Nash Bargaining Partitioning in Decentralized Portfolio Management

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Abstract

In the context of decentralized portfolio management, understanding how to distribute a fixed budget among decentralized intermediaries is a relevant question for financial investors. We consider the Nash bargaining partitioning for a class of decentralized investment problems, where intermediaries are in charge of the portfolio construction in heterogeneous local markets and act as risk/disutility minimizers. We propose a reformulation that is valid within a class of risk/disutility measures (that we call *quasi-homogeneous* measures) and allows the reduction of a complex bilevel optimization model to a convex separable knapsack problem. As numerically shown using stock returns data from U.S. listed enterprises, this modelling reduction of the Nash bargaining solution in decentralized investment (driven by the notion of *quasi-homogeneous* measures), allows solving the vast majority of large-scale investment instances in less than a minute.

Keywords: Financial intermediaries, Nash bargaining solution, Portfolio management, Bilevel optimization, Knapsack problem

1. Introduction

In modern theory on portfolio selection, there has been an increasing interest in the *decentralized investment problem*, where an individual investor delegates the construction of a portion of a global investment to financial intermediaries, who take decisions rationally and independently. They are frequently affiliates of large organizations, such as asset management firms, and implement investment strategies in disjoint sets of assets, under regulations from a common headquarter. Financial intermediaries can also act as expert advisors for asset allocation and compete for budget resources from investors (Greenbaum et al., 2015).[1]

An economic oriented strain of research has focused on compensation contracts between the individual investor and financial intermediaries, as well as their implications for the asset pricing (Bhattacharya and Pfleiderer, 1985; Ou-Yang, 2003; Nagurney and Ke, 2006; Stracca, 2006; Maug and Naik, 2011).[2] Others

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[1]Pension funds are appropriate examples, constituting 20.2% of the U.S. gross domestic product in 2017 (retrieved from the Organisation for Economic Co-operation and Development Statistics and Data Directorate).

[2]In a seminal study on delegated portfolio management, Bhattacharya and Pfleiderer (1985) proposed a model where the investor solicited higher-quality information on the rate of return on a risky asset from a better informed agent. The delegated portfolio management approach is a principal–agent relationship of hidden information between the investor (principal) and a better informed portfolio manager (agent).

studies have focused on the optimal design of delegation rules in principal-intermediary-agent hierarchy (Liang, 2013), as well as on the role of the topology of principals-agents linkages among market participants (Fainmesser, 2019). From the optimization perspective, recent contributions have studied alternative solution techniques, when the decentralized investment problem grows large and complex (Thi et al., 2012; Benita et al., 2019; Leal et al., 2020). On this respect, solving this class of problems by state-of-the-art approaches has been proved to be both theoretically (Stoughton, 1993; Stracca, 2006; Liang, 2013) and computationally challenging (Benita et al., 2019; Leal et al., 2020).

With a view to provide an efficient answer to the decentralized investment problem, this article identifies a class of portfolio optimization problems that can be solved as continuous convex separable knapsack problems. Specifically, this class consists in finding the Nash bargaining solution (hereafter referred to as NBS) to split a fixed investment budget into m portions (Nash Jr, 1950; Van Damme, 1986; Conley and Wilkie, 1996). Each portion is assigned to one intermediary aiming at minimizing *quasi-homogeneous* risk or disutility measures. The concept of *quasi-homogeneity* is introduced in this work to characterize the aforementioned class and constitutes the theoretical cornerstone of the proposed methodology.

For its desirable properties, Rocheteau et al. (2020) integrated and extended the NBS notion into models of decentralized investment. To justify its appropriateness in our context, we elucidate hereafter the financial implications of the four NBS axioms as a result of multilateral negotiations to build an equilibrium portfolio.[3]

- The invariance of the equilibrium portfolio to affine transformations implies that risk or disutility behave as preference representations over budget allocations. In the case of quasi-homogeneity, it shall be seen that the invariance to affine transformations translates into the invariance of the intermediary best responses to budget allocations.

- The symmetry entails the lack of bias or propensity for intermediaries having equal disagreement points and equal best responses to budget allocations.

- The Pareto efficiency implies that the assigned budget must be used, when it gives rise to a reduction in the risk or an increase in the utility at the intermediary level.

- If an equilibrium portfolio is independent of irrelevant alternatives then including financial constraints and regulations to the budget allocation doesn't impact the equilibrium portfolio, as long as the equilibrium portfolio remains feasible under these constraints.

Altogether these are desirable features for a centralized coordination of a decentralized portfolio selection. With a view to translate the search for a decentralized portfolios satisfying these axioms into a treatable optimization problems, the first aim of this work is to establish an algebraic characterization of the aforementioned class of quasi-homogeneous risk or disutility functions. We show that quasi-homogeneous risk or disutility generalize the portfolio moments, the conditional value at risk of portfolio losses (hereafter

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[3]Throughout this paper the terms *equilibrium portfolio*, *equilibrium investment*, *equilibrium budget partition*, and *equilibrium budget allocation* are used indistinctly.

referred to as CVaR), as well as other well-known disutility measures. Secondly, we prove that for this class of risk or disutility functions, the NBS can be efficiently found, reducing a complex bilevel optimization model (resulting from the extensive formulation of the NBS) to a continuous convex separable knapsack problem.

On the empirical side, we assess the correctness and efficiency of the proposed methodology by solving the aforementioned knapsack problem for large-scale portfolio instances of 7,256 U.S. listed enterprises within the period 1999–2014 (available from the *Center for Research in Security Prices*). We consider a collection of $m = 73$ industries, classified by the grouped-industry code from the Standard Industrial Classification (hereafter referred to as SIC), where intermediaries operate. Therefore, the resulting bargaining consists in partitioning a fixed budget among the 73 industries, while disutility levels are set as a best response of intermediaries to budget partition, resulting in a nested (bilevel) decision setting.

The rest of this paper is organized as follows. Section 2 introduces a bargaining game among m financial intermediaries and a characterization of a class of risk measures. Section 3 establishes the equivalence between this class of problems and a convex separable knapsack problem. Section 4 presents numerical tests using real financial data. Section 5 concludes this paper.

2. The decentralized portfolio as a bargaining game

We consider an investment context where a collection $\mathcal{M}$ (with $|\mathcal{M}| = m$) of financial intermediaries have to split a fixed budget. We define as $\mathcal{Z} \subseteq [0,1]^m$, the space of m-partitions of the interval $[0,1]$. Each intermediary aims at investing its portion of the budget within its local market, where a set $\mathcal{N}_k$ (with $|\mathcal{N}_k| = n_k$) of investment options are available. They have preferences represented by payoff functions $u_k : \mathcal{Z} \to \mathbb{R}$, for each $k \in \mathcal{M}$, which are constructed as the solution to the following disutility minimization problem within each local market:

$$
u_k(\mathbf{z}) = \begin{cases} -\min_{\mathbf{x}_k} G_k(\mathbf{x}_k) \ \text{ subj. to } \ \mathbf{x}_k \in \Omega_k(\mathbf{z}) & \text{when } G_k \text{ is a risk function,} \\[2ex] \max_{\mathbf{x}_k} G_k(\mathbf{x}_k) \ \text{ subj. to } \ \mathbf{x}_k \in \Omega_k(\mathbf{z}) & \text{when } G_k \text{ is a utility function,} \end{cases} \tag{1}
$$

where $\mathbf{z} = [z_1, \ldots, z_m]^\top \in \mathcal{Z}$ characterizes the m-partition; the objective function $G_k : \mathbb{R}^{n_k} \to \mathbb{R}$ captures the risk or disutility measure associated with the local investment $x_{1,k}, \ldots, x_{n_k,k}$ (namely, the portion invested in asset $i \in \mathcal{N}_k$ in the kth);[4] the strategy set (once the partitioning is established) is

$$
\Omega_k(\mathbf{z}) = \{\mathbf{x}_k \in [0,1]^{n_k} \ : \ \mathbf{x}_k^\top \mathbf{e} = z_k, \ \mathbf{x}_k^\top \hat{\mathbf{r}}_k \geq \rho_k z_k\}.
$$

The exogenous parameters $\hat{r}_{ik}$ and ρ_k are the expected rate of return of asset $i \in \mathcal{N}_k$ (with vector form $\hat{\mathbf{r}}_k = [\hat{r}_{1,k}, \ldots, \hat{r}_{n_k,k}]^\top$), and the margin requirement of expected return, as a proportion of the residual budget, respectively. Since Ω_k only depends on the kth component of $\mathbf{z}$, we can equivalently use the notation $\Omega_k(z_k)$ in place of $\Omega_k(\mathbf{z})$.

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[4]We sometimes refer G_k by simply G when no confusion is possible.

The game of decentralized investment bargaining (hereafter referred to as DIB) among the m financial intermediaries is defined as a pair $(\mathcal{U}, \mathbf{d})$, where

$$\mathcal{U} = \left\{ (v_1 \ldots v_m) \in \mathbb{R}^m \ : \ v_k = u_k(\mathbf{z}), \ k \in \mathcal{M}, \ \mathbf{z} \in \mathcal{Z} \right\},$$

and $\mathbf{d} = [d_1, \ldots, d_m]^\top$ denotes the vector of disagreement points, corresponding to the intermediaries' payoffs if negotiations break down.[5]

In what follows, we identify a class of risk and utility measures that will allow for a fundamental reduction of the DIB game, as explored in the next section. This is done by requiring G_k to have the following *quasi-homogeneous* behavior.

Assumption 1. (QUASI-HOMOGENEITY) *The risk (utility) functions $G_k : \mathbb{R}^{n_k} \to \mathbb{R}$ are strictly convex (concave), continuously differentiable, and verify the following property. There exists a vector-valued function $\zeta : \mathbb{R} \to \mathbb{R}^{n_k}$ and a real-valued function $\delta : \mathbb{R} \to \mathbb{R}$, such that for every $\mathbf{x} \in \mathbb{R}^{n_k}$ and for every $z > 0$,*

$$\nabla G(z\mathbf{x}) \ = \ \delta(z) \nabla G(\mathbf{x}) + \zeta(z) \mathbf{e}.$$

By letting R_{ik} be the random rate of return of asset $i \in \mathcal{N}_k$ in intermediary $k \in \mathcal{M}$ and $\mathbf{R}_k = [R_{1,k}, \ldots, R_{n,k}]^\top$, we note that the portfolio moments, the CVaR, the hyperbolic absolute risk aversion utility function (hereafter referred to as HARA), and the Cobb Douglass utility function (hereafter referred to as CD) satisfy Assumption 1.

Remark 1 (Moments). *Consider the jth moment (with $j \in \mathcal{J}$) of a portfolio return: $G(\mathbf{x}) = \mathbb{E}\left[(\mathbf{R}^\top \mathbf{x})^j \right]$ and note that $\nabla G(\mathbf{x}) = j \mathbb{E}\left[\mathbf{R}(\mathbf{R}^\top \mathbf{x})^{j-1} \right]$, so that when $\mathbf{x} = z\mathbf{y}$, we have*

$$\nabla G(z\mathbf{y}) \ = \ z^{j-1} j \mathbb{E}\left[\mathbf{R}(\mathbf{R}^\top \mathbf{y})^{j-1} \right] \ = \ z^{j-1} \nabla G(\mathbf{y}).$$

The convexity can be assessed by noticing that the Hessian matrix of $G(\mathbf{x})$ verifies the following:

$$\nabla^2 G(\mathbf{x}) \ = \ j(j-1) \mathbb{E}\left[\mathbf{R}\mathbf{R}^\top (\mathbf{R}^\top \mathbf{x})^{j-2} \right],$$

so that

$$\mathbf{u}^\top \nabla^2 G(\mathbf{x}) \mathbf{u} \ = \ j(j-1) \mathbb{E}\left[\mathbf{u}^\top \mathbf{R}\mathbf{R}^\top (\mathbf{R}^\top \mathbf{x})^{j-2} \mathbf{u} \right] \ = \ j(j-1) \mathbb{E}\left[W^2 \right] \ \geq \ 0,$$

where $W = (\mathbf{u}^\top \mathbf{R})(\mathbf{R}^\top \mathbf{x})^{(j-2)/2}$.

Remark 2 (Gaussian CVaR). *Consider the CVaR of the portfolio loss and define the intermediaries' objectives as $G(\mathbf{x}) = \mathbb{E}\left[z - \mathbf{R}^\top \mathbf{x} \mid z - \mathbf{R}^\top \mathbf{x} \geq F^{-1}(\mathbf{y}, \alpha) \right]$, where $F^{-1}(\mathbf{x}, \alpha) = \inf\{q \ : \ F(\mathbf{x}, q) \geq \alpha\}$ is the left-continuous quantile function of the random quantity $z - \mathbf{R}^\top \mathbf{x}$ (i.e. its α-quantile) and $F(\mathbf{x}, \eta) =$*

$\mathbb{P}\left\{z - \mathbf{R}^\top \mathbf{x} \leq \eta\right\}$ is its distribution function (based on a suitable definition of the probability measure $\mathbb{P}$), at the point η. When $\mathbf{R}$ is distributed as a multivariate Gaussian, we have

$$G(\mathbf{x}) = \mathbb{E}\left[z - \mathbf{R}^\top \mathbf{x} \mid z - \mathbf{R}^\top \mathbf{x} \geq F^{-1}(\mathbf{y}, \alpha)\right] = z - \hat{\mathbf{r}}^\top \mathbf{x} + \left(\mathbb{E}\left[((\mathbf{R} - \hat{\mathbf{r}})^\top \mathbf{x})^\top (\mathbf{R} - \hat{\mathbf{r}})^\top \mathbf{x})\right]\right)^{1/2} B, \quad (2)$$

where $\hat{\mathbf{r}}$ is the expectation of $\mathbf{R}$, $B = 1/(\alpha\sqrt{2\pi})\exp(-(\phi^{-1}(\alpha))^2)$, and $\phi^{-1}(\alpha)$ is the standard normal quantile. Therefore, we obtain the following:

$$\nabla G(z\mathbf{y}) = -\hat{\mathbf{r}} + \frac{1}{z\left(\mathbb{E}\left[(\mathbf{R} - \hat{\mathbf{r}})^\top \mathbf{y})^\top (\mathbf{R} - \hat{\mathbf{r}})^\top \mathbf{y})\right]\right)^{1/2}} z\mathbb{E}\left[(\mathbf{R} - \hat{\mathbf{r}})((\mathbf{R} - \hat{\mathbf{r}})^\top \mathbf{y})\right] B = \nabla G(\mathbf{y}).$$

Remark 3 (Expected HARA utility). *We consider the expected HARA utility of the portfolio return* $G(\mathbf{x}) = \mathbb{E}\left[(\zeta \mathbf{e}^\top \mathbf{x} + \mathbf{R}^\top \mathbf{x})^{1-\theta}/(1-\theta)\right]$ *and note that* $\nabla G(\mathbf{x}) = \mathbb{E}\left[(\zeta \mathbf{e} + \mathbf{R})((\zeta \mathbf{e} + \mathbf{R})^\top \mathbf{x})^{-\theta}\right]$, *so that when* $\mathbf{x} = z\mathbf{y}$, *we have*

$$\nabla G(z\mathbf{y}) = \frac{1}{z^\theta}\mathbb{E}\left[(\zeta \mathbf{e} + \mathbf{R})((c\mathbf{e} + \mathbf{R})^\top \mathbf{y})^{-\theta}\right] = \frac{1}{z^\theta}\nabla G(\mathbf{y}).$$

The concavity can be assessed by noticing that the Hessian matrix of $G(\mathbf{x})$ *verifies the following:*

$$\nabla^2 G(\mathbf{x}) = -\theta \mathbb{E}\left[\mathbf{R}\mathbf{R}^\top (1/\mathbf{R}^\top \mathbf{x})^{\theta+1}\right],$$

so that for any real vector $\mathbf{u}$, *we have*

$$\mathbf{u}^\top \nabla^2 G(\mathbf{x})\mathbf{u} = -\theta \mathbb{E}\left[\mathbf{u}^\top \mathbf{R}\mathbf{R}^\top (1/\mathbf{R}^\top \mathbf{x})^{\theta+1}\mathbf{u}\right] = -\theta \mathbb{E}\left[W^2\right] \leq 0,$$

where $W = (\mathbf{u}^\top \mathbf{R})(1/\mathbf{R}^\top \mathbf{x})^{(\theta+1)/2}$.

Remark 4 (Conditional expectation of the CD utility). *Consider the conditional expectation of a Cobb Douglas utility with random elasticities:*

$$G(\mathbf{x}) = \mathbb{E}\left[\prod_{i\in\mathcal{N}}(x_i)^{\hat{h}_{ik}} \Bigg| \sum_{i\in\mathcal{N}}\hat{h}_{ik} = r_k^h\right], \quad \nabla G(\mathbf{x}) = \mathbb{E}\left[\prod_{i\in\mathcal{N}}\hat{h}_{ik}(x_i)^{\hat{h}_{ik}-1} \Bigg| \sum_{i\in\mathcal{N}}\hat{h}_{ik} = r_k^h\right],$$

where $\hat{h}_{ik} = h(1 + R_{ki})$, *with* $h : \mathbb{R} \longrightarrow [0,1]$. *When* $\mathbf{x} = z\mathbf{y}$, *we have*

$$\nabla G(z\mathbf{y}) = z^{r_k^h}\mathbb{E}\left[\prod_{i\in\mathcal{N}}\hat{h}_{ik}(y_i)^{\hat{h}_{ik}-1} \Bigg| \sum_{i\in\mathcal{N}}\hat{h}_{ik} = r_k^h\right]. = z^{r_k^h}\nabla G(\mathbf{y}).$$

A sufficient condition for the concavity is of this specification of G *can be directly deduced from the deterministic case, studied by* Avvakumov et al. (2010) *and by* Kojić (2021). *In fact,* G *is concave if* $\sum_{i\in\mathcal{N}}\hat{h}_{ik} < 1$.

These four remarks show relevant cases that can be accommodated within the functional form imposed by Assumption 1. We refer to this functional form as *quasi-homogeneous* functions.

3. The NBS and the separable knapsack problem

The NBS has been extensively used to determine a unique partitioning that satisfies the axioms of scale invariance, symmetry, efficiency, and independence of irrelevant alternatives (Nash Jr, 1950; Van Damme, 1986; Conley and Wilkie, 1996).[6] We discussed in Section 1 the supporting arguments that make these properties being desirable features of a budget partitioning in the context of decentralized portfolio management. For this DIB problem, the NBS is obtained by solving the following bilevel problem:

$$
\begin{cases}
\max_{\tilde{\mathbf{x}}_1,\ldots,\tilde{\mathbf{x}}_m} \quad F(\tilde{\mathbf{x}}_1,\ldots,\tilde{\mathbf{x}}_m) & \text{(3a)} \\[2ex]
\text{s.t.} \quad \tilde{\mathbf{x}}_k \in \left\{ \underset{\mathbf{x}_k}{\text{optimize}}\, G_k(\mathbf{x}_k), \quad \text{s.t.} \quad \mathbf{x}_k \in \Omega_k\Big(1 - \sum_{h\neq k}\sum_{i\in\mathcal{N}_h}\tilde{x}_{ih}\Big) \right\}, & \forall k \in \mathcal{M}, \quad \text{(3b)} \\[2ex]
G_k(\tilde{\mathbf{x}}_k) \in [\ell_k, \upsilon_k], & \forall k \in \mathcal{M}, \quad \text{(3c)}
\end{cases}
$$

where *optimize* stands for either *argmin* or *argmax*, depending on whether G_k is a risk of utility function, respectively; $\Omega_1(\mathbf{z}) \times \Omega_2(\mathbf{z})\ldots \times \Omega_m(\mathbf{z})$ is closed, convex and bounded; and

$$
F(\tilde{\mathbf{x}}_1,\ldots,\tilde{\mathbf{x}}_m) \quad = \quad
\begin{cases}
\displaystyle\prod_{k\in\mathcal{M}} (d_k - G_k(\tilde{\mathbf{x}}_k)) & \text{when } G_k \text{ is a risk function,} \\[3ex]
\displaystyle\prod_{k\in\mathcal{M}} (G_k(\tilde{\mathbf{x}}_k) - d_k) & \text{when } G_k \text{ is a utility function,}
\end{cases}
$$

$$
[\ell_k,\ \upsilon_k] \quad \equiv \quad
\begin{cases}
[0,\ d_k] & \text{when } G_k \text{ is a risk function,} \\[2ex]
[d_k,\ 0] & \text{when } G_k \text{ is a utility function.}
\end{cases}
$$

As noted by Dempe and Dutta (2012), bilevel models such as (3) can be seen as non-convex programs with implicitly defined feasible regions, whose solution require single-level reformulations. Different available alternatives can be used for this purpose (Luo et al., 1996). It is shown hereafter that the quasi-homogeneity assumption allows approaching the optimal solution of (3) by solving a convex separable knapsack problem. This relies upon the linearity of the intermediaries' best responses, as established in the next proposition.

Proposition 1 (Linearity of intermediaries' best response to partitioning). *Consider problem* (1) *and suppose Assumption 1 holds. Let us define the intermediaries' best response to a partitioning* $\mathbf{z} \in \mathcal{Z}$ *as*

$$
\boldsymbol{\Psi}_k(\mathbf{z}) = \underset{\mathbf{x}_k}{optimize}\ \ G_k(\mathbf{x}_k)\ \ subj.\ to\ \ \mathbf{x}_k \in \Omega_k(\mathbf{z}).
$$

We have $\boldsymbol{\Psi}_k(\mathbf{z}) = z_k \boldsymbol{\Psi}_k(\mathbf{1})$, *with* $|\boldsymbol{\Psi}_k(\mathbf{1})| = 1$ *for all* $k \in \mathcal{M}$.

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[6]Different contributions tried to relax of replaced the independence of irrelevant alternative axiom. For instance, Peters and Van Damme (1991) provided a characterization of the NBS that does not require the independence of irrelevant alternative axiom, but relies upon changes in the disagreement point. This is relevant for the interpretation of the NBS in our DIB context. In fact, this axiom has two important implications: (i) the inclusion of a new asset in any of the intermediaries' local markets can only give rise to a change in the equilibrium partitioning if it is capable of reducing the risk at the corresponding market with the same budget allocation; (ii) including financial constraints and regulations to the budget allocation doesn't impact the equilibrium portfolio, as long as the equilibrium portfolio remains feasible under these constraints.

Proof:. In this proof we drop the subindex k, as the result is valid for every local market. We differentiate two cases. When $z = 0$, we can see that $\Psi(z) = \mathbf{0}$, so that the condition $\Psi(z) = z\Psi(\mathbf{1})$ is verified. When $z \in (0, 1]$, we consider the Karush-Kuhn Tucker conditions of problem (1) and define $\Psi(\mathbf{1})$ as the optimal solution of problem (1) for $\mathbf{z} = \mathbf{1}$. Then, $\Psi(\mathbf{1})$ must solve the following Karush-Kuhn Tucker conditions:

$$
KKT(1) \ : \ \begin{cases} \Psi(\mathbf{1})^\top \mathbf{e} = 1 \\ \Psi(\mathbf{1})^\top \hat{\mathbf{r}} \geq \rho \\ \nabla G^F(\Psi(\mathbf{1})) + \lambda - \gamma - \boldsymbol{\mu} = \mathbf{0} \\ \mu_i \Psi_i(1) = 0, \ i = 1, \ldots, n \\ \Psi(\mathbf{1}) \geq 0 \\ (\boldsymbol{\mu}, \gamma) \geq 0, \end{cases}
$$

where $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_n]^\top$ is the vector of Lagrangian multipliers of $\mathbf{x} \geq 0$, λ is the Lagrangian multiplier of $\mathbf{x}^\top \mathbf{e} = 1$, and γ is the Lagrangian multiplier of $\mathbf{x}^\top \hat{\mathbf{r}} \geq \rho$. We now show that $z\Psi_i(1)$ will necessarily solve $KKT(z)$ (the optimality conditions of $\Psi(z)$), for any $z \in (0, 1]$:

$$
KKT(z) \ : \ \begin{cases} z\Psi(\mathbf{1})^\top \mathbf{e} = z \\ z\Psi(\mathbf{1})^\top \hat{\mathbf{r}} \geq \rho z \\ \nabla G^F(z\Psi(\mathbf{1})) + \hat{\lambda} - \hat{\gamma} - \hat{\boldsymbol{\mu}} = 0 \\ \hat{\mu}_i z \Psi_i(1) = 0, \ i = 1, \ldots, n \\ \Psi(\mathbf{1})z \geq 0 \\ (\boldsymbol{\mu}, \gamma) \geq 0. \end{cases}
$$

Starting from the feasibility, it is sufficient to note that for any $z > 0$, we have that $\mathbf{x} \geq 0$ is equivalent to $z\mathbf{x} \geq 0$, and the following linear systems admit the same $\mathbf{x}$ solutions:

$$
\begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \hat{\mathbf{r}} \geq \rho, \end{cases} \quad \text{and} \quad \begin{cases} z\mathbf{x}^\top \mathbf{e} = z \\ z\mathbf{x}^\top \hat{\mathbf{r}} \geq \rho z. \end{cases}
$$

Concerning the complementarity, the existence of $\mu_i \geq 0$ satisfying

$$
\mu_i \Psi_i(1) = 0
$$

is equivalent to the existence of $\hat{\mu}_i \geq 0$ satisfying

$$
\hat{\mu}_i z \Psi_i(1) = 0.
$$

Finally, the stationarity conditions are verified by virtue of Assumption 1. In fact, the local markets' objective function is such that there must exist two real-valued functions $\zeta : \mathbb{R} \to \mathbb{R}$ and $\delta : \mathbb{R} \to \mathbb{R}$, such that for every $\Psi(\mathbf{1}) \in \mathbb{R}^{n_k}$ and for every $z > 0$:

$$
\nabla G(z\Psi(\mathbf{1})) + \hat{\lambda} - \hat{\gamma} - \hat{\boldsymbol{\mu}} \ = \ \nabla G(\Psi(\mathbf{1})) + \frac{1}{\delta(z)}\left(\hat{\lambda} + \zeta(z)\mathbf{e}\right) - \frac{1}{\delta(z)}\hat{\gamma} - \frac{1}{\delta(z)}\hat{\boldsymbol{\mu}} \ = \ \mathbf{0},
$$

which implies

$$\lambda = \frac{1}{\delta(z)}\left(\hat{\lambda} + \zeta(z)\mathbf{e}\right), \qquad \gamma = \frac{1}{\delta(z)}\hat{\gamma}, \quad \text{and} \quad \boldsymbol{\mu} = \frac{1}{\delta(z)}\hat{\boldsymbol{\mu}}.$$

We conclude that $\boldsymbol{\Psi}(\mathbf{z}) = z\boldsymbol{\Psi}(\mathbf{1})$. Note that the Lagrangian multipliers are not required to be the same, as they depends on z.

$\square$

From this result, we deduce an interesting reformulation of problem (3), as the feasibility conditions (3b)-(3c) can be replaced by the following functional relationship:

$$\tilde{\mathbf{x}}_k = \left(1 - \sum_{h \neq k}\sum_{i \in \mathcal{N}_h} \tilde{x}_{ih}\right)\boldsymbol{\Psi}_k(\mathbf{1}).$$

Therefore, problem (3) reduces to the following a convex separable knapsack problem:

$$\max_{\mathbf{z}} F\left(z_1\hat{\boldsymbol{\Psi}}_1, \ldots, z_m\hat{\boldsymbol{\Psi}}_m\right), \quad \text{subj. to} \quad \mathbf{z}^\top \mathbf{e} = 1, \quad \tilde{\mathbf{d}}_L \leq \mathbf{z} \leq \tilde{\mathbf{d}}_U, \tag{4}$$

where we used the shorter notation $\hat{\boldsymbol{\Psi}}_k = [\hat{\Psi}_{1,k}\ldots\hat{\Psi}_{n_k,k}]^\top$, instead of $\boldsymbol{\Psi}_k(\mathbf{1}) = [\Psi_{1,k}(1)\ldots\Psi_{n_k,k}(1)]^\top$, to denote the best response of the kth intermediary to a budget allocation $z_k = 1$. The variable bounds $\tilde{\mathbf{d}}_L$ and $\tilde{\mathbf{d}}_U$ are predefined by setting the lower and upper limits of z_k for which $G_k\left(z_k\hat{\boldsymbol{\Psi}}_k\right) \leq d_k$ (for the case of risk) or $G_k\left(z_k\hat{\boldsymbol{\Psi}}_k\right) \geq d_k$ (for the case of utility).

As noted by Rubinstein et al. (1992), the very simplicity of the NBS is in itself an attractive feature of this axiomatic approach. This claim is even more appropriate in the specific DBI context with quasi-homogeneous measures, as a result of the reduction obtained by virtue of Proposition 1. Additionally, based on the quasi-homogeneity, the invariance to affine transformation axiom translates into the invariance of the intermediary best responses to budget allocations. In fact, for any α_k and β_k, we have

$$\boldsymbol{\Psi}_k(\mathbf{z}) = \operatorname*{optimize}_{\mathbf{x}_k} \; \alpha_k + \beta_k G_k(\mathbf{x}_k) \;\; \text{subj. to} \;\; \mathbf{x}_k \in \Omega_k(\mathbf{z}).$$

The next proposition provides a characterization of the equilibrium allocation, based on the quasi-homogeneity of G_k, suggesting an alternative algorithmic resolution, next to well-known approaches to convex separable knapsack problems.

Proposition 2 (NBS characterization). *Let $\mathcal{M}_0 = \{k \in \mathcal{M} \; : \; \tilde{d}_{L,k} < z_k < \tilde{d}_{U,k}\}$ be a set of local markets. For all $i, i' \in \mathcal{M}_0$, the solution of problem (3) satisfies*

$$\frac{\delta(z_i)b_i + \zeta(z_i)}{\delta(z_{i'})b_{i'} + \zeta(z_{i'})} = \begin{cases} \dfrac{d_i - G_i(z_i\hat{\boldsymbol{\Psi}}_i)}{d_{i'} - G_{i'}(z_{i'}\hat{\boldsymbol{\Psi}}_{i'})} & \text{when } G_k \text{ is a risk function,} \\[4mm] \dfrac{G_i(z_i\hat{\boldsymbol{\Psi}}_i) - d_i}{G_{i'}(z_{i'}\hat{\boldsymbol{\Psi}}_{i'}) - d_{i'}} & \text{when } G_k \text{ is a utility function.} \end{cases} \tag{5}$$

where $b_i = \hat{\boldsymbol{\Psi}}_i^\top \nabla G_i(\hat{\boldsymbol{\Psi}}_i)$ and δ and ζ are known functions defined in Assumption 1.

Proof:. Firstly, using the equivalent problem (4) we consider

$$\max_{\mathbf{z}} \; \log F\left(z_1\hat{\boldsymbol{\Psi}}_1, \ldots, z_m\hat{\boldsymbol{\Psi}}_m\right), \quad \text{subj. to} \quad \mathbf{z}^\top \mathbf{e} = 1, \quad \tilde{\mathbf{d}}_L \leq \mathbf{z} \leq \tilde{\mathbf{d}}_U,$$

By virtue of Assumption 1, the Karush-Kuhn Tucker conditions corresponding to $k \in \mathcal{M}_0$ implies

$$
\begin{cases}
\dfrac{\delta(z_k)b_k + \zeta(z_k)}{d_k - G_k(z_k \hat{\boldsymbol{\Psi}}_k)} + \lambda = 0 & \text{when } G_k \text{ is a risk function,} \\[4mm]
\dfrac{\delta(z_k)b_k + \zeta(z_k)}{G_k(z_k \hat{\boldsymbol{\Psi}}_k) - d_k} + \lambda = 0 & \text{when } G_k \text{ is a utility function.}
\end{cases}
$$

Therefore, the equality (5) follows with few simplifications.

$\square$

To complement this closed-form analysis of the DBI, the following illustrative examples provide an explicit relationship between the returns distribution and the equilibrium budget allocation. The main focus is on the impact of stock correlations and the disagreement points on the equilibrium budget allocation, based on different returns distributions.

Example 1 (Illustrative example: Moments). *Consider two identical intermediaries with two stocks each, namely $m = 2$, $n_1 = 2$ and $n_2 = 2$, and assume that $\mathbf{R}_k \sim \mathcal{N}(\hat{\mathbf{r}}_k, \Sigma_k)$, with $\hat{r}_{1,k} > \hat{r}_{2,k}$ and $\rho_k \leq r_1$, for $k \in \{1, 2\}$. When G is taken to be the j central moment, the intermediaries problems become:*

$$
\Psi_k(z_k) = \begin{cases}
\underset{\mathbf{x}_k}{argmin} \ (j-1)!! \left(x_{1,k}^2 \sigma_{1,k}^2 + x_{2,k}^2 \sigma_{2,k}^2 + 2\sigma_{1,k}\sigma_{2,k}c_k x_{1,k} x_{2,k} \right)^{j/2} & \text{(6a)} \\[3mm]
subj. \ to \ \ x_{1,k} + x_{2,k} = z_k & \text{(6b)} \\[3mm]
\qquad\qquad x_{1,k}\hat{r}_{1,k} + x_{2,k}\hat{r}_{2,k} \geq \rho z_k & \text{(6c)} \\[3mm]
\qquad\qquad x_{1,k}, x_{2,k} \geq 0, & \text{(6d)}
\end{cases}
$$

where $(j-1)!!$ denotes the double factorial of $(j-1)$. Let us define $b_k = \sigma_{2,k}^2(1 - c_k\sigma_{1,k}^2)/(\sigma_{1,k}^2 + \sigma_{2,k}^2 - 2c_k\sigma_{1,k}\sigma_{2,k})$ and $a_k = ((\rho_k - r_2)/(r_{1,k} - r_{2,k}))^+$. The intermediary's best response is

$$
\hat{\Psi}_{1,k} = \begin{cases}
1, & if \ b_k \geq 1 \\
b_k, & if \ a_k < b_k < 1 \qquad and \qquad \hat{\Psi}_{2,k} = 1 - \hat{\Psi}_{1,k}. \\
a_k, & otherwise
\end{cases}
$$

The single level reformulation is constructed by defining

$$
H_k = (j-1)!! \left((\hat{\Psi}_{1,k})^2 \sigma_1^2 + (1 - \hat{\Psi}_{1,k})^2 \sigma_2^2 + 2\sigma_1\sigma_2 c \hat{\Psi}_{1,k}(1 - \hat{\Psi}_{1,k}) \right)^{j/2}
$$

and obtaining

$$
\max_{z_1 \in [0,1]} \left(d_1 - z_1^j H_1 \right) \left(d_2 - (1 - z_1)^j H_2 \right). \tag{7}
$$

Figure 1 depicts the relationship between the equilibrium budget z_1^* (obtained by solving (7)) and the correlation, c, between the two stocks. Not surprisingly, when the two stocks of each market are highly correlated, diversification requires a well-balanced budget allocation: $z_1^* = z_2^* = 0.5$. This is specially true when the two stocks have the same variances. Next, when stock returns are uncorrelated, the equilibrium budget favours the market with a higher disagreement point, so that $z_1^* < z_2^*$. This effect seems to be lightened when higher conditional moments are used ((b) panel of Figure 1).
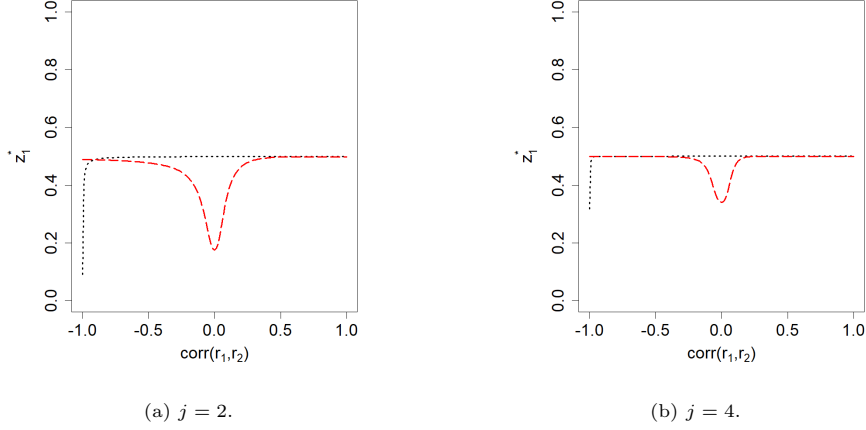
(a) $j = 2$. (b) $j = 4$.

Figure 1: The equilibrium budget z_1^* as a function of the correlation, c, between returns with $d_1 = 0.1$ and $d_2 = 1$. The black dotted line corresponds to the case $\sigma_1 = \sigma_2 = 40$, whereas the red dashed line to the case $\sigma_1 = 40$ and $\sigma_2 = 1$. The plot on the left panel differs from the one on the right panel with respect to the moment j.

Example 2 (Illustrative example: CVaR). *Consider two identical intermediaries with two stocks each, namely $m = 2$, $n_1 = 2$ and $n_2 = 2$. Assume that $\mathbf{R}_i \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, with $B^2\left(\sigma_1^2 + \sigma_2^2 - 2c\sigma_1\sigma_2\right) + \hat{r}_2 > \hat{r}_1 > \hat{r}_2$ and $\rho_k \leq r_1$, for $k \in \{1, 2\}$. When G is taken to be the α-CVaR. The intermediaries' problems become:*

$$\Psi_k(z_k) = \begin{cases} \underset{\mathbf{x}_k}{argmin} \; z - \hat{r}_1 x_{1,k} - \hat{r}_2 x_{2,k} + \left(x_{1,k}^2\sigma_1^2 + x_{2,k}^2\sigma_2^2 + 2\sigma_1\sigma_2 c x_{1,k} x_{2,k}\right)^{1/2} B & \text{(8a)} \\ subj. \; to \;\; x_{1,k} + x_{2,k} = z_k & \text{(8b)} \\ \qquad x_{1,k}\hat{r}_1 + x_{2,k}\hat{r}_2 \geq \rho z_k & \text{(8c)} \\ \qquad x_{1,k}, x_{2,k} \geq 0, & \text{(8d)} \end{cases}$$

To characterize $\boldsymbol{\Psi}_k(\mathbf{1})$, we solve

$$\min_{\mathbf{x}_k} \; -\hat{r}_1 x_{1,k} - \hat{r}_2(1 - x_{1,k}) + \left(x_{1,k}^2\sigma_1^2 + (1 - x_{1,k})^2\sigma_2^2 + 2\sigma_1^2\sigma_2^2 c x_{1,k}(1 - x_{1,k})\right)^{1/2} B \quad subj. \; to \; x_{1,k} \in [a_k, 1]$$

By the first order conditions

$$\left(\frac{B}{2}\right)\frac{2x_{1,k}\sigma_1^2 - 2(1 - x_{1,k})\sigma_2^2 + 2\sigma_1^2\sigma_2^2 c(1 - 2x_{1,k})}{\left(x_{1,k}^2\sigma_1^2 + (1 - x_{1,k})^2\sigma_2^2 + \sigma_1^2\sigma_2^2 c x_{1,k}(1 - x_{1,k})\right)^{1/2}} = \hat{r}_1 - \hat{r}_2.$$

The intermediaries' best response is

$$\hat{\Psi}_{1,k} = \begin{cases} 1, & \text{if } h \geq 1 \\ h, & \text{if } a_k < h < 1 \qquad \text{and} \qquad \hat{\Psi}_{2,k} = 1 - \hat{\Psi}_{1,k}, \\ a_k, & \text{otherwise} \end{cases}$$

where $a_k = ((\rho_k - r_2)/(r_1 - r_2))^+$ and

$$h = \frac{c\sigma_1\sigma_2 - \sigma_2^2}{2c\sigma_1\sigma_2 - \sigma_1^2 - \sigma_2^2} \pm \frac{\sqrt{F}}{\sigma_1^2 + \sigma_2^2 - 2c\sigma_1\sigma_2}, \qquad \text{with } F = \frac{(r_1 - r_2)(1 - c^2)}{B^2\left(\sigma_1^2 + \sigma_2^2 - 2c\sigma_1\sigma_2\right) + r_2 - r_1}.$$

Note that, since $B^2\left(\sigma_1^2 + \sigma_2^2 - 2c\sigma_1\sigma_2\right) + r_2 \geq \hat{r}_1 > \hat{r}_2$, we have that $F \geq 0$. The single level reformulation is constructed by defining

$$H_k = 1 - r_1\hat{\Psi}_{1,k} - r_2(1 - \hat{\Psi}_{1,k}) + \left((\hat{\Psi}_{1,k})^2\sigma_1^2 + (1 - \hat{\Psi}_{1,k})^2\sigma_2^2 + 2\sigma_1\sigma_2 c\hat{\Psi}_{1,k}(1 - \hat{\Psi}_{1,k})\right)^{j/2},$$

and obtaining

$$\max_{z_1 \in [0,1]} (d_1 - z_1 H_1)(d_2 - (1 - z_1)H_2).$$

The results in Figure 2 are consistent with the ones of Figure 1. Specifically, the diversification pattern is less sensitive to the correlation level when the variances between the two stocks are identical. In this case, only a highly negative correlation between stocks (close to -1) can push the equilibrium towards the intermediary with less tolerance d_k (more restrictive disagreement point). Alternatively, when the variance of one stock is larger than the one of the other, the disagreement points play a stronger role in determining the winning intermediary, so that $z_1^* < z_2^*$, when $d_1 < d_2$, and $z_2^* < z_1^*$, when $d_2 < d_1$.
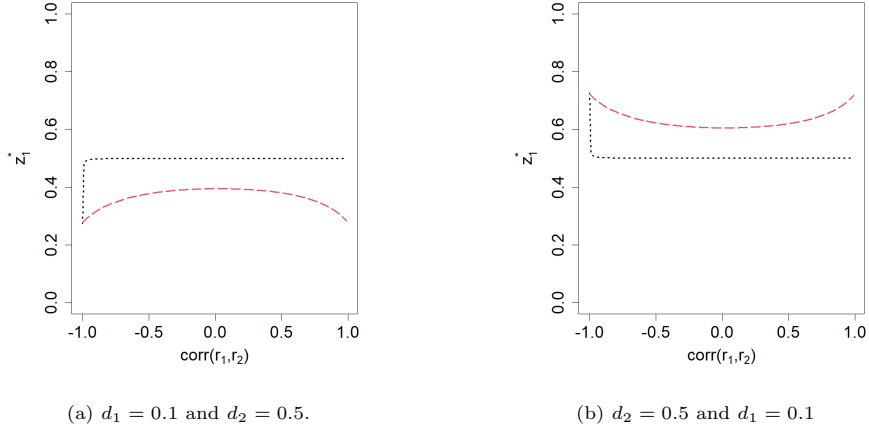

(a) $d_1 = 0.1$ and $d_2 = 0.5$.   (b) $d_2 = 0.5$ and $d_1 = 0.1$

Figure 2: The equilibrium budget z_1^* as a function of the correlation, c, between returns. We used $\alpha_k = 0.9$. The black dotted line corresponds to the case $\sigma_1 = \sigma_2 = 40$, whereas the red dashed line to the case $\sigma_1 = 40$ and $\sigma_2 = 1$. The plot on the left panel differs from the one on the right panel with respect to the disagreement points.

Example 3 (Illustrative example: HARA utility). *Consider the case of two identical intermediaries with two stocks each, namely $m = 2$, $n_1 = 2$ and $n_2 = 2$, and assume that $\mathbf{R}_k$ is uniformly distributed within the support $\{(x, y) \in [0,1]^2 \ : \ y \geq x\}$. Let $\kappa_k = \frac{2}{(2-\theta_k)(3-\theta_k)}$. When G is taken to be the HARA utility, using $x_{1,k} + x_{2,k} = 1$, we have*

$$\mathbb{E}\left[(\zeta + x_{1,k}\hat{r}_1 + x_{2,k}\hat{r}_2)^{1-\theta_k}\right] = \int_0^1 \int_{1-r_2}^1 2\left[(\zeta + x_{1,k}\hat{r}_1 + x_{2,k}\hat{r}_2)^{1-\theta_k}\right] dr_1 dr_2$$

$$= \kappa_k \left[\frac{(\zeta+1)^{3-\theta_k} - (\zeta + x_{1,k})^{3-\theta_k}}{x_{1,k}(1 - x_{1,k})} - \frac{(\zeta + 1 - x_{1,k})^{3-\theta_k} - (\zeta + x_{1,k})^{3-\theta_k}}{x_{1,k}(1 - 2x_{1,k})}\right]$$

when $x_{1,k} \neq 1/2$ and

$$\mathbb{E}\left[(\zeta + x_{1,k}\hat{r}_1 + x_{2,k}\hat{r}_2)^{1-\theta_k}\right] = \frac{2}{2^{1-\theta_k}} \int_0^1 \int_{1-r_2}^1 \left[(2\zeta + \hat{r}_1 + \hat{r}_2)^{1-\theta_k}\right] dr_1 dr_2$$

$$= \frac{2}{2^{1-\theta_k}(2-\theta_k)} \left[\frac{(2\zeta+2)^{3-\theta_k} - (2\zeta+1)^{3-\theta_k}}{3 - \theta_k} - (2\zeta+1)^{2-\theta_k}\right]$$

when $x_{1,k} = x_{2,k} = 1/2$. As long as $\rho_k \leq 1/2$, the followers problems become:

$$\Psi_{1,k}(1) = \operatorname*{argmax}_{\mathbf{x}_k} -\frac{1}{1-\theta_k}\mathbb{E}\left[(\zeta + x_{1,k}\hat{r}_1 + x_{2,k}\hat{r}_2)^{1-\theta_k}\right] \quad subj. \ to \ x_{1,k} \in [0,1].$$

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This problem attains its optimal solution when $x_{1,k} = 1/2$, independently from θ_k. The single level reformulation is constructed by defining

$$H_k = \frac{\mathbb{E}\left[(2\zeta + \hat{r}_1 + \hat{r}_2)^{1-\theta}\right]}{(1-\theta_k)2^{(1-\theta_k)}} = \frac{2^{\theta_k}}{(1-\theta_k)(2-\theta_k)}\left[\frac{(2\zeta+2)^{3-\theta_k} - (2\zeta+1)^{3-\theta_k}}{3-\theta_k} - (2\zeta+1)^{2-\theta_k}\right],$$

and obtaining

$$\max_{z_1 \in [0,1]} \left(z_1^{1-\theta_1} H_1 - d_1\right)\left((1-z_1)^{1-\theta_2} H_2 - d_2\right).$$

The optimal solution is either zero or one or the solution the first-order equation

$$H_2\frac{1}{(1-z_1)^{\theta_2}}\left((z_1)^{1-\theta_1} H_1 - d1\right) = H_1\frac{1}{(z_1)^{\theta}}\left((1-z_1)^{1-\theta_2} H_2 - d_2\right).$$

Figure 3 pinpoint how sensitive are HARA's bargaining outcomes to changes in disagreement payoffs.

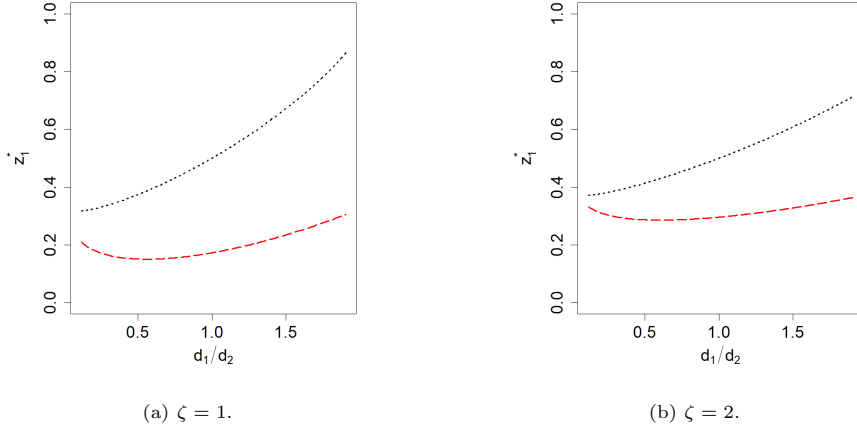(a) $\zeta = 1$.   (b) $\zeta = 2$.

Figure 3: The equilibrium budget z_1^* as a function of d_1/d_2. The black dotted line corresponds to the case $\theta_1 = \theta_2 = \frac{1}{3}$, whereas the red dashed line to the case $\theta_1 = \frac{1}{10}$ and $\theta_2 = \frac{2}{3}$. The plot on the left panel differs from the one on the right panel with respect to the utility parameter ζ.

Unmistakably, under identical intermediaries (i.e., $d_1 = d_2$ and $\theta_1 = \theta_2$) the equilibrium budget allocation is $z_1^* = z_2^* = 0.5$. The sensitivity to the disagreement point becomes higher when ζ is small (panel (a) in Figure 3), as H_k is decreasing in ζ.

Example 4 (Illustrative example: CD utility). *Consider two identical followers with two stocks each, namely $m = 2$, $n_1 = 2$ and $n_2 = 2$, and assume that $\mathbf{R}_k$ is uniformly distributed in $\{(x,y) \in [0,1]^2 \ : \ y \geq x\}$. When G is taken to be the conditional expectation of the CD utility, using $x_{1,k} + x_{2,k} = 1$, we have*

$$\mathbb{E}\left[(x_1)^{R_1}(1-x_1)^{R_2}\Big|\ R_1 + R_2 = r\right] = \frac{\displaystyle\int_0^1\int_{1-r_2}^1 \left[(x_1)^{r_1}(1-x_1)^{r_2}\right]dr_1 dr_2}{\displaystyle\int_0^r\int_{r-r_2}^r dr_1 dr_2}$$
$$= \frac{2}{r^2}\frac{1}{\log x_1}\left[\frac{1-2x_1}{\log x_1 - \log(1-x_1)} - \frac{x_1^2}{\log(1-x_1)}\right],$$

when $x_{1,k} \in (0, 1/2) \cup (1/2, 1)$,

$$\mathbb{E}\left[(x_1)^{R_1}(1-x_1)^{R_2}\Big|\ R_1 + R_2 = r\right] = 0,$$

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when $x_{1,k} \in \{0, 1\}$ and

$$\mathbb{E}\left[(x_1)^{R_1}(1 - x_1)^{R_2}\big| R_1 + R_2 = r\right] = \frac{1}{2^r},$$

when $x_{1,k} = 1/2$. As long as $\rho_k \leq 1/2$, the followers problems become:

$$\Psi_{1,k}(1) = \underset{\mathbf{x}_k}{argmax} - \mathbb{E}\left[(x_1)^{R_1}(1 - x_1)^{R_2}\big| R_1 + R_2 = r\right] \quad subj. \ to \ x_{1,k} \in [0, 1].$$

This problem attains its optimal solution when $x_{1,k} = 1/2$, independently from r. The single level reformulation is constructed as follows

$$\max_{z_1 \in [0,1]} \ (z_1^{r_1}/2^{r_1} - d_1)\left((1 - z_1)^{r_2}/2^{r_2} - d_2\right)$$

The optimal solution is either zero or one or the solution the first-order equation

$$\frac{r_2(1 - z_1)^{r_2 - 1}\left((z_1)^{r_1}/2^{r_2} - d_1\right)}{2^{r_2}} = \frac{r_1(z_1)^{r_1 - 1}\left((1 - z_1)^{r_2}/2^{r_2} - d_2\right)}{2^{r_1}}$$

Figure 4 depicts the relationships between the equilibrium budget z_1^* and the difference in disagreement points d_1 and d_2.

(a) $r_1 = r_2 = \frac{3}{4}$ ($r_1 = r_2 = 1$) in black dotted (red dashed).

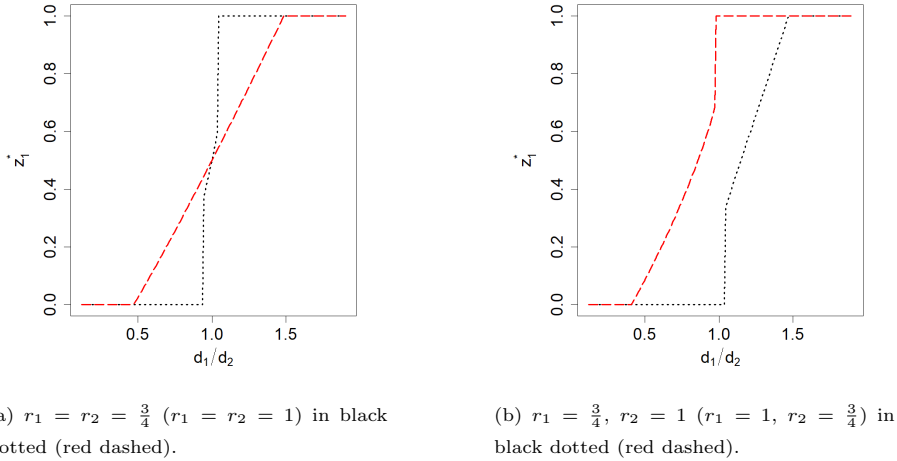(b) $r_1 = \frac{3}{4}$, $r_2 = 1$ ($r_1 = 1$, $r_2 = \frac{3}{4}$) in black dotted (red dashed).

Figure 4: The optimal budget z_1^* as a function of d_1/d_2. The plot on the left panel differs from the one on the right panel with respect to the combination of r values.

Differently from the pattern shown in Figure 3 for the HARA utility, the diversification pattern in the CD case is highly sensitive to the disagreement points, pushing the equilibrium allocation from one side to the opposite when d_1/d_2 varies.

These four illustrative examples provide stylized figures about how the dependencies in stocks correlations and intermediaries' disagreement points translate into Nash bargaining partitions of the global budget. In line with these illustrative figures, the analysis of large-scale financial investments with real stock returns remains to be performed, as explored next in Section 4.

4. Computational and empirical analysis

To characterize the solution in real investment settings, we take advantage of the continuous convex separable knapsack reformulation (4), and explore quasi-homogeneous measures in the form of the even

moments of the portfolio return, the CVaR of portfolio losses, the HARA utility, and the CD utility, using real data from U.S. listed enterprises.

Efficient algorithms for continuous convex separable knapsack problems have been proposed by the last decades (Bretthauer and Shetty, 2002). One of the most promising methods is the recent penalty approach by Hoto et al. (2020), which is applied hereafter to analyze a decentralized portfolio optimization context in which the fixed budget is slit into different industries. The data set of U.S. listed enterprises is presented in Section 4.1, while the numerical analysis of the resulting DIB problem is presented in 4.2, by applying the aforementioned penalty approach (a detailed description of its algorithmic applicability to problem (4) is provided in Appendix A). All optimization procedures are coded in MATLAB and CVX (Grant and Boyd, 2014). The experiments were run on a R5500 work-station with processor Intel(R) Xeon(R) CPU E5645 2.40 GHz, and 64 Gbytes of RAM.

4.1. Data set on stock prices

To test the proposed modelling framework, data from the *Center for Research in Security Prices* are considered for the entire population of 7,256 U.S. listed enterprises within the period 1999–2014.[7] We assume that a collection of decentralized intermediaries $\mathcal{M}$ is composed of $m = 73$ industries, classified by the grouped-industry code from the Standard Industrial Classification (SIC) in the Center for Research in Security Prices platform. In other words, intermediaries operate as expert advisors, each specialized in one of the 73 disjoint groups of stocks belonging to the different industries. Figure 5 displays the number of stocks per SIC industry. As we can observe, Banks (SIC code 401010) and Technology Hardware & Equipment 1 (SIC code 452040) are the intermediaries with the largest and smallest number of different stocks to invest in, respectively.
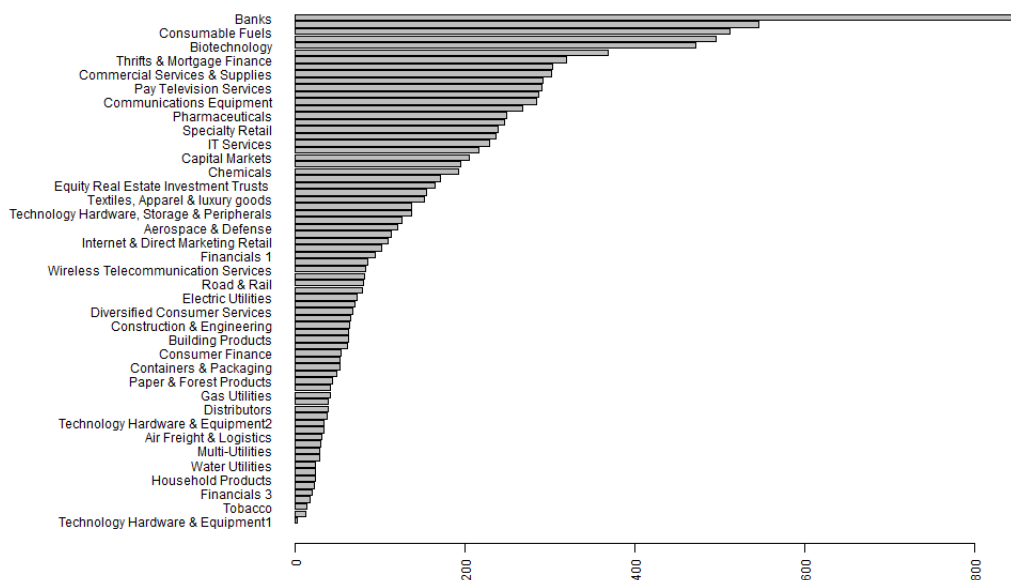

Figure 5: Number of stocks per follower; 73 SIC industries (only labels of 37 SIC codes) over 1999–2014.

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Figure 6 illustrates the dynamics of the stock returns in the data set in terms of the mean (black solid line) and plus/minus standard deviation (red dashed lines), within the period 1999–2014.
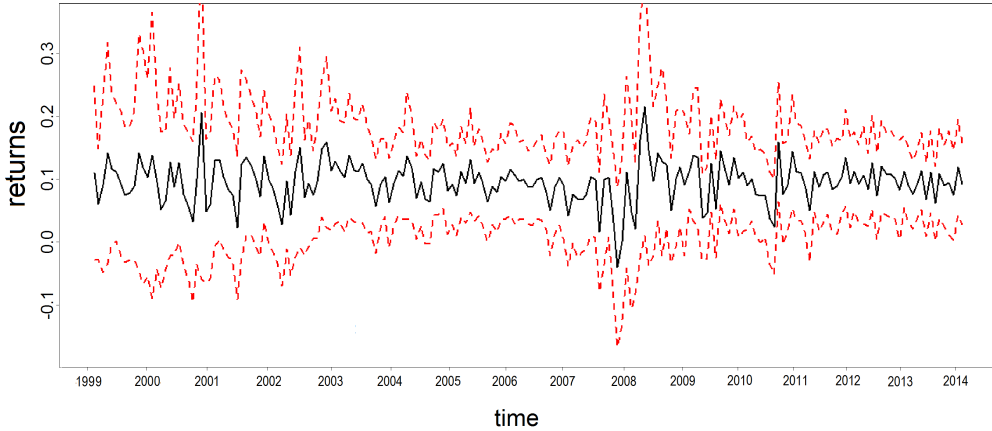

Figure 6: Average monthly stock returns along the observed time horizon.

These apparent variations motivate the use of a 16 year time window to assess the solution of the DIB problem, where the fixed budget is dynamically partitioned into the 73 industries.

4.2. Numerical tests

In this computational experiment, we explore the change in investment strategies for portfolios based on two settings for each risk and expected utility models. For investment portfolios minimizing risk we consider moments the second and fourth central moments ($j \in \{2, 4\}$). In CVaR we consider losses which are exceeded with 50% and 10% probability ($\alpha \in \{0.5, 0.9\}$). Alternatively, when the investor wants to maximize expected utility, we examine optimal portfolios for $\theta \in \{0.5, 0.9\}$ in the HARA utility, and elasticity $h \in \{0.001, 0.01\}$ in the CD utility. To generate instances, we set $\rho_k = \sum_i \frac{r_{ik}}{n_k} \hat{\rho}$ at different levels, with $\hat{\rho} \in \{0.7, 1.3\}$. For each year of the dataset and combination of hyperparameters, we solve one instance of the DIB by applying the recent penalty approach by Hoto et al. (2020). Details of the customized algorithm are given in the Appendix 5.

Table 1 compares the NBS (column *Optimal*) with two benchmark policies (budget partitioning), namely constant proportions (column *Constant*) and non-diversified (column *ND*). In the constant proportions policy, a uniform budget of $1/m$ is allocated to each intermediary, whereas in the ND selection criterion, the whole budget is assigned to the intermediary with the minimum risk or maximum utility. The latter requires solving problem (1) m times by alternatively fixing $z_k = 1$ for each $k \in \mathcal{M}$ and picking $k' \in \mathcal{M}$, for which either $G_{k'}(\hat{\boldsymbol{\Psi}}_{k'}) \leq G_k(\hat{\boldsymbol{\Psi}}_k)$ (risk minimization) or $G_{k'}(\hat{\boldsymbol{\Psi}}_{k'}) \geq G_k(\hat{\boldsymbol{\Psi}}_k)$ (utility maximization), for all $k \in \mathcal{M}$. For both *Optimal* and *ND*, the percentage of years in which the most recurrent intermediary was listed as the top industry is reported in parenthesis.

The managerial inside of Table 1 is the identification of the leading industries in receiving budget, under the NBS, as well as under the non-diversified strategy. We notice that the four risk and utility settings yield different patterns of equilibrium portfolios (optimal solution of the resulting knapsack problem (4)).

Table 1: Comparison of investment policies. The percentage of years in which the intermediary was listed as the top local market is reported in parenthesis.

		log F			Top intermediary	
$j/\alpha/\theta/h$	$\hat{\rho}$	Optimal	Constant	ND	Optimal	ND
					Moments	
2	0.7	-845.22	-839.95	-839.00	Specialty Retail (18.75%)	Ind. Power & Renew. Electr. Prod. (50%)
2	1.3	-799.32	-792.13	-791.18	Specialty Retail (18.75%)	Ind. Power & Renew. Electr. Prod. (50%)
4	0.7	-1492.90	-1492.90	-1491.69	Household Products (12.5%)	Ind. Power & Renew. Electr. Prod. (50%)
4	1.3	-1398.64	-1398.64	-1397.43	Specialty Retail (18.75%)	Ind. Power & Renew. Electr. Prod. (50%)
					CVaR	
0.5	0.7	-116.39	-109.81	-104.36	Health Care Tech. (18.75%)	Ind. Power & Renew. Electr. Prod. (50%)
0.5	1.3	-116.14	-109.59	-104.15	Health Care Tech. (12.5%)	Ind. Power & Renew. Electr. Prod. (50%)
0.9	0.7	-110.43	-95.77	-90.33	Diversified Telcom. (18.75%)	Ind. Power & Renew. Electr. Prod. (50%)
0.9	1.3	-111.37	-95.14	-89.69	Health Care Prov. & Svcs. (25%)	Ind. Power & Renew. Electr. Prod. (50%)
					HARA	
0.5	0.7	-236.99	-235.19	-257.29	Real Estate & Dev. (100%)	Energy Eqpt. & Svcs. (56.2%)
0.5	1.3	-236.99	-235.19	-257.29	Real Estate & Dev. (100%)	Energy Eqpt. & Svcs. (68.75%)
0.9	0.7	-259.10	-242.59	-345.61	Real Estate & Dev. (100%)	Ind. Power & Renew. Electr. Prod. (50%)
0.9	1.3	-259.10	-242.59	-345.61	Real Estate & Dev. (100%)	Ind. Power & Renew. Electr. Prod. (50%)
					CD	
0.001	0.7	560.42	564.30	319.44	Household Products (31.25%)	Ind. Power & Renew. Electr. Prod. (50%)
0.001	1.3	560.90	564.72	319.86	Household Products (31.25%)	Ind. Power & Renew. Electr. Prod. (50%)
0.010	0.7	3278.45	3279.18	3187.33	Software & Services (18.75%)	Ind. Power & Renew. Electr. Prod. (50%)
0.010	1.3	3282.63	3283.36	3191.52	Software & Services (18.75%)	Ind. Power & Renew. Electr. Prod. (50%)

For instance, the intermediary related to Specialty Retail (SIC code 255040) is the most recurrent leading industry, when even moments have to be minimized. In the expected HARA utility maximization, larger proportions of budget are allocated to the Real Estate Management & Development industry (SIC code 601020). Interestingly, regardless of the risk-based or expected utility models, partitions favoring markets that engage in the generation and distribution of electricity using renewable sources (SIC code 551050) will lead to optimal strategies under non-diversified policy investments (namely, when all the budget has to be assigned to a unique intermediary).

To provide a visual illustration, Figures 7 - 10 present stacked bar charts with the equilibrium budget partition in each year over the period 1999-2014. The red dotted line corresponds to the scaled expected returns (in $[0, 1]$) from solving (4). We notice a pattern in which larger proportions of the budget may be concentrated in few industries, when certain risk or utility measures are considered, while full diversification occurs when other measures are taken into account. For instance, the minimization of the fourth moment results in a stronger diversification than the second moment. Similarly, the maximization of the conditional expectation of the CD utility gives rise to a stronger diversification than the one of the expected HARA utility. Despite these differences, the order of the intermediaries with respect to the assigned budget is mostly stable when passing from the minimization of the moments and CVaR to the maximization of the expected HARA utility and conditional expectation of the CD utility.
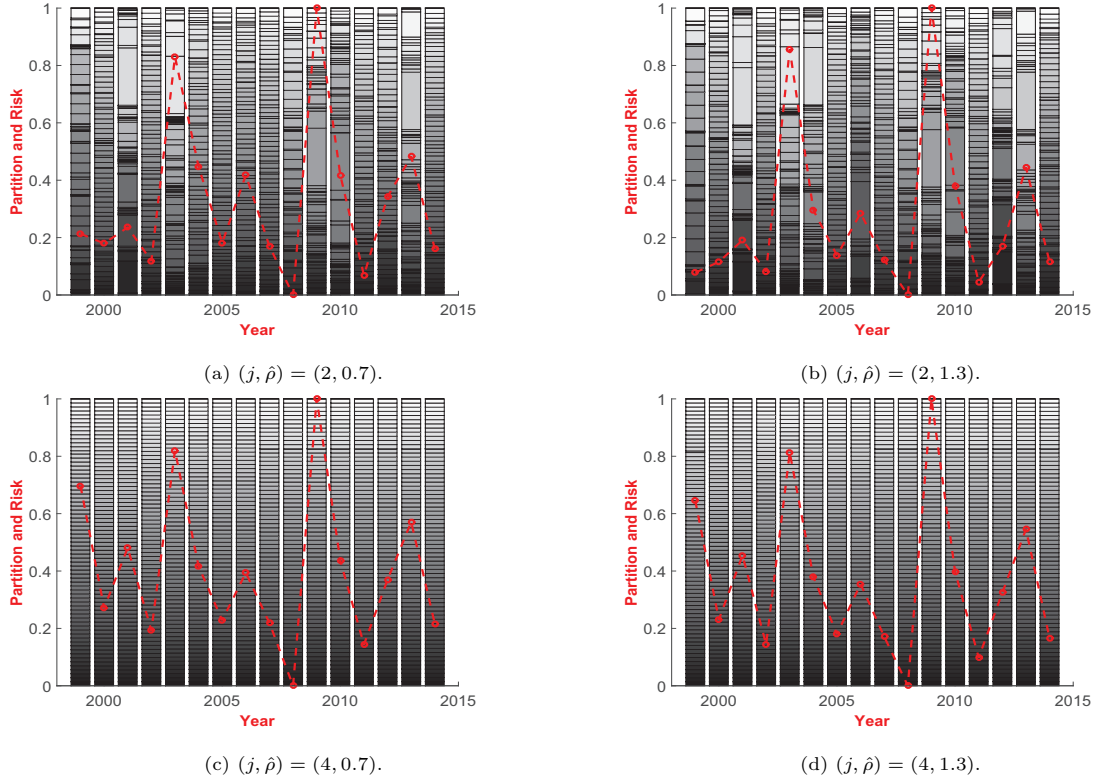
Figure 7: Minimization of higher order moments ($j \in \{2, 4\}$) of portfolio's risk with $\hat{\rho} \in \{0.7, 1.3\}$ and partitioning of 73 intermediaries. The expected returns (dotted line) is scaled into the interval $[0, 1]$.
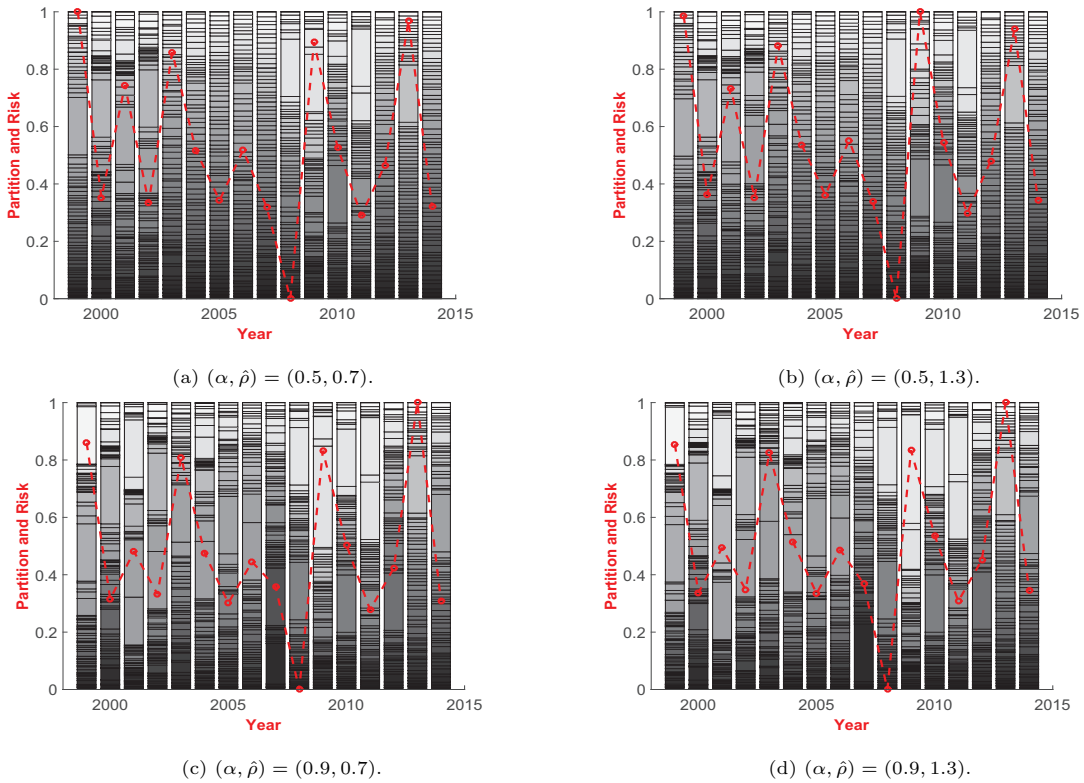

Figure 8: CVaR minimization with $\alpha \in \{0.5, 0.9\}$ for $\hat{\rho} \in \{0.7, 1.3\}$ and partitioning of 73 intermediaries. The expected returns (dotted line) is scaled into the interval $[0, 1]$.
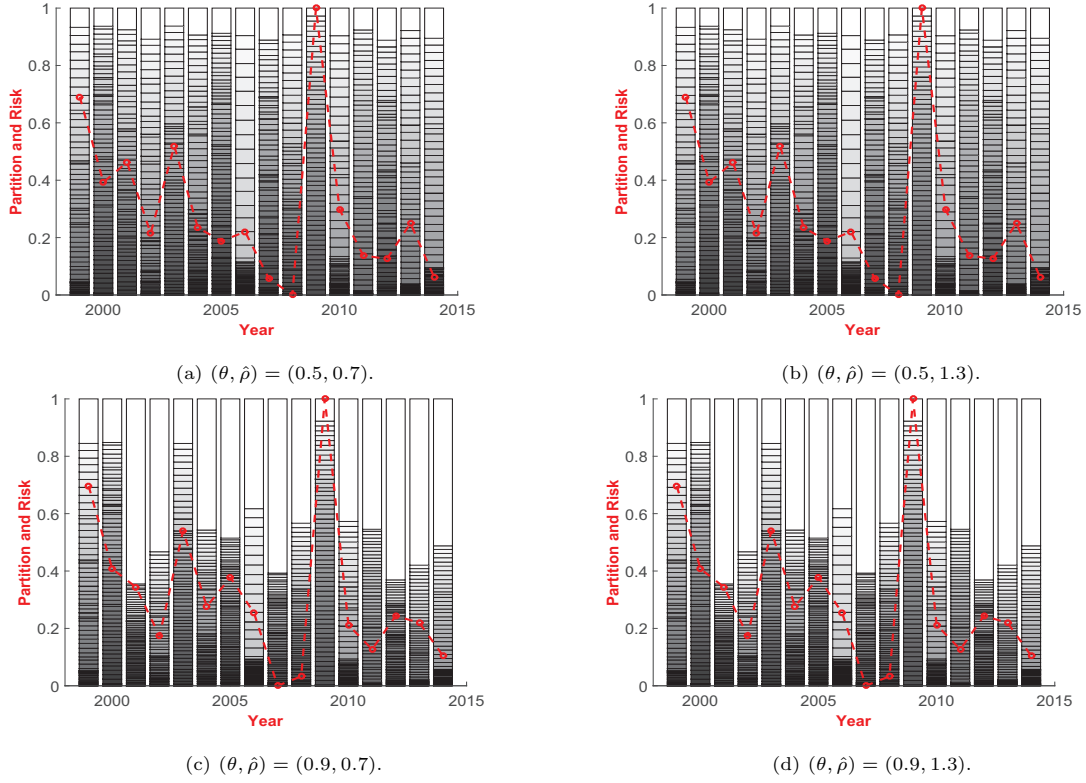
Figure 9: HARA maximization with $\theta \in \{0.5, 0.9\}$ for $\hat{\rho} \in \{0.7, 1.3\}$ and partitioning of 73 intermediaries. The expected returns (dotted line) is scaled into the interval $[0, 1]$.
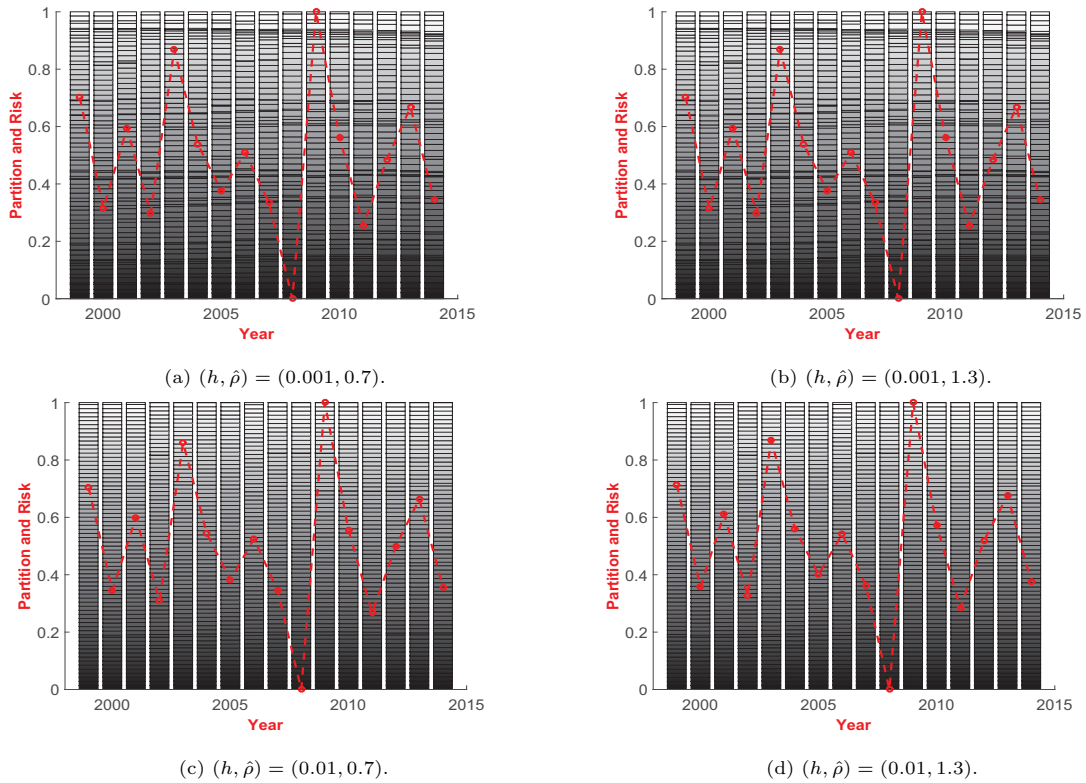

Figure 10: Conditional expected CD utility maximization with $h \in \{0.001, 0.01\}$ for $\hat{\rho} \in \{0.7, 1.3\}$ and partitioning of 73 intermediaries. The expected returns (dotted line) is scaled into the interval $[0, 1]$.

A key insight drawn from the figures is that our experimental setup captures the volatility of the 2007–2009 global financial crisis sparked by US subprime mortgage turmoil. We observe how all four instances illustrated in Figures 7–10 show the drop in expected returns and speculative activities in 2008 followed by a strongly rebound in the following year. Therefore, the equilibrium portfolio mirror the well expected returns dynamics.

Lastly, Table 2 summarizes average CPU time performance as well as the average iterations performed by the algorithm of Hoto et al. (2020), when solving these large instances of problem (4). The CPU time column includes the time taken for solving each intermediary independently (as stated in Proposition 1). Indeed, the vast majority of the computational effort is taken by the generation of $\hat{\mathbf{\Psi}}_1 \ldots \hat{\mathbf{\Psi}}_{73}$. The average values correspond to the arithmetic means of the yearly problems solved over the 16–year time span.

Table 2: Average CPU time and number of iterations per year.

$j/\alpha/\theta/h$	$\hat{\rho}$	CPU time (sec)	Iterations
		Moments	
2	0.7	21.38	47816.06
2	1.3	22.23	63673.44
4	0.7	25.94	307.81
4	1.3	28.15	639.56
		CVaR	
0.5	0.7	26.86	281727.0
0.5	1.3	31.67	282615.9
0.9	0.7	36.80	385150.7
0.9	1.3	34.27	366114.4
		HARA	
0.5	0.7	18.06	251.50
0.5	1.3	17.66	251.50
0.9	0.7	32.22	146.31
0.9	1.3	31.48	146.31
		CD	
0.001	0.7	545.79	25.62
0.001	1.3	554.73	23.69
0.010	0.7	530.74	50.88
0.010	1.3	515.18	49.25

For these large instances involving 7,256 U.S. listed enterprises groped into 73 industries, we clearly see that the minimization of the moments, the CVaR and the maximization of the expected HARA utility can be performed in less than a minute (and mostly in twenty of thirty seconds). Instead, the maximization of conditional expectation of the CD utility is slower in terms of CPU performance, mainly due to the time needed to solve each intermediary's problem in turn. Thus, parallel computing techniques are recommended

here to speed up computations. Overall, the computational performance is sensitive to the specific parameter configuration, but it remains limited to less than few minutes even for the most challenging instances.

5. Concluding remarks

We examined a new class of decentralized investment problems while a margin requirement of expected return is guaranteed. This class of problems, based on quasi-homogeneity, comprises the minimization of well-known risk functions (central moments and Conditional Value-at-Risk) as well as utility maximization (Hyperbolic Absolute Risk Aversion and Cobb–Douglas). In our formulation, financial intermediaries reach an equilibrium partitioning characterized by the Nash Bargaining Solution. We showed that the linearity of intermediaries' best response further leads to a reduction of a complex bilevel problem into a convex separable knapsack problem. This reformulation allows the implementation of numerically efficient algorithms to solve in less than one minute large-scale instances, e.g., dozens of intermediaries with hundreds of financial assets each. Our empirical evidence (based on monthly stock prices of 7,256 U.S. listed enterprises over the period 1999-2014) illustrates the operationality of our framework.

Our findings could be extended in many ways. For instance, we have assumed that the investor is able to recognize in advance the intermediaries' best responses. This an important limitation, as optimal delegated decentralized investment could also depend on intermediaries' differentiated advantageous position of trading. The inclusion of incomplete information would capture the belief of the investor on the actual returns distribution that intermediaries face. By including possible types of local markets per intermediary, the investor could describe the actual probability distribution of the individual rate of returns. Similarly, the inclusion of transaction costs of intermediaries meant to anticipate the rational behavior of investors may yield further extensions of this class of models.

Overall, the ambition of the proposed approach for the Nash bargaining partitioning in decentralized portfolio selection is to advance the frontiers of game theory and its applications to the domains of financial investment. Specifically, the reduction of this complex bilevel portfolio optimization problem to a convex separable knapsack problem opens the possibility of further reductions in the aforementioned directions, translating difficult leader-follower games into treatable optimization problems.

Appendix A: Application of the penalty algorithm for convex separable knapsack problem to the DIB game

The penalty algorithm proposed by Hoto et al. (2020), deals with convex separable knapsack problem in the following form:

$$\min_{z_1 \ldots z_m} \sum_{k \in \mathcal{M}} f_k(z_k, \hat{\boldsymbol{\Psi}}_1) \ \text{subj. to} \ \sum_{k \in \mathcal{M}} z_k = 1, \ z_k \in [0,1], \ \forall \ k \in \mathcal{M}.$$

Let us consider a optimization (3a)-(3c). We can rewrite the objective function as a logarithmic transformation for each of the four cases studied in Section 2:

$$\log F\left(z_1 \hat{\boldsymbol{\Psi}}_1, \ldots, z_m \hat{\boldsymbol{\Psi}}_m\right) = \sum_{k \in \mathcal{M}} f_k(z_k, \hat{\boldsymbol{\Psi}}_k)$$

Based on Proposition 1 (and the subsequent form of problem (4)) each term has the following form:

$$f_k(z_k, \hat{\boldsymbol{\Psi}}_k) = \begin{cases} \log\left(d_k - C_k(z_k)^{A_k}\right) & \text{when } G_k \text{ is a risk function,} \\ \\ -\log\left(C_k(z_k)^{A_k} - d_k\right) & \text{when } G_k \text{ is a utility function.} \end{cases}$$

where A_k and C_k are defined in Table 1A for the four specific cases of the risk and utility functions studied in remarks 1, 2, 3, and 4.

Table 1A: Parameter values for each of the four cases in remarks 1, 2, 3, and 4. (Note: a, for risk; b for utility.)

Type	C_k	A_k
Moments[a]	$\mathbb{E}\left[\left(\mathbf{R}_k^\top \hat{\boldsymbol{\Psi}}_k\right)^j\right]$	j
GCVaR[a]	$1 + \hat{\mathbf{r}}_k^\top \hat{\boldsymbol{\Psi}}_k + \mathbb{E}\left[\left(\mathbf{R}_k^\top \hat{\boldsymbol{\Psi}}_k\right)^2\right]^{1/2} B$	1
HARA[b]	$\frac{1}{1-\theta}\mathbb{E}\left[\left(\mathbf{R}_k^\top \hat{\boldsymbol{\Psi}}_k\right)^{1-\theta}\right]$	$1-\theta$
CD[b]	$\mathbb{E}\left[\prod_{i \in \mathcal{N}_k}(\hat{\Psi}_{ik})^{\hat{h}_{ik}} \Big| \sum_{i \in \mathcal{N}} \hat{h}_{ik} = r_k^h\right]$	r_k^h

The gradient of the objective function is:

$$\frac{d}{dz_k} f_k(z_k, \hat{\boldsymbol{\Psi}}_k) = \begin{cases} \dfrac{A_k C_k(z_k)^{A_k-1}}{d_k - C_k(z_k)^{A_k}} & \text{when } G_k \text{ is a risk function,} \\ \\ -\dfrac{A_k C_k(z_k)^{A_k-1}}{C_k(z_k)^{A_k} - d_k} & \text{when } G_k \text{ is a utility function.} \end{cases}$$

The application of the penalty algorithm to problem (3a)-(3c) proceeds in the following steps.

Initialization: Set $k' = 0$; Define $\{\rho_{k'}\}_{k' \in \mathbb{N}}$, $\{\beta_{k'}\}_{k' \in \mathbb{N}}$; Set $(z_1^{k'}, \dots, z_m^{k'}) = (1/m, \dots, 1/m)$; Set a Tolerance level.

Step 1. Compute the step-length

$$\alpha_{k'} = \frac{\beta_{k'}}{\max\left\{1, \|\nabla \log F\left(z_1^{k'} \hat{\boldsymbol{\Psi}}_1, \dots, z_m^{k'} \hat{\boldsymbol{\Psi}}_m\right)\|\right\}}$$

Step 2. Compute the projection:

$$z_l^{k'+1} = \frac{z_l^{k'} \exp\left\{-\alpha_{k'} [\nabla \log F\left(z_1^{k'} \hat{\boldsymbol{\Psi}}_1, \dots, z_m^{k'} \hat{\boldsymbol{\Psi}}_m\right)]_l\right\}}{\sum_{h \in \mathcal{M}} z_h^{k'} \exp\left\{-\alpha_{k'} [\nabla \log F\left(z_1^{k'} \hat{\boldsymbol{\Psi}}_1, \dots, z_m^{k'} \hat{\boldsymbol{\Psi}}_m\right)]_h\right\}}, \qquad l \in \mathcal{M}.$$

Step 3. Define the gap between subsequent solutions $z_k^{k'+1}$ and $z_k^{k'}$. If GAP < Tolerance level, then stop. Otherwise, set $k' = k' + 1$ and return to **Step 1**.

The notation $\nabla \log F\left(z_1^{k'} \hat{\boldsymbol{\Psi}}_1, \dots, z_m^{k'} \hat{\boldsymbol{\Psi}}_m\right)$ referees to the gradient vector of $\log F\left(z_1^{k'} \hat{\boldsymbol{\Psi}}_1, \dots, z_m^{k'} \hat{\boldsymbol{\Psi}}_m\right)$; for a vector $\mathbf{x}$, we denoted its two-norm as $\|\mathbf{x}\|_2$ and its ℓth position as $[\mathbf{x}]_\ell$. The convergence conditions for the penalty algorithm proposed by Hoto et al. (2020) require that the two positive sequences $\{\rho_{k'}\}_{k' \in \mathbb{N}} \subset \mathbb{R}^+$, $\{\beta_{k'}\}_{k' \in \mathbb{N}} \subset \mathbb{R}^+$ satisfy

$$\sum_{k'=1}^{+\infty} \beta_{k'} = +\infty, \qquad \sum_{k'=1}^{+\infty} \beta_{k'}^2 < \infty, \qquad \rho_{k'} > 0, \qquad \lim_{k' \to \infty} (\rho_k' \beta_k') = 0.$$

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