

EXACT CONVERGENCE RATES OF ALTERNATING PROJECTIONS FOR NONTRANSVERSAL INTERSECTIONS

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ABSTRACT. We study the exact convergence rate of the alternating projection method for the nontransversal intersection of a semialgebraic set and a linear subspace. If the linear subspace is a line, the exact rates are expressed by multiplicities of the defining polynomials of the semialgebraic set, or related power series. Our methods are also applied to give upper bounds for the case that the linear subspace has the dimension more than one. The upper bounds are shown to be tight by obtaining exact convergence rates for a specific semialgebraic set, which depend on the initial points.

1. INTRODUCTION

The alternating projection method is an algorithm for finding a point in the intersection of two sets, by iteratively projecting points to each of the two sets. The method has a variety of applications, such as image recovery [4], [1], phase retrieval [2], control theory [8] and factorization of completely positive matrices [6]. In general, if two sets are semialgebraic, [3] showed that the sequence constructed by the alternating projections converges to a point in the intersection without any regularity conditions. If two closed convex sets intersect transversely, then the convergence rate is linear [7], and the behavior of alternating projections is well-known; see, e.g. [12]. However, if the intersection is nontransversal, then the convergence rate is sub-linear, and the known upper bounds on the convergence rate are far from being tight as discussed in [3, Remark 4.5]. The convergence rate of alternating projections for a nontransversal intersection is also studied in [5], using Hölder regularity.

In this paper, we consider the *exact convergence rate* of the sequence $\{u_k\}$ constructed by the alternating projection method;

$$(1) \quad u_{k+1} = P_B \circ P_A(u_k),$$

where P_A and P_B are the projections onto convex sets A and B in \mathbb{R}^n , respectively. To argue exact convergence rates, we restrict ourselves to the case where A is a semialgebraic convex set defined by one or two polynomials, B is a linear subspace, and the intersection $A \cap B$ is nontransversal and a singleton. Even in this setting, if B has the dimension more than one, then the exact convergence rate depends on the initial point, and it seems to be hard to obtain for a general case as discussed in Section 5.2. However, when B is a line, we can directly analyze the recursive equations defining the sequence $\{u_k\}$ in (1), and obtain the exact convergence rate. Our results also improve corresponding estimates of the upper bounds on the convergence rate in [3] to our setting while showing the obtained estimates are tight.

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When the semialgebraic set A is defined by a single polynomial inequality, the recursive equations are analyzed rigorously by using ideals of the ring of convergent power series. Then we show that the exact rate is determined by the multiplicity of the defining polynomial of A at the intersecting point (Theorem 4.4). When A is defined by two polynomial inequalities and is in the three-dimensional space, the boundary of A is partitioned into three regions; two surfaces defined by each polynomial and a curve defined by the two polynomials. Then we obtain the exact rate by using a number that can be seen as a multiplicity of the curve defined by the two defining polynomials of A at the intersecting point if a point b on the line B is projected to the curve (Theorem 4.7). We also give sufficient conditions that the projection $P_A(b)$ is on the curve for each b on B sufficiently close to the intersecting point (Theorem 4.9). Moreover, we show that the tangent plane to A at the common point of A and B is explicitly partitioned into three regions; each of the two regions is projected to a surface defined by a single polynomial, and the other region is projected to the curve defined by the two polynomials (Theorem 4.11). Then the partitions are used to obtain the exact rates (Example 4.12, 4.13).

The arguments on the exact rates are then applied to obtain upper bounds of the rate for the case that A is defined by a single polynomial inequality and B is a linear subspace with the dimension more than one. For general cases, we use the Lojasiewicz exponent of the defining polynomial of A (Theorem 5.3), or that of the restriction of the polynomial to the linear subspace (Theorem 5.9), and give upper bounds. Furthermore, for a specific polynomial, we obtain the exact rate which depends on the initial point of the alternating projection method (Proposition 5.5). This specific case also shows that our upper bounds are tight.

The organization of the paper is the following. The basic notation and brief explanations on a projection onto a convex set and on the analytic implicit function theorem are given in Section 2. In Section 3, we obtain the convergence rate of the sequence defined by a special kind of a recursive equation, or inequality. In Section 4, we obtain exact convergence rates for intersections of semialgebraic sets and lines. Lastly, we give upper bounds for intersections of semialgebraic sets and subspaces with dimensions more than one in Section 5.

2. PRELIMINARIES

2.1. Notation and Definitions. Let $\|\cdot\|$ be the Euclidean norm on \mathbb{R}^n , $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ for $x, y \in \mathbb{R}^n$, and $[n] = \{1, \dots, n\}$. Let ∂A denote the boundary of a set $A \subset \mathbb{R}^n$. The distance $d(x, A)$ from a point $x \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ is defined by $d(x, A) = \inf_{a \in A} \|x - a\|$.

Let $\mathbb{R}[x]$ and $\mathbb{R}\{x\}$ be the set of polynomials and the set of convergent power series in the variables $x = (x_1, \dots, x_n)$ with coefficients in \mathbb{R} , respectively. For $f_1, \dots, f_m \in \mathbb{R}\{x\}$, the ideal generated by f_1, \dots, f_m is denoted by $\langle f_1, \dots, f_m \rangle$; i.e. $\langle f_1, \dots, f_m \rangle = \{\sum_{i=1}^m h_i f_i : h_i \in \mathbb{R}\{x\}\}$. For $f \in \mathbb{R}\{x\}$, the set of all the exponents of the monomials appearing in f is called the *support* of f and denoted by $\text{supp } f$. The convex hull of the union $\bigcup_{\kappa \in \text{supp}(f)} (\kappa + \mathbb{R}_{\geq 0}^n)$, which is denoted by $\Gamma_+(f)$, is called the *Newton diagram* of f , and $\Gamma(f) = \bigcup(\text{compact face of } \Gamma_+(f))$ is called the *Newton boundary* of f . A polynomial f is said to be *nondegenerate* if for each face

$\Delta \subset \Gamma(f)$,

$$\frac{\partial f_\Delta}{\partial x_1} = \dots = \frac{\partial f_\Delta}{\partial x_n} = 0$$

has no solution in $(\mathbb{R} \setminus \{0\})^n$.

For $f, g : \mathbb{R} \rightarrow \mathbb{R}$, we write $f(x) = O(g(x))$ as $x \rightarrow \infty$ if there exist $C, M > 0$ such that $|f(x)| \leq Cg(x)$ for all x with $|x| > M$. We also write $f(x) = \Theta(g(x))$ as $x \rightarrow \infty$ if there exist $C_1, C_2 > 0$ such that $C_1g(x) \leq f(x) \leq C_2g(x)$ for all x with $|x| > M$. The meaning of the statement $f(x) = O(g(x))$ as $x \rightarrow 0$ is defined similarly. If there is no ambiguity, we simply write $f(x) = O(g(x))$, or $f(x) = \Theta(g(x))$. For a sequence $\{u_k\} \subset \mathbb{R}^n$ with $\bar{u} = \lim_{k \rightarrow \infty} u_k$, we say that $\{u_k\}$ converges in the rate $O(g(k))$ if $\|u_k - \bar{u}\| = O(g(k))$ as $k \rightarrow \infty$, and in the *exact rate* $\Theta(g(k))$ if $\|u_k - \bar{u}\| = \Theta(g(k))$ as $k \rightarrow \infty$.

2.2. The projection and the implicit function theorem. We briefly review the projection and the implicit function theorem. Let $B = \{x \in \mathbb{R}^n : f_i(x) \geq 0, i \in [m]\}$ for $f_i \in \mathbb{R}[x]$ for $i \in [m]$. For $x \in B$, an index i is said to be *active* at x if $f_i(x) = 0$. We say that B is *smooth* if $\{\nabla f_i(x) : i \text{ is active at } x\}$ is linearly independent for all $x \in B$. In this paper, we define smoothness of B for the particular defining polynomials of B . For a closed convex set $A \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$, it is known that there exists a unique optimal solution to

$$(2) \quad \text{minimize} \{ \|x - p\|^2 : x \in A \}.$$

The optimal solution is called the *projection* of p onto A and denoted by $P_A(p)$.

Lemma 2.1. *Let $f_i \in \mathbb{R}[x]$, $i \in [m]$ and $A = \{x \in \mathbb{R}^n : f_i(x) \geq 0, i \in [m]\}$. Suppose that A is convex and $\{\nabla f_i(x) : i \text{ is active at } x\}$ is linearly independent for all $x \in B$. Then $u = P_A(p)$ if and only if there exist $c_i \in \mathbb{R}$ such that*

$$(3) \quad u - p = \sum_{i=1}^m c_i \nabla f_i(u), \quad f_i(u) \geq 0, \quad c_i \geq 0, \quad c_i f_i(u) = 0, \quad i \in [m].$$

Proof. Since the objective function of (2) and A are convex, we see that u is optimal if and only if $-(u - p) \in N_A(u)$, where $N_A(u)$ is the normal cone of A at u ; i.e. $N_A(u) = \{y \in \mathbb{R}^n : \langle y, u' - u \rangle \leq 0, u' \in A\}$; see, e.g. [13]. Then linear independence of $\{\nabla f_i(x) : i \text{ is active at } x\}$ ensures that the equality

$$N_A(u) = \left\{ \sum_{i=1}^m c_i \nabla f_i(u) : c_i \leq 0, \quad c_i f_i(u) = 0, \quad i \in [m] \right\}.$$

holds [13, Theorem 6.14]. □

Since we consider polynomial systems, the analytic implicit function theorem ensures that the solution functions are convergent power series; see, e.g. Theorem 6.1.2 and the following paragraph of [10].

Theorem 2.2 (The implicit function theorem). *Let $m \leq n$ and $f_1, \dots, f_m \in \mathbb{R}[x, y] := \mathbb{R}[x_1, \dots, x_m, y_1, \dots, y_{n-m}]$. Consider the system of equations*

$$f_1(x, y) = \dots = f_m(x, y) = 0.$$

For a solution (\bar{x}, \bar{y}) to the system above, if $\{\nabla f_{i,x}(\bar{x}, \bar{y}) : i \in [m]\}$ is linearly independent, then there exists a unique map $\varphi(y) = (\varphi_1(y), \dots, \varphi_m(y))$, where each

$\varphi_i(y)$ is a convergent power series around \bar{y} such that $\bar{x} = \varphi(\bar{y})$ and $f_i(\varphi(y), y) = 0$ for $i \in [m]$ and y close to \bar{y} .

Remark 2.3. Let $n \leq m$ and $A = \{x \in R^n : f_i(x) = 0, i \in [m]\}$. Suppose that the Jacobian matrix $\frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n)}(\bar{x})$ has full rank at $\bar{x} \in A$. Then Theorem 2.2 implies that for a tangent vector v to A at \bar{x} , there exists a convergent power series $\varphi(s)$ around \bar{s} such that $\varphi(\bar{s}) = \bar{x}$, $\varphi'(\bar{s}) = v$ and $\varphi(s) \in A$ for s close to \bar{s} ; see, e.g. [13, Exercise 6.7].

3. RECURSIVE EQUATION AND INEQUALITY

The following lemma and corollary give the convergence rate of the sequence defined by recursive equation and inequality. They are fundamental tools for our arguments and will be used repeatedly in the paper.

Lemma 3.1. *Suppose that the sequence $\{x_k\}$ satisfies $x_k > 0$, $x_k \rightarrow 0$, and*

$$x_{k+1} (1 + Cx_{k+1}^q + x_{k+1}^{q+1}h(x_{k+1})) = x_k \quad (k = 0, 1, \dots),$$

for some $C > 0$, $q \in \mathbb{N}$ and a convergent power series $h(x)$. Then

$$\lim_{k \rightarrow \infty} (qC)^{\frac{1}{q}} k^{\frac{1}{q}} x_k = 1.$$

Proof. First, we show that

$$g(x) = \frac{(1 + Cx^q + x^{q+1}h(x))^q - 1}{qCx^q}$$

is a convergent power series around $x = 0$, and $g(0) = 1$. In fact, we have

$$\begin{aligned} qCx^q g(x) &= \sum_{i=1}^q \binom{q}{i} (1 + Cx^q)^{q-i} (x^{q+1}h(x))^i + (1 + Cx^q)^q - 1 \\ &= x^{q+1}h(x) \sum_{i=1}^q \binom{q}{i} (1 + Cx^q)^{q-i} (x^{q+1}h(x))^{i-1} + \sum_{i=1}^q \binom{q}{i} (Cx^q)^i, \\ g(x) &= \frac{xh(x)}{qC} \sum_{i=1}^q \binom{q}{i} (1 + Cx^q)^{q-i} (x^{q+1}h(x))^{i-1} + \frac{1}{q} \sum_{i=2}^q \binom{q}{i} (Cx^q)^{i-1} + 1. \end{aligned}$$

Now we see that

$$\begin{aligned} qCx_k^q &= qCx_{k+1}^q (1 + Cx_{k+1}^q + x_{k+1}^{q+1}h(x_{k+1}))^q \\ &= qCx_{k+1}^q (1 + qCx_{k+1}^q g(x_{k+1})), \\ \frac{1}{qCx_{k+1}^q} - \frac{1}{qCx_k^q} &= \frac{1}{qCx_{k+1}^q} - \frac{1}{qCx_{k+1}^q (1 + qCx_{k+1}^q g(x_{k+1}))} \\ &= \frac{g(x_{k+1})}{1 + qCx_{k+1}^q g(x_{k+1})}. \end{aligned}$$

By summing the equation, we obtain

$$\frac{1}{qCx_k^q} - \frac{1}{qCx_0^q} = \sum_{i=1}^k \frac{g(x_i)}{1 + qCx_i^q g(x_i)},$$

and hence

$$\lim_{k \rightarrow \infty} \frac{1}{kqCx_k^q} = \lim_{k \rightarrow \infty} \frac{1}{kqCx_0^q} + \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \frac{g(x_i)}{1 + qCx_i^q g(x_i)} = 1,$$

since the last summation is a Cesàro mean and $x_k \rightarrow 0$. \square

Corollary 3.2. *Suppose that the sequence $\{x_k\}$ satisfies $x_k \geq 0$, $x_k \rightarrow 0$ and*

$$x_{k+1} (1 + Cx_{k+1}^q + x_{k+1}^{q+1}h(x_{k+1})) \leq x_k \quad (k = 0, 1, \dots),$$

for some $C > 0$, $q \in \mathbb{N}$ and a convergent power series $h(x)$. Then

$$\limsup_{k \rightarrow \infty} (qC)^{\frac{1}{q}} k^{\frac{1}{q}} x_k \leq 1.$$

Proof. Since $x_k \rightarrow 0$, there exists k_0 such that $1 + Cx_{k+1}^q + x_{k+1}^{q+1}h(x_{k+1}) > 0$ for $k \geq k_0$. If $x_k = 0$ for some $k > k_0$, we have $x_{k+1} = 0$. Then the desired inequality holds.

Thus, for each $k \geq k_0$, we assume $x_k > 0$. By using $g(x)$ in Lemma 3.1, we have

$$\begin{aligned} qCx_k^q &\geq qCx_{k+1}^q (1 + Cx_{k+1}^q g(x_{k+1})), \\ \frac{1}{qCx_{k+1}^q} - \frac{1}{qCx_k^q} &\geq \frac{1}{qCx_{k+1}^q} - \frac{1}{qCx_{k+1}^q (1 + qCx_{k+1}^q g(x_{k+1}))} \\ &= \frac{g(x_{k+1})}{1 + qCx_{k+1}^q g(x_{k+1})} \end{aligned}$$

Since the limit of a Cesàro mean is not affected by an absence of the finite number of terms, the similar arguments in Lemma 3.1 implies that $\liminf_{k \rightarrow \infty} \frac{1}{kqCx_k^q} \geq 1$, and

hence $\limsup_{k \rightarrow \infty} qCkx_k^q \leq 1$. Therefore, $\limsup_{k \rightarrow \infty} (qC)^{\frac{1}{q}} k^{\frac{1}{q}} x_k \leq 1$. \square

Remark 3.3. If a sequence $\{x_k\}$ satisfies $\lim_{k \rightarrow \infty} Ck^{\frac{1}{q}} x_k = 1$ for $C > 0$, then, for any $\varepsilon > 0$, we have $(1 - \varepsilon)C^{-1}k^{-\frac{1}{q}} \leq x_k \leq (1 + \varepsilon)C^{-1}k^{-\frac{1}{q}}$ for sufficiently large k . Thus $x_k = \Theta(k^{-\frac{1}{q}})$. Therefore, it is implied that the condition $\lim_{k \rightarrow \infty} Ck^{\frac{1}{q}} x_k = 1$ is a stronger property than the property $x_k = \Theta(k^{-\frac{1}{q}})$. Similarly, $\limsup_{k \rightarrow \infty} Ck^{\frac{1}{q}} x_k \leq 1$ implies $x_k = O(k^{-\frac{1}{q}})$.

4. INTERSECTIONS WITH LINES

We consider the intersection of a semialgebraic convex set A and a linear subspace B . Let P_A and P_B be projections to A and B respectively. We assume that the intersection is nontransversal and a singleton. By translation, we also assume $A \cap B = \{0\}$. The alternating projection method constructs a sequence $\{u_k\} \subset \mathbb{R}^{n+1}$ by (1). Note that u_k converges to 0; see, e.g. [3, Fact 2.14]. In this section, we investigate the exact convergence rate of the alternating projection method in the case that B is a line.

4.1. Hypersurfaces. When we consider the alternating projection method for semi-algebraic sets, only the boundaries have a crucial role. Thus, by an abuse of terminology, we call a semialgebraic set A a *hypersurface* if it is defined by a single polynomial. In this section, we consider the case that A is a hypersurface and B is a line. The following lemmas are stated in sufficient generality that they can be used in later sections.

Lemma 4.1. *For a nonnegative convex polynomial g , let $A = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z \geq g(x)\}$. For $(X, 0) \notin A$, we have $(x, z) = P_A(X, 0)$ if and only if the system*

$$x_i + g_{x_i}(x)g(x) = X_i, \quad i \in [n], \quad z = g(x)$$

holds.

Proof. Let $(X, 0) \notin A$. Then the projection of $(X, 0)$ onto A lies on the boundary of A by the definition. Since A is convex and $\nabla(z - g(x))$ is a nonzero vector for any (x, z) , Lemma 2.1 implies that $(x, z) = P_A(X, 0)$ if and only if

$$\begin{pmatrix} x \\ z \end{pmatrix} - \begin{pmatrix} X \\ 0 \end{pmatrix} = s \begin{pmatrix} -\nabla g(x) \\ 1 \end{pmatrix}, \quad z = g(x),$$

for some $s \geq 0$. Since $s = z = g(x)$, we obtain the desired system. \square

Lemma 4.2. *Let $I, \mathfrak{a}, \mathfrak{m}$ be ideals in $\mathbb{R}\{x_1, \dots, x_n\}$. If $I \subset \mathfrak{a} + \mathfrak{m}I$ and $\mathfrak{m}^s \subset \mathfrak{a}$ for some $s \in \mathbb{N}$, then $I \subset \mathfrak{a}$.*

Proof. We prove $I \subset \mathfrak{a} + \mathfrak{m}^k I$ for $k \in \mathbb{N}$ by induction. Suppose $I \subset \mathfrak{a} + \mathfrak{m}^k I$. Then we have

$$I \subset \mathfrak{a} + \mathfrak{m}^k(\mathfrak{a} + \mathfrak{m}I) = \mathfrak{a} + \mathfrak{m}^k \mathfrak{a} + \mathfrak{m}^{k+1}I = \mathfrak{a} + \mathfrak{m}^{k+1}I.$$

Thus the claim is proved. Since $\mathfrak{m}^s \subset \mathfrak{a}$, we obtain $I \subset \mathfrak{a}$. \square

Note that for a polynomial $g(x)$ with n variables, we say that $\Gamma(g)$ meets all the axes if g has a monomial $x_i^{d_i}$ with $d_i > 0$ for each $i = 1, \dots, n$.

Lemma 4.3. *For $1 \leq m \leq n - 1$, $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and a polynomial $g(x, y)$, we consider the system*

$$(*) \quad x_i + g_{x_i}(x, y)g(x, y) = 0, \quad i = 1, \dots, m.$$

Suppose that $g(0, 0) = g_{x_i}(0, 0) = 0$ and $\Gamma(g(0, y))$ meets all the axes. Define ideals $\mathfrak{m} = \langle y_1, \dots, y_{n-m} \rangle$, $\mathfrak{a} = \langle y^\alpha : \alpha \in \text{supp}(g(0, y)) \rangle$, and $I = \langle \varphi_1(y), \dots, \varphi_m(y) \rangle$ of $\mathbb{R}\{y\}$, where $\varphi_i(y)$ is the convergent power series which solves $()$ as $x_i = \varphi_i(y)$ and $\varphi_i(0) = 0$ for $i = 1, \dots, m$. Then we have $I \subset \mathfrak{m}\mathfrak{a}$.*

Proof. By the implicit function theorem (Theorem 2.2), there exist convergent power series $\varphi_i(y)$ which solve the equation $(*)$ as $x_i = \varphi_i(y)$ and $\varphi_i(0) = 0$ for $i = 1, \dots, m$. Let $\varphi(y) = (\varphi_1(y), \dots, \varphi_m(y))$. Then we have, for some polynomials p_j ,

$$\begin{aligned} \varphi_i(y) &= x_i = -g_{x_i}(\varphi(y), y)g(\varphi(y), y) \\ &= -g_{x_i}(\varphi(y), y) \left(g(0, y) + \sum_{j=1}^m \varphi_j(y)p_j(\varphi(y), y) \right) \\ &= -g_{x_i}(\varphi(y), y)g(0, y) - g_{x_i}(\varphi(y), y) \sum_{j=1}^m \varphi_j(y)p_j(\varphi(y), y) \end{aligned}$$

Since $g_{x_i}(0, 0) = 0$, we have $g_{x_i}(\varphi(y), y) \in \mathfrak{m}$. Thus the above equality implies that

$$I \subset \mathfrak{m}\mathfrak{a} + \mathfrak{m}I.$$

Since $\Gamma(g(0, y))$ meets all the axes, we have $\mathbf{m}^s \subset \mathbf{m}\mathbf{a}$ for $s = (n - m)(\deg g(0, y) + 1)$. By applying Lemma 4.2, we obtain $I \subset \mathbf{m}\mathbf{a}$. \square

For a convex polynomial g and $a \in \mathbb{R}^n \setminus \{0\}$, let

$$A = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z \geq g(x)\},$$

$$B = \{t(a, 0) \in \mathbb{R}^n \times \mathbb{R} : t \in \mathbb{R}\},$$

Suppose that $g(x) > 0$ for $x \neq 0$ and $g(0) = 0$.

Theorem 4.4. *Suppose that $g(\|a\|^{-1}at) = c_0t^d + O(t^{d+1})$ for some $c_0, d > 0$. Then the sequence $\{u_k\}$ constructed by the alternating projection method (1) converges to $A \cap B$ with the exact rate $\Theta(k^{\frac{1}{2d-2}})$. More precisely, we have*

$$(4) \quad \lim_{k \rightarrow \infty} ((2d - 2)dc_0^2)^{\frac{1}{2d-2}} k^{\frac{1}{2d-2}} \|u_k\| = 1.$$

Proof. By rotating about z axis, we may assume that $B = \{t(e_n, 0) \in \mathbb{R}^n \times \mathbb{R} : t \in \mathbb{R}\}$, where $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$. Then, we have

$$g(0, x_n) = c_0x_n^d + O(x_n^{d+1}).$$

Since $g(x) > 0$ for $x \neq 0$, we see $d \geq 2$.

For a point $u_0 = (te_n, 0)$ of B , let $u = (x, z) = P_A(u_0)$ and $u_1 = (\tilde{t}e_n, 0) = P_B(u)$. Then Lemma 4.1 implies that

$$(5) \quad x_i + g_{x_i}(x)g(x) = 0, \quad i \in [n - 1],$$

$$(6) \quad x_n + g_{x_n}(x)g(x) = t,$$

$$\tilde{t} = x_n.$$

By the implicit function theorem, there exist convergent power series $\varphi_i(x_n)$ which solve equations (5) as $x_i = \varphi_i(x_n)$ with $\varphi_i(0) = 0$ for $i = 1, \dots, n - 1$. Here we note that $g(0) = 0$, $g_{x_i}(0) = 0$ and $g(0, x_n) = c_0x_n^d + O(x_n^{d+1})$ with $c_0 \neq 0$. Then we can apply Lemma 4.3 to the equation (5), where $r = 1$, $\mathbf{m} = \langle x_n \rangle$, $\mathbf{a} = \langle x_n^d \rangle$, and $I = \langle \varphi_1(x_n), \dots, \varphi_{n-1}(x_n) \rangle$. Thus we obtain $\varphi_i(x_n) \in \langle x_n^{d+1} \rangle$, which means $\varphi_i(x_n) = O(x_n^{d+1})$ for $i = 1, \dots, n - 1$.

Next, we will modify the equation (6) to obtain a relation between $\|u_0\|$ and $\|u_1\|$. Let $\varphi(x_n) = (\varphi_1(x_n), \dots, \varphi_{n-1}(x_n))$. Now we have, for some polynomials p_i, r_i ,

$$g(\varphi(x_n), x_n) = g(0, x_n) + \sum_{i=1}^{n-1} \varphi_i(x_n)p_i(\varphi(x_n), x_n) = c_0x_n^d + O(x_n^{d+1}),$$

$$g_{x_n}(\varphi(x_n), x_n) = g_{x_n}(0, x_n) + \sum_{i=1}^{n-1} \varphi_i(x_n)r_i(\varphi(x_n), x_n) = dc_0x_n^{d-1} + O(x_n^d).$$

Then the equation (6) gives that

$$t = x_n + g_{x_n}(\varphi(x_n), x_n)g(\varphi(x_n), x_n) = x_n + dc_0^2x_n^{2d-1} + O(x_n^{2d}),$$

and hence

$$t = \tilde{t} + dc_0^2\tilde{t}^{2d-1} + O(\tilde{t}^{2d}).$$

Since $\|u_0\| = t$, $\|u_1\| = \tilde{t}$, Lemma 3.1 implies the equality (4). \square

Remark 4.5. In general, for a univariate convergent power series $f(x) = cx^d + O(x^{d+1})$ with $c \neq 0$, the lowest degree d of f is called the multiplicity of f at 0.

Example 4.6. Let $A = \{(x, y) \in \mathbb{R}^2 : z \geq g(x, y)\}$ for $g(x, y) = x^2 + y^4$ and $B = \{t(a, b) \in \mathbb{R}^2 : t \in \mathbb{R}\}$. Define $\{u_k\}$ by $u_{k+1} = P_B \circ P_A(u_k)$. Then we have

$$g(\|(a, b)\|^{-1}(a, b)t) = \frac{a^2}{a^2 + b^2}t^2 + \frac{b^4}{(a^2 + b^2)^2}t^4.$$

Let d be the lowest degree of the polynomial above and c_0 be its coefficient. If $a \neq 0$, then $d = 2$ and $c_0 = \frac{a^2}{a^2 + b^2}$. By Theorem 4.4, we have $\lim_{k \rightarrow \infty} \frac{2a^2}{a^2 + b^2} k^{\frac{1}{2}} \|u_k\| = 1$. On the other hand, if $a = 0$, then $d = 4$ and $c_0 = 1$. We have $\lim_{k \rightarrow \infty} (24)^{\frac{1}{6}} k^{\frac{1}{6}} \|u_k\| = 1$.

4.2. Sets Defined by Two Polynomials. We consider the case that $A, B \subset \mathbb{R}^3$ and that A is defined by two polynomials and B is a line. For convex polynomials f_1, f_2 and $(a, b) \neq (0, 0)$, let

$$\begin{aligned} A &= \{(x, y, z) \in \mathbb{R}^3 : z \geq f_1(x, y), z \geq f_2(x, y)\}, \\ B &= \{t(a, b, 0) \in \mathbb{R}^3 : t \in \mathbb{R}\}. \end{aligned}$$

Suppose that the intersection of A and xy -plane is $\{(0, 0, 0)\}$. We assume

$$C = \{(x, y, z) \in \mathbb{R}^3 : z = f_1(x, y) = f_2(x, y)\}$$

is smooth in the sense of Section 2.2. In this section, we first obtain the exact convergence rate under the assumption that all points on B which are sufficiently close to the origin are projected to C by P_A (Theorem 4.7). Then we discuss a sufficient condition that the assumption holds (Theorem 4.9).

Let $(\alpha_1, \alpha_2, \alpha_3)$ be a nonzero tangent vector to C at the origin, which is a generating vector of the kernel of the matrix $\begin{pmatrix} -f_{1,x}(0,0) & -f_{1,y}(0,0) & 1 \\ -f_{2,x}(0,0) & -f_{2,y}(0,0) & 1 \end{pmatrix}$. Since xy -plane is tangent to C there, we see that $\alpha_3 = 0$. By Remark 2.3, there exist convergent power series $\varphi_1(s), \varphi_2(s), \varphi_3(s)$ such that

$$\varphi(s) = \begin{pmatrix} \varphi_1(s) \\ \varphi_2(s) \\ \varphi_3(s) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} s + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} s^2 + O(s^3).$$

Theorem 4.7. Let $B = \{t(\alpha_1, \alpha_2, 0) \in \mathbb{R}^3 : t \in \mathbb{R}\}$. Suppose that $P_A(u_k) \in C$ for all sufficiently large k , and that d is the lowest degree of

$$(7) \quad \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}} ((\alpha_2 \varphi_1(s) - \alpha_1 \varphi_2(s))^2 + (\alpha_1^2 + \alpha_2^2) \varphi_3(s)^2)^{\frac{1}{2}} = c_0 s^d + O(s^{d+1}),$$

for some $c_0, d > 0$ as a power series in s . Then $\{u_k\}$ constructed by the alternating projection method (1) converges to 0 in the exact rate $\Theta\left(k^{\frac{-1}{2d-2}}\right)$. Moreover,

$$(8) \quad \lim_{k \rightarrow \infty} \left(\frac{(2d-2)dc_0^2}{(\alpha_1^2 + \alpha_2^2)^d} \right)^{\frac{1}{2d-2}} k^{\frac{1}{2d-2}} \|u_k\| = 1.$$

Proof. First, we calculate $d(\varphi(s), B)$. Let $t(\alpha_1, \alpha_2, 0) = P_B(\varphi(s))$. By the property of the projection, we have

$$(9) \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \cdot \varphi(x) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \cdot t \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix},$$

$$t = \frac{1}{\alpha_1^2 + \alpha_2^2} (\alpha_1 \varphi_1(s) + \alpha_2 \varphi_2(s)).$$

Thus we obtain

$$\begin{aligned} d(\varphi(s), B)^2 &= \|(\varphi_1(s), \varphi_2(s), \varphi_3(s)) - t(\alpha_1, \alpha_2, 0)\|^2 \\ &= \frac{1}{\alpha_1^2 + \alpha_2^2} ((\alpha_2 \varphi_1(s) - \alpha_1 \varphi_2(s))^2 + (\alpha_1^2 + \alpha_2^2) \varphi_3(s)^2). \end{aligned}$$

By the equation (7), we see $d(\varphi(s), B) = c_0 s^d + O(s^{d+1})$.

Next, we rotate the curve C about z -axis and reparametrize its parameter by a nonzero scalar multiple so that $(\alpha_1, \alpha_2, 0) = (1, 0, 0)$. Then the curve C is represented by $\psi(x) = (x, \psi_2(x), \psi_3(x))$ for some convergent power series ψ_2, ψ_3 . Suppose that $Q \xrightarrow{P_A} R := \psi(x) \xrightarrow{P_B} S$ is written as

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{P_A} \begin{pmatrix} x \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \xrightarrow{P_B} t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

for some $x, t, T \in \mathbb{R}$. By applying equation (9), we have $x = t$. Since R is the projection of Q onto C , we see that \overrightarrow{RQ} is orthogonal to C . Thus we have

$$(10) \quad \begin{aligned} &\left(\begin{pmatrix} x \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} - T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ \psi_2'(x) \\ \psi_3'(x) \end{pmatrix} = 0, \\ &t + \psi_2(t)\psi_2'(t) + \psi_3(t)\psi_3'(t) = T. \end{aligned}$$

Here, let $h(x) = \sqrt{\psi_2(x)^2 + \psi_3(x)^2}$. Since $P_B(\psi(x)) = (x, 0, 0)$, we have $d(\psi(x), B) = d(\psi(x), (x, 0, 0)) = h(x)$. By comparing the speed of the parametric curve $\varphi(s)$ with that of $\psi(x)$, we see that $d(\varphi(s), B) = c s^d + O(s^{d+1})$, where $c = \frac{c_0}{(\alpha_1^2 + \alpha_2^2)^{\frac{d}{2}}}$. Now we have

$$\frac{d}{dx} \left(\frac{1}{2} h(x)^2 \right) = \psi_2(x)\psi_2'(x) + \psi_3(x)\psi_3'(x).$$

Thus $\psi_2(t)\psi_2'(t) + \psi_3(t)\psi_3'(t) = dc^2 t^{2d-1} + O(t^{2d})$. Let u_k and u_{k+1} be the coordinate vectors of Q and S , respectively. Then the equation (10) can be written as

$$\|u_{k+1}\| + dc^2 \|u_{k+1}\|^{2d-1} + O(\|u_{k+1}\|^{2d}) = \|u_k\|.$$

Therefore, Lemma 3.1 implies the equality (8). \square

Example 4.8. Let $A = \{(x, y, z) \in \mathbb{R}^3 : z \geq f_1(x, y), z \geq f_2(x, y)\}$, where

$$\begin{cases} f_1(x, y) = x^2 + y^4, \\ f_2(x, y) = (x-1)^2 + (y-1)^4 - 2. \end{cases}$$

The tangent line to the curve C at the origin is given by $B = \{t(-2, 1, 0) \in \mathbb{R}^3 : t \in \mathbb{R}\}$. The curve C is written as

$$\varphi(y) = \begin{pmatrix} -2y + 3y^2 - 2y^3 \\ y \\ 4y^2 - 12y^3 + 18y^4 - 12y^5 + 4y^6 \end{pmatrix}$$

By Theorem 4.9 below, we can easily check that $P_A(t(-2, 1, 0)) \in C$ for all sufficiently small $t > 0$; see Example 4.10. Now the equation (7) is $\sqrt{\frac{89}{5}}y^2 + O(y^3)$. Thus, for $u_{k+1} = P_B \circ P_A(u_k)$, Theorem 4.7 implies

$$\lim_{k \rightarrow \infty} \sqrt{\frac{356}{125}} k^{\frac{1}{2}} \|u_k\| = 1,$$

and hence $\|u_k\| = \Theta(k^{-\frac{1}{2}})$.

Theorem 4.9. *Suppose that $(a, b, 0)$, $\nabla(z - f_1)(0, 0, 0)$, $\nabla(z - f_2)(0, 0, 0)$ are linearly independent.*

(i) *If $(a, b, 0) = c(\alpha_1, \alpha_2, 0)$ for some $c \neq 0$ and the solution $(\lambda_1, \lambda_2, \mu)$ to the system*

$$(11) \quad \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} -f_{1x}(0, 0) \\ -f_{1y}(0, 0) \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -f_{2x}(0, 0) \\ -f_{2y}(0, 0) \\ 1 \end{pmatrix} + \mu \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

satisfies $\lambda_1, \lambda_2 > 0$, then $P_A(p) \in C$ for each point p in B close to the origin.

(ii) *If $(a, b, 0) \neq c(\alpha_1, \alpha_2, 0)$ for any $c \neq 0$, or the solution $(\lambda_1, \lambda_2, \mu)$ to (11) satisfies $\lambda_1 \lambda_2 < 0$, then there exists $\varepsilon > 0$ such that $P_A(p) \notin C$ for any point p in $B \cap \varepsilon \mathbb{B}$.*

Proof. Suppose that $(a, b, 0) = c(\alpha_1, \alpha_2, 0)$ for some $c \neq 0$ and $(\beta_1, \beta_2, \beta_3)$ is written as (11). We will apply Lemma 2.1 to show that for each t sufficiently close to 0, there exist s such that $P_A(t(a, b, 0)) = (\varphi_1(s), \varphi_2(s), \varphi_3(s))$. The equation in the condition of Lemma 2.1 can be written as

$$(12) \quad \begin{aligned} c_1 \begin{pmatrix} -f_{1x} \\ -f_{1y} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -f_{2x} \\ -f_{2y} \\ 1 \end{pmatrix} &= \begin{pmatrix} \varphi_1(s) \\ \varphi_2(s) \\ \varphi_3(s) \end{pmatrix} - t \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \\ \begin{pmatrix} -f_{1x} & -f_{2x} & a \\ -f_{1y} & -f_{2y} & b \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ t \end{pmatrix} &= \begin{pmatrix} \varphi_1(s) \\ \varphi_2(s) \\ \varphi_3(s) \end{pmatrix}, \end{aligned}$$

where $f_{ix} = f_{ix}(\varphi(s))$, $f_{iy} = f_{iy}(\varphi(s))$. We put

$$M = \begin{pmatrix} -f_{1x} & -f_{2x} & a \\ -f_{1y} & -f_{2y} & b \\ 1 & 1 & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} -f_{1x}(0, 0) & -f_{2x}(0, 0) & a \\ -f_{1y}(0, 0) & -f_{2y}(0, 0) & b \\ 1 & 1 & 0 \end{pmatrix}.$$

Since the column vectors of M_0 are linearly independent, the linear equations (12) have a solution for s close to 0, and we have

$$\begin{aligned} t &= \frac{1}{|M|} \begin{vmatrix} -f_{1x} & -f_{2x} & \varphi_1(s) \\ -f_{1y} & -f_{2y} & \varphi_2(s) \\ 1 & 1 & \varphi_3(s) \end{vmatrix} \\ &= \frac{1}{|M_0|} \begin{vmatrix} -f_{1x}(0,0) & -f_{2x}(0,0) & \alpha_1 \\ -f_{1y}(0,0) & -f_{2y}(0,0) & \alpha_2 \\ 1 & 1 & 0 \end{vmatrix} s + O(s^2) = c^{-1}s + O(s^2), \\ c_1 &= \frac{1}{|M|} \begin{vmatrix} \varphi_1(s) & -f_{2x} & a \\ \varphi_2(s) & -f_{2y} & b \\ \varphi_3(s) & 1 & 0 \end{vmatrix} \\ &= \frac{1}{|M|} \begin{vmatrix} \alpha_1 & -f_{2x} & a \\ \alpha_2 & -f_{2y} & b \\ 0 & 1 & 0 \end{vmatrix} s + \frac{1}{|M|} \begin{vmatrix} \beta_1 & -f_{2x} & a \\ \beta_2 & -f_{2y} & b \\ \beta_3 & 1 & 0 \end{vmatrix} s^2 + O(s^3). \end{aligned}$$

Since the first term of c_1 is 0, we have

$$c_1 = \frac{1}{|M_0|} \begin{vmatrix} \beta_1 & -f_{2x}(0,0) & a \\ \beta_2 & -f_{2y}(0,0) & b \\ \beta_3 & 1 & 0 \end{vmatrix} s^2 + O(s^3).$$

By the condition (11), we have $c_1 = \lambda_1 s^2 + O(s^3)$. Similarly, we have $c_2 = \lambda_2 s^2 + O(s^3)$. Thus, if $\lambda_1, \lambda_2 > 0$, then for t sufficiently close to 0, there exists s such that $c_1, c_2 > 0$. Therefore, Lemma 2.1 ensures that $P_A(t(a, b, 0)) \in C$. If λ_1 and λ_2 have distinct signs, then so do c_1 and c_2 , and hence $P_A(t(a, b, 0)) \notin C$.

Lastly, we show the case that $(a, b, 0) \neq c(\alpha_1, \alpha_2, 0)$ for any $c \neq 0$. If we write

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} = d_1 \begin{pmatrix} -f_{1x}(0,0) \\ -f_{1y}(0,0) \\ 1 \end{pmatrix} + d_2 \begin{pmatrix} -f_{2x}(0,0) \\ -f_{2y}(0,0) \\ 1 \end{pmatrix} + \mu \begin{pmatrix} a \\ b \\ 0 \end{pmatrix},$$

then d_1 and d_2 have distinct signs. Now

$$c_1 = \frac{1}{|M_0|} \begin{vmatrix} \alpha_1 & -f_{2x}(0,0) & a \\ \alpha_2 & -f_{2y}(0,0) & b \\ 0 & 1 & 0 \end{vmatrix} s + O(s^2) = d_1 s + O(s^2).$$

Similarly we have $c_2 = d_2 s + O(s^2)$. Thus, for t sufficiently close to 0, we see that c_1 and c_2 have distinct signs. Therefore, Lemma 2.1 ensures the result. \square

Example 4.10. We consider $A = \{(x, y, z) \in \mathbb{R}^3 : z \geq f_1(x, y), z \geq f_2(x, y)\}$, $B = \text{Span}\{\nabla(z - f_1)(0, 0, 0), \nabla(z - f_2)(0, 0, 0)\}^\perp$ and $C = \{(x, y, z) \in \mathbb{R}^3 : z = f_1(x, y) = f_2(x, y)\}$.

(i) Let

$$\begin{cases} f_1(x, y) = x^2 + y^4, \\ f_2(x, y) = (x - 1)^2 + (y - 1)^4 - 2. \end{cases}$$

Then $\nabla(z - f_1)(0, 0, 0) = (0, 0, 1)$, $\nabla(z - f_2)(0, 0, 0) = (2, 4, 1)$, and $B = \{t(-2, 1, 0) : t \in \mathbb{R}\}$. The curve C is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y + 3y^2 - 2y^3 \\ y \\ 4y^2 - 12y^3 + 18y^4 - 12y^5 + 4y^6 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} y^2 + O(y^3).$$

Now we have

$$\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \frac{37}{10} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{3}{10} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} - \frac{6}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Then Theorem 4.9 guarantees that $P_A(t(-2, 1, 0)) \in C$ for all sufficiently small $t \geq 0$.

(ii) Let

$$\begin{cases} f_1(x, y) = x^2 + y^4, \\ f_2(x, y) = (x + \frac{1}{2})^2 + (y + \frac{1}{2})^4 - \frac{5}{16}. \end{cases}$$

Then $\nabla(z - f_1)(0, 0, 0) = (0, 0, 1)$, $\nabla(z - f_2)(0, 0, 0) = (-1, -1/2, 1)$, $B = \{t(1, -2, 0) : t \in \mathbb{R}\}$. The curve C is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}y - \frac{3}{2}y^2 - 2y^3 \\ y \\ \frac{1}{4}y^2 + \frac{3}{2} + \frac{21}{4}y^4 + 6y^5 + 4y^6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -\frac{3}{2} \\ 0 \\ \frac{1}{4} \end{pmatrix} y^2 + O(y^3).$$

Now we have

$$\begin{pmatrix} -\frac{3}{2} \\ 0 \\ \frac{1}{4} \end{pmatrix} = -\frac{19}{20} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{6}{5} \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} - \frac{3}{10} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

By Theorem 4.9, there exists $\varepsilon > 0$ such that $P_A(t(1, -2, 0)) \notin C$ for any $0 < t < \varepsilon$.

4.3. The Partition of the Region. The boundary of A consists of subsets of two surfaces $\bar{A}_1 = \{(x, y, z) \in \mathbb{R}^3 : z = f_1(x, y) > f_2(x, y)\}$, $\bar{A}_2 = \{(x, y, z) \in \mathbb{R}^3 : z = f_2(x, y) > f_1(x, y)\}$, and the curve $C = \{(x, y, z) \in \mathbb{R}^3 : z = f_1(x, y) = f_2(x, y)\}$. The xy -plane can be partitioned so that we can determine to which part of ∂A a point is mapped by P_A , as in Theorem 4.11.

We write $\{f_i - f_j * 0\} = \{(x, y) \in \mathbb{R}^2 : f_i(x, y) - f_j(x, y) * 0\}$ for $i, j \in \{1, 2\}$, where the symbol $*$ is $>$, \geq or $=$. Let $\Psi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$\Psi_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + f_{ix}(x, y)f_i(x, y) \\ y + f_{iy}(x, y)f_i(x, y) \end{pmatrix},$$

and $A_i = \{(x, y, z) \in \mathbb{R}^3 : z \geq f_i(x, y)\}$ for $i = 1, 2$.

Theorem 4.11. (i) $P_A \circ \Psi_1(\{f_1 - f_2 > 0\}) \subset \bar{A}_1$.

(ii) $P_A \circ \Psi_2(\{f_2 - f_1 > 0\}) \subset \bar{A}_2$.

(iii) $P_A(\Psi_1(\{f_2 - f_1 \geq 0\}) \cap \Psi_2(\{f_1 - f_2 \geq 0\})) \subset C$.

Proof. For $(X, Y) \in \mathbb{R}^2$, Lemma 4.1 implies that $(x, y, f_1(x, y)) = P_{A_1}(X, Y, 0)$ if and only if

$$x + f_{1x}(x, y)f_1(x, y) = X, \quad y + f_{1y}(x, y)f_1(x, y) = Y.$$

This means $(X, Y) = \Psi_1(x, y)$. Thus Ψ_1 is injective. Similarly, Ψ_2 is injective.

Since C is smooth, Lemma 2.1 gives that $(x, y, z) = P_A(X, Y, 0)$ if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$(13) \quad \begin{cases} - \begin{pmatrix} x - X \\ y - Y \\ z \end{pmatrix} = \lambda_1 \begin{pmatrix} -f_{1x}(x, y, z) \\ -f_{1y}(x, y, z) \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -f_{2x}(x, y, z) \\ -f_{2y}(x, y, z) \\ 1 \end{pmatrix}, \\ \lambda_1, \lambda_2 \leq 0, \quad \lambda_i(z - f_i(x, y)) = 0, \quad i = 1, 2. \end{cases}$$

If $(X, Y) \in \Psi_1(\{f_1 - f_2 > 0\})$, then there exists (x, y) such that $f_1(x, y) > f_2(x, y)$ and

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x + f_{1x}(x, y)f_1(x, y) \\ y + f_{1y}(x, y)f_1(x, y) \end{pmatrix}.$$

Since $f_1(x, y) > 0$, we have

$$- \begin{pmatrix} x - X \\ y - Y \\ f_1(x, y) \end{pmatrix} = \lambda_1 \begin{pmatrix} -f_{1x}(x, y, z) \\ -f_{1y}(x, y, z) \\ 1 \end{pmatrix},$$

where $\lambda_1 = -f_1(x, y)$. Thus $(x, y, f_1(x, y))$ satisfies (13) and hence $P_A(X, Y, 0) = (x, y, f_1(x, y)) \in \overline{A}_1$. Similarly, if $(X, Y) \in \Psi_2(\{f_2 - f_1 > 0\})$, then $P_A(X, Y, 0) \in \overline{A}_2$. Thus we have shown (i) and (ii).

Next, we will show (iii). Let $(X, Y) \in \Psi_1(\{f_2 - f_1 \geq 0\}) \cap \Psi_2(\{f_1 - f_2 \geq 0\})$ and $(x, y, z) = P_A(X, Y, 0)$. Then the system (13) is satisfied for some $\lambda_1, \lambda_2 \in \mathbb{R}$. If $(x, y, z) \in \overline{A}_1$, then we have $z = f_1(x, y) > f_2(x, y)$, and hence

$$- \begin{pmatrix} x - X \\ y - Y \\ z \end{pmatrix} = \lambda_1 \begin{pmatrix} -f_{1x}(x, y) \\ -f_{1y}(x, y) \\ 1 \end{pmatrix}.$$

Since $\lambda_1 = z = f_1(x, y)$, we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x + f_1(x, y)f_{1x}(x, y) \\ y + f_1(x, y)f_{1y}(x, y) \end{pmatrix}.$$

Thus $(X, Y) \in \Psi_1(\{f_1 - f_2 > 0\})$. Since Ψ_1 is injective, this contradicts to the inclusion $(X, Y) \in \Psi_1(\{f_2 - f_1 \geq 0\})$. Thus $(x, y, z) \in \partial A \setminus \overline{A}_1$. Similarly, we have $(x, y, z) \in \partial A \setminus \overline{A}_2$. \square

The software `Macaulay2`, for example, calculates the defining polynomial for the image of $\{f_i - f_j = 0\}$ under Ψ_i by the elimination theory as in Figure 1. In the following examples, let $A = \{(x, y) : z \geq f_1(x, y), z \geq f_2(x, y)\}$ and $C = \{(x, y, z) : z = f_1(x, y) = f_2(x, y)\}$.

Example 4.12. Let $f_1(x, y) = x^2 + y^4$ and $f_2(x, y) = (x - 1)^4 + (y - 1)^4 - 2$ as in Example 4.8. Let $B = \{(x, y, 0) : x + 2y = 0\}$; the tangent line to the curve C at $(0, 0, 0)$. In the left of Figure 1, the solid curve and the dotted curves are $\Psi_1(\{f_1 - f_2 = 0\})$ and $\Psi_2(\{f_2 - f_1 = 0\})$, respectively. The thin line is $\pi(B)$, where $\pi : (x, y, z) \mapsto (x, y)$. The line $\pi(B)$ passes through the origin while it keeps lying on the region $\Psi_1(\{f_2 - f_1 \geq 0\}) \cap \Psi_2(\{f_1 - f_2 \geq 0\})$. Thus all points on B are projected to C by P_A .

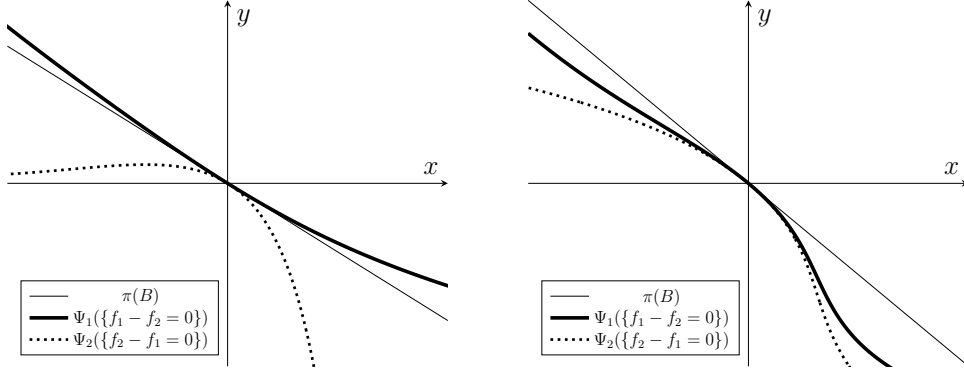


FIGURE 1. Example 4.12 (left) and Example 4.13 (right)

Next, we will show that the convergence rate may differ depending on the initial point. We consider the projection of a point on $B' = \{t(0, 1, 0) : t \in \mathbb{R}\}$ by P_A . Let $B'_+ = \{(0, y, 0) : y > 0\}$ and $B'_- = \{(0, y, 0) : y < 0\}$. For $(x, y) \in \Psi_1^{-1}(0, Y)$, we see that

$$(14) \quad x + 2x(x^2 + y^4) = 0, \quad y + 4y^3(x^2 + y^4) = Y.$$

Then we have $x = 0$. If $(0, Y, 0) \in B'_+$, then the second equation of (14) ensures $y > 0$. Since $f_1(0, y) - f_2(0, y) = y^4 - ((y-1)^4 - 1) = 2y(2(y - \frac{3}{4})^2 + \frac{1}{2}) > 0$, we have $(0, Y) \in \Psi_1(\{f_1 - f_2 > 0\})$. From (i) of Theorem 4.11, a point in B'_+ is projected to $\bar{A}_1 = \{(x, y, z) : z = f_1(x, y) > f_2(x, y)\}$ by P_A . If the initial point is taken from B'_+ , then the sequence $\{(x_k, y_k, 0)\}$ constructed by alternating projections between A and B' behaves like those between $A_1 = \{(x, y, z) : z \geq f_1(x, y)\}$ and B' . Since $f_1(t(0, 1)) = t^4$, Theorem 4.4 gives $\|(x_k, y_k, 0)\| = \Theta(k^{-\frac{1}{6}})$.

Similarly, if the initial point is taken from B'_- , then the sequence $\{(x_k, y_k, 0)\}$ constructed by alternating projections between A and B' behaves like those between $A_2 = \{(x, y, z) : z \geq f_2(x, y)\}$ and B' . Here, the lowest degree of $f_2(t(0, 1)) = -4t + 6t^2 - 4t^3 + t^4$ is equal to 1, and this means that B' intersects transversely with A_2 . Therefore, we see that $\{(x_k, y_k, 0)\}$ converges linearly from a well-known fact that the alternating projection method for a transversal intersection converges linearly; see, e.g. [7], [12].

Example 4.13. Let $f_1(x, y) = (x + \frac{1}{2})^2 + (y + \frac{1}{2})^4 - \frac{5}{16}$ and $f_2(x, y) = x^2 + y^4$ as in (ii) of Example 4.10. The tangent line to the curve C at $(0, 0, 0)$ is given by $B = \{t(1, -2, 0) : t \in \mathbb{R}\}$. Now, the right of Figure 1 corresponds to this case. We see that the thin line $\pi(B)$ is contained in $\Psi_1(\{f_1 - f_2 \geq 0\})$ around the origin. From (i) of Theorem 4.11, any point on $\pi(B)$ close to the origin is projected to $\bar{A}_1 = \{(x, y, z) : z = f_1(x, y) > f_2(x, y)\}$. A sequence $\{(x_k, y_k, 0)\}$ constructed by alternating projections between A and B behaves like those between $A_1 = \{(x, y, z) : z \geq f_1(x, y)\}$ and B . Since $f_1(t(1, -2)) = 7t^2 - 16t^3 + 16t^4$, Theorem 4.4 gives $\|(x_k, y_k, 0)\| = \Theta(k^{-\frac{1}{2}})$.

5. INTERSECTIONS WITH SUBSPACES

We return to the case that a semialgebraic set A is defined by a single polynomial. In Section 4.1, we have obtained the exact convergence rate of the alternating

projection method if the intersecting subspace has a dimension one. However, if the intersecting subspace has a dimension more than one, the convergence rate depends on the initial point. Section 5.2 is devoted to a specific hypersurface to explain this phenomenon. In the remaining sections, we give upper bounds on the convergence rates, by applying the arguments for the exact rates.

5.1. Hyperplanes. We consider an intersection of a hypersurface A defined by a convex polynomial g and a hyperplane B :

$$\begin{aligned} A &= \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z \geq g(x)\}, \\ B &= \{(x, 0) : x \in \mathbb{R}^n\}, \end{aligned}$$

where $A \cap B$ is a singleton. We note that $g(x) > 0$ ($x \neq 0$) and $g(0) = 0$.

It is known that for any convergent power series f with $f(x) > 0$ ($x \neq 0$), $f(0) = 0$, there are $C > 0$ and a rational number $\alpha \geq 1$ such that

$$f(x) \geq C\|x\|^\alpha \text{ for } x \text{ close to } 0,$$

see, e.g. [11, equality (G1)]. The smallest exponent α is called the *Łojasiewicz exponent* of f and denoted by $\mathcal{L}(f)$.

Example 5.1. Corollary 2.1 of [14] says that if f is convenient and nondegenerate, then $\mathcal{L}(f)$ is the maximum length from the origin to the intersection of $\Gamma(f)$ and each axis. For example, if $f(x_1, x_2, x_3) = x_1^6 + x_2^4 + x_3^2$, we have $\mathcal{L}(f) = 6$.

We consider the sequences $\{a_k\}$, $\{b_k\}$ constructed by alternating projections as

$$a_k \xrightarrow{P_B} b_k \xrightarrow{P_A} a_{k+1}.$$

The following lemma is implied by the inequality (4.3) of [3]. We give a proof for the reader's convenience.

Lemma 5.2. *Let A be a closed convex set, B be a linear subspace and $A \cap B = \{0\}$. Then we have*

$$\|a_{k+1}\|^2 + d(a_k, B)^2 \leq \|a_k\|^2.$$

Proof. If $b_k = P_B a_k = 0$, then $d(a_k, B) = \|a_k\|$ and $a_{k+1} = P_A b_k = 0$. Thus we obtain the desired inequality. So, we assume $b_k \neq 0$. Since B is a linear subspace, we have $\|a_k\|^2 = \|b_k\|^2 + \|b_k - a_k\|^2$. By the property of a projection, we see $b_k - a_{k+1} \in N_A(a_{k+1})$, which means $\langle b_k - a_{k+1}, a - a_{k+1} \rangle \leq 0$ for all $a \in A$. Since $0 \in A$, we obtain $0 \geq \langle b_k - a_{k+1}, -a_{k+1} \rangle = -\langle b_k, a_{k+1} \rangle + \|a_{k+1}\|^2 \geq -\|b_k\| \|a_{k+1}\| + \|a_{k+1}\|^2$. Thus we have $\|b_k\| \geq \|a_{k+1}\|$. Therefore $\|a_k\|^2 = \|b_k\|^2 + \|b_k - a_k\|^2 \geq \|a_{k+1}\|^2 + d(a_k, B)^2$. \square

Theorem 5.3. *Suppose $g(x)^2 \geq C(\sum_i x_i^2)^d$ for x close to 0. Then b_k converges to 0 in the rate $O\left(k^{\frac{-1}{2d-2}}\right)$. Moreover*

$$\limsup_{k \rightarrow \infty} ((d-1)C)^{\frac{1}{2d-2}} k^{\frac{1}{2d-2}} \|b_k\| \leq 1.$$

Proof. Let $a_k = (x_{1,k}, \dots, x_{n,k}, z_k)$ and $d_k := \|b_k\| = \|(x_{1,k}, \dots, x_{n,k})\|$. Since $d(a_k, B) = z_k$, Lemma 5.2 gives

$$\begin{aligned} \|(x_{1,k+1}, \dots, x_{n,k+1}, z_{k+1})\|^2 + z_k^2 &\leq \|(x_{1,k}, \dots, x_{n,k}, z_k)\|^2, \\ d_{k+1}^2 + z_{k+1}^2 &\leq d_k^2, \\ d_{k+1}^2 + g(x_{1,k+1}, \dots, x_{n,k+1})^2 &\leq d_k^2, \\ d_{k+1}^2 + C(d_{k+1}^2)^d &\leq d_k^2. \end{aligned}$$

Since $\mathcal{L}(g) \geq 1$, we have $2d = \mathcal{L}(g^2) \geq 2$. By Corollary 3.2, we have

$$\limsup_{k \rightarrow \infty} ((d-1)C)^{\frac{1}{d-1}} k^{\frac{1}{d-1}} d_k^2 \leq 1,$$

and thus $d_k = O\left(k^{\frac{-1}{2d-2}}\right)$. \square

Remark 5.4. Suppose that $A = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z \geq g(x)\}$, $B = \{(x, 0) \in \mathbb{R}^n \times \mathbb{R}\}$ and $(x_k, 0) \xrightarrow{P_A} (x, z) \xrightarrow{P_B} (x_{k+1}, 0)$. Then Lemma 4.1 implies that

$$\begin{aligned} x + g(x)\nabla g(x) &= x_k, \\ x_{k+1} &= x. \end{aligned}$$

Thus the sequence x_k is expected to follow the path defined by the gradient system

$$\frac{d}{dt}x(t) = -\nabla \frac{1}{2}g^2(x(t)).$$

The convergence rate of the gradient system is discussed in [9, Thm 1.6]. The exponent used in their result can be obtained with $\mathcal{L}(g^2)$ and is equal to the rate in this paper.

5.2. Exact Rates for a Specific Polynomial. We consider the specific polynomial

$$g(x, y) = x^2 + y^4.$$

Let $A = \{(x, y, z) \in \mathbb{R}^3 : z \geq g(x, y)\}$, $B = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$, and $b_k = \{(x_k, y_k, 0)\}$ be the sequence constructed by $b_{k+1} = P_B \circ P_A(b_k)$. Since $\mathcal{L}(g) = 4$, Theorem 5.3 shows $\limsup_{k \rightarrow \infty} Ck^{\frac{1}{6}} \|(x_k, y_k)\| \leq 1$, and thus the convergence rate has

the upper bound $O(k^{-\frac{1}{6}})$. On the other hand, the following proposition gives *exact* convergence rates, which depend on the initial points. Moreover, it shows that the exact rate achieves the upper bound for a generic initial point.

Proposition 5.5. *Let (x_0, y_0) be the initial point of a sequence constructed by the alternating projection method. If $y_0 \neq 0$, then (x_k, y_k) converges to 0 in the exact rate of $\Theta(k^{-\frac{1}{6}})$. If $y_0 = 0$, then (x_k, y_k) converges to 0 in the exact rate of $\Theta(k^{-\frac{1}{2}})$.*

The proof uses the following two technical lemmas. By Lemma 4.1, we have

$$(15) \quad \begin{cases} x_{k+1}(1 + 2(x_{k+1}^2 + y_{k+1}^4)) = x_k \\ y_{k+1}(1 + 4y_{k+1}^2(x_{k+1}^2 + y_{k+1}^4)) = y_k \end{cases}, \quad z_{k+1} = x_{k+1}^2 + y_{k+1}^4.$$

Lemma 5.6. *For sufficiently small $\varepsilon > 0$, if $0 < x_k < y_k^2 \leq \varepsilon$, then we have $0 < x_{k+1} < y_{k+1}^2 < \varepsilon$.*

Proof. Let $(X, Y) = (x_k, y_k)$, $(x, y) = (x_{k+1}, y_{k+1})$. By (15), we see that $x, y > 0$ and $x \leq X$, $y \leq Y$. Now we have for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} (1 + 4y^2(x^2 + y^4))^2 &\leq (1 + 4\varepsilon(x^2 + y^4))^2 \\ &= 1 + 8\varepsilon(x^2 + y^4) + 16\varepsilon^2(x^2 + y^4)^2 \\ &= 1 + 2(x^2 + y^4) + 16\varepsilon^2(x^2 + y^4)^2 + (8\varepsilon - 2)(x^2 + y^4) \\ &= 1 + 2(x^2 + y^4) + (x^2 + y^4)(32\varepsilon^4 + 8\varepsilon - 2) \\ &\leq 1 + 2(x^2 + y^4). \end{aligned}$$

Thus we obtain

$$1 < \frac{Y^2}{X} = \frac{y^2 (1 + 4y^2(x^2 + y^4))^2}{x (1 + 2(x^2 + y^4))} \leq \frac{y^2}{x}.$$

□

Lemma 5.7. *Suppose that (x_0, y_0) be a point which is sufficiently close to $(0, 0)$ and $x_0, y_0 > 0$. Then there exists k_0 such that $x_k < y_k^2$ for all $k > k_0$.*

Proof. By Lemma 5.6, it is enough to show there exists k_0 such that $x_{k_0} < y_{k_0}^2$. We show it by contradiction. Suppose that $x_k \geq y_k^2$ for all k . Since $x_k = x_{k+1}(1 + 2(x_{k+1}^2 + y_{k+1}^4))$, we have

$$x_{k+1}(1 + 2x_{k+1}^2) \leq x_k$$

By Corollary 3.2, the inequality implies that $\limsup_{k \rightarrow \infty} 2k^{\frac{1}{2}}x_k \leq 1$. Then $x_k^2 \leq \frac{C}{k}$ for $C > \frac{1}{4}$ and sufficiently large k .

Next, $y_k = y_{k+1}(1 + 4y_{k+1}^2(x_{k+1}^2 + y_{k+1}^4))$ gives that

$$y_{k+1}(1 + 4y_{k+1}^6) \leq y_k \leq y_{k+1}(1 + 8y_{k+1}^2x_{k+1}^2).$$

Since

$$(1 + 8y_{k+1}^2x_{k+1}^2)(1 - 8y_k^2x_{k+1}^2) = 1 - 8(y_k^2 - y_{k+1}^2)x_{k+1}^2 - 64y_{k+1}^2y_k^2x_{k+1}^4 \leq 1,$$

we obtain

$$\begin{aligned} y_{k+1} &\geq y_{k+1}(1 + 8y_{k+1}^2x_{k+1}^2)(1 - 8y_k^2x_{k+1}^2) \geq y_k(1 - 8y_k^2x_{k+1}^2), \\ y_{k+1}^2 &\geq y_k^2(1 - 8y_k^2x_{k+1}^2)^2 \geq y_k^2(1 - 16y_k^2x_{k+1}^2), \\ \frac{1}{y_{k+1}^2} &\leq \frac{1}{y_k^2(1 - 16y_k^2x_{k+1}^2)} \leq \frac{1}{y_k^2} + \frac{16x_{k+1}^2}{(1 - 16y_k^2x_{k+1}^2)}. \end{aligned}$$

By summing the last inequality, we have

$$\begin{aligned} \frac{1}{y_K^2} - \frac{1}{y_{K_0}^2} &\leq \sum_{k=K_0}^{K-1} \frac{16x_{k+1}^2}{(1 - 16y_k^2x_{k+1}^2)} \\ &\leq \sum_{k=K_0}^{K-1} \frac{16\frac{C}{k+1}}{1 - \frac{16C}{k+1}y_k^2} = \sum_{k=K_0}^{K-1} \frac{16C}{k+1 - 16Cy_k^2}. \end{aligned}$$

Since $y_k \rightarrow 0$, for sufficiently large K_0 , we see that $1 - 16C y_k^2 > 0$. Then we obtain

$$\begin{aligned} \frac{1}{y_K^2} - \frac{1}{y_{K_0}^2} &\leq \sum_{k=K_0}^{K-1} \frac{16C}{k+1-16C y_k^2} \leq \sum_{k=K_0}^{K-1} \frac{16C}{k}, \\ \frac{1}{y_K^2} &\leq 16C \log K + C_1, \\ y_K^2 \log K &\geq \frac{1}{16C + \frac{C_1}{\log K}} \geq C_2 \end{aligned}$$

for some positive constants C_1, C_2 . Thus, we have $x_k \leq \frac{C}{\sqrt{k}}$ and $y_k^2 \geq \frac{C_2}{\log k}$ for all sufficiently large k . This contradicts to the assumption that $x_k \geq y_k^2$ for all k . \square

Proof of Proposition 5.5. By symmetry, we may assume $x_0, y_0 \geq 0$. If $x_0, y_0 > 0$, then Lemma 5.7 implies that for any $\varepsilon > 0$, we have $0 < x_k < y_k^2 \leq \varepsilon$ for all sufficiently large k . Then

$$\|(x_k, y_k)\| \leq \sqrt{y_k^4 + y_k^2} = y_k \sqrt{1 + y_k^2}.$$

Since $y_k = y_{k+1}(1 + 4y_{k+1}^2(x_{k+1}^2 + y_{k+1}^4))$, we have

$$y_{k+1}(1 + 4y_{k+1}^6) \leq y_k \leq y_{k+1}(1 + 8y_{k+1}^6).$$

By Corollary 3.2, the first inequality implies $\limsup_{k \rightarrow \infty} 24^{\frac{1}{6}} k^{\frac{1}{6}} y_k \leq 1$,

and hence $\limsup_{k \rightarrow \infty} 24^{\frac{1}{6}} k^{\frac{1}{6}} \|(x_k, y_k)\| \leq 1$. By similar arguments to Corollary 3.2, the second inequality implies $\liminf_{k \rightarrow \infty} 48^{\frac{1}{6}} k^{\frac{1}{6}} \|(x_k, y_k)\| \geq 1$. If $x_0 = 0, y_0 > 0$, then $x_k = 0$ and $y_k = y_{k+1}(1 + 4y_{k+1}^6)$. Then Lemma 3.1 implies $\lim_{k \rightarrow \infty} 24^{\frac{1}{6}} k^{\frac{1}{6}} \|(x_k, y_k)\| = 1$. If $x_0 > 0, y_0 = 0$, then $y_k = 0$ and $x_k = x_{k+1}(1 + 2x_{k+1}^2)$. Then Lemma 3.1 implies $\lim_{k \rightarrow \infty} 2k^{\frac{1}{2}} \|(x_k, y_k)\| = 1$. \square

5.3. Subspaces with Dimensions More than One. We consider an intersection of a hypersurface A defined by a convex polynomial g and a subspace B :

$$\begin{aligned} A &= \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z \geq g(x)\}, \\ B &= \{(x, 0) \in \mathbb{R}^n \times \mathbb{R} : x \in B_0\}, \end{aligned}$$

where B_0 is an r -dimensional subspace of \mathbb{R}^n , where $1 \leq r \leq n-1$. We assume that $g(x) > 0$ ($x \neq 0$) and $g(0) = 0$.

By rotation about z -axis, we may assume

$$B = \{(x, y, 0) \in \mathbb{R}^{n-r} \times \mathbb{R}^r \times \mathbb{R} : x = 0\}.$$

We consider the sequences $\{a(k)\}, \{b(k)\}$ constructed by the alternating projections as

$$(16) \quad a(k) \xrightarrow{P_B} b(k) \xrightarrow{P_A} a(k+1).$$

The following lemma is an easy consequence of Lemma 3.4 of [14].

Lemma 5.8. *Let $f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha} \in \mathbb{R}\{x\}$ and $f_{\Gamma}(x) = \sum_{\alpha \in \Gamma(f)} f_{\alpha} x^{\alpha}$. If f is nonnegative and nondegenerate, then $\mathcal{L}(f) = \mathcal{L}(f_{\Gamma})$.*

Theorem 5.9. *Suppose $g(0, y)^2$ is nondegenerate and $d = \mathcal{L}(g(0, \cdot))$. Then $b(k)$ defined by (16) converges to 0 in the rate $O(k^{\frac{-1}{2d-2}})$.*

Proof. We write

$$a(k) = (x(k), y(k), z(k)) = (x_1(k), \dots, x_{n-r}(k), y_1(k), \dots, y_r(k), z(k)).$$

Then $b(k) = (0, y(k), 0)$, and Lemma 4.1 gives

$$(17) \quad \begin{aligned} x_i(k+1) + g_{x_i}(x(k+1), y(k+1))g(x(k+1), y(k+1)) &= 0, \quad i = 1, \dots, n-r, \\ y_j(k+1) + g_{y_j}(x(k+1), y(k+1))g(x(k+1), y(k+1)) &= y_j(k), \quad j = 1, \dots, r. \end{aligned}$$

Since $d(a(k), B)^2 = \|x(k)\|^2 + z(k)^2$ and $\|b(k)\| = \|y(k)\|$, Lemma 5.2 implies

$$\begin{aligned} \|(x(k+1), y(k+1), z(k+1))\|^2 + \|x(k)\|^2 + z(k)^2 &\leq \|(x(k), y(k), z(k))\|^2 \\ \|(x(k+1), y(k+1), z(k+1))\|^2 &\leq \|y(k)\|^2, \\ \|x(k+1)\|^2 + \|b(k+1)\|^2 + g(x(k+1), y(k+1))^2 &\leq \|b(k)\|^2. \end{aligned}$$

Here, we consider the system

$$x_i + g_{x_i}(x, y)g(x, y) = 0, \quad i = 1, \dots, n-r.$$

By implicit function theorem, there exist convergent power series $\varphi_i(y)$ which solve equation (17) as $x_i = \varphi_i(y)$ and $\varphi_i(0) = 0$ for $i = 1, \dots, n-r$. We will apply Lemma 4.3. We claim that the Newton boundary of $g(0, y)$ meets all the axes. In fact, if there exists j such that the j th axis has no exponent of the support of $g(0, y)$, then $g(0, \dots, 0, y_j, 0, \dots, 0) = 0$. It contradicts to $g(0, y) > 0$ for $y \neq 0$. Since $g_{x_i}(0, 0) = 0$, Lemma 4.3 implies that $I \subset \mathfrak{ma}$, where $I = \langle \varphi_1(y), \dots, \varphi_{n-r}(y) \rangle$, $\mathfrak{m} = \langle y_1, \dots, y_r \rangle$, and $\mathfrak{a} = \langle y^\alpha : \alpha \in \text{supp}(g(0, y)) \rangle$.

Let $\varphi(y) = (\varphi_1(y), \dots, \varphi_{n-r}(y))$. Then we have

$$\|b(k+1)\|^2 + g(\varphi(y(k+1)), y(k+1))^2 + \|\varphi(y(k+1))\|^2 \leq \|b(k)\|^2.$$

Here, there exist polynomials p_i such that

$$\begin{aligned} g(\varphi(y), y)^2 &= \left(g(0, y) + \sum_{i=1}^{n-r} \varphi_i(y) p_i(\varphi(y), y) \right)^2 \\ &= g(0, y)^2 + 2g(0, y) \left(\sum_{i=1}^{n-r} \varphi_i(y) p_i(\varphi(y), y) \right) + \left(\sum_{i=1}^{n-r} \varphi_i(y) p_i(\varphi(y), y) \right)^2 \end{aligned}$$

Since $I \subset \mathfrak{ma}$, we have $g(\varphi(y), y)^2 - g(0, y)^2 \in \mathfrak{ama} + \mathfrak{m}^2 \mathfrak{a}^2 \subset \mathfrak{ma}^2$. Thus the Newton boundary of $g(\varphi(y), y)^2 + \|\varphi(y)\|^2$ is equal to that of $g(0, y)^2$. Since $g(0, y)^2$ is nondegenerate, Lemma 5.8 implies that $g(\varphi(y), y)^2 + \|\varphi(y)\|^2$ and $g(0, y)^2$ have the same Łojasiewicz exponent. Thus we obtain

$$\|b(k+1)\|^2 + C\|b(k+1)\|^{2d} \leq \|b(k)\|^2$$

for some $C > 0$. By Corollary 3.2, we have $\|b(k)\| = O\left(k^{\frac{-1}{2d-2}}\right)$. \square

Example 5.10. Let $g = x_1^6 + x_2^4 + x_3^2$ as in Example 5.1, $A = \{(x, z) \in \mathbb{R}^3 \times \mathbb{R} : z \geq g(x)\}$ and $B = \{(x, z) \in \mathbb{R}^3 \times \mathbb{R} : x_1 = x_2 = 0\}$. Then $\mathcal{L}(g(0, 0, x_3)) = 2$ while $\mathcal{L}(g) = 6$. Theorem 5.9 implies that $b(k)$ defined by (16) converges to 0 in the rate $O(k^{\frac{-1}{2}})$.

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