Battery Storage Formulation and Impact on Day Ahead Security Constrained Unit Commitment

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***Abstract*— This paper discusses battery storage formulations and analyzes the impact of the constraints on the computational performance of security constrained unit commitment (SCUC). Binary variables are in general required due to mutual exclusiveness of charging and discharging modes. We use valid inequalities to improve the SOC constraints. Adding batteries to the MISO day ahead market clearing cases reveals the impact of binary variables and the valid inequalities on SCUC solving time. Warm start and lazy constraint techniques are applied to improve the performance and make the valid inequalities more effective, reducing computation time to acceptable levels for implementation.**

***Index Terms*— mixed-integer programming, storage, tight formulation.**

I. Introduction

Storage of generated energy has existed as a contributing resource within the power sector for over a century in the case of pumped hydro storage. However, the deployment of chemical battery storage, with different operating characteristics to pumped hydro storage, is increasing within the industry. In 2018, FERC issued Order 841 [1] that requires each Regional Transmission Organization (RTO) and Independent System Operator (ISO) to establish a wholesale market participation model for electric storage resources that recognizes their unique physical and operational characteristics and removes any barrier to their participation. The order does not specifically require RTOs/ISOs to optimize the state of the charge (SOC). Recently, storage-plus-generation co-located hybrid resources have also been increasing as a share of new proposed projects and participants are seeking self-optimizing opportunities [2].

Some RTOs/ISOs optimize pumped storage hydro in the day ahead market [10]. Even though most RTOs/ISOs do not currently optimize SOC for battery storage, the SOC is an essential aspect of the operating characteristics of storage. This paper investigates the impact of optimizing storage with explicit representation of SOC on computational performance and the potential economic benefit, with a case study based on MISO day ahead security-constrained-unit-commitment (SCUC) problems. Here, we focus on battery storage that can be continuously dispatched across 0 MW (i.e., smoothly dispatched across charging and discharging modes). Storage optimization with a discontinuous dispatch range, such as pumped storage hydro, is discussed in [3][4][11-14]. Given the trend of co-located hybrid resources, we added one battery to each wind generator location in several MISO day ahead cases for a future scenario case study.

Formulation of battery storage requires complementarity constraints due to the mutually exclusive charging and discharging modes, necessitating either non-convex continuous constraints or binary variables. References [5][6] prove sufficient conditions under which the battery storage formulation can be relaxed to the convex form.

In [4], valid inequalities are derived for pumped storage hydro (PSH) with two binary variables to represent mutual exclusivity amongst three PSH configurations (pumping, generating, and off), and represent SOC.

In this paper, we investigate the impact of battery storage binary variables, valid inequality constraints, and explicit representation of battery SOC on the performance of day ahead SCUC. MISO solves one of the largest and most complicated day ahead electricity market clearing problems [7]. MISO has been striving to improve the computational performance and develop market clearing systems to meet future market enhancement needs. For prototyping and benchmarking MISO production day ahead market clearing cases, MISO developed the “HIPPO” software [8]. The battery storage formulations are prototyped and studied on HIPPO.

The main contribution of this paper includes:

- The derivation of a more general form of the sufficient conditions for convex relaxation of the battery storage formulation. The impact of binary variables on day ahead SCUC performance is evaluated on MISO day ahead cases.
- Introduction and analysis of a single binary variable battery storage formulation with the conventional state of charge (SOC) formulation and using an enhanced SOC formulation with tighter valid inequality constraints.
- A summary of factors contributing to challenges with the battery storage model for the day ahead SCUC, based on MISO case studies.
- Results from the novel approach of applying the enhanced SOC formulation with warm start and lazy constraint techniques to speed up the performance and make the valid inequalities more effective.

II. Conventional Battery Storage Formulation

A. Conventional battery storage formulation with convex relaxation and properties

Assume that the battery storage is allowed to bid to buy energy at price C_t^p for charging and to offer to sell energy at price C_t^g for discharging for t = 1, …, T, with charging and discharging decisions also subject to SOC limits. Because of RTO/ISO rules on bids/offers being non-decreasing, we require that $C_t^p - C_t^g \leq 0$.

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Assume there are K other generators with cost of $C_k(g_{k,t})$ for k =1, …, K and t =1, …, T. Here we consider a unit commitment and economic dispatch (UCED) model with battery storage and generators. Define $g_{k,t}$ to be the output of generator k at time t, x_k and y_k to be the set of continuous and binary variables associated with generator k, $C_k(x_k, y_k)$ is its cost function, and $F_k(x_k, y_k) \le 0$ are the constraints specifying its feasible operating region. Throughout this paper, we assume that if the binary variables are fixed then these constraints are either linear in the continuous variables or, together with the other constraints in each formulation, satisfy a suitable constraint qualification so that strong duality holds.

Define g_t to be the variable for battery discharging with maximum value $\overline{g}$ and define p_t to be the variable for battery charging with maximum $\overline{p}$ for interval t. Assume battery initial SOC is s_0. Following [4], let $\beta > \alpha > 0$ be the coefficients reflecting the effect of charging and discharging on SOC, with $\alpha/\beta < 1$ being the roundtrip efficiency for the battery. Finally, $\overline{s}$ and $\underline{s}$ are the maximum and minimum SOC of the battery.

The UCED model with convex form for the battery storage is as follows. To focus on the discussion of battery storage, we assume the generator binary variables are fixed at pre-solved unit commitment values y_k^0. Also, we assume that $C_k(x_k, y_k^0)$ is convex. With those assumptions, the "convex form" of the battery storage can be formulated as a convex problem as follows (with dual variables in parenthesis):

UCED = $Min \sum_{t=1}^{T}[-C_t^p p_t + C_t^g g_t] + C_k(x_k, y_k^0)$

s.t. $F_k(x_k, y_k^0) \le 0 \qquad \forall k = 1, \dots, K$ (1)

$-\sum_{k=1}^{K} g_{k,t} - g_t + p_t = -D_t \qquad (\lambda_t)$ (2)

$-g_t \le 0 \qquad (\delta_t')$ (3)

$g_t - \overline{g} \le 0 \qquad (\delta_t'')$ (4)

$-p_t \le 0 \qquad (\gamma_t')$ (5)

$p_t - \overline{p} \le 0 \qquad (\gamma_t'')$ (6)

$s_0 + \sum_{\tau=1}^{t}(\alpha \cdot p_\tau - \beta \cdot g_\tau) \le \overline{s} \qquad (\sigma_t')$ (7)

$-s_0 - \sum_{\tau=1}^{t}(\alpha \cdot p_\tau - \beta \cdot g_\tau) \le -\underline{s} \qquad (\sigma_t'')$ (8)

$\forall t = 1, \dots, T$

In this UCED formulation with battery storage, the objective is to minimize production cost, (1) are the operating constraints for generators with fixed binary variables, (2) is power balance constraint with D_t as the fixed demand at interval t, (3)-(6) are the limit constraints for battery discharging and charging, and (7) and (8) are SOC constraints. For the purposes of discussing the implications of the battery formulation, we ignore transmission constraints, so all battery capacity is assumed to be aggregated into a single equivalent battery. (We will include transmission constraints in MISO case studies in section IV.)

Theorem 1. The necessary condition to clear a battery with simultaneous charging and discharging MW under the convex form is to have the energy clearing price λ_{t1} for some interval $t1$ be less than or equal to $\frac{\beta C_{t1}^p - \alpha C_{t1}^g}{(\beta-\alpha)}$.

Proof: The dual of UCED related to battery primal and dual variables is as follows:

DUUCED= $Max \sum_{t=1}^{T}[D_t \lambda_t - \overline{s}\sigma_t' + \underline{s}\sigma_t'' - \delta_t''\overline{g} - \gamma_t''\overline{p}]$

s.t.

$-\lambda_t - \delta_t' + \delta_t'' + \beta \sum_{\tau=t}^{T}(\sigma_\tau'' - \sigma_\tau') \ge -C_t^g \qquad (g_t)$ (9)

$\lambda_t - \gamma_t' + \gamma_t'' - \alpha \sum_{\tau=t}^{T}(\sigma_\tau'' - \sigma_\tau') \ge C_t^p \qquad (p_t)$ (10)

$\delta_t', \delta_t'', \gamma_t', \gamma_t'', \sigma_t', \sigma_t'' \ge 0$

$\forall t = 1, \dots, T$

The condition for charging and discharging at the same time is to have an optimal solution with $g_{t1} > 0$ and $p_{t1} > 0$ for some specific interval $1 \le t1 \le T$. By complementary slackness,

From (3), (5): $\delta_{t1}' = 0$, $\gamma_{t1}' = 0$

From (9): $-\lambda_{t1} + \delta_{t1}'' + \beta \sum_{\tau=t1}^{T}(\sigma_\tau'' - \sigma_\tau') = -C_{t1}^g$ (11)

From (10): $\lambda_{t1} + \gamma_{t1}'' - \alpha \sum_{\tau=t1}^{T}(\sigma_\tau'' - \sigma_\tau') = C_{t1}^p$ (12)

With $\alpha \cdot (11) + \beta \cdot (12)$ we have:

$$\lambda_{t1} = \frac{\beta C_{t1}^p - \alpha C_{t1}^g - \alpha\delta_{t1}'' - \beta\gamma_{t1}''}{(\beta-\alpha)} \le \frac{\beta C_{t1}^p - \alpha C_{t1}^g}{(\beta-\alpha)} \qquad (13)$$

QED

We observe that adding (11) and (12) yields:

$$\delta_{t1}'' + \gamma_{t1}'' + (\beta - \alpha)\sum_{\tau=t1}^{T}(\sigma_\tau'' - \sigma_\tau') = C_{t1}^p - C_{t1}^g \qquad (14)$$

By assumption, $C_t^p - C_t^g \le 0$ for a battery offered into the market. If $C_{t1}^p - C_{t1}^g < 0$, we must have $\sigma_\tau' > 0$ (i.e., binding at maximum SOC) for at least one of the intervals $\tau \ge t1$.

With transmission constraints and losses represented, we could analogously derive a condition similar to (14) by replacing λ_{t1} with the battery locational marginal price at t1. To summarize, simultaneous clearing of charging and discharging MW may happen if the price is less than or equal to $\frac{\beta C_{t1}^p - \alpha C_{t1}^g}{(\beta-\alpha)}$.

Even for a special case with a single interval, i.e., T =1, the storage may simultaneously clear with $g_t > 0$ and $p_t > 0$ as will be shown in section IV.A for a single interval example.

Hence, a sufficient condition for using the convex relaxation formulation is for the LMP to be higher than $\frac{\beta C_{t1}^p - \alpha C_{t1}^g}{(\beta-\alpha)}$. For batteries bidding and offering to arbitrage energy across different intervals with $C_t^p = C_t^g = 0$, respectively, the sufficient condition is for the LMP to be positive. This is consistent with Lemma 1 in [4]. This condition may be satisfied most of the time; however, negative prices increasingly occur with renewable integration and transmission congestion. Moreover, renewables and congestion are important drivers of battery installation. That is, even if zero and negative prices are relatively rare, they can be expected to at least occasionally be present with storage and so this issue must be tackled, as discussed in the next section.

B. Battery storage formulation with binary variables

Adding one binary variable u_t for each battery as in the following problem (UCED_BIN) can ensure mutual exclusivity of charging and discharging modes under any prices. To the best of the authors' knowledge, this single binary variable formulation is novel.

UCED_BIN=

$Min \sum_{t=1}^{T}[-C_t^p p_t + C_t^g g_t] + C_k(x_k, y_k^0)$

s.t. (1)(2)(3)(5)(7)(8)

$g_t - \overline{g}\, u_t \le 0$ (15.BIN)

$p_t - \overline{p}(1 - u_t) \le 0$ (16.BIN)

$u_t \in \{0,1\}$ (17.BIN)

The model with u_t relaxed to being a continuous variable is UCED_BIN_Rel:

$$Min\ \sum_{t=1}^{T}[-C_t^p p_t + C_t^g g_t] + C_k(x_k, y_k^0)$$

s.t. (1)(2)(3)(5)(7)(8)

$$g_t - \overline{g}\, u_t \le 0 \qquad (\delta_t'') \qquad (15)$$

$$p_t - \overline{p}(1-u_t) \le 0 \qquad (\gamma_t'') \qquad (16)$$

$$-u_t \le 0 \qquad (\rho_t') \qquad (17)$$

$$u_t \le 1 \qquad (\rho_t'') \qquad (18)$$

$$\forall t = 1, \dots, T$$

For the constraints associated with the individual battery: (3)(5)(7)(8) and (15)-(18), we observe that the relaxed feasible set with u_t continuous does not form the convex hull of the feasible set of (UCED_BIN) even for a single interval case *T*=1.

For example, for *T*=1, assume $s_0 = 0$, $\underline{s} = 0$, then there is a fractional solution $u_1 = \alpha\overline{p}/(\alpha\overline{p} + \beta\overline{g})$ corresponding to the following three linearly independent binding constraints: $g_1 = \overline{g}\, u_1$, $p_1 = \overline{p}(1-u_1)$, and $-s_0 - \alpha \cdot p_1 + \beta \cdot g_1 \le -\underline{s}$. This example shows that an extreme point with fractional u_t exists for this formulation. Hence, this set of constraints does not specify the convex hull for individual batteries even for *T*=1.

Next, we examine more generally the conditions under which u_t can be fractional.

Theorem 2. The necessary condition for the convex problem UCED_BIN_Rel to solve with fractional u_t for a battery is to have its price less than or equal to $\frac{\beta C_{t1}^p - \alpha C_{t1}^g}{(\beta-\alpha)}$.

Proof: For the binary relaxation problem, the optimality conditions include:

$$-\delta_t' + \delta_t'' + \beta\sum_{\tau=t}^{T}(\sigma_\tau'' - \sigma_\tau') \ge -C_t^g + \lambda_t \qquad (g_t) \qquad (19)$$

$$-\gamma_t' + \gamma_t'' - \alpha\sum_{\tau=t}^{T}(\sigma_\tau'' - \sigma_\tau') \ge C_t^p - \lambda_t \qquad (p_t) \qquad (20)$$

$$-\overline{g}\delta_t'' + \overline{p}\gamma_t'' - \rho_t' + \rho_t'' \ge 0 \qquad (u_t) \qquad (21)$$

$$\delta_t', \delta_t'', \gamma_t', \gamma_t'', \sigma_t', \sigma_t'', \rho_t', \rho_t'' \ge 0$$

$$\forall t = 1, \dots, T$$

Note that any solution of (UCED_BIN_Rel) satisfying g_t= 0 and $0 < u_t < 1$ has an equivalent binary solution of g_t= 0 and u_t=0. Similarly, any solution $p_t = 0$ and $0 < u_t < 1$ has an equivalent binary solution of p_t = 0 and u_t = 1. In both cases, we could say that the solution of (UCED_BIN_Rel) has been "cured," since from the perspective of (UCED_BIN), an optimal solution exists with the same values of pumping and generation. Problem (UCED_BIN_Rel) has a fractional optimal solution that cannot be cured to a solution of (UCED_BIN) when $0 < u_t < 1$, $g_t > 0$ and $p_t > 0$.

Similar to Theorem 1, we can prove that the necessary condition for $g_t > 0, p_t > 0$ and $0 < u_t < 1$ to happen is:

$$\lambda_t = \frac{\beta C_t^p - \alpha C_t^g - \alpha\delta_t'' - \beta\gamma_t''}{(\beta-\alpha)} \le \frac{\beta C_t^p - \alpha C_t^g}{(\beta-\alpha)}$$

It is the same condition as (13). QED

III. Tightened Battery Storage Formulation and the Sufficient Condition for Convex Relaxation

A. Tightened SOC formulation

The left-hand side of (7) and (8) is the SOC at interval t.

$$s_t = s_0 + \sum_{\tau=1}^{t}(\alpha \cdot p_\tau - \beta \cdot g_\tau) = s_{t-1} + \alpha \cdot p_t - \beta \cdot g_t$$

Conditions (7) and (8) are equivalent to $\underline{s} \le s_t \le \overline{s}$. As discussed in [4], with mutually exclusive charging and discharging, (7) and (8) can be tightened to:

$$s_{t-1} + \alpha \cdot p_t \le \overline{s} \qquad (22)$$

$$s_{t-1} - \beta \cdot g_t \ge \underline{s} \qquad (23)$$

Reference [4] discusses this formulation for pumped storage hydro, but (22) and (23) are also valid inequalities for any storage device with mutually exclusive modes. As observed in [4], (22) and (23) are tighter than (7) and (8). In this section, we examine the impact of (22) and (23) on battery storage.

B. Tightened battery storage formulation with convex relaxation

The convex form of the UCED optimization model with the tightened battery storage SOC formulation is as follows:

UCED_T = $Min\ \sum_{t=1}^{T}[-C_t^p p_t + C_t^g g_t] + \sum_{k=1}^{K}\sum_{t=1}^{T} C_k(g_{k,t})$

s.t. (1)-(6),

$$s_0 + \sum_{\tau=1}^{t-1}(\alpha \cdot p_\tau - \beta \cdot g_\tau) + \alpha \cdot p_t \le \overline{s} \qquad (\sigma_t') \qquad (24)$$

$$-s_0 - \sum_{\tau=1}^{t-1}(\alpha \cdot p_\tau - \beta \cdot g_\tau) + \beta \cdot g_t \le -\underline{s} \qquad (\sigma_t'') \qquad (25)$$

$$\forall t = 1, \dots, T$$

Theorem 3. The convex formulation UCED_T guarantees mutual exclusivity for the last interval *T*. For *t*<*T*, the necessary condition to clear a battery with simultaneous charging and discharging MW under the convex formulation UCED_T is to have its price less than or equal to $\frac{\beta C_{t1}^p - \alpha C_{t1}^g}{(\beta-\alpha)}$.

Proof: The condition for charging and discharging at the same time is to have an optimal solution with $g_{t1} > 0$ and $p_{t1} > 0$ for a specific interval $1 \le t1 \le T$.

From complementary slackness: $\delta_{t1}' = 0$, $\gamma_{t1}' = 0$.

The optimality conditions are:

$$-\lambda_t - \delta_t' + \delta_t'' + \beta\sigma_t'' + \beta\sum_{\tau=t+1}^{T}(\sigma_\tau'' - \sigma_\tau') = C_t^g \ (g_t) \ (26)$$

$$\lambda_t - \gamma_t' + \gamma_t'' + \alpha\sigma_t' - \alpha\sum_{\tau=t+1}^{T}(\sigma_\tau'' - \sigma_\tau') = C_t^p \ (p_t) \ (27)$$

$$\delta_t', \delta_t'', \gamma_t', \gamma_t'', \sigma_t', \sigma_t'' \ge 0$$

$$\forall t = 1, \dots, T$$

Summing (26) and (27) yields:

$$\delta_t'' + \beta\sigma_t'' + \gamma_t'' + \alpha\sigma_t' + (\beta-\alpha)\sum_{\tau=t+1}^{T}(\sigma_\tau'' - \sigma_\tau') = C_t^p - C_t^g \qquad (28)$$

Evaluating $\alpha \cdot (26) + \beta \cdot (27)$ yields:

$$(\beta-\alpha)\lambda_t + \alpha\delta_t'' + \beta\gamma_t'' + \alpha\,\beta(\sigma_t'' + \sigma_t') = \beta\, C_t^p - \alpha C_t^g \quad (29)$$

By assumption, $C_t^p \le C_t^g$. For the last interval *t* =*T*, we have from (28):

$$0 \le \delta_T'' + \beta\sigma_T'' + \gamma_T'' + \alpha\sigma_T' = C_T^p - C_T^g \le 0 \qquad (30)$$

This condition can only be true when $\lambda_T = C_T^p = C_T^g$ and $\delta_T'' = \sigma_T'' = \gamma_T'' = \sigma_T' = 0$, which means that the battery constraints are not binding on any MW or MWh limit and the battery is setting the price. In this case, if $g_T \ge p_T > 0$, we have $s_{T-1} + \alpha \cdot p_T \le \overline{s}$ and $-s_{T-1} + \beta \cdot g_t \le -\underline{s}$ under the tightened SOC formulation.

Consider $g_T' = g_T - p_T$ and $p_T' = 0$.

$$s_{T-1} + \alpha \cdot p_T{}' \le s_{T-1} + \alpha \cdot p_T \le \overline{s}$$

$$-s_{T-1} + \beta \cdot g_T{}' \le -s_{T-1} + \beta \cdot g_T \le -\underline{s}$$

Hence g_T' and p_T' is also an optimal solution with the same objective.

Similarly, if $p_T \ge g_T > 0$, the solution $p_T' = p_T - g_T$ and $g_T' = 0$ is also an optimal solution with the same objective.

Under all other conditions, (30) cannot be true. Hence, the last interval T always has an optimal solution that satisfies the exclusive charging and discharging condition.

Similar logic can be applied for time $t1$ if $\sigma'_\tau = 0$, which is satisfied if (24) are not binding for any intervals $\tau > t1$. Under this condition, mutually exclusive charging and discharging will be satisfied for $t1$.

However, if the SOC constraint (24) binds at an interval $t > t1$, then (28) may hold for $t < t1$ and so simultaneous clearing of $g_t > 0$ and $p_t > 0$ may happen. The necessary condition for simultaneous clearing of pumping and generation to happen is:

$$\lambda_t = \frac{\beta\, C_t^p - \alpha C_t^g - \alpha\delta_t'' - \beta\gamma_t'' - \alpha\, \beta(\sigma_t'' + \sigma_t')}{(\beta - \alpha)} \le \frac{\beta\, C_t^p - \alpha C_t^g}{(\beta - \alpha)}$$

It is again the same condition as (13). QED

C. Tightened battery storage formulation with binary variables

Adding one binary variable u_t for each battery (UCED_BIN_T) to the tightened battery storage model can ensure mutual exclusivity of charging and discharging modes. The model with u_t relaxed to a continuous variable is:

UCED_BIN_T_Rel =

$$Min\ \sum_{t=1}^{T}[-C_t^p p_t + C_t^g g_t] + C_k(x_k, y_k^0)$$

s.t. (1)(2)(3)(5), (15)-(18), (24) (25)

$$\forall t = 1, \dots, T$$

Next, we examine the condition under which u_t can be fractional.

Theorem 4. Under the formulation UCED_BIN_T_Rel, the last interval always has an optimal solution with u_T binary. For $t<T$, the necessary condition for the convex problem UCED_BIN_T_Rel to solve with fractional u_t for a battery is to have its price less than or equal to $\frac{\beta\, C_{t1}^p - \alpha C_{t1}^g}{(\beta - \alpha)}$.

Proof: Consider the profit maximization problem with λ_t as the price for the battery at time t.

$$Max\ \sum_{t=1}^{T} \lambda_t(-p_t + g_t) + C_t^p p_t - C_t^g g_t$$

s.t. (3)(5), (15)-(18), (24) (25)

For the binary relaxation problem, the optimality conditions are:

$$-\delta_t' + \delta_t'' + \beta\sigma_t'' + \beta \sum_{\tau=t+1}^{T}(\sigma_\tau'' - \sigma_\tau') = -C_t^g + \lambda_t \quad (g_t)\ (31)$$

$$-\gamma_t' + \gamma_t'' + \alpha\sigma_t' - \alpha \sum_{\tau=t+1}^{T}(\sigma_\tau'' - \sigma_\tau') \ge C_t^p - \lambda_t\ (p_t)\ \ (32)$$

$$-\overline{g}\delta_t'' + \overline{p}\gamma_t'' - \rho_t' + \rho_t'' \ge 0 \qquad (u_t)\ (33)$$

$$\delta_t', \delta_t'', \gamma_t', \gamma_t'', \sigma_t', \sigma_t'', \rho_t', \rho_t'' \ge 0$$

$$\forall t = 1, \dots, T$$

For the problem to have fractional solution, i.e., $0 < u_t < 1$, $g_t > 0$ and $p_t > 0$,

Similar to Theorem 3, we can derive that the last interval T always has an optimal solution that satisfies the mutually exclusive charging and discharging condition if $C_T^p \le C_T^g$. Note, the solution g_T=0 and $0 < u_T < 1$ can be cured to have an equivalent binary solution of g_T=0 and u_T=0. Similarly, p_T=0 and $0 < u_T < 1$ has an equivalent binary solution of p_T=0 and u_T=1. Hence, the last interval always has an optimal solution with u_T binary. To summarize, for the special case T =1, this tightened SOC formulation specifies convex hull for individual battery storage if $C_T^p \le C_T^g$.

For $T > 1$, we can also similarly prove that the necessary condition for $g_t > 0$, $p_t > 0$ and $0 < u_t < 1$ to occur is if:

$$\lambda_t = \frac{\beta\, C_t^p - \alpha C_t^g - \alpha\delta_t'' - \beta\gamma_t'' - \alpha\, \beta(\sigma_t'' + \sigma_t')}{(\beta - \alpha)} \le \frac{\beta\, C_t^p - \alpha C_t^g}{(\beta - \alpha)}$$

It is again the same condition as (13).

IV. Case Studies

A. One interval small example

Assume a battery with: $\overline{p} = \overline{g} = 6MW$, $\alpha = 0.9$, $\beta = 1/0.9$, $C_t^p = \$1/MWh$ and $C_t^g = \$3/MWh$, s_0=5MWh, $\overline{s} = 10MWh$, $\underline{s} = 0$ MWh. The cleared discharging MW is g_t, charging MW is p_t and SOC is s_t.

Assume the system has one thermal generator with offer C/MWh. The generator can be dispatched between 0 and 10 MW with cleared MW represented by G_t. Also assume load is 5 MW for all intervals.

First, assume single interval T=1. The sufficient condition for mutually exclusive charging and discharging mode in the clearing is for LMP to be above $\frac{\beta\, C_t^p - \alpha C_t^g}{(\beta - \alpha)} = -\$7.526$/MWh. In the following cases, we will consider the alternatives of the LMP being just slightly below and just slightly above this threshold.

1) Conventional battery storage formulation with convex form (model UCED). The single interval clearing result is shown in Table 1 for $C= -\$7.52$/MWh and $-\$7.53$/MWh.

Table 1 Single interval conventional ED model

C (\$/MWh)	G_1(MW)	g_1(MW)	p_1(MW)	s_1 (MWh)	LMP$_1$ (\$/MWh)
-7.52	10.56	0	5.56	10	-7.52
-7.53	10.64	0.36	6	10	-7.53

Table 1 shows that under model UCED, when LMP$= -\$7.53$/MWh $<\frac{\beta\, C_t^p - \alpha C_t^g}{(\beta - \alpha)}$, the storage may simultaneously clear non-zero g_1 and p_1 even under single interval. When LMP$= -\$7.52$/MWh $>\frac{\beta\, C_t^p - \alpha C_t^g}{(\beta - \alpha)}$, there is no simultaneous clearing of charging and discharging MW.

2) Relaxation of the conventional battery storage formulation with binary variable (model UCED_BIN_Rel). The single interval clearing result is shown in Table 2 under $C= -\$7.52$/MWh and $C= -\$7.53$/MWh.

Table 2 Single interval conventional EDBIN_Rel model

C (\$/MWh)	G_1(MW)	g_1(MW)	p_1(MW)	s_1 (MWh)	LMP$_1$ (\$/MWh)	u_1
-7.52	10.56	0	5.56	10	-7.52	0
-7.53	10.6	0.2	5.8	10	-7.53	0.033

Table 2 shows that under model UCED_BIN_Rel, when LMP$<\frac{\beta\, C_t^p - \alpha C_t^g}{(\beta - \alpha)}$, the storage may simultaneously clear non-zero g_1 and p_1 with u_1 at a fractional value even in a single interval case. When LMP$>\frac{\beta\, C_t^p - \alpha C_t^g}{(\beta - \alpha)}$, there is no simultaneous clearing of charging and discharging MW and u_1 is binary at the solution of the relaxed problem.

3) Tightened battery storage formulation with convex relaxation UCED_T:
With C= -\$100/MWh, the clearing result is $g_1 = 0$ and $p_1 = 5.56$ MW. $s_1 = 10MWh$. The generator output is $G_1 = 10.56MW$. LMP = -\$100/MWh.
Hence, with the UCED_T model, it does not clear non-zero g_1 and p_1 simultaneously for the single interval even with a very negative LMP. This demonstrates that the tightened formulation improves on the relaxation of the conventional battery formulation for this (special) single interval case.

4) Relaxation of the tightened battery storage formulation with binary variable (model UCED_BIN_T_Rel:
With C= -\$100/MWh, the clearing result is $g_1 = 0$ and $p_1 = 5.56$ MW. $u_1 = 0,\ s_1 = 10MWh$. The generator output is $G_1 = 10.56MW$. LMP = -\$100/MWh.
Hence, with the EDBIN_T model, it does not clear non-zero g_1 and p_1 simultaneously for the single interval even with a very negative LMP if u_1 is required to be binary.

B. Two interval small example

In this example we examine the tightened battery storage formulation with T=2. Assume the same battery and generator offer parameters for both intervals as for the previous example.

1) Tightened battery storage formulation with convex relaxation UCED_T. The 2-interval clearing result is shown in Table 3 under $C = -$\$7.52/MWh and $C = -$\$7.53/MWh.

Table 3 Two-interval enhanced ED_T model

C (\$/MWh)	G_1(MW)	g_1(MW)	p_1(MW)	s_1 (MWh)	LMP$_1$ (\$/MWh)
-7.52	5	0	0	5	-7.52
-7.53	5.6	4.5	5.1	4.6	-7.53
C (\$/MWh)	G_2(MW)	g_2(MW)	p_2(MW)	s_2(MWh)	LMP2 (\$/MWh)
-7.52	10.56	0	5.56	10	-7.52
-7.53	11	0	6	10	-7.53

Table 3 demonstrates that with the tightened model, when LMP<$\frac{\beta\, c_t^p - \alpha c_t^g}{(\beta-\alpha)}$, the storage may simultaneously clear non-zero g_1 and p_1 if the SOC upper limit binds for t >1. When the LMP> $\frac{\beta\, c_t^p - \alpha c_t^g}{(\beta-\alpha)}$, battery storage does not clear simultaneously for charging and discharging.

2) Relaxation of the enhanced battery storage formulation with binary variable (model UCEDBIN_T_Rel). The 2-interval clearing result is shown in Table 4 under C= $-$\$7.52/MWh and C= $-$\$7.53/MWh.
Table 4 demonstrates that with the enhanced model, when LMP<$\frac{\beta\, c_t^p - \alpha c_t^g}{(\beta-\alpha)}$, the storage may simultaneously clear g_1 and p_1 with u_1 at fractional if the SOC upper limit binds for t >1. When LMP>$\frac{\beta\, c_t^p - \alpha c_t^g}{(\beta-\alpha)}$, battery storage clears with exclusive charging and discharging, and also solves with binary value of u_t.
The last interval always clears with mutually exclusive charging and discharging as well as a binary value of u_T.

Table 4 Two-interval enhanced UCED_BIN_E_Rel model

C (\$/MWh)	G_1(MW)	g_1(MW)	p_1(MW)	s_1 (MWh)	LMP$_1$ (\$/MWh)	u_1
-7.52	10.56	0	5.56	10	-7.52	0
-7.53	5.232	2.884	3.116	4.6	-7.53	0.48
C (\$/MWh)	G_2(MW)	g_2(MW)	p_2(MW)	s_2(MWh)	LMP2 (\$/MWh)	u_2
-7.52	5	0	0	10	-7.52	0
-7.53	11	0	6	10	-7.53	0

C. MISO day ahead cases – normal cases

In this study, we simulated batteries co-located with each wind generator offered into the Midcontinent ISO (MISO) day ahead market. We studied 10 day ahead cases from 2019 that were normal or typical in terms of their computation time before the addition of batteries. The numbers of batteries added for each case range from 169 to 203. The power capacity of the battery is the lesser of 20 MW or the maximum limit of the wind generator and with a duration of 4 hours of storage and $\alpha = 0.9$ and $\beta = 1/\alpha$. All batteries are offered with $C_t^p = C_t^g = \$0$ for energy arbitrage. MISO production day ahead SCUC MIP stops when one of the following three criteria is satisfied: 1) the solving time limit of 1,200 seconds is reached, 2) the MIP relative gap tolerance is 0.1%, 3) the MIP absolute gap tolerance is \$24,000. The cases are run on a Linux server with 32 Intel Core Processor (Haswell, no TSX, IBRS) and 528GB memory. With the same stopping criteria, Fig. 1 compares the MIP solving time under three different battery storage models:

- Conventional battery storage formulation with convex relaxation (UCED). Under this model, batteries with negative LMP may simultaneously clear charging and discharging MW. But there are no binary variables for the batteries so solving time *might* be expected to be less than for the models with binary variables.
- Conventional battery storage formulation with binary (UCED_BIN). With the binary variables, the mutual exclusivity of charging and discharging mode is always enforced.
- Tightened battery storage formulation with binary (UCED_BIN_T). With the binary variables, the mutual exclusivity of charging and discharging mode is always enforced.

To reduce the number of non-zeros in the constraint matrix of the models, we use a compact form for the SOC constraints. In particular, for the conventional formulation, (7) and (8) are replaced by:

$$s_t = s_{t-1} + \alpha \cdot p_t - \beta \cdot g_t \qquad (7')$$

$$s_t \textit{ has a lower bound } \underline{s} \textit{ and upper bound } \overline{s} \qquad (8')$$

For the tightened formulation, (28’) and (27’) are replaced by (7’), (27), and (28).

The results in Fig. 1 (“No Battery” and “Battery – UCED”) show that adding batteries can increase MIP solving time significantly. However, somewhat contrary to expectations, the solving times under “Battery – UCED_BIN” are mostly similar to or less than the solving time under ED. Hence, removing the binaries in the storage formulation does not lead to faster solution.

Even though the tightened formulation is a closer approximation to the convex hull than the conventional formulation, the solving time (“Battery – UCED_BIN_T”) is not necessarily faster. Part of the reason is because the tightened model UCED_BIN_T introduces more rows and non-zeros to the constraint matrix of the MIP model than the conventional model UCED_BIN. Table 5 compares the MIP problems size for the day ahead SCUC model without batteries and with batteries under UCED, UCED_BIN and UCED_BIN_T models.

To address the issue of adding too many rows and non-zeros under the tightened storage model UCED_BIN_T, we experimented with “lazy” constraint techniques.

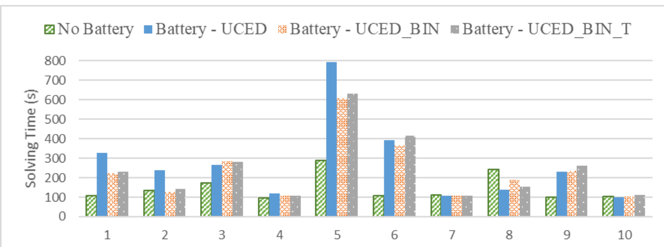


Fig. 1 Normal case solving time (second) comparison

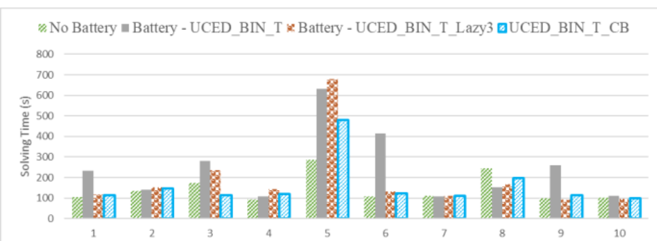


Fig. 2 Normal case solving time (second) comparison – lazy constraint

Gurobi provides a constraint attribute “Lazy” [9]. A constraint whose Lazy attribute is set to 1, 2, or 3 (the default value is 0) is removed from the model and placed in the lazy constraint pool. When a new solution is found, it is checked against the lazy constraint pool and violated lazy constraints may be added to the active model. With a value of 3, lazy constraints that cut off the relaxation solution at the root node are also incorporated.

We first use Gurobi “lazy” attribute on UCED_BIN_T. We built UCED_BIN_T with SOC constraints (7’)(8’)(22) and (23). We then set all constraints (22) and (23) as lazy=3. Gurobi started with removing (22) and (23) from the model and placed them in the lazy pool. With binary variables, p_t and g_t should not clear simultaneously and (7’)(8’) should be satisfied for any new incumbent solutions. Hence, (22) and (23) should also be satisfied. However, (22) and (23) may be violated in the relaxation solution in the case that the binary variables have fractional values in the relaxation solution. Fig. 2 shows that the lazy setting (“Battery – UCED_BIN_T_Lazy3”) is very effective for case 1, 6, and 9. For most of the other cases, the solving times with lazy=3 are similar to the default (“Battery – UCED_BIN_E”). However, lazy setting slightly increases the solving time for case 5.

We then experiment to start with EDBIN and then add (22) and (23) through call back (“UCED_BIN_T_CB”). At each new Gurobi MIPNODE relaxation solution, we check if (22) or (23) is violated under the relaxation solution. We only add the constraints that are violated by over 1% of the maximum SOC of the battery ($i.e., 0.01 \cdot \overline{s}$) and add no more than 200 in total. This approach is very effective for case 5 and 3. This approach has similar computation time to Gurobi Lazy3 for other cases.

To summarize, with UCED_BIN_T_Lazy3 and UCED_BIN_T_CB, the solving times for 9 out of 10 cases are similar to the “No Battery” model. The solving time of case 5 increases by about 200-300s. It is still within 700 seconds and is still considered normal from the perspective of computation time. In the next section, we build on this experience for several hard MISO cases.

Table 5 Problem size comparison

		Before Pre-solve				
		Rows	Columns	non-zeros	continuous variables	binary variables
No battery (a)	[(a)-(a)]/(a)	0.00%	0.00%	0.00%	0.00%	0.00%
Battery UCED (b)	[(b)-(a)]/(a)	0.98%	2.00%	1.08%	2.32%	0.00%
Battery UCED_BIN (c)	[(c)-(a)]/(a)	2.95%	2.67%	1.62%	2.32%	4.88%
Battery UCED_BIN_T (d)	[(d)-(a)]/(a)	4.91%	2.67%	2.16%	2.32%	4.88%
		After Pre-solve				
		Rows	Columns	non-zeros	continuous variables	binary variables
No battery (a)	[(a)-(a)]/(a)	0.00%	0.00%	0.00%	0.00%	0.00%
Battery UCED (b)	[(b)-(a)]/(a)	2.83%	3.35%	1.97%	3.70%	0.00%
Battery UCED_BIN (c)	[(c)-(a)]/(a)	8.09%	4.36%	2.83%	3.63%	11.34%
Battery UCED_BIN_T (d)	[(d)-(a)]/(a)	13.04%	4.30%	3.75%	3.56%	11.34%

D. MISO day ahead cases – hard cases

In this section, we show the results from 9 day ahead cases from 2014 that were hard in terms of the computation time even *before* the addition of optimization for batteries. The numbers of batteries added for each case range from 129 to 176. The power capacity of the battery is the lesser of 20 MW or the maximum limit of the wind generator with a duration of 4 hours of storage and $\alpha = 0.9$ and $\beta = 1/\alpha$. All batteries are offered with $C_t^p = C_t^g = \$0$.

This set of cases can solve with below 1% gap at 1,200 seconds without the batteries. But most of them cannot reach 0.1% or $24,000 MIP gap tolerance before timing out at 1,200 seconds. A solution below 1% MIP gap is considered acceptable and is refined through a polishing procedure [7].

With the added batteries, it can take Gurobi8.0 up to 1,400 seconds to find the first incumbent solution with all three models. Table 6 compares MIP gap and solving time for the model with no battery, with conventional battery non-binary formulation (UCED), conventional battery binary formulation (UCED_BIN), tightened battery non-binary formulation (UCED_T) and tightened battery binary formulation (UCED_BIN_T). The solving time increases significantly by adding the batteries. Similar to the normal cases, adding binary variables to the non-binary formulation typically only has minor impact on the solving time.

We then focused on the model with binary variables for the remaining of the hard case study.

1) MIP cold start

Most cases with battery cannot solve to MIP gap tolerance within 3,000 seconds under the conventional battery formulation. The tightened formulation has more rows and non-zeros. Again, even though it is a closer approximation to the convex hull, the overall performance is not necessarily better.

Table 7 compares MIP gap, solving time and objective differences under no battery, UCED_BIN, UCED_BIN_T and UCED_BIN_T_Lazy3.

First compare UCED_BIN_T to UCED_BIN. Three cases are solved with either improved time or improved objective under UCED_BIN_T. However, the average solving times and objectives increase slightly.

Secondly, compare UCED_BIN_T_Lazy3 to UCED_BIN, 6 cases are solved with either improved time or improved objective under UCED_BIN_T_Lazy3. However, case 7 objective increases significantly. A possible explanation for this observation is that lazy constraint settings and callback may turn off advanced features in commercial solvers.

Overall, the tightened formulation under MIP cold start shows some benefit. But the impact is not consistent. To try to harness the benefits of the tightened formulation without greatly increasing the computation time, we consider a hybrid warm start approach in the next section.

2) MIP warm start

Most of the cases without batteries can solve to 3% gap very fast. In this section, we first solve the model without battery with 500 seconds and 3% MIP gap tolerance. We then use the generator commitment decisions as the MIP start for the model with battery (i.e., warm start).

For most cases, the commitment decisions from the non-battery model result in a very good initial solution for the battery model as shown in Table 8 column “Battery model MIP start initial gap”. Starting from that, most cases can reach below 1% gap in 1800s with the UCED_BIN, UCED_BIN_T and UCED_BIN_Lazy3 models.

Compared to UCED_BIN, UCED_BIN_T solutions have lower objectives for all cases. The solving time for case 2 increases by about 400s with UCED_BIN_T and the time for all other cases are similar between UCED_BIN_T and UCED_BIN.

UCED_BIN_T_Lazy3 is significantly better than UCED_BIN_T for case 1, 2 and 9 and worse than UCED_BIN_T and UCED_BIN for case 6.

In [8], the concurrent SCUC solver is developed to solve multiple solution methods in parallel under HIPPO. The best upper bound and lower bound solution is sent to HIPPO master and the solution process stops when it meets the stopping criteria. The enhanced battery formulation, warm start, and lazy constraint settings can be applied with various strategies to achieve the best outcome using such a parallel approach.

Fig. 3 shows production cost reduction from each of the 19 MISO day ahead cases after adding battery storage to the wind farms. Each case has 36 hourly intervals. This represents the system benefit from RTO/ISO optimizing the batteries globally. For the 2014 hard cases, the average daily benefit (prorated to 24 hours) is about $290,000 (about 4% of the average daily objective value). For 2019 normal cases, the average daily benefit is about $147,000 (about 3% of the average daily objective value). The overall benefit to the system could be significant. Batteries combined with renewables may also provide additional benefit in real time in firming capacity and providing flexibility.

V. Conclusion

In this paper, we analyzed the conventional and the tighter SOC formulations with and without binary variables for battery storage. Conditions are derived for mutual exclusivity of charging and discharging when binary variables are relaxed or omitted. Case studies validated the conditions. A MISO case study of both hard and normal day ahead market clearing cases show significant computational challenges introduced from SOC constraints and show the benefit from adding the tightened SOC formulation with lazy constraints or callback. The study also revealed that binary variables on battery models may not add more challenges, while warm start from the solution without batteries may significantly improve the solving time.

Acknowledgements

The authors would like to thank Jessica Harrison for the support and review of the paper.

Table 6 MIP gap and solving time comparison – hard cases

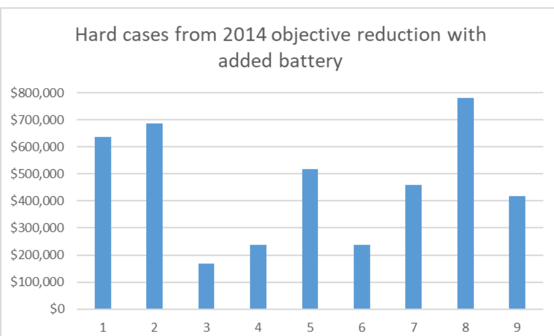
	No battery		Battery - UCED		Battery - UCED_BIN		Battery - UCED_T		Battery - UCED_BIN_T	
	MIP Gap	time (s)	MIP Gap	time (s)	MIP Gap	time (s)	MIP Gap	time (s)	MIP Gap	time (s)
case 1	0.16%	1207	0.09%	2067	0.16%	3001	0.10%	2040	0.10%	2474
case 2	0.39%	767	0.43%	1018	0.37%	1148	0.44%	1556	0.30%	3025
case 3	0.89%	1213	0.24%	2964	0.55%	3001	0.37%	3001	0.56%	3003
case 4	0.34%	1214	0.26%	3001	0.23%	3000	0.27%	3001	0.22%	3000
case 5	0.11%	809	0.10%	903	0.13%	1123	0.10%	1009	0.13%	1168
case 6	2.69%	1224	2.32%	3000	2.68%	3003	2.20%	3027	2.01%	3027
case 7	0.58%	1206	1.07%	3001	0.78%	3000	0.71%	3000	0.77%	3000
case 8	3.85%	234	4.10%	1238	4.62%	1147	3.15%	1228	6.71%	1375
case 9	0.90%	1211	3.56%	3001	0.32%	2972	1.22%	3008	1.01%	3001
Average		1010		2244		2377		2319		2564

Table 7 Comparison of MIP solving time – hard case cold start

Time Limit 3000s	No battery		UCED_BIN		UCED_BIN_T			UCED_BIN_T_Lazy3		
	MIP Gap	time (s)	MIP Gap	time (s)	MIP Gap	time (s)	Objective relative to UCED_BIN ("-" means reduction)	MIP Gap	time (s)	Objective relative to UCED_BIN ("-" means reduction)
case 1	0.16%	1207	0.16%	3000	0.10%	2595	-$15,337	0.10%	2179	-$19,664
case 2	0.39%	767	0.37%	1185	0.53%	3026	$11,093	0.44%	1089	$3,613
case 3	0.89%	1213	0.55%	3003	0.56%	3000	$763	0.51%	3000	-$3,475
case 4	0.34%	1214	0.23%	3000	0.22%	3000	-$1,262	0.28%	3000	$7,617
case 5	0.11%	809	0.13%	1199	0.13%	1146	-$391	0.15%	3023	-$3,478
case 6	2.69%	1224	2.68%	3003	2.01%	3023	-$4,425	2.48%	3003	-$2,046
case 7	0.58%	1206	0.78%	3000	0.77%	3000	$13,326	1.64%	3000	$113,668
case 8	3.85%	234	4.62%	1156	6.71%	1337	$7,646	3.66%	1137	-$1,143
case 9	0.90%	1211	0.32%	2912	1.01%	3001	$28,731	0.80%	3000	$20,849
Average		**1010**		**2384**		**2570**	**$4,460**		**2492**	**$12,882**

Table 8 Comparison of MIP solving time – hard case warm start

Non-battery 3% gap or 500s			UCED_BIN		UCED_BIN_T			UCED_BIN_T_Lazy3		
Time Limit 1800s MIP start from non-battery	time (s)	Battery model MIP start initial gap	MIP Gap	time (s)	MIP Gap	time (s)	Objective relative to UCED_BIN ("-" means reduction)	MIP Gap	time (s)	Objective relative to UCED_BIN ("-" means reduction)
case 1	127	0.59%	0.19%	1800	0.19%	1801	-$13	0.13%	1800	-$23,825
case 2	129	2.61%	0.44%	1347	0.41%	1763	-$1,214	0.36%	943	-$5,377
case 3	268	1.82%	1.70%	1800	1.68%	1800	-$1,185	1.71%	1800	-$1,185
case 4	258	0.51%	0.28%	1802	0.28%	1800	-$1,113	0.27%	1800	$159
case 5	122	0.99%	0.10%	929	0.13%	1080	-$54	0.14%	1529	-$2,817
case 6	506	6.77%	3.49%	1801	2.84%	1801	-$6,280	4.72%	1816	$16,855
case 7	265	2.06%	1.71%	1801	1.01%	1800	-$95,604	1.43%	1803	-$38,955
case 8	209	17.60%	6.76%	597	6.72%	655	-$489	6.35%	719	-$7
case 9	424	1.78%	0.80%	1800	0.83%	1801	-$707	0.56%	1800	-$12,696
Average	**257**			**1520**		**1589**	**-$11,851**		**1557**	**-$7,539**



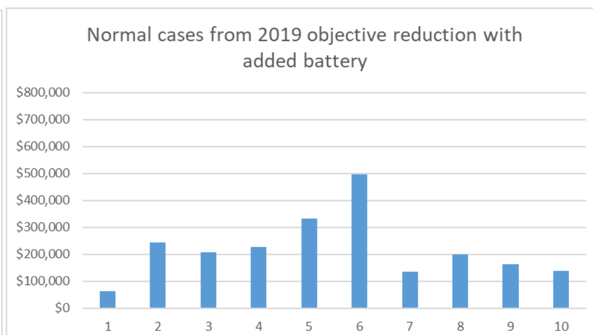


Fig. 3 Objective reduction with added batteries

REFERENCES

[1] FERC Order 841 "Electric Storage Participation in Markets Operated by Regional Transmission Organizations and Independent System Operators", February 15, 2018.

[2] Energy Storage Association Report, "Enabling Versatility: Allowing Hybrid Resources to Deliver Their Full Value to Customers", September 19, 2019.

[3] B. Huang, Y. Chen, R. Baldick, "A Configuration Based Pumped Storage Hydro Model in MISO Day-Ahead Market", working paper, available online: https://arxiv.org/abs/2009.04944.

[4] R. Baldick, Y. Chen, B. Huang, "Optimization formulations for storage devices", working paper, March 2021. Available online http://www.optimization-online.org/DB_FILE/2021/07/8502.pdf.

[5] Z. Li, Q. Guo, H. Sun and J. Wang, "Sufficient Conditions for Exact Relaxation of Complementarity Constraints for Storage-Concerned Economic Dispatch," in *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1653-1654, March 2016

[6] Z. Li, Q. Guo, H. Sun and J. Wang, "Further Discussions on Sufficient Conditions for Exact Relaxation of Complementarity Constraints for Storage-Concerned Economic Dispatch", available online: https://arxiv.org/abs/1505.02493.

[7] Y. Chen, A. Casto, F. Wang, Q. Wang, X. Wang, and J. Wan, "Improving large scale day-ahead security constrained unit commitment performance," *IEEE Trans. Power Syst*., vol. 31, no. 6, pp. 4732-4743, Nov. 2016.

[8] Y. Chen, F. Pan, J. Holzer, E. Rothberg, Y. Ma, and A. Veeramany. A high performance computing based market economics driven neighborhood search and polishing algorithm for security constrained unit commitment. IEEE Transactions on Power Systems, vol. 36, no. 1, pp. 292-302, Jan. 2021.

[9] https://www.gurobi.com/documentation/9.1/refman/lazy.html

[10] A. Giacomoni, Q. Gu, B. Gisin, Optimizing hydroelectric pumped storage in PJM's day-ahead energy market, FERC Technical Conference, June, 2020. Available online: https://www.ferc.gov/sites/default/files/2020-06/T2-3_Giacomoni_et_al.pdf.

[11] Slobodan Ruzic, Nikola Rajakovic, and Aca Vuckovic. A flexible approach to short-term hydro-thermal coordination, part i: Problem formulation and general solution procedure. IEEE Transactions on Power Systems, 11(3):1564-1571, August 1996.

[12] Slobodan Ruzic, Aca Vuckovic, and Nikola Rajakovic. A flexible approach to short-term hydro-thermal coordination, part ii: Dual problem solution procedure. IEEE Transactions on Power Systems, 11(3):1572-1578, August 1996.

[13] E. D. Castronuovo and J. A. P. Lopes, "Optimal operation and hydro storage sizing of a wind–hydro power plant," International Journal of Electrical Power & Energy Systems, vol. 26, no. 10, pp. 771–778, 2004.

[14] P. D. Brown, J. P. Lopes, and M. A. Matos, "Optimization of pumped storage capacity in an isolated power system with large renewable penetration," IEEE Transactions on Power systems, vol. 23, no. 2, pp. 523–531, 2008.