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Worst-case evaluation complexity of derivative-free nonmonotone line search methods for solving nonlinear systems of equations

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Abstract In this paper we study a class of derivative-free nonmonotone line search methods for solving nonlinear systems of equations, which includes the method N-DF-SANE proposed in (IMA J. Numer. Anal. 29: 814–825, 2009). These methods correspond to derivative-free optimization methods applied to the minimization of a suitable merit function. Assuming that the mapping defining the system of nonlinear equations has Lipschitz continuous Jacobian, we show that the methods in the referred class need at most $\mathcal{O}\left(|\log(\epsilon)|\epsilon^{-2}\right)$ function evaluations to generate an ϵ-approximate stationary point to the merit function. For the case in which the mapping is strongly monotone, we present two methods with evaluation-complexity of $\mathcal{O}\left(|\log(\epsilon)|\right)$.

Keywords Nonlinear systems of equations · Nonmonotone line search · Global convergence · Worst-case complexity

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1 Introduction

1.1 Motivation and Contributions

In this paper we study methods to solve nonlinear systems of equations of the form

$$F(x)=0, \tag{1}$$

where $F:\mathbb{R}^n\to\mathbb{R}^n$ is a continuously differentiable mapping. Methods to solve (1) have a central role in a wide range of computational techniques to solve real problems (see, e.g., [14,18,19,26,28,29,30]). The main iterative method for solving (1) is the Newton's method [24]. However, when the dimension n is large, the computational cost of the Newton's method can be prohibitive, since its execution requires the computation of the Jacobian matrix of $F(\,\cdot\,)$ and also the solution of a large-scale linear system at each iteration. These difficulties motivated the development of derivative-free methods for solving (1), i.e., methods that do not require the use of the Jacobian of $F(\,\cdot\,)$ neither the solution of linear systems. In this context, La Cruz, Martínez and Raydan [16] proposed the *Derivative-Free Spectral Algorithm for Nonlinear Equations* (DF-SANE), in which the iterates are defined as follows:

$$x_{k+1}=x_k+\alpha_k d_k,\quad d_k\in\left\{-\sigma_k F(x_k),\sigma_k F(x_k)\right\},$$

with $|\sigma_k|\in[\sigma_{min},\sigma_{max}]$, and $\alpha_k>0$ satisfying

$$f(x_k+\alpha_k d_k)\leq \max_{0\leq j\leq \min\{k,M-1\}}\left[f(x_{k-j})\right]+\theta_k-\rho\alpha_k^2 f(x_k), \tag{2}$$

where $f(x)\equiv\|F(x)\|_2^p$ for $p\in\{1,2\}$, $\{\theta_k\}_{k\geq 0}$ is a summable sequence of positive numbers, and M is a positive integer. An accelerated variant of DF-SANE was recently addressed in [2,3]. The line search condition (2) allows the selection of a stepsize α_k for which $f(x_{k+1})=f(x_k+\alpha_k d_k)>f(x_k)$ and, therefore, the sequence $\{f(x_k)\}$ may be nonmonotone. As mentioned in [16], one of the inspirations for (2) was the nonmonotone line search condition proposed by Grippo, Lampariello and Lucidi [11] in the context of unconstrained optimization. Another well-known line search strategy for unconstrained optimization is the one proposed by Zhang and Hager [31]. As a natural development for nonlinear equations, Cheng and Li [5] presented N-DF-SANE, a variant of DF-SANE inspired by the Zhang and Hager's nonmonotone line search. The main difference is that, at the k-th iteration of N-DF-SANE, the stepsize $\alpha_k>0$ is chosen such that

$$f(x_k+\alpha_k d_k)\leq C_k+\theta_k-\rho\alpha_k^2 f(x_k), \tag{3}$$

where $f(x)\equiv(1/2)\|F(x)\|_2^2$,

$$C_0=f(x_0),\quad C_{k+1}=\frac{\eta_k Q_k(C_k+\theta_k)+f(x_{k+1})}{Q_{k+1}},$$

$Q_0 = 1$, $Q_{k+1} = \eta_k Q_k + 1$ and $\eta_k \in [0,1]$. In the numerical experiments described in [5], N-DF-SANE was competitive with DF-SANE, requiring less function evaluations in the majority of problems tested.

In this work we propose a general class of derivative-free nonmonotone methods for solving (1), which includes N-DF-SANE. Specifically, we consider the nonmonotone line search condition:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \nu_k + \theta_k - \rho\alpha_k^2 f(x_k), \tag{4}$$

where $\{\nu_k\}_{k\geq 0}$ is a sequence of nonnegative numbers. Similarly to (2) and (3), condition (4) is also motivated by a nonmonotone strategy originally developed for unconstrained optimization. In this case, our inspiration is the nonmonotone line search proposed by Sachs and Sachs [25]. Notice that different choices for $\{\nu_k\}_{k\geq 0}$ lead to different methods[1]. The goal of our analysis is to establish worst-case complexity bounds for the number of function evaluations that these methods need to generate an ϵ-approximate stationary point of the merit function $f(\,\cdot\,)$. Assuming that the mapping $F(\,\cdot\,)$ has Lipschitz continuous Jacobian, we show that the methods in the referred class need at most $\mathcal{O}\left(|\log(\epsilon)|\epsilon^{-2}\right)$ evaluations of $F(\,\cdot\,)$ to generate x_k such that $\psi(x_k) \leq \epsilon$, where $\psi(\,\cdot\,)$ is defined as

$$\psi(x) = \begin{cases} \min\left\{\|F(x)\|, \dfrac{|\langle \nabla f(x), F(x)\rangle|}{\|F(x)\|}\right\}, & \text{whenever } F(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For the case in which $F(\,\cdot\,)$ is also strongly monotone, we present two particular variants that need only $\mathcal{O}\left(|\log(\epsilon)|\right)$ function evaluations to generate x_k such that $f(x_k) = (1/2)\|F(x_k)\|_2^2 \leq \epsilon$.

1.2 Related Literature

In the context of derivative-based methods[2], iteration-complexity bounds of $\mathcal{O}\left(\epsilon^{-2}\right)$ have been proved for several Levenberg-Marquardt (LM) methods with respect to the stropping criterion

$$\|J(x_k)^T F(x_k)\|_2 \leq \epsilon, \tag{6}$$

where $J(x)$ denotes the Jacobian matrix of $F(\,\cdot\,)$ at point x (see, e.g., [27,13, 32,12]). In particular, an improved iteration-complexity bound of $\mathcal{O}\left(|\log(\epsilon)|\right)$ was obtained in [12] with respect to the stopping criterion

$$\|F(x_k)\|_2 \leq \epsilon,$$

under an additional regularity assumption about the Jacobian of $F(\,\cdot\,)$. More recently, an evaluation-complexity bound of $\mathcal{O}\left(|\log(\epsilon)|\epsilon^{-2}\right)$ was established

[1] For example, condition (3) for N-DF-SANE is obtained taking $\nu_k = C_k - f(x_k)$

[2] By *derivative-based methods* we mean methods that make explicit use of the Jacobian matrix of $F(\,\cdot\,)$.

in [1] for a LM variant with respect to the stopping criterion (6). Regarding different types of methods, evaluation-complexity bounds of $\mathcal{O}\left(\epsilon^{-2}\right)$ were obtained in [4] for trust-region and quadratic regularization methods that can be applied to the minimization of merit functions of the form $f(x) \equiv \|F(x)\|_p$ with $p \in \{1, 2, \infty\}$. In the more restricted setting of Jacobian-free methods[3], an evaluation-complexity bound of $\mathcal{O}\left(\epsilon^{-2}\right)$ was proved in [21] for a nonmonotone method based on structured diagonal Hessian approximations. On the other hand, in the context of derivative-free methods based on interpolation models, evaluation-complexity bounds of $\mathcal{O}\left(n^2|\log(\epsilon)|\epsilon^{-2}\right)$ and of $\mathcal{O}\left(n^2\epsilon^{-2}\right)$ were obtained in [10] and [6], respectively, for derivative-free trust-region methods that can be applied to the minimization of merit functions of the form $f(x) \equiv \|F(x)\|_p$ with $p \in \{1, 2, \infty\}$.

To the best of our knowledge, the present work is the first one to consider the worst-case evaluation complexity of a class of derivative-free nonmonotone line search methods for nonlinear systems of equations including the method N-DF-SANE [5].

1.3 Contents

The paper is organized as follows. In Section 2, we analyze the worst-case complexity of our general class of derivative-free nonmonotone methods. In Section 3, we present a subclass of methods that includes N-DF-SANE. In Section 4, we introduce two nonmonotone methods with improved evaluation complexity for strongly monotone mappings. Finally, in Section 5, we report numerical results that illustrate our theoretical findings.

1.4 Notations

The symbol $\|\cdot\|$ denotes the 2-norm for vectors or matrices (depending on the context). The Euclidian inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$. Given a set $X \subset \mathbb{R}^n$, $\mathrm{co}\,(X)$ denotes de convex hull of X.

2 General Class of Non-Monotone Algorithms

In what follows, we will consider the merit function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \frac{1}{2}\|F(x)\|^2. \tag{7}$$

Let us consider the following general algorithm.

[3] By *Jacobian-free* methods we mean methods that only require the results of matrix-vector products envolving $J(x)$ $(x \in \mathbb{R}^n)$, without the need to store $J(x)$.

Algorithm 1. (General Non-Monotone Method)

Step 0. Given a starting point $x_0 \in \mathbb{R}^n$ and constants $\beta \in (0,1)$, $0 < \sigma_{min} \leq \sigma_0 < \sigma_{max}$ and $\rho > 0$, choose a sequence $\{\theta_k\}_{k\geq 0}$ of positive numbers satisfying

$$\sum_{k=0}^{+\infty} \theta_k \leq \theta < +\infty, \tag{8}$$

and set $k := 0$.
Step 1. Compute $\sigma_k \neq 0$ such that $|\sigma_k| \in [\sigma_{min}, \sigma_{max}]$.
Step 2.1. Set $\ell := 0$ and choose $\nu_k \geq 0$.
Step 2.2. If

$$f\left(x_k - \beta^\ell \sigma_k F(x_k)\right) \leq f(x_k) + \nu_k + \theta_k - \rho \left(\beta^\ell\right)^2 f(x_k), \tag{9}$$

then, set $\ell_k = \ell$, $\alpha_k = \beta^{\ell_k}$, $d_k = -\sigma_k F(x_k)$ and go to Step 3. Otherwise, go to Step 2.3.
Step 2.3. If

$$f\left(x_k + \beta^\ell \sigma_k F(x_k)\right) \leq f(x_k) + \nu_k + \theta_k - \rho \left(\beta^\ell\right)^2 f(x_k), \tag{10}$$

then set $\ell_k = \ell$, $\alpha_k = \beta^{\ell_k}$, $d_k = \sigma_k F(x_k)$ and go to Step 3. Otherwise, set $\ell := \ell + 1$, and go to Step 2.2.
Step 3. Set $x_{k+1} = x_k + \alpha_k d_k$, $k := k + 1$ and go to Step 1.

Remark 1 Notice that Step 1 allows the choice of $\sigma_k < 0$. A concrete choice for σ_k is described in Section 5.

Our analysis will be carried out under the assumptions:

A1 The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and its Jacobian $J : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is L_J-Lipschitz continuous.
A2 $\sum_{k=0}^{+\infty} \nu_k \leq \nu < +\infty$.
A3 The level set $\mathcal{L}_f(x_0) := \{x \in \mathbb{R}^n : f(x) \leq f(x_0) + \nu + \theta\}$ is bounded as follows

$$\sup\{\|x - x_0\| : x \in \mathcal{L}_f(x_0)\} \equiv D_0 < +\infty.$$

Remark 2 Under A3, if $\bar{x} \in \mathrm{co}\,(\mathcal{L}_f(x_0))$, then $\|\bar{x} - x_0\| \leq D_0$.

Remark 3 Combining (9), (10), (8) and A2, if $\{x_k\}_{k=0}^T$ is well-defined, then

$$f(x_k) \leq f(x_0) + \sum_{i=0}^{k-1} \nu_i + \sum_{i=0}^{k-1} \theta_i \leq f(x_0) + \nu + \theta,$$

for all $k \in \{1, \ldots, T\}$. Thus, we have $\{x_k\}_{k=0}^T \subset \mathcal{L}_f(x_0)$. Consequently, if A3 holds it follows that $\|x_k - x_0\| \leq D_0$ for all $k \geq 0$.

Our first result provides a quadratic overestimation for $f(\cdot)$ on the level set $\mathcal{L}_f(x_0)$.

Lemma 1 *Suppose that A1 and A3 hold. Then, for function $f(\,\cdot\,)$ in (7) we have*

$$|f(w)-f(z)-\langle \nabla f(z), w-z\rangle| \leq \frac{L}{2}\|w-z\|^2, \quad \forall z, w \in \mathcal{L}_f(x_0), \tag{11}$$

where

$$L = \left(L_J D_0 + \|J(x_0)\|\right)^2 + L_J\left[\left(L_J D_0 + \|J(x_0)\|\right) D_0 + \|F(x_0)\|\right]. \tag{12}$$

Proof To obtain (11), it is enough to show that $\nabla f(\,\cdot\,)$ is L-Lipschitz continuous on $\mathrm{co}\,(\mathcal{L}_f(x_0))$. Given $x, y \in \mathrm{co}\,(\mathcal{L}_f(x_0))$,

$$\begin{aligned}\|\nabla f(x)-\nabla f(y)\| &= \|J(x)^T F(x) - J(y)^T F(y)\| \\ &\leq \|J(x)^T F(x) - J(x)^T F(y)\| + \|J(x)^T F(y) - J(y)^T F(y)\| \\ &\leq \|J(x)\|\|F(x)-F(y)\| + \|J(x)-J(y)\|\|F(y)\|. \end{aligned} \tag{13}$$

From A1 and Remark 2, we get

$$\begin{aligned}\|J(\bar{x})\| &\leq \|J(\bar{x}) - J(x_0)\| + \|J(x_0)\| \leq L_J\|\bar{x}-x_0\| + \|J(x_0)\| \\ &\leq L_J D_0 + \|J(x_0)\|, \quad \forall \bar{x} \in \mathrm{co}\,(\mathcal{L}_f(x_0)). \end{aligned} \tag{14}$$

Thus, by the Mean Value Inequality and (14), we have

$$\|F(x)-F(y)\| \leq \left(L_J D_0 + \|J(x_0)\|\right)\|x-y\|, \tag{15}$$

and

$$\|F(y)\| \leq \|F(y)-F(x_0)\| + \|F(x_0)\| \leq \left(L_J D_0 + \|J(x_0)\|\right) D_0 + \|F(x_0)\|. \tag{16}$$

Finally, combining (13)-(16) and A1, we obtain

$$\begin{aligned}\|\nabla f(x)-\nabla f(y)\| \leq &\left(L_J D_0 + \|J(x_0)\|\right)^2\|x-y\| \\ &+ L_J\left[\left(L_J D_0 + \|J(x_0)\|\right) D_0 + \|F(x_0)\|\right]\|x-y\|, \end{aligned}$$

and so, (11) holds for L given in (12). □

The next lemma guarantees that the iterates of Algorithm 1 are well-defined.

Lemma 2 *Suppose that A1-A3 hold and let x_k ($k \geq 0$) be an iterate in Algorithm 1. If $\langle \nabla f(x_k), F(x_k)\rangle \neq 0$ and*

$$0 < \alpha \leq \frac{2|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2}, \tag{17}$$

then

$$\min\left\{f(x_k + \alpha\sigma_k F(x_k)), f(x_k - \alpha\sigma_k F(x_k))\right\} \leq f(x_k) + \nu_k + \theta_k - \rho\alpha^2 f(x_k). \tag{18}$$

Proof Let us divide the proof in two cases.

Case I: $\langle \nabla f(x_k), \sigma_k F(x_k)\rangle < 0$.

In this case, we will show that

$$f(x_k + \alpha\sigma_k F(x_k)) \leq f(x_k) + \nu_k + \theta_k - \rho\alpha^2 f(x_k). \tag{19}$$

For that, assume by contradiction that (19) is not true, that is,

$$f(x_k + \alpha\sigma_k F(x_k)) > f(x_k) + \nu_k + \theta_k - \rho\alpha^2 f(x_k). \tag{20}$$

Let us define

$$\xi_1(t) = f(x_k + t\sigma_k F(x_k)) - \left[f(x_k) + \nu_k + \theta_k - \rho t^2 f(x_k)\right].$$

Then, by (20), we have $\xi_1(0) = -\nu_k - \theta_k < 0 < \xi_1(\alpha)$. Consequently, by the Intermediate Value Theorem, there exists $\hat{\alpha} \in (0, \alpha)$ such that $\xi_1(\hat{\alpha}) = 0$, that is,

$$f(x_k + \hat{\alpha}\sigma_k F(x_k)) = f(x_k) + \nu_k + \theta_k - \rho(\hat{\alpha})^2 f(x_k). \tag{21}$$

By Remark 3 and (21), we have $x_k, x_k + \hat{\alpha}\sigma_k F(x_k) \in \mathcal{L}_f(x_0)$. Then, combining (21) and Lemma 1, it follows that

$$\begin{aligned}
-\rho(\hat{\alpha})^2 f(x_k) &\leq f(x_k + \hat{\alpha}\sigma_k F(x_k)) - f(x_k)\\
&\leq \hat{\alpha}\langle \nabla f(x_k), \sigma_k F(x_k)\rangle + \frac{L(\hat{\alpha})^2}{2}\|\sigma_k F(x_k)\|^2\\
\Longrightarrow -\rho\hat{\alpha} f(x_k) &\leq \langle \nabla f(x_k), \sigma_k F(x_k)\rangle + \frac{L\hat{\alpha}\sigma_{max}^2}{2}\|F(x_k)\|^2\\
\Longrightarrow -\langle \nabla f(x_k), \sigma_k F(x_k)\rangle &\leq \hat{\alpha}\left(\frac{\rho + L\sigma_{max}^2}{2}\right)\|F(x_k)\|^2\\
\Longrightarrow \hat{\alpha} \geq -\frac{2\langle \nabla f(x_k), \sigma_k F(x_k)\rangle}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2} &= \frac{2|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2}.
\end{aligned}$$

Since $\alpha > \hat{\alpha}$, it follows that

$$\alpha > \frac{2|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2},$$

contradicting (17). Thus, (19) must be true.

Case II: $\langle \nabla f(x_k), \sigma_k F(x_k)\rangle > 0$.

In this case, we will show that

$$f(x_k - \alpha\sigma_k F(x_k)) \leq f(x_k) + \nu_k + \theta_k - \rho\alpha^2 f(x_k). \tag{22}$$

For that, assume by contradiction that (22) is not true, that is,

$$f(x_k - \alpha\sigma_k F(x_k)) > f(x_k) + \nu_k + \theta_k - \rho\alpha^2 f(x_k). \tag{23}$$

Let us define

$$\xi_2(t) = f(x_k - t\sigma_k F(x_k)) - \left[f(x_k) + \nu_k + \theta_k - \rho t^2 f(x_k)\right].$$

Then, by (23), we have $\xi_2(0) = -\nu_k - \theta_k < 0 < \xi_2(\alpha)$. Consequently, by the Intermediate Value Theorem, there exists $\bar{\alpha} \in (0, \alpha)$ such that $\xi_2(\bar{\alpha}) = 0$, that is,

$$f(x_k - \bar{\alpha}\sigma_k F(x_k)) = f(x_k) + \nu_k + \theta_k - \rho(\bar{\alpha})^2 f(x_k). \tag{24}$$

As in Case I, by Remark 3, (24) and Lemma 1, we obtain

$$\begin{aligned}
-\rho(\bar{\alpha})^2 f(x_k) &\leq f(x_k - \bar{\alpha}\sigma_k F(x_k)) - f(x_k) \\
&\leq -\bar{\alpha}\langle \nabla f(x_k), \sigma_k F(x_k)\rangle + \frac{L(\bar{\alpha})^2}{2}\|\sigma_k F(x_k)\|^2 \\
\Longrightarrow -\rho\bar{\alpha} f(x_k) &\leq -\langle \nabla f(x_k), \sigma_k F(x_k)\rangle + \frac{L\bar{\alpha}\sigma_{max}^2}{2}\|F(x_k)\|^2 \\
\Longrightarrow \langle \nabla f(x_k), \sigma_k F(x_k)\rangle &\leq \bar{\alpha}\left(\frac{\rho + L\sigma_{max}^2}{2}\right)\|F(x_k)\|^2 \\
\Longrightarrow \bar{\alpha} &\geq \frac{2\langle \nabla f(x_k), \sigma_k F(x_k)\rangle}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2} = \frac{2|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2}.
\end{aligned}$$

Since $\alpha > \bar{\alpha}$, it follows that

$$\alpha > \frac{2|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2},$$

contradicting (17). Thus, (22) must be true. □

From Lemma 2, we can obtain a lower bound for α_k in Algorithm 1.

Lemma 3 *Suppose that A1-A3 hold and let x_k be an iterate in Algorithm 1. If $\langle \nabla f(x_k), F(x_k)\rangle \neq 0$, then*

$$\alpha_k \geq \min\left\{1, \frac{2\beta\sigma_{min}|\langle \nabla f(x_k), F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2}\right\}. \tag{25}$$

Proof If $\ell_k = 0$, then $\alpha_k = 1$ and (25) holds. If $\ell_k > 0$, it follows from Step 2 of Algorithm 1 that

$$\min\left\{f(x_k + \beta^{\ell_k - 1}\sigma_k F(x_k)), f(x_k - \beta^{\ell_k - 1}\sigma_k F(x_k))\right\} > f(x_k) + \nu_k + \theta_k - \rho\left(\beta^{\ell_k - 1}\right)^2 f(x_k)$$

In view of Lemma 2, we must have

$$\beta^{\ell_k - 1} > \frac{2|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2}$$

and so

$$\alpha_k = \beta^{\ell_k - 1}\beta > \frac{2\beta|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2}.$$

In this case, the conclusion (25) folows from the inequality above and $|\sigma_k| \geq \sigma_{min}$. □

The theorem below establishes that, given $\epsilon > 0$, Algorithm 1 takes at most $\mathcal{O}\left(\epsilon^{-2}\right)$ iterations to generate x_k such that $\psi(x_k) \leq \epsilon$.

Theorem 1 *Suppose that A1-A3 hold and let* $\{x_k\}_{k\geq 0}$ *be generated by Algorithm 1. Given* $\epsilon > 0$*, the number of elements of the set*

$$\Omega(\epsilon) = \{k \,:\, \psi(x_k) > \epsilon\} \tag{26}$$

is bounded as follows

$$|\Omega(\epsilon)| \leq \frac{2(f(x_0)+\nu+\theta)}{\rho \min\left\{1, \dfrac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right\}^2}\epsilon^{-2}. \tag{27}$$

Proof By Steps 2 and 3 of Algorithm 1, Lemma 3 and (5), if $k \in \Omega(\epsilon)$, we have

$$\begin{aligned}
\theta_k + \nu_k + f(x_k) - f(x_{k+1}) &\geq \rho\alpha_k^2 f(x_k) \\
&\geq \rho \min\left\{1, \left(\frac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right)^2 \frac{|\langle \nabla f(x_k), F(x_k)\rangle|^2}{\|F(x_k)\|^4}\right\} f(x_k) \\
&= \rho \min\left\{\frac{1}{2}\|F(x_k)\|^2, \left(\frac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right)^2 \frac{|\langle \nabla f(x_k), F(x_k)\rangle|^2}{2\|F(x_k)\|^2}\right\} \\
&\geq \frac{\rho}{2} \min\left\{1, \left(\frac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right)^2\right\} \min\left\{\|F(x_k)\|, \frac{|\nabla f(x_k), F(x_k)\rangle|}{\|F(x_k)\|}\right\}^2 \\
&= \frac{\rho}{2} \min\left\{1, \left(\frac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right)^2\right\} \psi(x_k)^2 && (28) \\
&> \frac{\rho}{2} \min\left\{1, \left(\frac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right)^2\right\} \epsilon^2. && (29)
\end{aligned}$$

Then, combining (29), (8) and A2, it follows that

$$\begin{aligned}
\frac{\rho}{2} \min\left\{1, \left(\frac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right)^2\right\} \epsilon^2 |\Omega(\epsilon)| &= \sum_{k\in\Omega(\epsilon)} \frac{\rho}{2} \min\left\{1, \left(\frac{2\beta\sigma_{min}}{\rho + L\sigma_{max}^2}\right)^2\right\} \epsilon^2 \\
&\leq \sum_{k=0}^{+\infty} f(x_k) - f(x_{k+1}) + \nu_k + \theta_k \\
&\leq f(x_0) + \sum_{k=0}^{+\infty} \nu_k + \sum_{k=0}^{+\infty} \theta_k \\
&\leq f(x_0) + \nu + \theta.
\end{aligned}$$

Therefore, $\Omega(\epsilon)$ satisfies (27). □

Remark 4 Let N_k be the number of function evalutions at the k-th iteration of Algorithm 1. Note that $N_k \leq 2(\ell_k+1)$. If $\psi(x_k)>\epsilon$ with $\epsilon\in(0,1)$, then, by Lemma 3 and (16), we have

$$\begin{aligned}
\alpha_k=\beta^{\ell_k} &\geq \min\left\{1,\frac{2\beta\sigma_{min}|\langle\nabla f(x_k),F(x_k)\rangle|}{(\rho+L\sigma_{max}^2)\|F(x_k)\|^2}\right\}\\
&\geq \min\left\{1,\frac{2\beta\sigma_{min}}{(\rho+L\sigma_{max}^2)\left[(L_JD_0+\|J(x_0)\|)D_0+\|F(x_0)\|\right]}\frac{|\langle\nabla f(x_k),F(x_k)\rangle|}{\|F(x_k)\|}\right\}\\
&> \min\left\{1,\frac{2\beta\sigma_{min}\epsilon}{(\rho+L\sigma_{max}^2)\left[(L_JD_0+\|J(x_0)\|)D_0+\|F(x_0)\|\right]}\right\}\\
&\geq \min\left\{1,\frac{2\beta\sigma_{min}}{(\rho+L\sigma_{max}^2)\left[(L_JD_0+\|J(x_0)\|)D_0+\|F(x_0)\|\right]}\right\}\epsilon\equiv\kappa_c\epsilon.
\end{aligned}$$

Consequently,

$$N_k\leq 2(\ell_k+1)\leq 2+\frac{2\log(\kappa_c\epsilon)}{\log(\beta)}.$$

Combining this result with Theorem 1 it follows that Algorithm 1 performs at most $\mathcal{O}\left(|\log(\epsilon)|\epsilon^{-2}\right)$ evaluations of $f(\,\cdot\,)$ to generate the first iterate x_k for which $\psi(x_k)\leq\epsilon$. Notice that the factor $|\log(\epsilon)|$ comes from the fact that the lower bound obtained for α_k depends on ϵ, namely $\alpha_k=\beta^{\ell_k}\geq\kappa_c\epsilon$. This logarithmic factor can be avoided when $F(\,\cdot\,)$ is strongly monotone. Indeed, under this assumption there exists a real number $\mu>0$ such that

$$F(x_k)^TJ(x_k)F(x_k)\geq\mu\|F(x_k)\|^2\quad\forall k.$$

Then, by Lemma 3 we get

$$\begin{aligned}
\alpha_k=\beta^{\ell_k} &\geq \min\left\{1,\frac{2\beta\sigma_{min}|\langle\nabla f(x_k),F(x_k)\rangle|}{(\rho+L\sigma_{max}^2)\|F(x_k)\|^2}\right\}\\
&= \min\left\{1,\frac{2\beta\sigma_{min}|F(x_k)^TJ(x_k)F(x_k)|}{(\rho+L\sigma_{max}^2)\|F(x_k)\|^2}\right\}\\
&\geq \min\left\{1,\frac{2\beta\sigma_{min}\mu}{(\rho+L\sigma_{max}^2)}\right\}\equiv\tilde{\alpha},
\end{aligned}$$

that is, we get a lower bound for α_k independent of ϵ. In this case, we obtain the following upper bound for the number of function evalutions at the k-th iteration:

$$N_k\leq 2(\ell_k+1)\leq 2+\frac{2\log(\tilde{\alpha})}{\log(\beta)},$$

which combined with Theorem 1 gives an improved evaluation complexity bound of $\mathcal{O}\left(\epsilon^{-2}\right)$ for Algorithm 1. In fact, under the same strong monotonicity assumption, a much better complexity bound can be obtained with suitable choices of $\{\theta_k\}$ and $\{\nu_k\}$. This is the case addressed in Section 4.

Corollary 1 *Suppose that A1-A3 hold and let $\{x_k\}_{k\geq 0}$ be generated by Algorithm 1. Then $\{x_k\}$ has a limit point x^* such that $\psi(x^*)=0$.*

Proof First, let us show that

$$\lim_{k\to+\infty} \psi(x_k) = 0. \tag{30}$$

Indeed, if we assume that (30) does not hold, then there exists $\epsilon > 0$ and a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ of $\{x_k\}$ such that

$$\psi(x_{k_j}) > \epsilon, \quad \forall j \in \mathbb{N}.$$

Consequently, for this ϵ, we would have $|\Omega(\epsilon)| = +\infty$, contradicting Theorem 1. Therefore, (30) is true. Since $\{x_k\} \subset \mathcal{L}_f(x_0)$, it follows from A3 that $\{x_k\}_{k\in\mathbb{N}}$ is bounded. Thus, $\{x_k\}$ possess a subsequence $\{x_{k_\ell}\}_{\ell\in\mathbb{N}}$ that is convergent, let us say

$$\lim_{\ell\to+\infty} x_{k_\ell} = x^*. \tag{31}$$

Notice that $\psi(\cdot)$ is continuous. Hence, combining (30) and (31) we conclude that $\psi(x^*) = 0$. □

Remark 5 If $F(x^*) = 0$ then, by (5) we have $\psi(x^*) = 0$. However, the converse is not necessarily true. For example, if we consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x) = \begin{bmatrix} \sin^2(x_1) & x_2 - 1 \end{bmatrix}^T$, for the point $x^* = \begin{bmatrix} \frac{3\pi}{4} & \frac{3}{2} \end{bmatrix}^T$ we have $\psi(x^*) = 0$ but $F(x^*) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \neq \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

Remark 6 If $F(\cdot)$ is strictly monotone, then $J(x)$ is positive definite for all $x \in \mathbb{R}^n$ and, consequently, $\psi(x^*) = 0$ implies that $F(x^*)$. Therefore, when $F(\cdot)$ is strictly monotone, it follows from Corollary 1 that at least one limit point of any sequence $\{x_k\}_{k\geq 0}$ generated by Algorithm 1 is a zero of $F(\cdot)$.

3 A Subclass of Non-Monotone Algorithms

Let us consider now the following algorithmic framework:

Algorithm 2.

Step 0. Given a starting point $x_0 \in \mathbb{R}^n$ and constants $\delta_{min}, \beta \in (0, 1)$, $0 < \sigma_{min} \leq \sigma_0 < \sigma_{max}$ and $\rho > 0$, choose a sequence $\{\theta_k\}_{k\geq 0}$ of positive numbers satisfying $\sum_{k=0}^{+\infty} \theta_k \leq \theta < +\infty$. Set $C_0 = f(x_0)$ and $k := 0$.
Step 1. Compute $\sigma_k \neq 0$ such that $|\sigma_k| \in [\sigma_{min}, \sigma_{max}]$.
Step 2.1. Set $\ell := 0$.
Step 2.2. If

$$f\left(x_k - \beta^\ell \sigma_k F(x_k)\right) \leq C_k + \theta_k - \rho\left(\beta^\ell\right)^2 f(x_k), \tag{32}$$

then set $\ell_k = \ell$, $\alpha_k = \beta^{\ell_k}$, $d_k = -\sigma_k F(x_k)$ and go to Step 3. Otherwise, go to Step 2.3.

Step 2.3. If

$$f\left(x_k + \beta^{\ell}\sigma_k F(x_k)\right) \leq C_k + \theta_k - \rho\left(\beta^{\ell}\right)^2 f(x_k), \tag{33}$$

then set $\ell_k = \ell$, $\alpha_k = \beta^{\ell_k}$, $d_k = \sigma_k F(x_k)$ and go to Step 3. Otherwise, set $\ell := \ell + 1$ and go to Step 2.2.
Step 3. Set $x_{k+1} = x_k + \alpha_k d_k$, compute $\delta_{k+1} \in [\delta_{min}, 1]$, set

$$C_{k+1} = (1 - \delta_{k+1})(C_k + \theta_k) + \delta_{k+1} f(x_{k+1}), \tag{34}$$

$k := k + 1$ and go to Step 1.

In Algorithm 2, different choices for δ_{k+1}, give different non-monotone terms C_k and, consequently, different nonmonotone algorithms. For example, consider the choice

$$\delta_{k+1} = \frac{1}{\eta_k Q_k + 1},$$

where $Q_0 = 1$, $Q_{k+1} = \eta_k Q_k + 1$ and $\eta_k \in [\eta_{min}, \eta_{max}]$ with $0 \leq \eta_{min} \leq \eta_{max} < 1$. In this case, we have

$$Q_{k+1} = 1 + \sum_{j=0}^{k} \Pi_{i=0}^{j} \eta_{k-i} \leq 1 + \sum_{j=0}^{k} \eta_{max}^{j+1} \leq \sum_{j=0}^{+\infty} \eta_{max}^{j} = \frac{1}{1 - \eta_{max}},$$

which gives

$$\delta_{k+1} = \frac{1}{Q_{k+1}} \geq 1 - \eta_{max} \equiv \delta_{min}.$$

Moreover, the corresponding update rule for the nonmonotone terms is

$$\begin{aligned} C_{k+1} &= (1 - \delta_{k+1})(C_k + \theta_k) + \delta_{k+1} f(x_{k+1}) \\ &= \frac{\eta_k Q_k}{\eta_k Q_k + 1}(C_k + \theta_k) + \frac{f(x_{k+1})}{\eta_k Q_k + 1} \\ &= \frac{\eta_k Q_k (C_k + \theta_k) + f(x_{k+1})}{Q_{k+1}}. \end{aligned}$$

This is exactly how the nonmonotone terms are defined in the Algorithm N-DF-SANE proposed by Cheng and Li [5]. Therefore, N-DF-SANE is a particular instance of Algorithm 2.

Our next lemma establishes that Algorithm 2 is a particular case of Algorithm 1 with the corresponding sequence $\{\nu_k\}$ satisfying A2. The proof is an adaptation of the proof of Theorem 4 in [9].

Lemma 4 *Let* $\{C_k\}_{k \geq 0}$ *be generated by Algorithm 2. Then,*

$$C_k = f(x_k) + \nu_k, \quad \forall k, \tag{35}$$

with

$$\nu_0 = 0 \quad \textit{and} \quad \nu_{k+1} = (1 - \delta_{k+1})(f(x_k) + \nu_k + \theta_k) + (\delta_{k+1} - 1) f(x_{k+1}). \tag{36}$$

Moreover, the sequence $\{\nu_k\}_{k\geq 0}$ *defined in (36) satisfies*

$$\sum_{k=0}^{+\infty} \nu_k \leq \left(\frac{1-\delta_{min}}{\delta_{min}}\right)(f(x_0)+\theta) \equiv \nu. \tag{37}$$

Proof Since

$$C_0 = f(x_0) = f(x_0) + \nu_0,$$

it follows that (35) holds for $k=0$. Assume that (35) is true for some $k \geq 0$. Then, by the induction assumption and (36) we have

$$\begin{aligned} C_{k+1} &= (1-\delta_{k+1})(C_k+\theta_k) + \delta_{k+1} f(x_{k+1}) \\ &= (1-\delta_{k+1})(f(x_k)+\nu_k+\theta_k) + \delta_{k+1} f(x_{k+1}) \\ &= f(x_{k+1}) + [(1-\delta_{k+1})(f(x_k)+\nu_k+\theta_k) + (\delta_{k+1}-1) f(x_{k+1})] \\ &= f(x_{k+1}) + \nu_{k+1}, \end{aligned}$$

that is, (35) also holds for $k+1$. Therefore, (35) is true.

On the other hand, since

$$f(x_{k+1}) + \nu_{k+1} = C_{k+1} = f(x_k) + \nu_k + \theta_k - \delta_{k+1}\left[f(x_k) + \nu_k + \theta_k - f(x_{k+1})\right]$$

we have

$$\delta_{k+1}\left[f(x_k)+\nu_k+\theta_k - f(x_{k+1})\right] \leq (f(x_k)+\nu_k+\theta_k) - (f(x_{k+1})+\nu_{k+1}).$$

Then, summing up the above inequalities for $k = 0, \ldots, N-1$ and using $f(x_N) \geq 0$ and $\sum_{k=0}^{+\infty} \theta_k \leq \theta$, we obtain

$$\begin{aligned} \sum_{k=0}^{N-1} \delta_{k+1}\left[f(x_k)+\nu_k+\theta_k - f(x_{k+1})\right] &\leq \sum_{k=0}^{N-1} f(x_k) - f(x_{k+1}) + \sum_{k=0}^{N-1} \theta_k \\ &\quad + \sum_{k=0}^{N-1} \nu_k - \nu_{k+1} \\ &= f(x_0) - f(x_N) + \sum_{k=0}^{N-1} \theta_k + \nu_0 - \nu_N \\ &\leq f(x_0) + \sum_{k=0}^{+\infty} \theta_k \\ &\leq f(x_0) + \theta. \end{aligned} \tag{38}$$

Combining (36), (38) and $\delta_{k+1} \geq \delta_{min}$, it follows that

$$\begin{aligned} \sum_{k=0}^{N} \nu_k = \sum_{k=0}^{N-1} \nu_{k+1} &= \sum_{k=0}^{N-1} \left(\frac{1-\delta_{k+1}}{\delta_{k+1}}\right) \delta_{k+1}\left[f(x_k)+\nu_k+\theta_k - f(x_{k+1})\right] \\ &\leq \left(\frac{1-\delta_{min}}{\delta_{min}}\right) \sum_{k=0}^{N-1} \delta_{k+1}\left[f(x_k)+\nu_k+\theta_k - f(x_{k+1})\right] \\ &\leq \left(\frac{1-\delta_{min}}{\delta_{min}}\right)(f(x_0)+\theta). \end{aligned}$$

Because $N \geq 0$ is arbitrary, we conclude that $\sum_{k=0}^{+\infty} \nu_k \leq \left(\frac{1-\delta_{min}}{\delta_{min}}\right)(f(x_0)+\theta)$.
□

By Lemma 4, Algorithm 2 is a particular case of Algorithm 1 with $\{\nu_k\}$ satisfying A2. Combining this fact with Theorem 1 and Remark 4 we obtain the following result.

Theorem 2 *Suppose that A1 holds. If A3 holds for ν given in (37) then, given $\epsilon > 0$, Algoritm 2 needs at most $\mathcal{O}(|\log(\epsilon)|\epsilon^{-2})$ evaluations of $f(\,\cdot\,)$ to generate the first x_k such that $\psi(x_k) \leq \epsilon$.*

4 An Algorithm for Strongly Monotone Mappings

Now, let us consider the following instance of Algorithm 1 (with $\nu_k = 0$ for all k):

Algorithm 3.

Step 0. Given a starting point $x_0 \in \mathbb{R}^n$ and constants $\gamma, \beta \in (0,1)$, $0 < \sigma_{min} \leq \sigma_0 \leq \sigma_{max}$ and $\theta_0, \rho > 0$, set $k := 0$.
Step 1. Compute $\sigma_k \neq 0$ such that $|\sigma_k| \in [\sigma_{min}, \sigma_{max}]$.
Step 2.1. Set $\ell := 0$.
Step 2.2. If

$$f\left(x_k - \beta^{\ell}\sigma_k F(x_k)\right) \leq f(x_k) + \theta_k - \rho\left(\beta^{\ell}\right)^2 f(x_k),$$

then set $\ell_k := \ell$, $\alpha_k = \beta^{\ell_k}$, $d_k = -\sigma_k F(x_k)$ and go to Step 3. Otherwise, go to Step 2.3.
Step 2.3. If

$$f\left(x_k + \beta^{\ell}\sigma_k F(x_k)\right) \leq f(x_k) + \theta_k - \rho\left(\beta^{\ell}\right)^2 f(x_k),$$

then set $\ell_k := \ell$, $\alpha_k = \beta^{\ell_k}$, $d_k = \sigma_k F(x_k)$ and go to Step 3. Otherwise, set $\ell := \ell + 1$ and go to Step 2.2.
Step 3. Set $x_{k+1} = x_k + \alpha_k d_k$, $\theta_{k+1} = \gamma\theta_k$, $k := k+1$ and go to Step 1.

To obtain improved complexity results for Algorithm 3, we will use the additional assumption:

A4. The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone, i.e., there exists a real number $\mu > 0$ such that

$$\left(F(x) - F(y)\right)^T (x - y) \geq \mu\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n.$$

Under assumption A4 we can establish an iteration complexity bound of $\mathcal{O}\left(|\log(\epsilon)|\right)$ for Algorithm 3.

Theorem 3 *Suppose that A1-A4 hold and let $\{x_k\}_{k\geq 0}$ be generated by Algorithm 3 such that*

$$f(x_k) > 0, \quad \forall k \geq 0. \tag{39}$$

Given $\epsilon > 0$, if $\theta_0 \leq (1-\gamma)\epsilon/2$ and $\rho \in (0,1)$, then

$$f(x_k) \leq (1-\kappa_f)^k f(x_0) + \frac{\epsilon}{2}, \quad \forall k \geq 1, \tag{40}$$

where

$$\kappa_f = \rho \min\left\{1, \left(\frac{2\beta\sigma_{min}\mu}{\rho + L\sigma_{max}^2}\right)^2\right\}. \tag{41}$$

Consequently,

$$f(x_k) \leq \epsilon, \quad \forall k \geq \frac{\log\left(2f(x_0)\epsilon^{-1}\right)}{|\log(1-\kappa_f)|} \tag{42}$$

Proof By A4 and A1, we have[4]

$$v^T J(x) v \geq \mu \|v\|_2^2, \quad \forall x, v \in \mathbb{R}^n. \tag{43}$$

Combining (39), (43) and Lemma 3, it follows that

$$\alpha_k \geq \min\left\{1, \frac{2\beta\sigma_{min}|F(x_k)^T J(x_k) F(x_k)|}{(\rho + L\sigma_{max}^2)\|F(x_k)\|^2}\right\} \geq \min\left\{1, \frac{2\beta\sigma_{min}\mu}{(\rho + L\sigma_{max}^2)}\right\} \tag{44}$$

In view of Steps 2 and 3 of Algorithm 3, by (44) and (41) we obtain

$$\theta_k + f(x_k) - f(x_{k+1}) \geq \rho\alpha_k^2 f(x_k) \geq \kappa_f f(x_k), \quad \forall k \geq 0.$$

Therefore,

$$f(x_{k+1}) \leq (1-\kappa_f) f(x_k) + \theta_k, \quad \forall k \geq 0. \tag{45}$$

From (45) and $\theta_{k+1} = \gamma\theta_k$, we can see that

$$f(x_k) \leq (1-\kappa_f)^k f(x_0) + \theta_0 \sum_{j=0}^{k-1} (1-\kappa_f)^{k-1-j}\gamma^j, \quad \forall k \geq 1.$$

Since $(1-\kappa_f), \gamma \in (0,1)$ and $\theta_0 \leq (1-\gamma)\epsilon/2$, it follows that

$$\begin{aligned} f(x_k) &\leq (1-\kappa_f)^k f(x_0) + \theta_0 \sum_{j=0}^{k-1} \gamma^j \leq (1-\kappa_f)^k f(x_0) + \theta_0 \sum_{j=0}^{+\infty} \gamma^j \\ &= (1-\kappa_f)^k f(x_0) + \frac{\theta_0}{1-\gamma} \leq (1-\kappa_f)^k f(x_0) + \frac{\epsilon}{2}, \end{aligned}$$

that is, (40) holds. Finally, note that

$$(1-\kappa_f)^k f(x_0) \leq \frac{\epsilon}{2}, \quad \forall k \geq \frac{\log(2f(x_0)\epsilon^{-1})}{|\log(1-\kappa_f)|}. \tag{46}$$

Thus, combining (46) and (40), we obtain (42). □

[4] See, e.g., Section 2 in [7].

Remark 7 By (44), we have

$$\alpha_k = \beta^{\ell_k} \geq \min\left\{1, \frac{2\beta\sigma_{min}\mu}{\rho + L\sigma_{max}^2}\right\}$$

and, consequently, the number N_k of evaluation of $f(\,\cdot\,)$ at the k-th iteration of Algorithm 3 is bounded as follows:

$$N_k \leq 2 + \frac{2\log\left(\min\left\{1, \frac{2\beta\sigma_{min}\mu}{\rho + L\sigma_{max}^2}\right\}\right)}{\log(\beta)} \tag{47}$$

Note that the upper bound above is independent of ϵ. Combining this result with Theorem 3, it follows that Algorithm 3 performs at most $\mathcal{O}\left(|\log(\epsilon)|\right)$ evaluations of $f(\,\cdot\,)$ to generate the first iterate x_k such that $f(x_k) \leq \epsilon$.

Under Assumption A4, if $F(x_k) \neq 0$ and $\sigma_k > 0$, we have

$$\langle \nabla f(x_k), \sigma_k F(x_k)\rangle > 0, \quad \forall k.$$

Then, by the Case II in the proof of Lemma 2, it follows that

$$f(x_k - \alpha\sigma_k F(x_k)) \leq f(x_k) + \theta_k - \rho\alpha^2 f(x_k) \tag{48}$$

whenever

$$0 < \alpha \leq \frac{2|\langle \nabla f(x_k), \sigma_k F(x_k)\rangle|}{(\rho + L\sigma_{max}^2)\,\|F(x_k)\|^2}.$$

In particular,

$$0 < \alpha \leq \frac{2\sigma_{min}\mu}{\rho + L\sigma_{max}^2} \implies (48). \tag{49}$$

This means that, under A4, Step 2.3 in Algorithm 3 becomes unnecessary, since we only need to find α_k such that

$$f(x_k - \alpha_k\sigma_k F(x_k)) \leq f(x_k) + \theta_k - \rho\alpha_k^2 f(x_k).$$

On the other hand, in view of inequality (44), each stepsize α_k in Algorithm 3 may be interpreted as an estimate to the constant

$$\tilde{\alpha} = \min\left\{1, \frac{2\beta\sigma_{min}\mu}{(\rho + L\sigma_{max}^2)}\right\}.$$

Since $\alpha_k = \beta^{\ell_k}$ with $\ell_k \geq 0$, the selection of this stepsize completely ignores the previous stepsize α_{k-1}. This memoryless approach make possible the acceptance of stepsizes potentially bigger than $\tilde{\alpha}$, such as $\alpha_k = 1$, which may accelerate the convergence of $f(x_k)$. However, when $\tilde{\alpha}$ is vey small, it follows from (47) that each iteration of Algorithm 3 may require a very large number of evaluations of $f(\,\cdot\,)$.

These remarks motivate our next algorithm, in which $d_k = -\sigma_k F(x_k)$ for all k, and the selection of the stepsize at the k-th iteration takes into account the stepsize from the $(k-1)$-th iteration.

Algorithm 4.

Step 0. Given a starting point $x_0 \in \mathbb{R}^n$ and constants $\gamma, \beta \in (0,1)$, $0 < \sigma_{min} \leq \sigma_0 \leq \sigma_{max}$ and $\alpha_0, \theta_0, \rho > 0$, set $k := 0$.
Step 1. Compute $\sigma_k \neq 0$ such that $|\sigma_k| \in [\sigma_{min}, \sigma_{max}]$, and define $d_k = -\sigma_k F(x_k)$.
Step 2.1. Set $\ell := 0$.
Step 2.2. If $f\left(x_k + \alpha_k \beta^\ell d_k\right) \leq f(x_k) + \theta_k - \rho\left(\alpha_k \beta^\ell\right)^2 f(x_k)$,
then set $\ell_k := \ell$ and go to Step 3. Otherwise, set $\ell := \ell + 1$ and go to Step 2.2.
Step 3. Set $x_{k+1} = x_k + \alpha_k \beta^{\ell_k} d_k$, $\alpha_{k+1} = \alpha_k \beta^{\ell_k - 1}$, $\theta_{k+1} = \gamma \theta_k$, $k := k+1$ and go to Step 1.

Remark 8 It is worth mentioning that the line search condition in Step 2.2 of Algorithm 4 is similar to the one used in the variant of DF-SANE proposed by La Cruz [17].

From (49), we can obtain the following lower bound for α_k in Algorithm 4.

Lemma 5 *Suppose that A1-A4 hold and let $\{\alpha_k\}_{k\geq 0}$ be generated by Algorithm 4. If*

$$f(x_k) > 0 \quad \text{for } k = 0, \ldots, T-1, \tag{50}$$

then

$$\alpha_k \geq \min\left\{\alpha_0, \frac{2\sigma_{min}\mu}{\rho + L\sigma_{max}^2}\right\} \equiv \check{\alpha} \tag{51}$$

for $k = 0, \ldots, T$.

Proof Inequality (51) is obviously true for $k = 0$. Suppose that (51) is also true for some k, with $0 \leq k \leq T-1$. If $\ell_k = 0$, then by the induction assumption we have

$$\alpha_{k+1} = \beta^{-1}\alpha_k > \alpha_k \geq \check{\alpha},$$

that is, (51) holds for $k+1$. Now, assume that $\ell_k \geq 1$. In this case, by the definition of ℓ_k we have

$$f(x_k + \alpha_{k+1} d_k) > f(x_k) + \theta_k - \rho\alpha_{k+1}^2 f(x_k). \tag{52}$$

By (50) and Assumption A4, we also have

$$\langle \nabla f(x_k), F(x_k)\rangle \neq 0. \tag{53}$$

Then, it follows from A4 and (49) that

$$\alpha_{k+1} > \frac{2\sigma_{min}\mu}{\rho + L\sigma_{max}^2}.$$

Therefore, (51) also holds for $k+1$. □

In view of Lemma 5, we also can establish an iteration complexity bound of $\mathcal{O}\left(|\log(\epsilon)|\right)$ for Algorithm 4.

Theorem 4 *Suppose that A1-A4 hold and let* $\{x_k\}_{k=0}^{T}$ *be generated by Algorithm 4 such that*

$$f(x_k) > \epsilon \quad \textit{for } k = 0, \ldots, T, \tag{54}$$

with $\epsilon > 0$. *If* $\alpha_0 = 1$, $\theta_0 \leq (1-\gamma)\epsilon/2$ *and* $\rho \in (0,1)$, *then*

$$T < \frac{\log\left(2f(x_0)\epsilon^{-1}\right)}{|\log(1-\breve{\kappa}_f)|}, \tag{55}$$

where

$$\breve{\kappa}_f = \rho \min\left\{\alpha_0\beta, \left(\frac{2\beta\sigma_{min}\mu}{\rho + L\sigma_{max}^2}\right)\right\}^2. \tag{56}$$

Proof Combining Steps 2 and 3 of Algorithm 4, Lemma 5, (54) and (56), we have

$$\theta_k + f(x_k) - f(x_{k+1}) \geq \rho\left(\alpha_{k+1}\beta\right)^2 f(x_k) \geq \breve{\kappa}_f f(x_k), \quad \text{for } k = 0, \ldots, T-1. \tag{57}$$

From (57), the bound (55) follows as in the proof of Theorem 3. □

Remark 9 Note that at its i-th iteration, Algorithm 4 performs at most $\ell_i + 1$ evaluations of $f(\,\cdot\,)$. From the definition of α_{i+1}, we have

$$\ell_i + 1 = 2 + \frac{\log(\alpha_{i+1}) - \log(\alpha_i)}{\log(\beta)}.$$

Thus, denoting by $T_f(k)$ the total number of evaluations of $f(\,\cdot\,)$ performed up to the k-th iteration of Algorithm 4 with $\alpha_0 = 1$, it follows that

$$T_f(k) = \sum_{i=0}^{k} \ell_i + 1 \leq 2(k+1) + \frac{\log(\alpha_{k+1}) - \log(\alpha_k)}{\log(\beta)} \leq 2(k+1) + \frac{\log(\breve{\alpha})}{\log(\beta)}. \tag{58}$$

In particular, by Theorem 4, if $\bar{k}$ is the first iteration for which $f(x_{\bar{k}}) \leq \epsilon$, then $T_f(\bar{k}) \leq \mathcal{O}\left(|\log(\epsilon)|\right)$, i.e., Algorithm 4 has an evaluation complexity of $\mathcal{O}\left(|\log(\epsilon)|\right)$. Moreover, it also follows from (58) that

$$\frac{1}{k} T_f(k) \leq 2\left(1 + \frac{1}{k}\right) + \frac{1}{k}\frac{\log(\breve{\alpha})}{\log(\beta)}.$$

Therefore, for Algorithm 4, the average number of function evaluations per iteration is asymptotically bounded by 2, which can be significantly smaller than N_k for Algorithm 4 (please, recall Remark 7).

5 Illustrative Numerical Results

We performed preliminary numerical experiments comparing the following Octave implementations:

- **DF-SANE**: the non-monotone algorithm in [16] that corresponds to Algorithm 1 with $\theta_k = \dfrac{\|F(x_0)\|}{(1+k)^2}$ and
$$\nu_k = \max_{0\leq j \min\{k,M-1\}} [f(x_{k-j})] - f(x_k),$$
with $M = 10$.
- **N-DF-SANE**: the non-monotone algorithm in [5] that corresponds to Algorithm 2 with $\theta_k = \dfrac{\|F(x_0)\|}{(1+k)^2}$ and
$$\delta_{k+1} = \frac{1}{\eta_k Q_k + 1},$$
where $Q_0 = 1$, $Q_{k+1} = \eta_k Q_k + 1$ and $\eta_k = 0.85$.
- **NM1**: Algorithm 3 with $\theta_0 = (1-\gamma)\epsilon/2$ and $\gamma = 0.5$.
- **NM2**: Algorithm 4 with $\theta_0 = (1-\gamma)\epsilon/2$ and $\gamma = 0.5$.

In all implementations we consider parameters $\sigma_{min} = 10^{-1}$, $\sigma_{max} = 10^{10}$, $\alpha_0 = \sigma_0 = 1$, $\beta = 0.5$ and $\rho = 10^{-4}$. The spectral stepsize σ_k is computed as in [16]. Specifically, let
$$\tilde{\sigma}_k = \frac{\langle s_k, s_k\rangle}{\langle s_k, y_k\rangle},$$
where $s_k = x_k - x_{k-1}$ and $y_k = F(x_k) - F(x_{k-1})$. We set $\sigma_k = \tilde{\sigma}_k$ whenever $|\tilde{\sigma}_k| \in [\sigma_{min}, \sigma_{max}]$. Otherwise, we set
$$\sigma_k = \begin{cases} 1, & \text{if} \quad \|F(x_k)\| > 1, \\ \|F(x_k)\|^{-1}, & \text{if} \quad 10^{-5} \leq \|F(x_k)\| \leq 1, \\ 10^5, & \text{if} \quad \|F(x_k)\| < 10^{-5}. \end{cases}$$

In our first experiment, we applied codes DF-SANE, N-DF-SANE and NM1 (with $\epsilon = 10^{-7}$) to a set of 33 unconstrained optimization problems from [22], using the same dimensions as in [9]. Specifically, n ranges from 2 to 20. In our tests, we applied the codes to find zeros of the gradients of the objective functions. The comparison between the codes was done using *data profiles* proposed in [23]. Specifically, let $\{x_k^s\}_{k\geq 0}$ be the sequence generated by solver $s \in \mathcal{S}$ applied to a certain problem, where $\mathcal{S}$ is the set of solvers being compared. The smallest value of $f(\cdot)$ obtained by any solver in $\mathcal{S}$ within a given budget of μ_f function evaluations is
$$f_L = \min_{k=0,\ldots,\mu_f} \left(\min_{s\in\mathcal{S}} \{f(x_k^s)\} \right).$$

We declare that the problem was solved by the solver s with tolerance $\tau \in (0,1)$, when this solver generates a point x_k^s such that

$$f(x_0^s) - f(x_k^s) \geq (1-\tau)(f(x_0^s) - f_L). \tag{59}$$

Let $t_{p,s}$ be the number of function evaluations required by solver s to generate x_k^s satisfying (59) for problem p. Then, the percentage of problems solved by solver s with α function evaluations is given by

$$d_s(\alpha) = \frac{\text{number of problems for which } t_{p,s} \leq \alpha}{\text{total number of problems}}.$$

Figure 1 contains the graphs of $d_s(\,\cdot\,)$ for the solvers DF-SANE, N-DF-SANE and NM1 with respect to $\tau = 10^{-5}$ and a budget of $\mu_f = 1000$ function evaluations. As we can see, NM1 solved more problems than DF-SANE and N-DF-SANE using the same number of function evaluations.

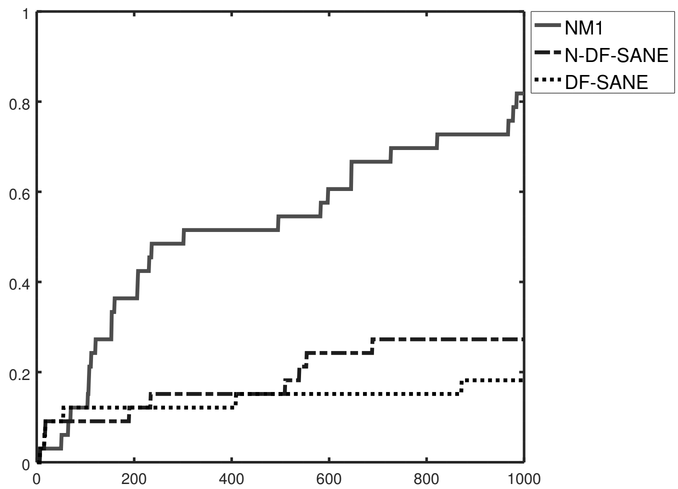

Fig. 1 Percentage of problems solved as a function of the number of function evaluations.

In our second experiment, we applied the same three codes to the set of 20 test problem described in [15]. For all problems we considered $n = 540$. Moreover, for NM1 we used $\epsilon = 10^{-7}$. Figure 2 contains the data profies for $\tau = 10^{-5}$ and a budget of $\mu_f = 1000$ function evaluations. Again, NM1 solved more problems than DF-SANE and N-DF-SANE with the same budget of function evaluations.

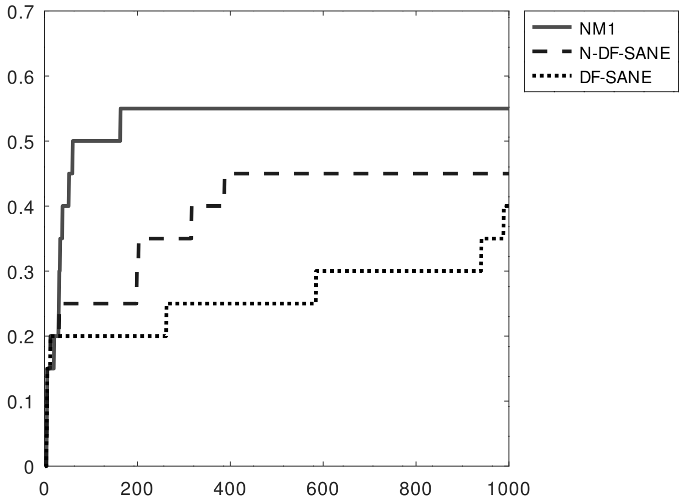

Fig. 2 Percentage of problems solved as a function of the number of function evaluations.

In our third experiment, we applied codes NM1 and NM2 to solve a nonlinear equation of the form

$$\nabla g(x) = 0, \tag{60}$$

with $g : \mathbb{R}^n \to \mathbb{R}$ given by

$$g(x) = -\sum_{i=1}^{m} \left[b^{(i)} \log(m_x(a^{(i)})) + (1 - b^{(i)}) \log(1 - m_x(a^{(i)})) \right] + \frac{\mu}{2}\|x\|_2^2,$$

where $\left\{(a^{(i)}, b^{(i)})\right\}_{i=1}^{m} \subset \mathbb{R}^n \times \{0, 1\}$ is the dataset, $m_x(a) \equiv 1/(1 + e^{-\langle a,x \rangle})$ is the logistic model, and $\mu > 0$ is the regularization parameter. Since $g(\,\cdot\,)$ is μ-strongly convex, it follows that $F(x) = \nabla g(x)$ satisfies

$$\langle F(y) - F(x), y - x \rangle \geq \mu \|y - x\|_2^2, \quad \forall x, y \in \mathbb{R}^n,$$

i.e., $F(\,\cdot\,)$ is strongly monotone. Specifically, we considered the Sonar dataset [8] , with $n = 61$, $m = 208$, $a_1^{(i)} = 1$ for $i = 1, \ldots, m$ and $\mu = 1$. As starting point we used $x_0 = \left[\, 0 \ldots 0 \,\right]^T \in \mathbb{R}^{61}$ in both codes. The results are presented in Table 1, where $\text{IT}(\epsilon)$ and $\text{FE}(\epsilon)$ represent the number of iterations and the number of function evaluations, respectively, required by the codes to generate the first iterate x_k such that $f(x_k) \leq \epsilon$.
As we can see in Table 1, for both codes we have

$$\text{IT}(10^{-q}) \leq q \times \text{IT}(10^{-1}) \quad \text{and} \quad \text{FE}(10^{-q}) \leq q \times \text{FE}(10^{-1}) \quad \text{for } q = 1, \ldots, 10.$$

This result is in accordance with Theorems 3 and 4. Moreover, for NM2 we also have the ralation

$$\text{FE}(10^{-q}) \approx 2 \times \text{IT}(10^{-q}) \quad \text{for } q = 1, \ldots, 10,$$

which was anticipated in Remark 9.

	NM1		NM2	
ϵ	IT(ϵ)	FE(ϵ)	IT(ϵ)	FE(ϵ)
10^{-1}	223	3178	177	359
10^{-2}	325	4630	277	560
10^{-3}	446	6431	395	794
10^{-4}	592	8379	530	1074
10^{-5}	734	10411	721	1449
10^{-6}	872	12555	860	1737
10^{-7}	1034	14727	1032	2068
10^{-8}	1173	17148	1158	2321
10^{-9}	1334	19343	1384	2774
10^{-10}	1483	21596	1606	3216

Table 1 Numerical results for equation (60) with the Sonar dataset [8].

6 Conclusion

In this paper we investigated the worst-case evaluation complexity of a wide class of derivative-free nonmonotone methods for nonlinear systems of equations of the form $F(x) = 0$. In these methods, the nonmonotonicity is controlled by two summable sequences, $\{\nu_k\}_{k\geq 0} \subset \mathbb{R}_+$ and $\{\theta_k\}_{k\geq 0} \subset \mathbb{R}_+ \setminus \{0\}$ that define the line search procedure. We proved that if the Jacobian of the mapping $F(\,\cdot\,)$ is Lipschitz continuous, then the methods in the referred class take at most $\mathcal{O}\left(\epsilon^{-2}\right)$ iterations to generate x_k such that $\psi(x_k) \leq \epsilon$, where $\psi(\,\cdot\,)$ is a stationarity measure for the merit function $f(x) = (1/2)\|F(x)\|_2^2$. From this iteration-complexity bound we obtained a lim-type global convergence result and also an evaluation-complexity bound of $\mathcal{O}\left(|\log(\epsilon)|\epsilon^{-2}\right)$. We showed that our results also apply to the N-DF-SANE method proposed in [5]. Moreover, for the case in which the mapping $F(\,\cdot\,)$ is strongly monotone, we propose two methods (namely, Algorithms 3 and 4) with evaluation-complexity of $\mathcal{O}\left(|\log(\epsilon)|\right)$. Finally, we presented preliminary numerical results comparing implementations of Algorithms 3 and 4, and implementations of DF-SANE [16] and N-DF-SANE [5]. These numerical results confirm our theoretical findings. The generality of our analysis in terms of possible choices for $\{\nu_k\}_{k\geq 0}$ and $\{\theta_k\}_{k\geq 0}$ allow more freedom for the design of new derivative-free nonmonotone methods for nonlinear systems of equations, with worst-case complexity guarantees.

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