star} = 0$: Suppose that (38) and (40) were both binding with strictly positive values of Lagrange multipliers. Then, since by assumption $v_{t_k}^{\star\star} = 0$ and since (38) is binding we have that $u_{t_k}^{\star\star} = 1$. Consequently, since $u_{t_k}^{\star\star} = 1$ and since (40) is binding, we also have that $p_{t_k}^{\star\star} = \overline{p}u_{t_k}^{\star\star} = \overline{p}$, contradicting that $p_{t_k}^{\star\star} < \overline{p}$. Therefore, (38) and (40) could not have been both binding with strictly positive values of Lagrange multipliers and instead we must have that $\mu_{38t_k}^{\star\star} = \mu_{40t_k}^{\star\star} = 0$.

$v_{t_k}^{\star\star} > 0$: Again suppose that (38) and (40) were both binding with strictly positive values of Lagrange multipliers. Since $g_{t_k}^{\star\star} = 0$ and $v_{t_k}^{\star\star} > 0$ we have that $0 = g_{t_k}^{\star\star} < \overline{g}v_{t_k}^{\star\star}$, so that $\mu_{41t_k}^{\star\star} > 0$ and $\mu_{42t_k}^{\star\star} = 0$. Moreover, since $v_{t_k}^{\star\star} > 0$, we have that $\mu_{37t_k}^{\star\star} = 0$. From (46),

$$\begin{aligned} \mu_{38t_k}^{\star\star} &= \mu_{37t_k}^{\star\star} + \overline{g}\mu_{42t_k}^{\star\star}, \\ &= 0, \end{aligned}$$

contradicting the supposition that $\mu_{38t_k}^{\star\star} > 0$. We conclude that, in fact, $\mu_{38t_k}^{\star\star} = \mu_{40t_k}^{\star\star} = 0$.
To summarize, in both cases we have that that $\mu_{38t_k}^{\star\star} = \mu_{40t_k}^{\star\star} = 0$. Substituting into (47), and noting that:

- $\mu_{39t_k}^{\star\star} = 0$,
- $\mu_{40t_k}^{\star\star} = 0$, and
- $\mu_{43\tau}^{\star\star} = \mu_{44\tau}^{\star\star} = 0$, for $\tau \in \{e_{\ell-1}+1, \ldots, e_\ell - 1\}$,

we obtain, for each case 1 uneven interval $t_k \in \{e_{\ell-1}+1, \ldots, e_\ell - 1\}$ that:

$$\begin{aligned} D_{t_k} &= \mu_{39t_k}^{\star\star} - \mu_{40t_k}^{\star\star} - \alpha\mu_{43t_k}^{\star\star} - \alpha \sum_{\tau=t_k+1}^{T} (\mu_{43\tau}^{\star\star} - \mu_{44\tau}^{\star\star}), \\ &= -\alpha \sum_{\tau=e_\ell}^{T} (\mu_{43\tau}^{\star\star} - \mu_{44\tau}^{\star\star}). \end{aligned}$$

We now consider the situation where the last interval in the ℓ-th epoch, namely e_ℓ, is a case 1 uneven interval, so that $t_K = e_\ell$. By assumption 1, at most one of (43) and (44) can be binding during interval $t_K = e_\ell$. Consequently, if $t_K = e_\ell$ is a case 1 uneven interval then, since $p^{\star\star}_{t_K} > 0$, only (43) could binding, while (44) cannot be binding, and so $\mu^{\star\star}_{44t_K} = 0$. Therefore, we have:

$$\begin{aligned}
D_{t_K} &= -\alpha\mu^{\star\star}_{43t_K} - \alpha \sum_{\tau=t_K+1}^{T} (\mu^{\star\star}_{43\tau} - \mu^{\star\star}_{44\tau}), \\
&= -\alpha\mu^{\star\star}_{43t_K} + \beta\mu^{\star\star}_{44t_K} - \alpha \sum_{\tau=t_K+1}^{T} (\mu^{\star\star}_{43\tau} - \mu^{\star\star}_{44\tau}), \\
&= -\alpha \sum_{\tau=t_K}^{T} (\mu^{\star\star}_{43\tau} - \mu^{\star\star}_{44\tau}), \\
&= -\alpha \sum_{\tau=e_\ell}^{T} (\mu^{\star\star}_{43\tau} - \mu^{\star\star}_{44\tau}).
\end{aligned}$$

Note that this means that D_{t_k} has the same value independent of t_k for all of the case 1 uneven intervals in the ℓ-th epoch, and its common value is $-\alpha\sum_{\tau=e_\ell}^{T}(\mu^{\star\star}_{43\tau} - \mu^{\star\star}_{44\tau})$. For convenience, write D for this common value. In all such case 1 uneven intervals t_k we have that $0 < p^{\star\star}_{t_k} < \overline{p}$, $u^{\star\star}_{t_k} > 0$, and $v^{\star\star}_{t_k} = 0$.

Similarly, C_{t_k} has the same value independent of t_k for all of the case 2 uneven intervals in the ℓ-th epoch, and is equal to $-\beta\sum_{\tau=e_\ell}^{T}(\mu^{\star\star}_{43\tau} - \mu^{\star\star}_{44\tau})$. For convenience, write C for this common value, and note that $C/\beta = D/\alpha$. In all such case 2 uneven intervals t_k we have that $0 < g^{\star\star}_{t_k} < \overline{g}$, $v^{\star\star}_{t_k} > 0$, and $u^{\star\star}_{t_k} = 0$.

Evaluating the contribution of uneven intervals to the objective Consider any specification of the entries of $x^\star$ in epoch ℓ for intervals $t_1, \ldots, t_K$ such that:

- case 1 intervals are assigned zero generation in $x^\star$, but possibly assigned non-zero pumping,
- case 2 intervals are assigned zero pumping in $x^\star$, but possibly assigned non-zero generation, and
- the change in SOC due to the non-zero pumping and generation in intervals $t_1, \ldots, t_K$ in $x^\star$ is the same as for the intervals $t_1, \ldots, t_K$ in $x^{\star\star}$.

That is, the last condition means that the pumping and generation $p_{t_k}, g_{t_k}, k = 1, \ldots, K$, in the intervals $t_k, k = 1, \ldots, K$, are chosen to satisfy:

$$\begin{aligned}
\sum_{k=1}^{K}(\alpha p^\star_{t_k} - \beta g^\star_{t_k}) &= \sum_{k=1}^{K}(\alpha p^{\star\star}_{t_k} - \beta g^{\star\star}_{t_k}), \\
&= \Delta s.
\end{aligned}$$

Consider the contribution to the objective of these intervals $t_k, k = 1, \ldots, K$. The contribution is:

$$\sum_{k=1}^{K}(C_{t_k} g^\star_{t_k} - D_{t_k} p^\star_{t_k}) = \sum_{k=1}^{K}(C g^\star_{t_k} - D p^\star_{t_k}),$$

$$
\begin{aligned}
&\quad \text{noting for case 1 intervals } t_k \text{ that } g^\star_{t_k} = 0, \text{ and} \\
&\qquad\quad \text{for case 2 intervals } t_k \text{ that } p^\star_{t_k} = 0, \\
&= (C/\beta) \sum_{k=1}^{K} (\beta g^\star_{t_k} - \alpha p^\star_{t_k}), \\
&\quad \text{recalling that } C/\beta = D/\alpha \\
&= (C/\beta) \sum_{k=1}^{K} (\beta g^{\star\star}_{t_k} - \alpha p^{\star\star}_{t_k}), \\
&\quad \text{by assumption on the change in the SOC }, \\
&= \sum_{k=1}^{K} (C_{t_k} g^{\star\star}_{t_k} - D_{t_k} p^{\star\star}_{t_k}),
\end{aligned}
$$

noting for case 1 intervals t_k that $g^{\star\star}_{t_k} = 0$, and for case 2 intervals t_k that $p^\star_{t_k} = 0$. That is, the contribution to the objective from the intervals $t_k, k = 1, \ldots, K$, in $x^\star$ is the same as that for $x^{\star\star}$.

Specifying the remaining entries of $x^\star$ Using the result about the contribution to the objective, we define the remaining unspecified entries of $x^\star$ to consolidate pumping and generation into a subset of the uneven intervals so that:

- in each case 1 uneven interval of $x^{\star\star}$ the pumping in that interval of $x^\star$ is either equal to 0 or $\overline{p}$,
- in each case 2 uneven interval of $x^{\star\star}$ the generation in that interval of $x^\star$ is either equal to 0 or $\overline{g}$,
- the sum of the changes in SOC in $s^\star$ due to all the intervals $t_1, \ldots, t_K$ is the same as the change in SOC for those intervals in $s^{\star\star}$, and
- $s^\star$ satisfies the SOC limits.

First consider epochs such that Δs is evenly divisible by $\breve{s}$. This applies to all epochs except possibly the last. Consider the following construction. Each specification of entries of $x^\star$ for an uneven interval t_k will change the SOCs $s^\star_{t_k}, s^\star_{t_k+1}, \ldots, s^\star_T$ compared to $s^{\star\star}_{t_k}, s^{\star\star}_{t_k+1}, \ldots, s^{\star\star}_T$. For each successive uneven interval t_k, we choose the specification that minimizes the deviation of $s^\star_{t_k}$ from $s^{\star\star}_{t_k}$, breaking ties arbitrarily, and we claim that this will maintain satisfaction of (43)–(44). For example, if the first uneven interval in the epoch, say interval t_1, has $p^{\star\star}_{t_1} = 0.6\overline{p}$ then set $u^\star_{t_1} = 1$ and $p^\star_{t_1} = \overline{p}$. If the second uneven interval in the epoch, t_2, has $p^{\star\star}_{t_2} = 0.3\overline{p}$ then set $u^\star_{t_2} = 0$ and $p^\star_{t_2} = 0$. Note that simpler approaches, such as assigning all pumping to the earliest uneven intervals will not generally maintain feasibility with respect to the SOC limits.

More generally, consider:

$$\Delta_k = \sum_{j=1}^{k} \alpha(p^\star_{t_j} - p^{\star\star}_{t_j}) - \beta(g^\star_{t_j} - g^{\star\star}_{t_j}).$$

If t_k is an uneven case 1 interval, then choose $p^\star_{t_k}$ to be either 0 or $\overline{p}$ so as to result in the smallest value for $|\Delta_{k-1} + \alpha(p^\star_{t_k} - p^{\star\star}_{t_k})|$, where, for convenience, we define $\Delta_0 = 0$ and where ties can be broken arbitrarily. If t_k is an uneven case 2 interval, choose $g^{\star}_{t_k}$ to be either 0 or $\overline{g}$ so as to result in the smallest value for $|\Delta_{k-1} - \beta(p^{\star}_{t_k} - p^{\star\star}_{t_k})|$, again breaking ties arbitrarily. Since $\Delta_0 = 0$, we observe that, by construction, $|\Delta_k| \leqslant \breve{s}/2 = \alpha\overline{p}/2 = \beta\overline{g}/2$.

We now prove that this specification also results in the entries of the SOC $s^{\star}$ being within limits for all intervals in the epoch. To see this, note that if an uneven interval had state $s^{\star\star}_{t_k}$ that was within $\breve{s}$ of $\overline{s}$ then, by construction, $s^{\star}_{t_k}$ is either $\overline{s}$ or $\overline{s} - \breve{s}$. For subsequent even intervals t until the next uneven interval, observe that $s^{\star\star}_t$ must have been no higher than $s^{\star\star}_{t_k}$ since otherwise it would equal $s^{\star\star}_{t_k} + \breve{s}$ and the SOC limits would have been violated by $x^{\star\star}$. Consequently, the corresponding values of $s^{\star}_t$ until the next uneven interval will also satisfy the SOC constraints. A similar argument applies for each uneven interval that had state $s^{\star\star}_{t_k}$ that was within $\breve{s}$ of $\underline{s}$.

Moreover, for t_K, the last uneven interval in the epoch, we must have that $\Delta_K = 0$, since

$$\begin{aligned}\Delta_K &= \sum_{j=1}^{K} \alpha(p^{\star}_{t_j} - p^{\star\star}_{t_j}) - \beta(g^{\star}_{t_j} - g^{\star\star}_{t_j}),\\ &= -\sum_{j=1}^{K}(\alpha p^{\star\star}_{t_j} - \beta g^{\star\star}_{t_j}) + \sum_{j=1}^{K}(\alpha p^{\star}_{t_j} - \beta g^{\star}_{t_j}),\\ &= -\Delta s + \sum_{j=1}^{K}(\alpha p^{\star}_{t_j} - \beta g^{\star}_{t_j}),\end{aligned}$$

and Δs is evenly divisible by $\breve{s}$ and the terms $(\alpha p^{\star}_{t_j} - \beta g^{\star}_{t_j})$ in the last summation are all equal to $\pm\breve{s}$ by construction. Consequently, $\Delta_K = 0, \pm\breve{s}, \pm 2\breve{s}, \ldots$. However, since $|\Delta_K| \leqslant \breve{s}/2$ by construction, this means that $\Delta_K = 0$. That is, the change in SOC due to the intervals $t_1, \ldots, t_K$ in $x^{\star}$ is the same as the change in SOC due to the intervals $t_1, \ldots, t_K$ in $x^{\star\star}$, and so the contribution to the objective is the same for these intervals.

Turning to the last epoch, $\ell = L$, note that if $s_{e_L} - s_{e_{L-1}}$ is not evenly divisible by $\breve{s}$ then for all uneven case 1 intervals t_k in the L-th epoch we must have $D_{t_k} = D = 0$ and for all uneven case 2 intervals t_k in the L-th epoch we must have $C_{t_k} = C = 0$. Otherwise, we could consider a small change either in $p^{\star\star}_{t_k}$ and $u^{\star\star}_{t_k}$, or in $g^{\star\star}_{t_k}$ and $v^{\star\star}_{t_k}$, respectively, that would improve the objective without violating any constraints, which contradicts optimality of $x^{\star\star}$.

To summarize, for the last epoch, if $s_{e_L} - s_{e_{L-1}}$ is not evenly divisible by $\breve{s}$ then because $C = D = 0$ we can arbitrarily assign the pumping for case 1 and the generation for case 2 intervals, $t_k, k = 1, \ldots, K$, without affecting the objective. We can choose this assignment to maintain feasibility with respect to the SOC constraints by following the same construction as for the other epochs.

Endnotes

1. It is acknowledged that uncertainty in renewable production from solar and wind resources is an important motivation for increasing utilization of storage capacity in electricity systems. For example, see [**?**, 4, 9] for discussion of coordinated dispatch of renewables and PSH. However, even in a formulation of hydro-thermal commitment that included randomness of solar and wind, the PSH could still be represented deterministically in the weekly or daily timeframe [4].

2. As discussed further in Section 5, the asset owner will still need to forecast the value of water after the end of the unit commitment horizon; however, this value is mostly affected by longer-term macroeconomic issues and longer-term weather forecasts. Furthermore, ISOs typically have access to region-wide renewable forecasts that may improve on the forecasts available to standalone asset owners, and thereby also reduce the first source of uncertainty compared to standalone asset optimization.

3. We ignore issues such as change in pumping power and round trip efficiency with head or with SOC. Piecewise linearized approximations to these issues can, in principle, be included in the model following the approaches in [3, 4, 5, 6, 42]. We also ignore cascaded reservoirs [4].

4. That is, the basic model does not represent *ternary* PSH systems that can pump and generate simultaneously [5][22, section 3.3] and, moreover, we are considering only a single turbine in a reservoir. A model of ternary PSH is considered in Section 4.3.3.

5. We ignore variation of efficiency with head or with SOC and we ignore self-discharge due to evaporation [4, 29, 46]. Our convention for the parameter β is the inverse of that in [46, 47]. Self-discharge could be incorporated as in [29].

6. It should be noted that there is no mutual exclusivity constraint for a ternary PSH. Consequently, the standard formulation (12) is appropriate for ternary PSH. See Section 4.3.3.

7. It is convenient to remove s from the formulation for proving theoretical results; however, the resulting formulation is dense. From a numerical implementation perspective, it is typically better to maintain s explicitly as a variable and maintain the sparse equality constraints (10) and this approach is taken in Section 6. In this context, it should be noted that the tighter

formulation requires more non-zero entries in the coefficient matrix than the standard formulation. Consequently, moving from the standard to the tighter formulation involves two changes with countervailing effects on computational efficiency: a tighter formulation, but a coefficient matrix that has more non-zeros. This is investigated in the large-scale test cases in Section 6.

8. In practice, PSH and reservoir hydroelectric resources may be able to affect prices through their bids and offers [24]. However, we ignore this issue here.

9. Also note that if $\underline{p} = 0$ then the binary variables can be completely eliminated resulting in a convex formulation. The corresponding case for PSH, with lower limits on pumping and generating both zero, still requires at least one binary variable for each interval in order to represent a round-trip efficiency less than one.

10. For example, the argument in [40, section III.D] regarding the complementarity formulation implicitly assumes that prices are positive.

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