

# Subgradient methods near active manifolds: saddle point avoidance, local convergence, and asymptotic normality

Damek Davis<sup>1</sup>      Dmitriy Drusvyatskiy<sup>2</sup>      Liwei Jiang<sup>3</sup>

Last updated: August 26, 2021

<sup>1</sup>School of ORIE, Cornell University, Ithaca, NY 14850, USA; [people.orie.cornell.edu/dsd95/](http://people.orie.cornell.edu/dsd95/). Research of Davis supported by an Alfred P. Sloan research fellowship and NSF DMS award 2047637.

<sup>2</sup>Department of Mathematics, U. Washington, Seattle, WA 98195; [www.math.washington.edu/~ddrusv](http://www.math.washington.edu/~ddrusv). Research of Drusvyatskiy was supported by NSF DMS-1651851 and CCF-2023166 awards.

<sup>3</sup>School of ORIE, Cornell University. Ithaca, NY 14850, USA; [orie.cornell.edu/research/grad-students/liwei-jiang](http://orie.cornell.edu/research/grad-students/liwei-jiang)

## Abstract

Nonsmooth optimization problems arising in practice, whether in signal processing, statistical estimation, or modern machine learning, tend to exhibit beneficial smooth substructure: their domains stratify into “active manifolds” of smooth variation, which common proximal algorithms “identify” in finite time. Identification then entails a transition to smooth dynamics, and permits the use of second-order information for acceleration. While identification is clearly useful algorithmically, empirical evidence suggests that even those algorithms that do not identify the active manifold in finite time—notably the subgradient method—are nonetheless affected by it. This work seeks to explain this phenomenon, asking: how do active manifolds impact the subgradient method in nonsmooth optimization?

To answer this question, our approach posits two algorithmically useful properties that link the behavior of the function on and off the active manifold. The first, which we call *aiming*, asserts that subgradients point towards the manifold. This property ensures that, though identification fails, the subgradient iterates steadily approach the manifold. The second property states that subgradients on and off the manifold are close in tangent directions up to a linear error. This property ensures that the nonsmooth dynamics of the subgradient method are well-approximated by their smooth shadow along the manifold, with a controlled error. We show that these properties, while not automatic, hold for a wide class of problems, including cone reducible/decomposable functions and generic semialgebraic problems. Moreover, we develop a thorough calculus, proving such properties are preserved under smooth deformations and spectral lifts.

We then turn to algorithmic consequences. Here, the two pillars—aiming and subgradient approximation—fully expose the smooth substructure of the problem, implying that the shadow of the (stochastic) subgradient method along the active manifold is precisely an inexact Riemannian gradient method with an implicit retraction. This viewpoint leads to several consequences that parallel results in smooth optimization, despite the nonsmoothness of the problem: local rates of convergence, asymptotic normality, and saddle point avoidance. The asymptotic normality results appear to be new even in the most classical setting of stochastic nonlinear programming. The results culminate in the following observation: the perturbed subgradient method on generic, Clarke regular semialgebraic problems, converges only to local minimizers.

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Chapter 2: Strong (a) regularity and (b) regularity. . . . .	6
1.2	Chapter 3: Algorithmic consequences . . . . .	6
1.2.1	Local rates of convergence . . . . .	7
1.2.2	Asymptotic normality . . . . .	8
1.2.3	Saddle point avoidance . . . . .	9
1.3	Outline . . . . .	10
1.4	Notation and basic constructions . . . . .	10
1.4.1	Manifolds . . . . .	11
1.4.2	Normal cones . . . . .	12
1.4.3	Subdifferentials . . . . .	13
1.4.4	Active manifolds . . . . .	13
<b>2</b>	<b>Regularity along a manifold: conditions of Whitney and Verdier in nonsmooth optimization</b>	<b>17</b>
2.1	The four fundamental regularity conditions . . . . .	17
2.2	Relation between the four conditions . . . . .	23
2.3	Basic examples . . . . .	26
2.4	Preservation of regularity under preimages by smooth transversal maps . . .	28
2.5	Preservation of regularity under spectral lifts . . . . .	30
2.6	Regularity of functions along manifolds . . . . .	33
2.7	Generic regularity along active manifolds . . . . .	35
<b>3</b>	<b>Algorithmic Consequences: local rates of convergence, asymptotic normality, and saddle point avoidance</b>	<b>38</b>
3.1	Algorithm and main assumptions . . . . .	38
3.2	Example algorithms satisfying Assumption A . . . . .	40
3.2.1	Projected subgradient method . . . . .	41
3.2.2	Proximal gradient method . . . . .	42
3.3	The two pillars . . . . .	44
3.3.1	Pillar I: Aiming towards the manifold . . . . .	45
3.3.2	Pillar II: The shadow iteration . . . . .	46
3.4	Local convergence guarantees . . . . .	47
3.5	Asymptotic normality . . . . .	48
3.5.1	Asymptotic normality in nonlinear programming . . . . .	50

3.6	Avoiding saddle points . . . . .	52
3.6.1	Consequences for generic semialgebraic functions . . . . .	54
3.7	Outline . . . . .	56
3.8	Proofs of the two pillars . . . . .	56
3.8.1	Proof of Proposition 3.3.1: aiming towards the manifold . . . . .	56
3.8.2	Proof of Proposition 3.3.2: the shadow iteration . . . . .	58
3.9	Proofs of the main theorems . . . . .	61
3.9.1	Proof of Theorem 3.4.1: general rates . . . . .	62
3.9.2	Proof of Theorem 3.4.2: rates near strong local minimizers . . . . .	63
3.9.3	Proof of Theorem 3.5.1: asymptotic normality . . . . .	66
3.9.4	Proof of Theorem 3.6.1: nonconvergence of stochastic process . . . . .	73
3.9.5	Proof of Theorem 3.6.2: nonconvergence to saddle points . . . . .	80
3.10	Appendix to Part 1 . . . . .	88
3.10.1	Proof of Proposition 2.3.3 . . . . .	88
3.11	Appendix to Part 2 . . . . .	89
3.11.1	Proof of Proposition 3.2.3: the projected gradient method . . . . .	89
3.11.2	Proof of Proposition 3.2.5: the proximal gradient method . . . . .	92
3.11.3	Proof of Corollary 3.5.2: asymptotic normality in nonlinear programming	96
3.11.4	Proof of Corollary 3.6.3: avoiding active strict saddle via projected subgradient method . . . . .	96
3.11.5	Proof of Corollary 3.6.4: avoiding active strict saddle via proximal gradient method . . . . .	97
3.11.6	Proofs of Corollaries 3.6.5, 3.6.6, and 3.6.7: saddle point avoidance for generic semialgebraic problems. . . . .	97
3.11.7	Sequences and Stochastic Processes . . . . .	97
3.11.8	Proof of Lemma 3.11.6 . . . . .	100

# Chapter 1

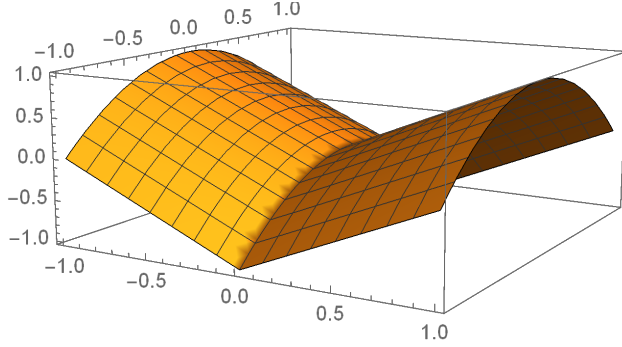
## Introduction

The notion of a “smooth active manifold” appears throughout the nonsmooth optimization literature, leading to transparent views of optimality conditions, sensitivity analysis, and algorithmic behavior. Formal models of active manifolds include identifiable surfaces [95], partly smooth manifolds [57],  $\mathcal{UV}$ -structures [54, 67], decomposable functions [82], and minimal identifiable sets [32]. Active manifolds pervade modern problems in signal recovery, robust statistics, and deep learning, and classically feature in “active set” methods of nonlinear programming. In these problems nonsmoothness is not pathological, but is highly-structured: domains stratify into manifolds of smooth variation, and under strict complementarity conditions, the manifolds become “active,” that is, quickly identifiable by common proximal methods. Once identified, smooth dynamics of proximal methods automatically ensue along the “active manifold,” and accommodate second-order acceleration techniques if the manifold is known [60, 67]. While identification has clear utility, empirical evidence suggests that non-identifying algorithms, such as the subgradient method, continue to benefit from the mere existence of active manifolds. This work seeks to explain this phenomenon, asking:

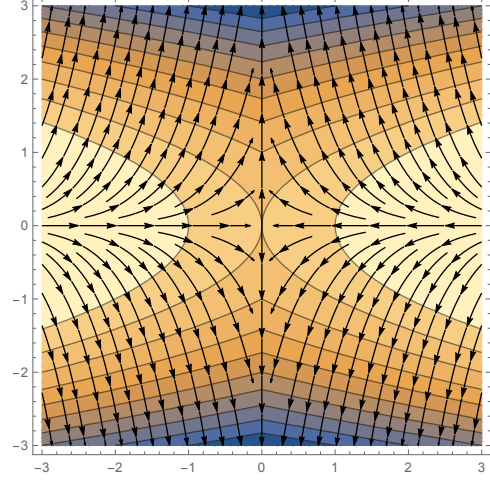
How do active manifolds impact subgradient methods?

Setting the stage, consider a locally Lipschitz function  $f$  on  $\mathbf{R}^d$  and let  $\bar{x}$  be a first-order critical point of  $f$  in the sense that the directional derivative of  $f$  at  $\bar{x}$  is nonnegative in every direction. Following [32], a smooth submanifold  $\mathcal{M}$  of  $\mathbf{R}^d$  is called *active* for  $f$  at  $\bar{x}$  if (i) the restriction of  $f$  to  $\mathcal{M}$  is smooth near  $\bar{x}$ , and (ii) all points  $x$  near  $\bar{x}$  with small Clarke subgradients  $w \in \partial_c f(x)$  must lie in  $\mathcal{M}$  (see Figure 1.1).

The algorithmic importance of active manifolds stems from the following observation: any algorithm that generates a sequence of points  $x_k$  converging to  $\bar{x}$ , along with “dual certificates”  $w_k \in \partial_c f(x_k)$  tending to zero, must eventually identify the manifold  $\mathcal{M}$  in the sense that all the iterates  $x_k$  eventually lie in  $\mathcal{M}$ . Once  $\mathcal{M}$  is identified, the nonsmoothness of the problem is largely irrelevant, since all future iterates lie on a smooth manifold along which  $f$  is smooth. Not all algorithms generate vanishing dual certificates, but several important examples exist, such as the proximal gradient method [42], the dual averaging procedure [53], and various operator-splitting schemes [58, 62]. An important requirement of all such methods, however, is that there exist an explicit decomposition of the problem into smooth and proximable terms.



(a) The function  $f(x, y)$



(b) Subgradient flow  $\dot{\gamma} \in -\partial f(\gamma)$

Figure 1.1: The  $y$ -axis is an active manifold for the function  $f(x, y) = |x| - y^2$  at the origin.

The subgradient method, on the other hand, is an important algorithm that does not generate asymptotically vanishing dual certificates. Given a control sequence  $\alpha_k > 0$  the method repeats the steps:

$$x_{k+1} = x_k - \alpha_k w_k \quad \text{where } w_k \in \partial_c f(x_k). \quad (1.0.1)$$

Though the subgradient method does not identify the active manifold in finite time, it is empirically strongly influenced by it. As an illustration, Figure 1.1 depicts a nonsmooth function, having the  $x$ -axis as its active manifold. The function has a unique critical point at the origin and directions of second-order negative curvature along the active manifold—a so-called strict saddle point. As one would hope for, Figure 1.1b shows the continuous time analogue of the subgradient method, when randomly initialized, almost surely avoids the origin, and moreover, one can check that similar behavior persists in discrete time. Thus, although the subgradient method never reaches the active manifold, it nonetheless inherits desirable properties from the function along the manifold, e.g., saddle point avoidance.

The purpose of this work is to offer a revealing explanation of the dynamics of the subgradient method near active manifolds. Our central observation is that under two mild regularity conditions, which we will describe shortly, the following is true:

the shadow of the (stochastic) subgradient method along the active manifold is precisely an inexact Riemannian gradient method with an implicit retraction.

More formally, we will find that the shadow sequence  $y_k = P_{\mathcal{M}}(x_k)$ , satisfies a recursion

$$y_{k+1} = y_k - \alpha_k \nabla_{\mathcal{M}} f(y_k) + O(\alpha_k \text{dist}(x_k, \mathcal{M}) + \alpha_k^2), \quad (1.0.2)$$

near  $\bar{x}$ , where  $\nabla_{\mathcal{M}} f$  denotes the *covariant gradient* of  $f$  along  $\mathcal{M}$ .<sup>1</sup> This observation allows us to infer desirable properties for the subgradient sequence  $x_k$  from those of its shadow  $y_k$ .

<sup>1</sup>The covariant gradient  $\nabla_{\mathcal{M}} f(y)$  is the projection onto  $T_{\mathcal{M}}(y)$  of  $\nabla \hat{f}(y)$  where  $\hat{f}$  is any  $C^1$  smooth function defined on a neighborhood  $U$  of  $\bar{x}$  and that agrees with  $f$  on  $U \cap \mathcal{M}$ .

Concretely, such desirable properties include local rates of convergence, asymptotic normality of the iterates, and saddle point avoidance.

To link the iterates of the subgradient method with a Riemannian gradient sequence along the active manifold, we require two regularity properties that we now describe.

**Regularity property I: aiming towards the manifold.** Although the subgradient method fails to identify the active manifold, the iterates it generates often steadily approach the manifold. In this work, we will show that a sufficient condition for this behavior is the following *proximal aiming* inequality:

$$\langle v, x - P_{\mathcal{M}}(x) \rangle \geq c \cdot \text{dist}(x, \mathcal{M}) \quad \text{for all } x \text{ near } \bar{x} \text{ and } v \in \partial f_c(x). \quad (1.0.3)$$

In words, the proximal aiming inequality asserts that subgradients are well aligned with directions pointing towards the nearest point on the manifold. A natural question is when should we expect (1.0.3) to hold. For example, it is not hard to see that (1.0.3) holds if  $f$  is weakly convex;<sup>2</sup> indeed, this follows directly from [25, Theorem D.2]. Weak convexity is not essential, however; instead (1.0.3) holds under the following property that is unrelated to convexity:

$$f(x) \geq f(P_{\mathcal{M}}(x)) + \langle v, x - P_{\mathcal{M}}(x) \rangle + o(\text{dist}(x, \mathcal{M})) \quad \text{as } x \rightarrow \bar{x} \text{ with } v \in \partial_c f(x). \quad (1.0.4)$$

We call this condition *(b)-regularity of  $f$  along  $\mathcal{M}$  at  $\bar{x}$* , for reasons that will become apparent shortly. In the simplest setting when  $\mathcal{M}$  is the singleton  $\{\bar{x}\}$ , this condition is precisely (one-sided) semi-smoothness of the function  $f$  in the sense of [66]. More generally, the condition (1.0.4) is a uniformization of semi-smoothness relative to  $\mathcal{M}$ . Summarizing, (b)-regularity along the active manifold implies the key aiming condition (1.0.3).

**Regularity property II: subgradients on and off the manifold.** The second regularity property posits that subgradients on and off the manifold are aligned in tangent directions up to a linear error, that is, there exists  $C > 0$  satisfying

$$\|P_{T_{\mathcal{M}}(y)}(\partial_c f(x) - \nabla_{\mathcal{M}} f(y))\| \leq C \cdot \|x - y\| \quad \text{for all } x \in \mathbf{R}^d \text{ and } y \in \mathcal{M} \text{ near } \bar{x}. \quad (1.0.5)$$

Whenever (1.0.5) holds, we say that  $f$  is *strongly (a)-regular along  $\mathcal{M}$* , for reasons that will become apparent shortly. An illustrative example is shown in Figure 1.1: there the function  $f$  is smooth in tangent directions, implying the relation between subgradients on and off the manifold. In general, strong (a) regularity requires  $f$  to vary smoothly in tangent directions to the manifold only up to a linear error.

The rest of this introduction discusses the main contents of this work, which are broken down into two chapters. Chapter 2 develops the strong (a) and (b) regularity properties including basic examples, calculus, and genericity guarantees. Chapter 3 of the paper focuses on algorithmic consequences. We now describe these parts in detail.

---

<sup>2</sup>A function  $f$  is weakly convex if the perturbed function  $x \mapsto f(x) + \frac{\rho}{2}\|x\|^2$  is convex for some  $\rho > 0$ .

## 1.1 Chapter 2: Strong (a) regularity and (b) regularity.

Conditions (b) and strong (a) play a central role in our work and are explored in detail in Chapter 2. The two properties, both in content and name, are entirely motivated by classical regularity conditions in stratification theory due to Whitney [92–94], Kuo [46], and Verdier [90]. There is an important distinction, however, that is worth emphasizing. Regularity conditions in stratification theory deal with compatibility between two smooth manifolds; the central goal is to prove that reasonable sets (e.g. semi-algebraic) can be partitioned into smooth manifolds so that the regularity condition holds for every adjacent pair of manifolds. In contrast, we will be concerned with compatibility between a *specific nonsmooth set*—the epigraph of  $f$ —and the *specific manifold*—the graph of the restriction of  $f$  to the active manifold  $\mathcal{M}$ .

The broad goal of the chapter is to convince the reader that conditions (b) and strong (a) are common in optimization problems and are easy to work with. To that end, we begin by defining regularity conditions for sets, focusing on building geometric intuition, and then pass to functions by means of epigraphs. A basic question arises immediately: are conditions (b) and strong (a) related? We provide a satisfying answer generalizing the results of Kuo [45], Verdier [90], and Ta Le Loi [48] in stratification theory. We will show that if  $f$  and  $\mathcal{M}$  are definable in an o-minimal structure, then strong (a) regularity implies condition (b). Thus, in most interesting examples, condition (b) follows automatically from strong (a).

Having developed basic definitions and explored their interplay, we present a number of examples that are relevant to optimization. In particular, we show that a sublinear function is both (b) and strongly (a) regular along its lineality space. We then develop a thorough calculus, which in particular implies that the regularity conditions are preserved under a transversal pre-composition with a smooth map. Thus, decomposable functions of [82], under a transversality condition, are both (b) and strongly (a) regular along their active manifolds. We moreover argue that the two conditions are common in eigenvalue problems because they satisfy the so-called transfer principle. Namely, any orthogonally invariant function of symmetric matrices will satisfy the regularity condition, as long as its restriction to diagonal matrices satisfies the analogous property. We end Chapter 2 of the paper by showing that if  $f$  is a definable function, then for almost all perturbations  $v \in \mathbf{R}^d$ , the tilted function  $f_v(x) = f(x) - \langle v, x \rangle$  admits at most finitely many first-order critical points, each lying on an active manifold along which  $f_v$  is both (b) and strongly (a) regular. Summarizing the chapter, typical functions, whether built from concrete structured examples or from unstructured linear perturbations, admit an active manifold around each critical point along which the objective function is both (b) and strongly (a) regular.

## 1.2 Chapter 3: Algorithmic consequences

Chapter 3 of the paper develops algorithmic consequences of strong (a)-regularity and (b) regularity along the active manifold: local rates of convergence, asymptotic normality, and saddle-point avoidance. Throughout the paper we consider a broad family of algorithms,



but our guarantees are easiest to state for the constrained minimization problem

$$\min_{x \in \mathcal{X}} f(x), \quad (1.2.1)$$

where  $\mathcal{X} \subseteq \mathbb{R}^d$  is closed and  $f$  is locally Lipschitz. Active manifolds  $\mathcal{M} \subseteq \mathcal{X}$  and the two regularity conditions naturally extend to this setting by adding to  $f$  the indicator function of  $\mathcal{X}$ ; see section 1.4.4 for details. For this problem class, we consider the stochastic projected subgradient method, which repeats the steps:

$$x_{k+1} = P_{\mathcal{X}}(x_k - \alpha_k(w_k + \nu_k)) \quad \text{where } w_k \in \partial_c f(x_k) \quad (1.2.2)$$

and  $\nu_k$  is zero mean stochastic error and  $\alpha_k = 1/k^\gamma$  for  $\gamma \in (1/2, 1)$ . For this algorithm, we will show that the shadow sequence

$$y_k = P_{\mathcal{M}}(x_k)$$

is still locally an inexact stochastic Riemannian gradient sequence with implicit retraction as in (1.0.2). Building on this observation, we extend several classical properties of (stochastic) gradient method sequences to our setting.

### 1.2.1 Local rates of convergence

The most immediate consequence of (1.0.2) is that the covariant gradient tends to zero at a controlled rate along the shadow of the iterate sequence: around every critical point  $\bar{x}$  contained in an active manifold, if  $x_k$  enters and remains indefinitely in a small neighborhood of  $\bar{x}$ , then

$$\min_{i=1, \dots, k} \text{dist}(x_i, \mathcal{M}) = O\left(\frac{1}{k^{\gamma-\epsilon}}\right) \quad \text{and} \quad \min_{i=1, \dots, k} \|\nabla_{\mathcal{M}} f(y_i)\| = O\left(\frac{1}{k^{(1-\gamma)/2}}\right).$$

for every  $\epsilon > 0$ .<sup>3</sup> If the problem (1.2.1) satisfies further regularity properties, we prove stronger rates of convergence for the distance  $\|x_k - \bar{x}\|$  itself. For example, the standard second-order sufficient condition for optimality asks that  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  is positive definite on the tangent space  $T_{\mathcal{M}}(\bar{x})$ : there exists  $\sigma > 0$  satisfying

$$u^\top \nabla_{\mathcal{M}}^2 f(\bar{x}) u \geq \sigma \|u\|^2 \quad \text{for all } u \in T_{\mathcal{M}}(\bar{x}). \quad (1.2.3)$$

Here  $\nabla_{\mathcal{M}}^2 f$  denotes the covariant Hessian of  $f$ .<sup>4</sup> In this setting, we show that if  $x_k$  enters and remains indefinitely in a small neighborhood of  $\bar{x}$ , then

$$\min_{i=1, \dots, k} \|x_i - \bar{x}\|^2 = O\left(\frac{1}{k^{\gamma-\epsilon}}\right).$$

for all  $\epsilon > 0$ . Intriguingly, this rate is near optimal, even for stochastic gradient methods in smooth and strongly convex optimization problems [71].

---

<sup>3</sup>The constants in the  $O$  term also depend on the sample path throughout.

<sup>4</sup>When  $f$  is  $C^3$ -smooth,  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  is simply the compression of  $\nabla^2(f \circ P_{\mathcal{M}})(\bar{x})$  to the tangent space  $T_{\mathcal{M}}(\bar{x})$ .

## 1.2.2 Asymptotic normality

Polyak and Juditsky [78] famously showed that the stochastic gradient method for minimizing smooth and strongly convex functions enjoys a central limit theorem: the error sequence  $\sqrt{k}(\bar{x}_k - \bar{x})$ , where  $\bar{x}_k := \frac{1}{k} \sum_{i=1}^k x_i$  is an average iterate, converges in distribution to a normal random vector. Moreover, the covariance matrix of the limiting distribution depends on the Hessian of  $f$  and the covariance of the stochastic gradient at the minimizer. In this section, we show that an analogous property holds under reasonable assumptions in nonsmooth optimizations, with the Hessian of  $f$  replaced by its covariant counterpart.

While we prove a more general result in Chapter 3, the guarantees are already interesting in the case of constrained smooth minimization:

$$\min_{x \in \mathcal{X}} f(x) = \mathbb{E}_{z \sim P}[f(x; z)] \quad (1.2.4)$$

where  $P$  is a fixed, unknown probability distribution, and for each  $z$  the function  $x \mapsto f(x; z)$  is  $C^1$ . Algorithm 1.2.2 then becomes the stochastic projected gradient method:

$$\begin{aligned} \text{Sample: } z_k &\sim P \\ \text{Update: } x_{k+1} &\in P_{\mathcal{X}}(x_k - \alpha_k \nabla f(x_k; z_k)). \end{aligned} \quad (1.2.5)$$

For the class of problems (1.2.4), we show that under classical second order sufficient optimality condition (1.2.3) and other mild technical assumptions, the following holds:

If  $x_k$  converges to  $\bar{x}$  with probability 1, the average iterate  $\bar{x}_k = \frac{1}{k} \sum_{i=1}^k x_i$  satisfies

$$\sqrt{k}(\bar{x}_k - \bar{x}) \xrightarrow{d} N(0, \nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger \text{Cov}(\nabla f(\bar{x}, z)) \nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger).$$

Convergence with probability 1 in turn is automatic if  $f$  has a unique critical point  $\bar{x}$  and  $\sup_k \|x_k\| < \infty$  almost surely. To the best of our knowledge, this is the first asymptotic normality guarantee for the standard stochastic projected gradient method even in the non-smooth convex setting.

The result appears to be new and interesting for classical nonlinear programming problems under linear independence of the active constraints, strict complementarity, and strong second order sufficient conditions for optimality. Note that in this case, the covariant Hessian coincides with the Hessian of the Lagrangian pre and post multiplied by the projection onto the tangent space at  $\bar{x}$ . In this context, according to [33, Theorem 1] the result is essentially unimprovable and matches the estimation quality of empirical risk minimization methods [81, Theorem 3.3]. Interestingly, it is known that even for the problem of minimizing a the expectation of a linear function over a ball, dual averaging [72] procedures may achieve suboptimal covariance [33, Section 5.2]. This is surprising to us since (stochastic) dual averaging procedures provably identify the active manifold in finite time [52] (also see [33, Section 4.1]), which intuitively suggests a transition to smooth dynamics, a setting where asymptotic normality traditionally holds with the optimal covariance. In contrast, the projected stochastic gradient method does not identify constraints, and yet exhibits optimal behavior.

### 1.2.3 Saddle point avoidance

The seminal papers [49, 50] prove that simple iterative methods (e.g., gradient descent) for  $C^2$  optimization avoid all *strict* saddle points (critical points that have negative curvature), when randomly initialized. If all saddle points are strict, such methods converge to local minimizers. Leveraging this result, further works (e.g., [4, 37, 38, 85, 86]) verified that a wealth of concrete statistical estimation and learning problems possessed this *strict-saddle property* and also had no spurious local minimizers, implying that simple randomly initialized methods converge to global minimizers. Recent work extended these results to  $C^2$  smooth manifold constrained optimization [17, 36, 87]. Other extensions to *nonsmooth* convex constraint sets are second-order, requiring at every step to minimize a nonconvex quadratic over a convex set, which is NP hard in general [41, 69, 73].

A long-standing open question is how to extend these results to simple iterative methods for nonsmooth optimization. In general, this is a difficult task, since even gradient methods on  $C^1$  functions may converge to saddle points from a positive measure set of initializations, as shown in [24]. To overcome this issue, the recent work [24] provided a new approach, introducing a novel saddle-point concept called *an active strict saddle*. These are critical points  $\bar{x}$  which admit an active manifold  $\mathcal{M}$  for which  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  has a negative eigenvalue. The work [24] then proved that several randomly initialized *proximal algorithms* avoid active strict saddles of *weakly convex* functions with probability 1. Following [24], the works [21, 44] showed how to escape active strict saddles of weakly convex functions at a controlled rate, using randomly perturbed proximal methods. The work [21] in addition treats a variant of the subgradient method that is based on inexact, high-accuracy evaluations of the proximal operator of  $f$ , which is computationally expensive.

While interesting, these works leave open the question

Do standard projected subgradient methods avoid active strict saddle points?

To understand why this question is difficult, let us recall the argument of [24]: Near active strict saddle points of weakly convex functions, the update mapping of common proximal methods inherit the smoothness of  $f$  along the active manifold and have an “unstable fixed-point.” Consequently, the classical center-stable manifold [43, 83, 84] may be applied and ensures nonconvergence. In contrast, the update mapping of the projected subgradient method does not inherit the smoothness of  $f$  along the manifold, rendering the argument of [24] inapplicable. One may hope that the existence of the shadow iteration (3.3.2) might allow one to apply a “center-stable manifold theorem with errors,” but to the best of our knowledge, such theorems (e.g., [91]) require errors to behave in a Lipschitz fashion, which appears untrue of the error in the shadow iteration (1.0.2). Thus, we resort to another technique from the stochastic approximation literature: random perturbations [2, 3, 13, 74].

A central consequence of the results [2, 3, 13, 74] is that gradient descent with uniform random perturbations converge to strict saddle points only with probability zero. In this work, we extend these results to the shadow sequence  $y_k$ , showing that, under reasonable assumptions (e.g., Clarke regularity)

for problems (1.2.2) that are (b) regular and strongly (a)-regular, the perturbed projected subgradient method with noise  $\nu_k$  uniformly distributed in the ball, converges to an active strict saddle point only with probability zero.

Moreover, based on the techniques of Part I, we are able to show that for generic semialgebraic Clarke regular problems, every (composite) critical point is either an active strict saddle point or a local minimizer. As a consequence, the paper culminates in the observation:

The perturbed projected subgradient method, on generic Clarke regular semialgebraic problems, converges only to local minimizers.

This result is striking when contrasted with the state-of-the-art results on the projected subgradient method, which only ensure convergence to critical points [26, 64].

In the final stages of completing this manuscript we became aware of the concurrent and independent work [5], which proves a similar result. The two papers, in large part, share the same core ideas, rooted in strong (a) regularity and proximal aiming.

## 1.3 Outline

The remainder of the paper is organized as follows. In the rest of this chapter, we introduce all the necessary preliminaries that will be used in the the rest of the paper. These preliminaries include sections on manifolds 1.4.1, normal cones 1.4.2, subdifferentials 1.4.3, and active manifolds 1.4.4. The results in these sections will be routinely used in the rest of the paper.

Chapter 2 discusses regularity properties of sets and functions along active manifolds. The goal of the section is to connect strong (a) and (b) regularity to classical “compatibility” conditions of stratification theory, as well as to develop a robust calculus. The section closes with a theorem asserting that conditions (b) and strong (a) hold along the active manifold around any limiting critical of generic semialgebraic problems.

Chapter 3 presents algorithmic consequences of (b) and strong (a) regularity: local rates of convergence, asymptotic normality, and saddle-point avoidance. The first several sections of the chapter outlines the main results, while the remaining sections consist of detailed proofs.

## 1.4 Notation and basic constructions

We follow standard terminology and notation of variational analysis, following mostly closely the monograph of Rockafellar-Wets [80]. Other influential treatments of the subject include [10, 15, 70, 75]. Throughout, we let  $\mathbf{E}$  and  $\mathbf{Y}$  denote Euclidean spaces with inner products denoted by  $\langle \cdot, \cdot \rangle$  and the induced norm  $\|x\| = \sqrt{\langle x, x \rangle}$ . The symbol  $\mathbf{B}$  will stand for the closed unit ball in  $\mathbf{E}$ , while  $B_r(x)$  will denote the closed ball of radius  $r$  around a point  $x$ . The closure of any set  $Q \subset \mathbf{E}$  will be denoted by  $\text{cl } Q$ , while its convex hull will be denoted by  $\text{conv } Q$ . The relative interior of a convex set  $Q$  will be written as  $\text{ri } Q$ . The *lineality space* of any convex cone is the linear subspace  $\text{Lin}(Q) := Q \cap -Q$ .

For any function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{+\infty\}$ , the *domain*, *graph*, and *epigraph* are defined as

$$\begin{aligned} \text{dom } f &:= \{x \in \mathbf{E} : f(x) < \infty\}, \\ \text{gph } f &:= \{(x, f(x)) \in \mathbf{E} \times \mathbf{R} : x \in \text{dom } f\}, \\ \text{epi } f &:= \{(x, r) \in \mathbf{E} \times \mathbf{R} : r \geq f(x)\}, \end{aligned}$$

respectively. If  $\mathcal{M}$  is some subset of  $\mathbf{E}$ , the symbol  $f|_{\mathcal{M}}$  denotes the function obtained by restricting  $f$  to  $\mathcal{M}$  and we set  $\text{gph } f|_{\mathcal{M}} := (\text{gph } f) \cap (\mathcal{M} \times \mathbf{R})$ . We say that  $f$  *sublinear* if its epigraph is a convex cone, and we then define the *lineality space* of  $h$  to be  $\text{Lin}(h) := \{x : h(x) = -h(-x)\}$ . The graph of  $h$  restricted to  $\text{Lin}(h)$  is precisely the lineality space of  $\text{epi } h$ .

The *distance* and the *projection* of a point  $x \in \mathbf{E}$  onto a set  $Q \subset \mathbf{E}$  are

$$d(x, Q) := \inf_{y \in Q} \|y - x\| \quad \text{and} \quad P_Q(x) := \operatorname{argmin}_{y \in Q} \|y - x\|,$$

respectively. The indicator function of a set  $Q$ , denoted by  $\delta_Q : \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$ , is defined to be zero on  $Q$  and  $+\infty$  off it.

For any closed two cones  $U, V \subset \mathbf{E}$ , we define the *gap of  $U$  to  $V$*  as

$$\Delta(U, V) := \sup\{\text{dist}(u, V) : u \in U, \|u\| = 1\}.$$

In particular, the containment  $U \subset V$  holds if and only if the gap  $\Delta(U, V)$  is zero. If  $U$  and  $V$  are linear subspaces, one may equivalently write:

$$\Delta(U, V) = \|P_{V^\perp} P_U\|_{\text{op}} = \|P_U P_{V^\perp}\|_{\text{op}} = \Delta(V^\perp, U^\perp),$$

where  $\|\cdot\|_{\text{op}}$  denotes the operator norm and  $\perp$  denotes the orthogonal complement.

A set-valued map  $F : \mathbf{E} \rightrightarrows \mathbf{Y}$  is an assignment of points  $x \in \mathbf{E}$  to subsets  $F(x) \subset \mathbf{Y}$ . The map  $F$  is called *inner-semicontinuous* at a point  $\bar{x} \in \mathbf{E}$  if for any vector  $\bar{y} \in F(\bar{x})$  and any sequence  $x_i \rightarrow \bar{x}$ , there exists a sequence  $y_i \in F(x_i)$  converging to  $\bar{y}$ .

### 1.4.1 Manifolds

We next recall a few basic properties of smooth embedded submanifolds of  $\mathbf{E}$ . For details, we refer the reader to the recent monograph on manifold optimization [11] and the classical text on smooth manifolds [51]. A set  $\mathcal{M} \subset \mathbf{E}$  is called a  $C^p$  *manifold* (with  $p \geq 1$ ) if around any point  $x \in \mathcal{M}$  there exists an open neighborhood  $U \subset \mathbf{E}$  and a  $C^p$ -smooth map  $F$  from  $U$  to some Euclidean space  $\mathbf{Y}$  such that the Jacobian  $\nabla F(x)$  is surjective and equality  $\mathcal{M} \cap U = F^{-1}(0)$  holds. Then the tangent and normal spaces to  $\mathcal{M}$  at  $x$  are defined as  $T_{\mathcal{M}}(x) := \text{Null}(\nabla F(x))$  and  $N_{\mathcal{M}}(x) := (T_{\mathcal{M}}(x))^\perp$ , respectively. Note that for  $C^p$  manifolds  $\mathcal{M}$  with  $p \geq 1$ , the projection  $P_{\mathcal{M}}$  is  $C^{p-1}$ -smooth on a neighborhood of each point  $x$  in  $\mathcal{M}$ , and is  $C^p$  smooth on the tangent space  $T_{\mathcal{M}}(x)$  [68]. Moreover, we have  $\text{range}(\nabla P_{\mathcal{M}}(x)) \subseteq T_{\mathcal{M}}(x)$  for all  $x$  near  $\mathcal{M}$  and the equality  $\nabla P_{\mathcal{M}}(x) = P_{T_{\mathcal{M}}(x)}$  holds for all  $x \in \mathcal{M}$ . That is, the Jacobian of the projection onto a manifold at a point on the manifold is simply the orthogonal projection onto the tangent space.

Let  $\mathcal{M} \subset \mathbf{E}$  be a  $C^p$ -manifold for some  $p \geq 1$ . Then a function  $f : \mathcal{M} \rightarrow \mathbf{R}$  is called  $C^p$ -smooth around a point  $x \in \mathcal{M}$  if there exists a  $C^p$  function  $\hat{f} : U \rightarrow \mathbf{R}$  defined on an open neighborhood  $U$  of  $x$  and that agrees with  $f$  on  $U \cap \mathcal{M}$ . Then the *covariant gradient of  $f$  at  $x$*  is defined to be the vector

$$\nabla_{\mathcal{M}} f(x) := P_{T_{\mathcal{M}}(x)}(\nabla \hat{f}(x)).$$

When  $f$  and  $\mathcal{M}$  are  $C^2$ -smooth, the *covariant Hessian of  $f$  at  $x$*  is defined to be the unique self-adjoint bilinear form  $\nabla_{\mathcal{M}}^2 f(x): T_{\mathcal{M}}(x) \times T_{\mathcal{M}}(x) \rightarrow \mathbf{R}$  satisfying

$$\langle \nabla_{\mathcal{M}}^2 f(x)u, u \rangle = \frac{d^2}{dt^2} f(P_{\mathcal{M}}(x + tu)) \big|_{t=0} \quad \text{for all } u \in T_{\mathcal{M}}(x).$$

If  $\mathcal{M}$  is a  $C^3$ -smooth manifold, then the composition  $F := f \circ P_{\mathcal{M}}$  is  $C^2$ -smooth near  $x$  and we can identify  $\nabla_{\mathcal{M}}^2 f(x)$  with the matrix  $P_{T_{\mathcal{M}}(x)} \nabla^2 F(x) P_{T_{\mathcal{M}}(x)}$ .

### 1.4.2 Normal cones

The symbol “ $o(h)$  as  $h \rightarrow 0$ ” stands for any univariate function  $o(\cdot)$  satisfying  $o(h)/h \rightarrow 0$  as  $h \searrow 0$ . The *Fréchet normal cone* to a set  $Q \subset \mathbf{E}$  at a point  $x \in \mathbf{E}$ , denoted  $\hat{N}_Q(x)$ , consists of all vectors  $v \in \mathbf{E}$  satisfying

$$\langle v, y - x \rangle \leq o(\|y - x\|) \quad \text{as } y \rightarrow x \text{ in } Q. \quad (1.4.1)$$

The set-valued map  $x \mapsto \hat{N}_Q$  does not have a closed graph, in general. With this in mind, the *limiting normal cone* to  $Q$  at  $x \in Q$ , denoted by  $N_Q(x)$ , is defined to consist of all vectors  $v \in \mathbf{E}$  for which there exist sequences  $x_i \in Q$  and  $v_i \in \hat{N}_Q(x_i)$  satisfying  $(x_i, v_i) \rightarrow (x, v)$ . The *Clarke normal cone* is the closed convex hull  $N_Q^c(x) = \text{cl conv } N_Q(x)$ . Thus the inclusions

$$\hat{N}_Q(x) \subset N_Q(x) \subset N_Q^c(x), \quad (1.4.2)$$

hold for all points  $x \in Q$ . The set  $Q$  is called *Clarke regular* at  $\bar{x} \in Q$  if  $Q$  is locally closed around  $\bar{x}$  and equality  $N_Q^c(x) = \hat{N}_Q(\bar{x})$  holds. In this case, all the inclusions in (1.4.2) hold with equalities.

A particular large class of Clarke regular sets consists of those called prox-regular. Following [16, 76], a locally closed set  $Q \subset \mathbf{E}$  is called *prox-regular at  $\bar{x} \in Q$*  if the projection  $P_Q(x)$  is a singleton set for all points  $x$  near  $\bar{x}$ . The result [76, Theorem 1.3] shows that a locally closed set  $Q$  is prox-regular at  $\bar{x} \in Q$  if and only if there exist constants  $\epsilon, \rho > 0$  satisfying

$$\langle v, y - x \rangle \leq \frac{\rho}{2} \|y - x\|^2,$$

for all  $y, x \in Q \cap B_{\epsilon}(\bar{x})$  and all normal vectors  $v \in N_Q(x) \cap \epsilon \mathbf{B}$ . Consequently, prox-regularity amounts to a possibility of replacing the little- $o$  term in (1.4.1) with a simple quadratic, whose amplitude is independent of the base-point  $x$  and the normal vector  $v$ . If  $Q$  is prox-regular at  $\bar{x}$ , then the projection  $P_Q(\cdot)$  is automatically locally Lipschitz continuous around  $\bar{x}$  [76, Theorem 1.3]. Prox-regularity admits a convenient calculus. In particular, common examples of prox-regular sets are convex sets and  $C^2$  manifolds, as well as sets cut out by finitely many  $C^2$  inequalities under the Mangasarian-Fromovitz constraint qualification [77]. Prox-regular sets are closely related to proximally smooth sets of [16] and sets with positive reach of [35].

### 1.4.3 Subdifferentials

Generalized gradients of functions can be defined through the normal cones to epigraphs. Namely, consider a function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  and a point  $x \in \text{dom } f$ . The *Fréchet*, *limiting*, and *Clarke* subdifferentials of  $f$  at  $x$  are defined, respectively, as

$$\begin{aligned}\hat{\partial}f(x) &:= \{v \in \mathbf{E} : (v, -1) \in \hat{N}_{\text{epi } f}(x, f(x))\}, \\ \partial f(x) &:= \{v \in \mathbf{E} : (v, -1) \in N_{\text{epi } f}(x, f(x))\}, \\ \partial_c f(x) &:= \{v \in \mathbf{E} : (v, -1) \in N_{\text{epi } f}^c(x, f(x))\}.\end{aligned}\tag{1.4.3}$$

Explicitly, the inclusion  $v \in \hat{\partial}f(x)$  amounts to requiring the lower-approximation property:

$$f(y) \geq f(x) + \langle v, y - x \rangle + o(\|y - x\|) \quad \text{as } y \rightarrow x.$$

Moreover, a vector  $v$  lies in  $\partial f(x)$  if and only if there exist sequences  $x_i \in \mathbf{E}$  and Fréchet subgradients  $v_i \in \hat{\partial}f(x_i)$  satisfying  $(x_i, f(x_i), v_i) \rightarrow (x, f(x), v)$  as  $i \rightarrow \infty$ . If  $f$  is locally Lipschitz continuous around  $x$ , then equality  $\partial_c f(x) = \text{conv } \partial f(x)$  holds. A point  $\bar{x}$  satisfying  $0 \in \partial f(x)$  is called *critical* for  $f$ , while a point satisfying  $0 \in \partial_c f(x)$  is called *Clarke critical*. Clearly, the latter requirement may be much weaker than the former. The distinction disappears for subdifferentially regular functions. We say that  $f$  is *subdifferentially regular* at  $x \in \text{dom } f$  if the set  $\text{epi } f$  is Clarke regular at  $(x, f(x))$ .

The three subdifferentials defined in (1.4.3) fail to capture the horizontal normals to the epigraph—meaning those of the form  $(v, 0)$ . Such horizontal normals play an important role in variational analysis, in particular for subdifferential calculus. Consequently, we define the *limiting* and *Clarke horizon subdifferentials*, respectively, by:

$$\begin{aligned}\partial^\infty f(x) &:= \{v \in \mathbf{E} : (v, 0) \in N_{\text{epi } f}(x, f(x))\}, \\ \partial_c^\infty f(x) &:= \{v \in \mathbf{E} : (v, 0) \in N_{\text{epi } f}^c(x, f(x))\}.\end{aligned}$$

A function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  is called  *$\rho$ -weakly convex* if the quadratically perturbed function  $x \mapsto f(x) + \frac{\rho}{2}\|x\|^2$  is convex. Weakly convex functions are subdifferentially regular. Indeed, the subgradients of a  $\rho$ -weakly convex function yield quadratic minorants, meaning

$$f(y) \geq f(x) + \langle v, y - x \rangle - \frac{\rho}{2}\|y - x\|^2$$

all points  $x, y \in \text{dom } f$  and all subgradients  $v \in \partial f(x)$ . The epigraph of any weakly convex function is a prox-regular set at each of its points. Weakly convex functions are widespread in applications, with the primary example being compositions of Lipschitz continuous convex functions with  $C^1$  maps. We refer the reader to [23, 30] for an extensive discussion of this function class in contemporary applications.

### 1.4.4 Active manifolds

Critical points of typical nonsmooth functions lie on a certain manifold that captures the activity of the problem in the sense that critical points of slight linear tilts of the function do not leave the manifold. Such active manifolds have been modeled in a variety of ways,

including identifiable surfaces [95], partly smooth manifolds [57],  $\mathcal{UV}$ -structures [54, 67],  $g \circ F$  decomposable functions [82], and minimal identifiable sets [32].

In this work, we adopt the following formal model of activity, explicitly used in [32], where the only difference is that we focus on the Clarke subdifferential instead of the limiting one.

**Definition 1.4.1** (Active manifold). Consider a closed function  $f: \mathbf{R}^d \rightarrow \mathbf{R} \cup \{\infty\}$  and fix a set  $\mathcal{M} \subseteq \mathbf{R}^d$  containing a point  $\bar{x}$  satisfying  $0 \in \partial_c f(\bar{x})$ . Then  $\mathcal{M}$  is called an *active  $C^p$ -manifold around  $\bar{x}$*  if there exists a constant  $\epsilon > 0$  satisfying the following.

- **(smoothness)** The set  $\mathcal{M} \cap U$  is a  $C^p$ -smooth manifold and the restriction of  $f$  to  $\mathcal{M} \cap U$  is  $C^p$ -smooth.
- **(sharpness)** The lower bound holds:

$$\inf\{\|v\| : v \in \partial_c f(x), x \in U \setminus \mathcal{M}\} > 0,$$

where we set  $U = \{x \in B_\epsilon(\bar{x}) : |f(x) - f(\bar{x})| < \epsilon\}$ .

The sharpness condition simply means that the subgradients of  $f$  must be uniformly bounded away from zero at points off the manifold that are sufficiently close to  $\bar{x}$  in distance and in function value. The localization in function value can be omitted for example if  $f$  is weakly convex or if  $f$  is continuous on its domain; see [32] for details.

Intuitively, the active manifold has the distinctive feature that the the function grows linearly in normal directions to the manifold; see Figure 1.1a for an illustration. This is summarized by the following theorem from [25, Theorem D.2].

**Proposition 1.4.2** (Identification implies sharpness). *Suppose that a closed function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  admits an active manifold  $\mathcal{M}$  at a point  $\bar{x}$  satisfying  $0 \in \hat{\partial} f(\bar{x})$ . Then there exist constants  $c, \epsilon > 0$  such that*

$$f(x) - f(P_{\mathcal{M}}(x)) \geq c \cdot \text{dist}(x, \mathcal{M}), \quad \forall x \in B_\epsilon(\bar{x}). \quad (1.4.4)$$

Notice that there is a nontrivial assumption  $0 \in \hat{\partial} f(\bar{x})$  at play in Proposition 1.4.2. Indeed, under the weaker inclusion  $0 \in \partial_c f(\bar{x})$  the growth condition (1.4.4) may easily fail, as the univariate example  $f(x) = -|x|$  shows. It is worthwhile to note that under the assumption  $0 \in \hat{\partial} f(\bar{x})$ , the active manifold is locally unique around  $\bar{x}$ . The following theorem is proved in [32, Proposition 8.2].

**Theorem 1.4.3** (Local uniqueness of the active manifold). *Suppose that a closed function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  admits two  $C^1$  active manifolds  $\mathcal{M}$  and  $\mathcal{L}$  at a point  $\bar{x}$  satisfying  $0 \in \hat{\partial} f(\bar{x})$ . Then there exists a neighborhood  $U$  of  $\bar{x}$  such that the equality holds:*

$$\mathcal{M} \cap U = \mathcal{L} \cap U.$$

Active manifolds are useful in optimization because they allow to reduce many questions about nonsmooth functions to a smooth setting. In particular, the notion of a strict saddle point of smooth functions naturally extends to a nonsmooth setting. The following definition is taken from [22].



**Definition 1.4.4** (Active strict saddle). Fix an integer  $p \geq 2$  and consider a closed function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  and a point  $\bar{x}$  satisfying  $0 \in \partial_c f(\bar{x})$ . We say that  $\bar{x}$  is a  $C^p$  *strict active saddle point* of  $f$  if  $f$  admits a  $C^p$  active manifold  $\mathcal{M}$  at  $\bar{x}$  such that the inequality  $\langle \nabla_{\mathcal{M}}^2 f(\bar{x})u, u \rangle < 0$  holds for some  $u \in T_{\mathcal{M}}(\bar{x})$ .

Figure 1.1 presents an example of an active strict saddle point of a nonsmooth function and its subgradient flow. Notice that the subgradient flow, when initialized uniformly at random, almost surely escapes the strict saddle point. The recent work [22] established this phenomenon for discrete proximal type methods, whereas in this work we investigate such behavior for the pure subgradient algorithm—a much more nuanced task.

It is often convenient to think about active manifolds of slightly tilted functions. Therefore, we say that  $\mathcal{M}$  is an *active  $C^p$  manifold of  $f$  at  $\bar{x}$  for  $v \in \partial_c f(\bar{x})$*  if  $\mathcal{M}$  is an active  $C^p$  manifold for the tilted function  $x \mapsto f(x) - \langle v, x \rangle$  at  $\bar{x}$ . Active manifolds for sets are defined through their indicator functions. Namely a set  $\mathcal{M} \subset Q$  is an *active  $C^p$  manifold of  $Q$  at  $\bar{x} \in Q$  for  $v \in N_Q^c(\bar{x})$*  if it is an active  $C^p$  manifold of the indicator function  $\delta_Q$  at  $\bar{x}$  for  $v$ .

A large class of sets admitting active manifolds is comprised of cone-reducible sets, introduced in [9]. Roughly speaking, these sets are smooth deformations of closed convex cones.

**Definition 1.4.5** (Cone reducible sets). A set  $Q \subset \mathbf{E}$  is  $C^p$  *cone-reducible* at  $\bar{x} \in Q$  if there exists a neighborhood  $U$  of  $\bar{x}$ , a  $C^p$  map  $F$  from  $U$  to some Euclidean space  $\mathbf{Y}$  with surjective Jacobian  $\nabla F(\bar{x})$  and  $F(\bar{x}) = 0$ , and a pointed closed convex cone  $C \subset \mathbf{Y}$  satisfying

$$Q \cap U = \{x \in U : F(x) \in C\}.$$

We then say that  $Q$  is  $C^p$  *cone reducible to  $C$  by  $F$  at  $\bar{x}$* .

We refer the reader to [9] for the numerous examples of cone reducible sets in optimization. Active manifolds of cone reducible sets admit the following simple description [57, Theorem 4.2].

**Theorem 1.4.6** (Active manifolds of cone reducible sets). *Suppose that a set  $Q$  is  $C^1$  cone-reducible to  $C$  by  $F$  at  $\bar{x}$ . Then the pair  $F^{-1}(0)$  is an active manifold at  $\bar{x}$  for any  $v \in \text{ri } N_Q(x)$ .*

A functional analogue of cone reducible sets consists of decomposable functions, introduced in [82]. Roughly speaking; these functions are smooth deformations of sublinear functions.

**Definition 1.4.7** (Decomposable functions). A function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  is called *properly  $C^p$  decomposable* at  $\bar{x}$  as  $h \circ c$  if on a neighborhood of  $\bar{x}$  it can be written as

$$f(x) = f(\bar{x}) + h(c(x))$$

for some  $C^p$ -smooth mapping  $c: \mathbf{E} \rightarrow \mathbf{Y}$  satisfying  $c(\bar{x}) = 0$  and some proper, closed sublinear function  $h: \mathbf{Y} \rightarrow \mathbf{R}$  satisfying the transversality condition:

$$\text{Lin}(h) + \text{Range}(\nabla c(\bar{x})) = \mathbf{Y}.$$

Any  $C^1$  decomposable function admits active manifolds in the following sense [82, p 683].

**Lemma 1.4.8** (Decomposable functions admit active manifolds). *Suppose that  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  is properly  $C^p$  decomposable at  $\bar{x}$  as  $h \circ c$ . Then the set  $\mathcal{M} = c^{-1}(\text{Lin}(h))$  is a  $C^p$ -active manifold around  $\bar{x}$  for any subgradient  $v \in \text{ri } \partial f(\bar{x})$ .*

Cone reducible sets and properly decomposable functions form widespread examples for which all of our techniques apply.

# Chapter 2

## Regularity along a manifold: conditions of Whitney and Verdier in nonsmooth optimization

### 2.1 The four fundamental regularity conditions

This section introduces compatibility conditions between two sets, motivated by the pioneering works of Whitney [92–94], Kuo [46], and Verdier [90]. Our discussion builds on the recent survey of Trotman [88]. We illustrate the definitions with examples and prove basic relations between them. It is important to note that these classical works focused on compatibility conditions between smooth manifolds, wherein primal (tangent) and dual (normal) based characterizations are equivalent. In contrast, it will be more expedient for us to base definitions on normal vectors instead of tangents. The reason is that when applied to epigraphs, such conditions naturally imply some regularity properties for the subgradients, which in turn underpin all algorithmic consequences in Chapter 3.

Throughout this section, we fix two sets  $\mathcal{X}$  and  $\mathcal{Y}$  that are locally closed around a point  $\bar{x} \in \mathcal{Y}$ . The reader should keep in my mind the most important setting when  $\mathcal{Y}$  is a smooth manifold contained in the closure of  $\mathcal{X}$ . We begin with the two classical conditions, introduced by Whitney in [93, 94].

**Definition 2.1.1** (Whitney conditions). Define the following two properties.

1. The pair  $(\mathcal{X}, \mathcal{Y})$  is *(a)-regular* at  $\bar{x}$  if for any sequence  $x_i \in \mathcal{X}$  converging to  $\bar{x}$  and any convergent sequence of normals  $v_i \in N_{\mathcal{X}}(x_i)$  the limit  $\lim_{i \rightarrow \infty} v_i$  lies in  $N_{\mathcal{Y}}(\bar{x})$ .
2. The pair  $(\mathcal{X}, \mathcal{Y})$  is *(b)-regular* at  $\bar{x}$  if for any sequence  $x_i \in \mathcal{X}$  converging to  $\bar{x}$ , any sequence  $y_i \in \mathcal{Y}$  converging to  $\bar{x}$ , and any unit vectors  $v_i \in N_{\mathcal{X}}(x_i)$ , we have  $\lim_{i \rightarrow \infty} \langle v_i, \frac{x_i - y_i}{\|x_i - y_i\|} \rangle = 0$ .

Both conditions (a) and (b) are geometrically transparent. Condition (a) simply asserts that “limits of normals to  $\mathcal{X}$  are normal to  $\mathcal{Y}$ ”—clearly a desirable property. Figure 2.1a illustrates how condition (a) may fail using the classical example of the Cartan umbrella

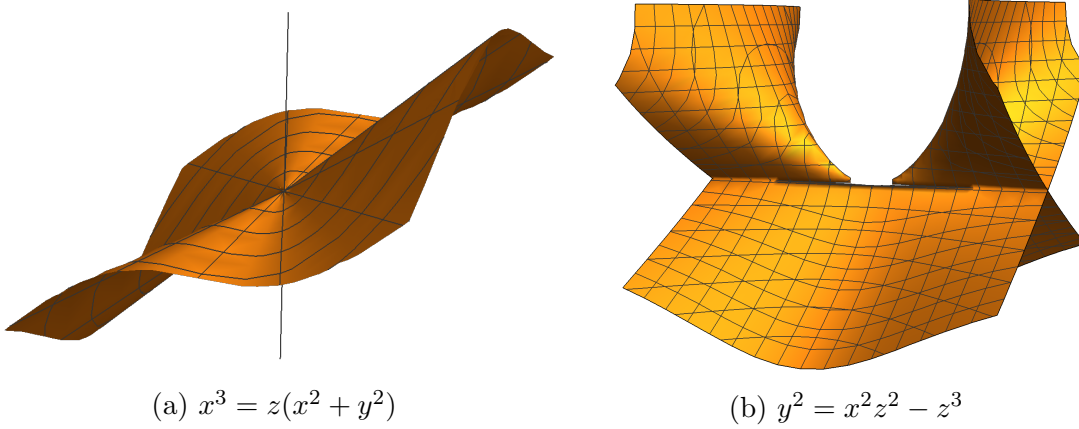


Figure 2.1: Illustrations of conditions (a) and (b).

$\mathcal{X} = \{(x, y, z) : z(x^2 + y^2) = x^3\}$ . If  $\mathcal{Y}$  is the  $z$ -axis, a subset of  $\mathcal{X}$ , then condition (a) fails for the pair  $(\mathcal{X}, \mathcal{Y})$  at the origin.

Condition (b) is more subtle, and is in essence a “restricted smoothness condition”. Namely, any  $C^1$ -smooth manifold  $\mathcal{M}$  containing a point  $\bar{x}$  satisfies the estimate:

$$\langle v(x), y - x \rangle = o(\|y - x\|) \quad \text{as } y, x \rightarrow \bar{x} \text{ in } \mathcal{M},$$

where  $v(x) \in N_{\mathcal{M}}(x)$  is any selection of unit normal vectors. Condition (b) asserts exactly this estimate but only along points  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  near  $\bar{x}$ . This “restricted smoothness” viewpoint will become even more clear at the end of the section, when we interpret conditions (a) and (b) for epigraphs of functions. Optimization experts might recognize condition (b) in the setting  $\mathcal{Y} = \{\bar{x}\}$  as semismoothness of the distance function  $\text{dist}_{\mathcal{X}}(\cdot)$  at  $\bar{x}$  in the sense of [39, 66]. In this vein, when  $\mathcal{Y}$  is any subset of  $\mathcal{X}$ , we can interpret condition (b) for the pair  $(\mathcal{X}, \mathcal{Y})$  as a kind of uniform semismoothness of  $\mathcal{X}$  relative to  $\mathcal{Y}$ .

A useful consequence of condition (a) is the following inclusion of normal cones. This observation in the smooth category played a fundamental role in the work [8], underpinning their projection formula and its numerous consequences for subgradient dynamics (e.g. [6, 26, 31]).

**Lemma 2.1.1.** *Suppose that the inclusion  $\mathcal{Y} \subset \mathcal{X}$  holds and that the pair  $(\mathcal{X}, \mathcal{Y})$  is (a)-regular at a point  $\bar{x} \in \mathcal{Y}$ . Then the inclusion  $N_{\mathcal{X}}(\bar{x}) \subset N_{\mathcal{Y}}(\bar{x})$  holds.*

*Proof.* This is almost tautological. The inclusion  $\hat{N}_{\mathcal{X}}(\bar{x}) \subset \hat{N}_{\mathcal{Y}}(\bar{x})$  holds since  $\mathcal{Y}$  is contained in  $\mathcal{X}$ . For any vector  $v \in N_{\mathcal{X}}(\bar{x})$ , we may find a sequence  $x_i \in \mathcal{X}$  converging to  $\bar{x}$  and vectors  $v_i \in \hat{N}_{\mathcal{X}}(x_i)$  converging to  $\bar{v}$ . Condition (a) therefore guarantees  $\bar{v} \in N_{\mathcal{Y}}(\bar{x})$ , as claimed.  $\square$

John Mather in his lecture notes [65] pointed out that condition (b) for two smooth manifolds  $\mathcal{X}$  and  $\mathcal{Y}$  implies condition (a). The following simple lemma shows that this is true for general locally closed sets  $\mathcal{X}$  and  $\mathcal{Y}$ .

**Lemma 2.1.2.** *The implication (b)  $\Rightarrow$  (a) holds.*

*Proof.* Suppose that the pair  $(\mathcal{X}, \mathcal{Y})$  is  $(b)$ -regular at  $\bar{x}$ . Let  $x_i \in \mathcal{X}$  be a sequence converging to  $\bar{x}$  and  $v_i \in N_{\mathcal{X}}(x_i)$  be a sequence converging to some vector  $v$ . It suffices to argue that the inclusion  $v \in \hat{N}_{\mathcal{Y}}(\bar{x})$  holds. To this end, consider an arbitrary sequence  $y_j \in \mathcal{Y} \setminus \{\bar{x}\}$  converging to  $\bar{x}$ . Choosing a subsequence  $i_j$ , we may ensure  $\|x_{i_j} - \bar{x}\| \leq \frac{\|y_j - \bar{x}\|}{j}$ . We therefore deduce

$$\limsup_{j \rightarrow \infty} \frac{\langle v, y_j - \bar{x} \rangle}{\|y_j - \bar{x}\|} = \limsup_{j \rightarrow \infty} \frac{\langle v_{i_j}, y_{i_j} - x_{i_j} \rangle}{\|y_j - x_{i_j}\|} = 0,$$

where the last inequality follows from condition  $(b)$ . Therefore the inclusion  $v \in \hat{N}_{\mathcal{Y}}(\bar{x})$  holds as claimed.  $\square$

**Example 2.1.1** (Condition  $(b)$  is strictly stronger than condition  $(a)$ ). Whitney showed that condition  $(b)$  may be strictly stronger than condition  $(a)$  even for smooth manifolds  $\mathcal{X}$  and  $\mathcal{Y}$ . This distinction is worth highlighting with Whitney's original example. Namely, define the set  $Q = \{(x, y, z) : y^2 = x^2 z^2 - z^3\}$ , depicted in Figure 2.1b. Define now two smooth manifolds  $\mathcal{X} = Q \cap \{z > 0\}$  and  $\mathcal{Y} = Q \cap \{z = 0\}$ ; note that  $\mathcal{Y}$  is just the  $x$ -axis. It is straightforward to see that the pair  $(\mathcal{X}, \mathcal{Y})$  satisfies condition  $(a)$  at the origin. Condition  $(b)$ , on the other hand, fails at the origin. To see this, we may define the sequences  $u_i = (i^{-1}, 0, 0)$  lying in  $\mathcal{Y}$  and  $w_i = (i^{-1}, 0, i^{-2})$ ; in the figure, the points  $w_i \in \mathcal{X}$  are lying on the parabola above the points  $u_i \in \mathcal{Y}$ . It is clear from the picture, and can be formally verified, that the secant line joining  $u_i$  with  $w_i$  and the normal line to  $\mathcal{X}$  at  $w_i$  become collinear in the limit as  $i \rightarrow \infty$ .

Notice that condition  $(a)$  does not specify the rate at which the gap  $\Delta(N_{\mathcal{X}}(x_i), N_{\mathcal{Y}}(\bar{x}))$  tends to zero as  $x_i \in \mathcal{X}$  tends to  $\bar{x}$ . A natural strengthening of the condition, introduced by Verdier [90], requires the gap to be linearly bounded by  $\|x_i - \bar{x}\|$ , with a coefficient that is uniform over all small perturbation of the base point  $\bar{x} \in \mathcal{Y}$ .<sup>1</sup> Condition  $(b)$  can be similarly strengthened. The following definition records the resulting two properties.

**Definition 2.1.2** (Strong  $(a)$  and strong  $(b)$  conditions). Define the following two properties.

1. The pair  $(\mathcal{X}, \mathcal{Y})$  is *strongly (a)-regular* at  $\bar{x}$  if there exists a constant  $C > 0$  satisfying

$$\Delta(N_{\mathcal{X}}(x), N_{\mathcal{Y}}(y)) \leq C \cdot \|x - y\|, \quad (2.1.1)$$

for all  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  sufficiently close to  $\bar{x}$ .

2. The pair  $(\mathcal{X}, \mathcal{Y})$  is *strongly (b)-regular* at  $\bar{x}$  if there exists a constant  $C > 0$  satisfying

$$|\langle v, x - y \rangle| \leq C \|x - y\|^2, \quad (2.1.2)$$

for all  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  sufficiently close to  $\bar{x}$  and all unit vectors  $v \in N_{\mathcal{X}}(x)$ .

Summarizing, we have defined four fundamental regularity conditions quantifying the compatibility of two sets  $\mathcal{X}$  and  $\mathcal{Y}$  near a point  $\bar{x}$ . The most important situation for our purposes is when  $\mathcal{Y}$  is a smooth manifold contained in  $\mathcal{X}$ . The algorithmic importance of these conditions becomes clear when we interpret what they mean for epigraphs of functions. With this in mind, we introduce the following definition.

---

<sup>1</sup>What we call strong  $(a)$  is often called condition  $(w)$ , the Verdier condition, or the Kuo-Verdier  $(kw)$  condition in the stratification literature.

**Definition 2.1.3** (Regularity of a function along a set). Consider a function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  and a set  $\mathcal{M} \subset \text{dom } f$ . We say that  $f$  is *(a)-regular along  $\mathcal{M}$  at a point  $\bar{x} \in \mathcal{M}$*  if the pair  $(\text{epi } f, \text{gph } f|_{\mathcal{M}})$  is *(a)-regular at  $(\bar{x}, f(\bar{x}))$* . Regularity of type (b), strong (a), and strong (b) are defined similarly.

The following theorem reinterprets the four geometric regularity conditions for the epi-graph in analytic terms.

**Theorem 2.1.4** (From geometry to analysis). *Consider a function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  that is locally Lipschitz continuous on its domain and fix a set  $\mathcal{M} \subset \text{dom } f$ . Let  $\bar{x} \in \mathcal{M}$  be a point such that near  $\bar{x}$ , the set  $\mathcal{M}$  is a  $C^1$ -smooth manifold and the restriction of  $f$  to  $\mathcal{M}$  is  $C^1$ -smooth. Then the following claims are true.*

1. **(condition (a))** *If  $f$  is (a)-regular along  $\mathcal{M}$  at  $\bar{x}$ , then the inclusions hold:*

$$P_{T_{\mathcal{M}}(\bar{x})}(\partial f(\bar{x})) \subset \{\nabla_{\mathcal{M}} f(\bar{x})\} \quad \text{and} \quad P_{T_{\mathcal{M}}(\bar{x})}(\partial^\infty f(\bar{x})) = \{0\}.$$

2. **(strong (a))** *If  $f$  is strongly (a)-regular along  $\mathcal{M}$  at  $\bar{x}$ , then there exists a constant  $C > 0$  such that the conditions:*

$$\|P_{T_{\mathcal{M}}(y)}(v - \nabla_{\mathcal{M}} f(y))\| \leq C\sqrt{1 + \|v\|^2}\|x - y\|, \quad (2.1.3)$$

$$\|P_{T_{\mathcal{M}}(y)}(w)\| \leq C\|w\|\|x - y\|, \quad (2.1.4)$$

*hold for  $x \in \text{dom } f$  and  $y \in \mathcal{M}$  close to  $\bar{x}$ , and all  $v \in \partial f(x)$  and all  $w \in \partial^\infty f(x)$ .*

3. **(condition (b))** *If  $f$  is (b)-regular along  $\mathcal{M}$  at  $\bar{x}$ , then for every  $\delta > 0$ , there exists  $\epsilon > 0$  such that the estimates*

$$|f(y) - f(x) - \langle v, y - x \rangle| \leq \delta\sqrt{1 + \|v\|^2} \cdot \|x - y\|, \quad (2.1.5)$$

$$|\langle w, y - x \rangle| \leq \delta\|w\|\|y - x\|, \quad (2.1.6)$$

*hold for all  $x \in (\text{dom } f) \cap B_\epsilon(\bar{x})$ ,  $y \in \mathcal{M} \cap B_\epsilon(\bar{x})$ ,  $v \in \partial f(x)$ , and  $w \in \partial^\infty f(x)$ .*

4. **(strong (b))** *If  $f$  is strongly (b)-regular along  $\mathcal{M}$  at  $\bar{x}$ , then there exists a constant  $C > 0$  such that the estimates*

$$|f(y) - f(x) - \langle v, y - x \rangle| \leq C\sqrt{1 + \|v\|^2} \cdot \|x - y\|^2,$$

$$|\langle w, y - x \rangle| \leq C\|w\|\|y - x\|^2,$$

*hold for all  $x \in \text{dom } f$  and  $y \in \mathcal{M}$  sufficiently close to  $\bar{x}$ , and for all  $v \in \partial f(x)$  and  $w \in \partial^\infty f(x)$ .*

*Proof.* The proof of the first claim is identical to the proof of the projection formula in [8]. We prove the rest of the claims in order. Without loss of generality, we suppose  $\bar{x} = 0$ . Throughout we fix points  $x \in \text{dom } f$  and  $y \in \mathcal{M}$  near  $\bar{x}$ , define the lifted points  $X = (x, f(x))$  and  $Y = (y, f(y))$ , and set  $\mathcal{X} = \text{epi } f$  and  $\mathcal{Y} = \text{gph } f|_{\mathcal{M}}$ . We let  $L$  be a Lipschitz constant of the restriction of  $f$  to  $\text{dom } f$  on some neighborhood of  $\bar{x}$ .

Next, suppose that  $f$  is strongly (a)-regular along  $\mathcal{M}$  at  $\bar{x}$ . Clearly the inclusion

$$\partial f(x) \times \{-1\} \subset N_{\mathcal{X}}(X),$$

holds. Therefore, for any vector  $v \in \partial f(x)$ , strong (a) regularity and Lipschitz continuity of  $f$  on  $\text{dom } f$  imply

$$(v, -1) \subset N_{\mathcal{Y}}(Y) + C\sqrt{1 + \|v\|^2}\|x - y\|\mathbf{B}. \quad (2.1.7)$$

Classical arguments yield the description of the tangent space

$$T_{\mathcal{Y}}(Y) = \{(u, \langle \nabla_{\mathcal{M}} f(y), u \rangle) : u \in T_{\mathcal{M}}(y)\}.$$

Therefore taking dot products of the inclusion (2.1.7) with such tangent vectors, we deduce

$$\langle v - \nabla_{\mathcal{M}} f(y), u \rangle \leq \sqrt{1 + \|v\|^2} C \|x - y\| \|(u, \langle \nabla_{\mathcal{M}} f(y), u \rangle)\|.$$

Note that if  $u \in T_{\mathcal{M}}(y)$  has unit length, then we may upper bound  $\|(u, \langle \nabla_{\mathcal{M}} f(y), u \rangle)\|$  by  $\sqrt{1 + L^2}$ . Therefore for such vectors  $u$  we deduce

$$\langle v - \nabla_{\mathcal{M}} f(y), u \rangle \leq \sqrt{(1 + \|v\|^2)(1 + L^2)} C \|x - y\|.$$

Setting  $u$  to be the normalized projection of  $v - \nabla_{\mathcal{M}} f(y)$  onto  $T_{\mathcal{M}}(y)$  immediately yields

$$\|P_{T_{\mathcal{M}}(y)}(v - \nabla_{\mathcal{M}} f(y))\| \leq \sqrt{(1 + \|v\|^2)(1 + L^2)} C \|x - y\|.$$

Thus 2.1.3 holds. Next, note the inclusion

$$\partial^\infty f(x) \times \{0\} \subset N_{\mathcal{X}}(X).$$

By the same argument as above, we deduce that for any  $w \in \partial^\infty f(x)$  we have

$$\langle w, u \rangle \leq (1 + L^2) C \|w\| \|x - y\|.$$

for all unit vectors  $u \in T_{\mathcal{M}}(y)$ . Choosing  $u$  to be the normalized projection of  $w$  onto  $T_{\mathcal{M}}(y)$  completes the proof of (2.1.4).

Next, suppose that  $f$  is (b)-regular along  $\mathcal{M}$  at  $\bar{x}$ . Then there exists a function  $C(t, s)$  that tends to zero as  $(t, x) \rightarrow 0$  such that for any vector  $v \in \partial f(x)$ , we have

$$\begin{aligned} |f(y) - f(x) - \langle v, y - x \rangle| &= |\langle (v, -1), X - Y \rangle| \\ &\leq C(\|x\|, \|y\|) \sqrt{\|x - y\|^2 + (f(x) - f(y))^2} \sqrt{1 + \|v\|^2} \\ &\leq C(\|x\|, \|y\|) \sqrt{1 + L^2} \sqrt{1 + \|v\|^2} \|x - y\| \end{aligned}$$

Thus (2.1.5) holds. For any vector  $w \in \partial^\infty f(x)$ , we similarly compute

$$\begin{aligned} |\langle w, y - x \rangle| &= |\langle (w, 0), X - Y \rangle| \\ &\leq C(\|x\|, \|y\|) \sqrt{\|x - y\|^2 + (f(x) - f(y))^2} \|w\| \\ &\leq C(\|x\|, \|y\|) \sqrt{1 + L^2} \|w\| \|x - y\|, \end{aligned}$$

thereby establishing (2.1.6). The final claim for strong (b)-regularity is completely analogous to (b)-regularity.  $\square$

The conditions in Theorem 2.1.4 are particularly transparent when  $f$  is Lipschitz continuous near  $\bar{x}$ . Then  $\partial f(\bar{x})$  is nonempty and condition (a) implies that the projection  $P_{T_{\mathcal{M}}(\bar{x})}(\partial f(\bar{x}))$  is a single point—the covariant gradient  $\nabla_{\mathcal{M}}f(\bar{x})$ . This guarantee is called the projection formula in [8]. Strong (a) provides a “stable improvement” over the projection formula wherein the deviation  $\partial f(x) - \nabla_{\mathcal{M}}f(y)$  in tangent directions  $T_{\mathcal{M}}(y)$  is linearly bounded by  $\|x - y\|$ , for points  $x \in \mathbf{E}$  and  $y \in \mathcal{M}$  near  $\bar{x}$ .

Condition (b) has a different flavor: it ensures a restricted Taylor approximation property

$$f(y) - f(x) - \langle v, y - x \rangle = o(\|x - y\|)$$

as  $x \in \mathbf{E}$  and  $y \in \mathcal{M}$  tend to  $\bar{x}$  and  $v \in \partial f(x)$  are arbitrary. In particular, in the setting  $\mathcal{M} = \{\bar{x}\}$ , this is exactly the semismoothness condition of [1].<sup>2</sup> Strong (b)-regularity, in turn, replaces the little-o term with the squared norm:

$$f(y) - f(x) - \langle v, y - x \rangle = O(\|x - y\|^2)$$

as  $x \in \mathbf{E}$  and  $y \in \mathcal{M}$  tend to  $\bar{x}$  and  $v \in \partial f(x)$  are arbitrary. In the setting  $\mathcal{M} = \{\bar{x}\}$ , this condition is called strong semi-smoothness in the optimization literature.

Condition (b) becomes particularly useful algorithmically when the inclusion  $0 \in \hat{\partial}f(\bar{x})$  holds and  $\mathcal{M}$  is a  $C^1$  active manifold of  $f$  around  $\bar{x}$ . Indeed, condition (b) along with the sharp growth guarantee of Theorem 1.4.2 then imply that there exists a constant  $c > 0$  such that the estimate

$$\langle v, x - P_{\mathcal{M}}(x) \rangle \geq c \cdot \text{dist}(x, \mathcal{M}), \quad (2.1.8)$$

holds for all  $y \in \mathcal{M}$  near  $\bar{x}$  and for all  $v \in \partial f(x)$ . In words, this means that negative subgradients of  $f$  at  $x$  always point towards the active manifold. The angle condition (2.1.8) together with strong (a) regularity will form the core of the algorithmic development in Chapter 3. For ease of reference, we slight generalization of the angle condition (2.1.8) when  $f$  is not necessarily locally Lipschitz around  $\bar{x}$  and can even be infinite-valued.

**Corollary 2.1.5** (Proximal aiming). *Consider a closed function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  that admits an active  $C^1$ -manifold  $\mathcal{M}$  at a point  $\bar{x}$  and suppose that  $f$  is prox-regular at  $\bar{x}$  or (b)-regular along  $\mathcal{M}$  at  $\bar{x}$ . Then, there exists a constant  $c > 0$  such that for any  $\delta > 0$ , there exists  $\epsilon > 0$  satisfying*

$$\langle v, x - P_{\mathcal{M}}(x) \rangle \geq (c - \delta \sqrt{1 + \|v\|^2}) \cdot \text{dist}(x, \mathcal{M}), \quad (2.1.9)$$

for all  $x \in B_{\epsilon}(\bar{x})$  and all  $v \in \partial f(x)$ . Moreover, if  $f$  is locally Lipschitz around  $\bar{x}$ , the same statement holds with  $\partial f(x)$  replaced by  $\partial_c f(x)$  and with  $\delta = 0$ .<sup>3</sup>

The rest of the chapter is devoted to exploring the relationship between the four basic regularity conditions, presenting examples, proving calculus rules, and justifying that these conditions hold “generically” along active manifolds. Chapter 3 will in turn use these conditions to analyze subgradient type algorithms.

<sup>2</sup>ignoring a directional differentiability requirement

<sup>3</sup>The last claim follows immediately from (3.11.5) by choosing  $\delta < \frac{c}{2\sqrt{1+L^2}}$ , where  $L$  is a local Lipschitz constant of  $f$  near  $\bar{x}$ , and taking convex combinations of limiting subgradients.



## 2.2 Relation between the four conditions

The goal of this section is to explore the relationship between the four regularity conditions. Recall that Lemma 2.1.2 already established the implication  $(b) \Rightarrow (a)$ . More generally, the goal of this section is to show in reasonable settings the string of implications:

$$\boxed{(a) \Leftarrow (b) \Leftarrow \text{strong}(a) \Leftarrow \text{strong}(b)}. \quad (2.2.1)$$

Before passing to formal statements, we require some preparation. Namely, the task of verifying conditions  $(b)$ ,  $\text{strong}(a)$ , and  $\text{strong}(b)$  requires considering arbitrary points  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ , which are a priori unrelated. We now show that it essentially suffices to set  $y$  to be the projection of  $x$  onto  $\mathcal{Y}$ . In this way, we may remove one degree of flexibility for the question of verification. We begin by defining the projected variants of conditions  $(b)$ ,  $\text{strong}(a)$ , and  $\text{strong}(b)$ .

**Definition 2.2.1** (Projected conditions). Fix two sets  $\mathcal{X}, \mathcal{Y} \subset \mathbf{E}$  and a point  $\bar{x} \in \mathcal{Y}$  around which  $\mathcal{Y}$  is prox-regular. The pair  $(\mathcal{X}, \mathcal{Y})$  is called  $(b^\pi)$ -regular if it satisfies condition  $(b)$  in the restricted setting  $y_i = P_{\mathcal{Y}}(x_i)$ . Conditions  $\text{strong}(a^\pi)$  and  $\text{strong}(b^\pi)$  are defined analogously.

The following theorem allows one to reduce the question of verifying regularity conditions to the setting  $y \in P_{\mathcal{Y}}(x)$ .

**Theorem 2.2.2.** Fix two sets  $\mathcal{X}, \mathcal{Y} \subset \mathbf{E}$  and a point  $\bar{x} \in \mathcal{Y}$ . Suppose in addition that  $\mathcal{Y}$  is a  $C^2$ -smooth manifold around  $\bar{x}$ . Then the following equivalences hold:

1.  $\text{strong}(a) \Leftrightarrow \text{strong}(a^\pi)$
2.  $(a) \text{ and } (b^\pi) \Leftrightarrow (b)$

Moreover, if  $\mathcal{Y}$  is  $C^3$ -smooth, then the equivalence holds:

$$(a) \text{ and } \text{strong}(b^\pi) \Leftrightarrow \text{strong}(b).$$

*Proof.* Suppose that the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly  $(a^\pi)$ -regular at  $\bar{x}$ . Thus there exists a constant  $C_1 > 0$  such that

$$\Delta(N_{\mathcal{X}}(x), N_{\mathcal{Y}}(P_{\mathcal{Y}}(x))) \leq C_1 \text{dist}(x, \mathcal{Y}) \leq C_1 \|x - y\|, \quad (2.2.2)$$

for all  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  sufficiently close to  $\bar{x}$ . On the other hand, since  $\mathcal{Y}$  is a  $C^2$ -manifold, there exists some constant  $C_2$  such that for any  $x \in \mathcal{X}, y \in \mathcal{Y}$  sufficiently close to  $\bar{x}$  we have

$$\begin{aligned} \Delta(N_{\mathcal{Y}}(P_{\mathcal{Y}}(x)), N_{\mathcal{Y}}(y)) &\leq C_2 \|P_{\mathcal{Y}}(x) - y\| \\ &\leq C_2 (\|P_{\mathcal{Y}}(x) - x\| + \|x - y\|) \\ &\leq 2C_2 \|x - y\|. \end{aligned} \quad (2.2.3)$$

Combining (2.2.2) and (2.2.3), and using the triangle inequality, we conclude  $\Delta(N_{\mathcal{X}}(x), N_{\mathcal{Y}}(y)) \leq (C_1 + 2C_2) \|x - y\|$ , for all  $x \in \mathcal{X}, y \in \mathcal{Y}$  sufficiently close to  $\bar{x}$ . Thus the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly  $(a)$ -regular at  $\bar{x}$  as claimed.

Next, suppose that the pair  $(\mathcal{X}, \mathcal{Y})$  satisfies both conditions (a) and  $(b^\pi)$  at  $\bar{x}$ . Let  $x_i \in \mathcal{X}$  and  $y_i \in \mathcal{Y}$  be sequences converging to  $\bar{x}$  and let  $v_i \in N_{\mathcal{X}}(x_i)$  be arbitrary. Let us write

$$\langle v_i, x_i - y_i \rangle = \langle v_i, x_i - P_{\mathcal{Y}}(x_i) \rangle + \langle v_i, P_{\mathcal{Y}}(x_i) - y_i \rangle,$$

We analyze each term on the right side separately. To this end, observe

$$\left\langle v_i, \frac{x_i - P_{\mathcal{Y}}(x_i)}{\|x_i - y_i\|} \right\rangle = \left\langle v_i, \frac{x_i - P_{\mathcal{Y}}(x_i)}{\|x_i - P_{\mathcal{Y}}(x_i)\|} \right\rangle \cdot \frac{\|x_i - P_{\mathcal{Y}}(x_i)\|}{\|x_i - y_i\|}.$$

Taking into account condition  $(b^\pi)$  and the inequality  $\|x_i - P_{\mathcal{Y}}(x_i)\| \leq \|y_i - x_i\|$ , we deduce  $\lim_{i \rightarrow \infty} \langle v_i, \frac{x_i - P_{\mathcal{Y}}(x_i)}{\|x_i - y_i\|} \rangle = 0$ .

Next since the projection  $P_{\mathcal{Y}}$  is  $C^1$ -smooth near  $\bar{x}$ , it holds:

$$\lim_{i \rightarrow \infty} \frac{\|P_{\mathcal{Y}}(y_i) - P_{\mathcal{Y}}(x_i) - \nabla P_{\mathcal{Y}}(x_i)(y_i - x_i)\|}{\|y_i - x_i\|} = 0. \quad (2.2.4)$$

Therefore taking into account  $y_i = P_{\mathcal{Y}}(y_i)$  we deduce

$$\limsup_{i \rightarrow \infty} \left| \left\langle v_i, \frac{P_{\mathcal{Y}}(x_i) - y_i}{\|x_i - y_i\|} \right\rangle \right| \leq \limsup_{i \rightarrow \infty} \left| \left\langle v_i, \frac{\nabla P_{\mathcal{Y}}(x_i)(y_i - x_i)}{\|y_i - x_i\|} \right\rangle \right|. \quad (2.2.5)$$

Passing to a subsequence, we may assume  $\frac{y_i - x_i}{\|y_i - x_i\|}$  tends to some vector  $w \in \mathbf{E}$  and that  $v_i$  converge to some vector  $v$ . Taking into account the equality  $\nabla P_{\mathcal{Y}}(\bar{x}) = P_{T_{\mathcal{Y}}(\bar{x})}$  and that condition (a) guarantees  $v \in N_{\mathcal{Y}}(\bar{x})$  implies that the right-side of (2.2.5) tends to zero. Thus condition (b) holds as claimed.

Lastly, suppose that  $\mathcal{Y}$  is a  $C^3$  smooth manifold and that the pair  $(\mathcal{X}, \mathcal{Y})$  satisfies both conditions (a) and strong  $(b^\pi)$  at  $\bar{x}$ . Note that the projection  $P_{\mathcal{Y}}$  is  $C^2$ -smooth. The proof proceeds exactly in the same way as the previous implication with (2.2.4) replaced by  $\limsup_{i \rightarrow \infty} \frac{\|P_{\mathcal{Y}}(y_i) - P_{\mathcal{Y}}(x_i) - \nabla P_{\mathcal{Y}}(x_i)(y_i - x_i)\|}{\|y_i - x_i\|^2} < \infty$ .  $\square$

With Theorem 2.2.2 at hand, we may now establish the remaining implications in (2.2.1), beginning with strong (b) implies strong (a).

**Proposition 2.2.3** (Strong (b) implies strong (a)). *Consider a  $C^3$  manifold  $\mathcal{Y}$  that is contained in a set  $\mathcal{X} \subset \mathbf{E}$ . Suppose that  $\mathcal{X}$  is prox-regular at a point  $\bar{x} \in \mathcal{Y}$ . Then the following implication holds for the pair  $(\mathcal{X}, \mathcal{Y})$  and any point  $\bar{x} \in \mathcal{Y}$ :*

$$\text{strong (b)} \Rightarrow \text{strong (a)}.$$

*Proof.* Suppose that the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (b)-regular at  $\bar{x}$ . In light of Theorem 2.2.2, it suffices to prove that the strong  $(a^\pi)$  condition holds. Since  $\mathcal{X}$  is a  $C^3$  manifold, the projection  $P_{\mathcal{Y}}$  is  $C^2$ -smooth. Therefore, there exist constants  $\epsilon, L > 0$  satisfying

$$\|P_{\mathcal{Y}}(y + h) - P_{\mathcal{Y}}(y) - \nabla P_{\mathcal{Y}}(y)h\| \leq L\|h\|^2 \quad (2.2.6)$$

for all  $y \in B_\epsilon(\bar{x})$  and  $h \in \epsilon\mathbf{B}$ . Fix now two points  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  and a unit vector  $v \in N_{\mathcal{X}}(x)$ . Clearly, we may suppose  $v \notin N_{\mathcal{Y}}(y)$ , since otherwise the claim is trivially true.

Define the normalized vector  $w := -\frac{P_{T_{\mathcal{Y}}(y)}(v)}{\|P_{T_{\mathcal{Y}}(y)}(v)\|}$ . Noting the equality  $\nabla P_{\mathcal{Y}}(y) = P_{T_{\mathcal{Y}}(y)}$  and appealing to (3.10.1), we deduce the estimate

$$\|P_{\mathcal{Y}}(y - \alpha w) - (y - \alpha w)\| \leq L\|\alpha w\|^2 = L\alpha^2,$$

for all  $y \in B_\epsilon(\bar{x})$  and  $\alpha \in (0, \epsilon)$ . Shrinking  $\epsilon > 0$ , prox-regularity yields the estimate

$$\langle v, P_{\mathcal{Y}}(y - \alpha w) - x \rangle \leq \frac{\rho}{2}\|x - P_{\mathcal{Y}}(y - \alpha w)\|^2,$$

for some constant  $\rho > 0$ . Therefore, we conclude

$$\begin{aligned} \alpha\|P_{T_{\mathcal{Y}}(y)}v\| &= -\alpha\langle v, w \rangle = \langle v, x - y \rangle + \langle v, P_{\mathcal{Y}}(y - \alpha w) - x \rangle + \langle v, (y - \alpha w) - P_{\mathcal{Y}}(y - \alpha w) \rangle \\ &\leq C\|x - y\|^2 + \frac{\rho}{2}\|x - P_{\mathcal{Y}}(y - \alpha w)\|^2 + L\alpha^2, \end{aligned}$$

where the last inequality follows from the strong (b) condition. Note that the middle term is small:

$$\|P_{\mathcal{Y}}(y - \alpha w) - x\|^2 \leq 2\|P_{\mathcal{Y}}(y - \alpha w) - (y - \alpha w)\|^2 + 2\|y - \alpha w - x\|^2 \leq 2L^2\alpha^4 + 4\|y - x\|^2 + 4\alpha^2.$$

Thus, we have

$$\alpha\|P_{T_{\mathcal{Y}}(y)}v\| \leq C\|x - y\|^2 + \rho L^2\alpha^4 + 2\rho\|x - y\|^2 + 2\rho\alpha^2 + L\alpha^2.$$

Dividing both sides by  $\alpha$  and setting  $\alpha = \|x - y\|$  completes the proof.  $\square$

Next we prove the last implication, strong (a)  $\Rightarrow$  (b), in the definable category. This result thus generalizes the theorems of Kuo [45], Verdier [90], and Ta Le Loi [48]. The proof technique we present is different from those in the earlier works on the subject and will be based on an application of the Kurdyka-Łojasiewicz inequality [8].

**Theorem 2.2.4** (Strong (a) implies (b)). *Fix two definable sets  $\mathcal{X}, \mathcal{Y} \subset \mathbf{E}$  and a point  $\bar{x} \in \mathcal{Y}$ . Suppose in addition that  $\mathcal{Y}$  is a  $C^2$ -smooth manifold around  $\bar{x}$  and that  $\mathcal{X}$  is a locally closed set. Then the following implication holds at  $\bar{x}$  for the pair  $(\mathcal{X}, \mathcal{Y})$ :*

$$\text{strong (a)} \quad \Rightarrow \quad (b).$$

We note that the theorem may easily fail for general  $C^\infty$ -manifolds  $\mathcal{X}$  and  $\mathcal{Y}$ , without some extra “tameness” assumption such as definability. See the discussion in [48] for details.

*Proof.* Suppose that a pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (a)-regular at  $\bar{x}$ . In light of Theorem 2.2.2, it suffices to show that  $(\mathcal{X}, \mathcal{Y})$  is  $(b^\pi)$ -regular at  $\bar{x}$ . To this end, define the function

$$g(x, v) = |\langle v, x - P_{\mathcal{Y}}(x) \rangle| + \delta_{\text{cl } \mathcal{X}}(x).$$

Fix a compact neighborhood  $U$  of  $\{\bar{x}\} \times \mathbb{B}$ . Then the KL-inequality [8, Theorem 11] ensures that there exists  $\eta > 0$  and a continuous function  $\psi: [0, \eta) \rightarrow \mathbf{R}$  satisfying  $\psi(0) = 0$  and  $\psi'(0) = 0$  such that

$$g(x, v) \leq \psi(\text{dist}(0, \partial g(x, v))). \tag{2.2.7}$$

for any  $(x, v) \in U$  with  $g(x, v) \leq \eta$ . It suffices now to show that  $\text{dist}(0, \partial g(x, v))$  is linearly upper bounded by  $\text{dist}(x, \mathcal{Y})$  for all  $x \in \mathcal{X}$  near  $\bar{x}$  and all unit vectors  $v \in N_{\mathcal{X}}(x)$ . To this end, fix any point  $(x, v)$ . Clearly, we may assume  $g(x, v) \neq 0$ , since otherwise there is nothing to prove. We compute

$$\partial g(x, v) = \{(I - \nabla P_{\mathcal{Y}}(x))v + N_{\text{cl}\mathcal{X}}(x)\} \times \{x - P_{\mathcal{Y}}(x)\}.$$

Therefore as long as  $v \in N_{\mathcal{X}}(x)$  we have

$$\text{dist}(0, \partial g(x, v)) \leq \|\nabla P_{\mathcal{Y}}(x)v\| + \text{dist}(x, \mathcal{Y}). \quad (2.2.8)$$

Since  $\mathcal{Y}$  is a  $C^2$ -manifold near  $\bar{x}$ , there exists a constant  $L > 0$  such that the inequality  $\|\nabla P_{\mathcal{Y}}(x)\| \leq L$  holds for all  $x$  near  $\bar{x}$ . Further, let  $C > 0$  be the constant from the defining property (2.1.1) of strong (a) regularity. Thus, as long as  $x \in \mathcal{X}$  is sufficiently close to  $\bar{x}$ , there exists a vector  $w \in N_{\mathcal{Y}}(P_{\mathcal{Y}}(x))$  satisfying  $\|v - w\| \leq C \text{dist}(x, \mathcal{Y})$ . Therefore, continuing with 2.2.8 we deduce

$$\text{dist}(0, \partial g(x, v)) \leq \|\nabla P_{\mathcal{Y}}(x)w\| + (1 + CL)\text{dist}(x, \mathcal{Y}).$$

To complete the proof, note that  $\nabla P_{\mathcal{Y}}(x)w = 0$  since  $\text{range}(\nabla P_{\mathcal{Y}}(x)) \subseteq T_{\mathcal{Y}}(P_{\mathcal{Y}}(x))$ .  $\square$

## 2.3 Basic examples

Having a clear understanding of how the four regularity conditions are related, we now present a few interesting examples of sets that are regular along a distinguished submanifold. More interesting examples can be constructed with the help of calculus rule, discussed at the end of the section. We begin with the following simple example showing that any convex cone is regular along its lineality space.

**Proposition 2.3.1** (Cones along the lineality space). *Let  $X \subset \mathbf{E}$  be a convex cone and let  $Y = \mathcal{X} \cap (-\mathcal{X})$  denote its lineality space. Then the pair  $(\mathcal{X}, \mathcal{Y})$  is both strongly (a) and strongly (b) regular at any point  $\bar{x} \in \mathcal{Y}$ .*

*Proof.* For any point  $x \in \mathcal{X}$ , the inclusion  $x + \mathcal{Y} \subset \mathcal{X}$  holds. Therefore we compute  $N_{\mathcal{X}}(x) \subset N_{x+\mathcal{Y}}(x) = \mathcal{Y}^\perp = N_{\mathcal{Y}}(y)$ , for any points  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ . Thus the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (a)-regular at all points  $\bar{x} \in \mathcal{Y}$ . Next, fix any points  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  and a vector  $v \in N_{\mathcal{X}}(x)$ . Translating by  $y$ , we see that  $v$  is a normal vector to  $\mathcal{X} - y$  at  $x - y$ . Taking into account  $\mathcal{X} - y = \mathcal{X}$ , we deduce  $\langle v, x - y \rangle = 0$ . Thus the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (b)-regular at all points  $\bar{x} \in \mathcal{Y}$ .  $\square$

More generally, any convex set is strongly (a)-regular along any affine space contained in it.

**Proposition 2.3.2** (Affine subsets of convex sets). *Consider a convex set  $\mathcal{X} \subset \mathbf{E}$  and a subset  $\mathcal{Y} \subset \mathcal{X}$  that is locally affine around a point  $\bar{x} \in \mathcal{Y}$ . Then the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (a)-regular at  $\bar{x}$ .*

*Proof.* Translating the sets we may suppose  $\bar{x} = 0$  and therefore that  $\mathcal{Y}$  coincides with a linear subspace near the origin. Fix now points  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  and a unit vector  $v \in N_{\mathcal{X}}(x)$ . Clearly, we may suppose  $v \notin N_{\mathcal{Y}}(y)$ , since otherwise the claim is trivially true. Define the normalized vector  $w := -\frac{P_{\mathcal{Y}}(v)}{\|P_{\mathcal{Y}}(v)\|}$ . Then for all  $y \in \mathcal{Y}$  near  $\bar{x}$  and all small  $\alpha > 0$ , using the linearity of the projection  $P_{\mathcal{Y}}$  we compute

$$\begin{aligned} \alpha\|P_{T_{\mathcal{Y}}(y)}v\| &= \alpha\|P_{\mathcal{Y}}v\| = -\alpha\langle v, w \rangle = \langle v, x - y \rangle + \langle v, P_{\mathcal{Y}}(y - \alpha w) - x \rangle \\ &\leq \|x - y\|, \end{aligned}$$

where the last inequality follows from convexity of  $\mathcal{X}$ . This completes the proof.  $\square$

Not surprisingly, the conclusion of Theorem 2.3.2 can easily fail if  $\mathcal{X}$  is prox-regular (instead of convex) or if  $\mathcal{Y}$  is a smooth manifold (instead of affine). This is the content of the following example.

**Example 2.3.1** (Failure of strong (a)-regularity). Define  $\mathcal{X}$  to be the epigraph of the function  $f(x, y) = \max\{0, y - x^2\}$  and set  $\mathcal{Y}$  to be the  $x$ -axis  $Y = \mathbf{R} \times \{0\} \times \{0\}$ . See Figure 2.2 for an illustration. Consider the sequence  $y_k = (1/k, 0, 0)$  in  $\mathcal{Y}$  and  $x_k = (1/k, 1/k^2, 0)$  in  $\mathcal{X}$  converging to the origin. Fix the sequence of normal vectors  $v_k = (-2/k, 1, -1) \in N_{\mathcal{X}}(x_k)$  and note  $N_{\mathcal{Y}}(y_k) = \{0\} \times \mathbf{R} \times \mathbf{R}$ . A quick computation shows

$$\Delta\left(\frac{v_k}{\|v_k\|}, N_{\mathcal{Y}}(y_k)\right) = \frac{2/k}{\sqrt{2 + 4/k^2}} \geq \frac{2}{k\sqrt{6}} = \frac{2}{\sqrt{6}} \sqrt{\|x_k - y_k\|}.$$

Therefore the pair  $(\mathcal{X}, \mathcal{Y})$  is not strongly (a)-regular at  $\bar{x}$ .

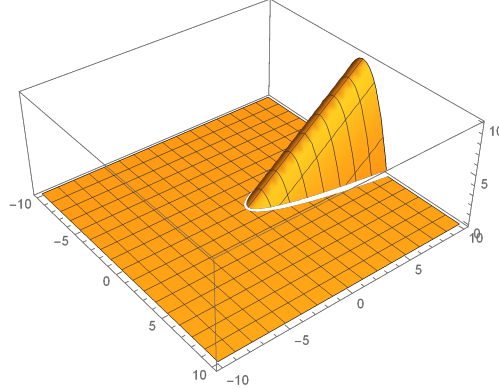


Figure 2.2: Graph of the function  $f(x, y) = \max\{0, y - x^2\}$ .

Strong (a)-regularity fails in the above example “by a square root factor in the distance to  $\mathcal{Y}$ .” The following theorem shows a surprising fact: the estimate (2.1.1) is guaranteed to hold up to a square root for any proximal smooth set along a smooth submanifold. Since we will not use this result and the proof is very similar to that of Proposition 2.2.3, we have placed the argument in the appendix.

**Proposition 2.3.3** (Strong (a) up to square root). *Consider a  $C^3$  manifold  $\mathcal{Y}$  that is contained in a set  $\mathcal{X} \subset \mathbf{E}$ . Suppose that  $\mathcal{X}$  is prox-regular around a point  $\bar{x} \in \mathcal{Y}$ . Then there exists a constant  $C > 0$  satisfying*

$$\Delta(N_{\mathcal{X}}(x), N_{\mathcal{Y}}(y)) \leq C \cdot \sqrt{\|x - y\|}, \quad (2.3.1)$$

for all  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  sufficiently close to  $\bar{x}$ .

The following example connects (b)-regularity to inner-semicontinuity of the normal cone map. In particular, any proximally smooth set is (b)-regular along any of its partly smooth submanifolds in the sense of Lewis [57].

**Proposition 2.3.4** (Condition (b) and inner semicontinuity). *Consider a proximally smooth set  $\mathcal{X}$  and a set  $\mathcal{Y} \subset \mathcal{X}$  such that the normal cone map  $N_{\mathcal{X}}$  is inner-semicontinuous on  $\mathcal{Y}$ . Then the pair  $(\mathcal{X}, \mathcal{Y})$  satisfies the (b)-condition.*

*Proof.* Consider sequences  $x_i \in \mathcal{X} \setminus \mathcal{Y}$  and  $y_i \in \mathcal{Y}$  converging to a point  $\bar{x} \in \mathcal{Y}$ . Let  $v_i \in N_{\mathcal{X}}(x_i)$  be arbitrary unit normal vectors. Passing to a subsequence we may assume that  $v_i$  converge to some unit normal vector  $\bar{v} \in N_{\mathcal{X}}(\bar{x})$ . By inner semicontinuity, there exist unit vectors  $w_i \in N_{\mathcal{X}}(y_i)$  converging to  $\bar{v}$ . Define the unit vectors  $u_i := \frac{x_i - y_i}{\|x_i - y_i\|}$ . Proximal smoothness of  $\mathcal{X}$  therefore guarantees  $\langle v_i, u_i \rangle \geq -\frac{\rho}{2}\|x_i - y_i\|$  and  $\langle w_i, u_i \rangle \leq \frac{\rho}{2}\|x_i - y_i\|$ . We conclude

$$-\frac{\rho}{2}\|x_i - y_i\| \leq \langle v_i, u_i \rangle = \langle w_i, u_i \rangle + \langle v_i - w_i, u_i \rangle \leq \frac{\rho}{2}\|x_i - y_i\| + \|v_i - w_i\|.$$

Noting that the left and right sides both tend to zero completes the proof.  $\square$

## 2.4 Preservation of regularity under preimages by smooth transversal maps

More interesting examples may be constructed through calculus rules. The next theorem shows that the four regularity conditions are preserved by taking preimages of smooth maps under a transversality condition.

**Theorem 2.4.1** (Smooth preimages). *Consider a  $C^1$ -map  $F: \mathbf{Y} \rightarrow \mathbf{E}$  and an arbitrary point  $\bar{x} \in \mathbf{Y}$ . Let  $\mathcal{X}, \mathcal{Y} \subset \mathbf{E}$  be two locally closed sets with  $\mathcal{Y}$  Clarke regular. Suppose that  $(\mathcal{X}, \mathcal{Y})$  is (a)-regular (respectively (b)-regular) at the point  $F(\bar{x}) \in \mathcal{Y}$  and that the transversality condition holds:*

$$N_{\mathcal{Y}}(F(\bar{x})) \cap \text{Null}(\nabla F(\bar{x})^*) = \{0\}. \quad (2.4.1)$$

*Then the pair  $(F^{-1}(\mathcal{X}), F^{-1}(\mathcal{Y}))$  is (a)-regular (respectively (b)-regular) at  $\bar{x}$ . Analogous statements hold for strong (a) and strong (b) conditions, provided  $F$  is in addition  $C^2$ -smooth.*

*Proof.* Transversality ensures that there exists a constant  $\tau > 0$  and a neighborhood  $U$  of  $\bar{x}$  satisfying

$$\|\nabla F(y)^* v\| \geq \tau \|v\| \quad \text{for all } y \in F^{-1}(\mathcal{Y}) \cap U, \ v \in N_{\mathcal{Y}}(F(y)).$$

Moreover, shrinking  $U$ , we may assume that  $F$  is  $\ell$ -Lipschitz continuous on  $U$ . We prove the theorem in the order: (a), strong (a), (b), strong (b).

Condition (a): Suppose that the pair  $(\mathcal{X}, \mathcal{Y})$  is (a)-regular at  $F(\bar{x})$ . Then, shrinking  $\eta, \tau > 0$  and  $U$ , we may ensure:

$$\|\nabla F(x)^* v\| \geq \tau \|v\| \quad \text{for all } x \in F^{-1}(\mathcal{X}) \cap U, \ v \in N_{\mathcal{X}}(F(x)). \quad (2.4.2)$$

Transversality and Clarke regularity of  $\mathcal{Y}$  imply [80, Theorem 10.6]

$$N_{F^{-1}(\mathcal{Y})}(y) = \nabla F(y)^* N_{\mathcal{Y}}(F(y)) \quad \text{and} \quad N_{F^{-1}(\mathcal{X})}(x) \subset \nabla F(x)^* N_{\mathcal{X}}(F(x)) \quad (2.4.3)$$

for all  $y \in F^{-1}(\mathcal{Y})$  and  $x \in F^{-1}(\mathcal{X})$  sufficiently close to  $\bar{x}$ .

Consider now a sequence  $x_i \in F^{-1}(\mathcal{X})$  converging to  $\bar{x}$  and a sequence of unit normal vectors  $w_i \in N_{F^{-1}(\mathcal{X})}(x_i)$  converging to some vector  $w$ . Using (2.4.3), we may write  $w_i = \nabla F(x_i)^* v_i$  for some vectors  $v_i \in N_{\mathcal{X}}(F(x_i))$ . Note that due to (2.4.2), the sequence  $v_i$  is bounded. Indeed, the norm of  $v_i$  is upper bounded by a constant that is independent of  $x_i$  and  $y_i$ . Therefore passing to a subsequence we may suppose  $v_i$  converges to some vector  $v$ . Since the pair  $(\mathcal{X}, \mathcal{Y})$  is (a)-regular at  $F(\bar{x})$ , the inclusion  $v \in N_{\mathcal{Y}}(F(\bar{x}))$  holds. Therefore using (2.4.3) we deduce  $w = \lim_{i \rightarrow \infty} \nabla F(x_i)^* v_i = \nabla F(\bar{x})^* v \in N_{F^{-1}(\mathcal{Y})}(F(\bar{x}))$ . Thus the pair  $(F^{-1}(\mathcal{X}), F^{-1}(\mathcal{Y}))$  is (a)-regular at  $\bar{x}$ .

Before moving on to the next three regularity conditions, note that each of them implies condition (a) (Lemma 2.1.2) and therefore we can be sure that the expressions (2.4.2) and (2.4.3) hold. Therefore, fix for the rest of the proof the sequence  $x_i, v_i$ , and  $w_i$  as above, and let  $y_i \in \mathcal{Y}$  be an arbitrary sequence converging to  $\bar{x}$ .

Condition strong (a): Suppose that  $F$  is  $C^2$ -smooth and the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (a)-regular at  $F(\bar{x})$  and let  $C > 0$  be the corresponding constant in (2.1.1). Shrinking  $U$  we may assume  $\nabla F$  is  $L$ -Lipschitz continuous on  $U$ . We successively compute

$$\begin{aligned} \text{dist}(w_i, N_{F^{-1}(\mathcal{Y})}(y_i)) &= \text{dist}(\nabla F(x_i)^* v_i, N_{F^{-1}(\mathcal{Y})}(y_i)) \\ &\leq \|\nabla F(x_i) - \nabla F(y_i)\|_{\text{op}} \|v_i\| + \text{dist}(\nabla F(y_i)^* v_i, N_{F^{-1}(\mathcal{Y})}(y_i)) \end{aligned} \quad (2.4.4)$$

$$= \|\nabla F(x_i) - \nabla F(y_i)\|_{\text{op}} \|v_i\| + \text{dist}(v_i, N_{\mathcal{Y}}(F(y_i))) \quad (2.4.5)$$

$$\leq L \|v_i\| \|x_i - y_i\| + C \|v_i\| \|F(x_i) - F(y_i)\| \quad (2.4.6)$$

$$\begin{aligned} &\leq (L + C\ell) \|v_i\| \|x_i - y_i\| \\ &\leq (L + C\ell) \tau^{-1} \|\nabla F(x_i)^* v_i\| \|x_i - y_i\| \end{aligned} \quad (2.4.7)$$

$$= (L + C\ell) \tau^{-1} \|w_i\| \|x_i - y_i\|,$$

where (2.4.4) follows from the triangle inequality, (2.4.5) follows from (2.4.3), the estimate (2.4.6) follows from strong (a)-regularity, and (2.4.7) follows from (2.4.2).

Setting the stage for the remainder of the proof, we compute

$$\begin{aligned}
\left| \left\langle w_i, \frac{y_i - x_i}{\|y_i - x_i\|} \right\rangle \right| &= \left| \left\langle v_i, \frac{\nabla F(x_i)(y_i - x_i)}{\|y_i - x_i\|} \right\rangle \right| \\
&\leq \left| \left\langle v_i, \frac{F(y_i) - F(x_i)}{\|y_i - x_i\|} \right\rangle \right| + \|v_i\| \cdot \frac{\|F(y_i) - F(x_i) - \nabla F(x_i)(y_i - x_i)\|}{\|x_i - y_i\|}.
\end{aligned} \tag{2.4.8}$$

Condition (b): Suppose that the pair  $(\mathcal{X}, \mathcal{Y})$  is (b)-regular at  $F(\bar{x})$ . Let us look at the two terms on the right side of (2.4.8). Local Lipschitz continuity of  $F$  and (b)-regularity imply that the first term tends to zero while  $C^1$ -smoothness of  $f$  ensures that the second term also tends to zero.

Condition strong (b): Suppose that  $F$  is  $C^2$ -smooth and the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (b)-regular at  $F(\bar{x})$ . Shrinking  $U$  we may assume  $\nabla F$  is  $L$ -Lipschitz continuous on  $U$ . Let us look again at the two terms on the right side of (2.4.8). Local Lipschitz continuity of  $F$  and strong (b)-regularity imply that the first term can be upper bounded as

$$\left| \left\langle v_i, \frac{F(y_i) - F(x_i)}{\|y_i - x_i\|} \right\rangle \right| \leq C \frac{\|F(y_i) - F(x_i)\|^2}{\|y_i - x_i\|} \leq C\ell^2 \|y_i - x_i\|,$$

where  $C$  is the constant in (2.1.2). The second term may be upper bounded as  $\|v_i\| \cdot \frac{\|F(y_i) - F(x_i) - \nabla F(x_i)(y_i - x_i)\|}{\|x_i - y_i\|} \leq L\|v_i\|\|y_i - x_i\|$ . Noting that the vectors  $v_i$  are bounded in norm by a constant that is independent of  $x_i$  and  $y_i$  completes the proof.  $\square$

The calculus rule just developed along with basic examples can be used to justify the four regularity conditions for many examples. In particular, cone reducible sets, introduced by [82], form a large class of sets are regular along their active manifolds. We refer the reader to [82] for numerous examples of cone reducible sets in optimization.

**Corollary 2.4.2** (Cone reducible sets are regular along the active manifold). *Suppose that a set  $\mathcal{X}$  is  $C^2$  cone reducible to  $C$  by  $F$  at  $\bar{x}$ . Then the pair  $(\mathcal{X}, F^{-1}(0))$  is strongly (a) and strongly (b)-regular at  $\bar{x}$ .*

## 2.5 Preservation of regularity under spectral lifts

In this section, we study the prevalence of the four regularity conditions in eigenvalue problems. We begin with some notation. The symbol  $\mathbf{S}^n$  will denote the Euclidean space of symmetric matrices, endowed with the trace inner product  $\langle A, B \rangle = \text{tr}(AB)$  and the induced Frobenius norm  $\|A\| = \sqrt{\text{tr}(A^2)}$ . The symbol  $O(n)$  will denote the set of  $n \times n$  orthogonal matrices. The eigenvalue map  $\lambda: \mathbf{S}^n \rightarrow \mathbf{R}^n$  assigns to every matrix  $X$  its ordered list of eigenvalues

$$\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X).$$

The following class of sets will be the subject of the study.



**Definition 2.5.1.** A set  $\mathcal{X} \subset \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$  is called *symmetric* if it satisfies

$$\pi\mathcal{X} \subset \mathcal{X} \quad \text{for all } \pi \in \Pi(n).$$

**Definition 2.5.2.** A set  $\mathcal{Q} \subset \mathbf{S}^n$  is called *spectral* if it satisfies

$$U\mathcal{Q}U^T \subset \mathcal{Q} \quad \text{for all } U \in O(n).$$

Thus a set in  $\mathbf{R}^n$  is symmetric if it is invariant under reordering of the coordinates. For example, all  $\ell_p$ -norm balls, the nonnegative orthant, and the unit simplex are symmetric. A set in  $\mathbf{S}^n$  is spectral if it is invariant under conjugation of its argument by orthogonal matrices. Spectral sets are precisely those that can be written as  $\lambda^{-1}(\mathcal{X})$  for some symmetric set  $\mathcal{X} \subset \mathbf{R}^n$ . See figure 2.3 for an illustration.

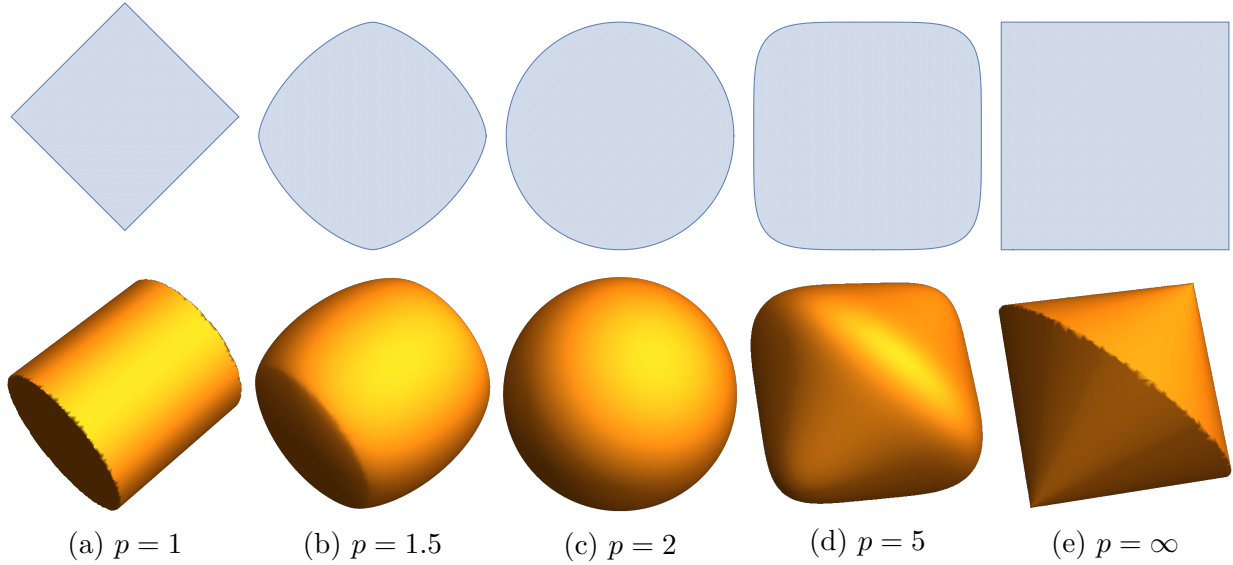


Figure 2.3: Unit  $\ell_p$  balls in  $\mathbf{R}^2$  (top row) and unit balls of Schatten  $\ell_p$ -norms  $\|A\|_p = \|\lambda(A)\|_p$  over  $\mathbf{S}^2$  (bottom row).

A prevalent theme in variational analysis is that a variety of geometric properties of a symmetric set  $\mathcal{X}$  and those of its induced spectral set  $\lambda^{-1}(\mathcal{X})$  are in one-to-one correspondence. Notable examples include convexity [20, 55], smoothness [59, 61], prox-regularity [19], and partial smoothness [18]. In this section, we add to this list the four regularity conditions. The key idea of the arguments is to pass through the projected conditions (Definition 2.2.1) and then invoke Theorem 2.2.2.

We will use the following expressions for the normal cone and the projection map to spectral sets  $\mathcal{X}$ :

$$\begin{aligned} P_{\lambda^{-1}(\mathcal{X})}(X) &= \{U\text{Diag}(w)U^T : w \in P_{\mathcal{X}}(\lambda(X)), U \in O_X\} \\ N_{\lambda^{-1}(\mathcal{X})}(X) &= \{U\text{Diag}(y)U^T : y \in N_{\mathcal{X}}(\lambda(X)), U \in O_X\}. \end{aligned} \quad (2.5.1)$$

where for any matrix  $X$ , we define the set of diagonalizing matrices

$$O_X := \{U \in O(n) : X = U\text{Diag}(\lambda(X))U^T\}.$$

The expression for the proximal map was established in [29] while the subdifferential formula was proved in [56]. A short and elementary proof of the subdifferential formula appears in [29].

**Theorem 2.5.3** (Spectral preservation of projected regularity). *Let  $\bar{X} \in \mathbf{S}^n$  be a symmetric matrix and consider two locally closed symmetric sets  $\mathcal{X}, \mathcal{Y} \subseteq \mathbf{R}^n$  such that  $\mathcal{Y}$  is prox-regular at  $\lambda(\bar{X})$ . If the pair  $(\mathcal{X}, \mathcal{Y})$  satisfies condition  $\#$  at  $\lambda(\bar{X}) \in \mathcal{X}$ , then the pair  $(\lambda^{-1}(\mathcal{X}), \lambda^{-1}(\mathcal{Y}))$  satisfies condition  $\#$  at  $\bar{X}$ , where  $\#$  can stand for strong  $(a^\pi)$ ,  $(b^\pi)$ , and strong  $(b^\pi)$ .*

*Proof.* The result for  $(a)$ -regularity holds trivially from (2.5.1). For the rest of the proof suppose that  $\mathcal{Y}$  is prox-regular at  $\lambda(\bar{X})$ . The result [19] therefore guarantees that  $\lambda^{-1}(\mathcal{Y})$  is prox-regular at  $\bar{X}$ .

Consider now an arbitrary matrix  $X \in \lambda^{-1}(\mathcal{X})$  near  $\bar{X}$  and a normal vector  $V \in N_{\lambda^{-1}(\mathcal{X})}(X)$  with unit Frobenius length. We may then write

$$V = U \text{Diag}(v) U^T,$$

for some unit vector  $v \in N_{\mathcal{X}}(\lambda(X))$  and orthogonal matrix  $U \in O_X$ . Define the projection  $Y = P_{\lambda^{-1}(\mathcal{Y})}(X)$ . Using (2.5.1), we may write

$$Y = U \text{Diag}(P_{\mathcal{Y}}(\lambda(X))) U^T.$$

Notice that because the coordinates of  $\lambda(X)$  are decreasing and  $\mathcal{Y}$  is symmetric, the coordinates of  $P_{\mathcal{Y}}(\lambda(X))$  are also decreasing; otherwise, one may reorder  $P_{\mathcal{Y}}(\lambda(X))$  and find a vector closer to  $\lambda(X)$  in  $\mathcal{Y}$ . Consequently, we have

$$\lambda(Y) = P_{\mathcal{Y}}(\lambda(X)) \quad \text{and} \quad U \in O_Y. \quad (2.5.2)$$

Suppose now that the pair  $(X, Y)$  is strongly  $(a^\pi)$  regular at  $\lambda(\bar{X})$  and let  $C$  be the corresponding constant in (2.1.1). Condition strong  $(a^\pi)$  ensures that there exists  $w \in N_{\mathcal{Y}}(P_{\mathcal{Y}}(\lambda(X)))$  satisfying

$$\|v - w\| = \text{dist}(v, N_{\mathcal{Y}}(P_{\mathcal{Y}}(\lambda(X)))) \leq C \|\lambda(X) - P_{\mathcal{Y}}(\lambda(X))\| = C \|X - Y\|,$$

where the last equation follows  $X$  and  $Y$  being simultaneously diagonalizable. Taking into account (2.5.1) and (2.5.2), we deduce that  $W := U \text{Diag}(w) U^T$  lies in  $N_{\lambda^{-1}(\mathcal{Y})}(Y)$ . Therefore we compute

$$\text{dist}(V, N_{\lambda^{-1}(\mathcal{Y})}(Y)) \leq \|V - W\| = \|v - w\| \leq C \|X - Y\|.$$

We therefore conclude that the pair  $(\lambda^{-1}(\mathcal{X}), \lambda^{-1}(\mathcal{Y}))$  is strongly  $(a^\pi)$  regular at  $\bar{X}$  as claimed.

Next moving onto conditions  $(b^\pi)$  and strong  $(b^\pi)$ , we compute

$$\langle V, X - Y \rangle = \langle v, \lambda(X) - P_{\mathcal{Y}}(\lambda(X)) \rangle.$$

The claimed results now follow immediately by noting  $\|\lambda(X) - P_{\mathcal{Y}}(\lambda(X))\| = \|X - Y\|$ .  $\square$

Combining Theorems 2.5.3, 2.2.2, and spectral preservation of smoothness [18] yields the main result of the section.

**Proposition 2.5.4** (Spectral Lifts). *Let  $\bar{X} \in \mathbf{S}^n$  be a symmetric matrix. Let  $\mathcal{X}, \mathcal{Y} \subseteq \mathbf{R}^n$  be locally closed symmetric sets with  $\mathcal{Y}$  a  $C^2$ -smooth manifold around  $\lambda(\bar{X})$ . If the pair  $(\mathcal{X}, \mathcal{Y})$  satisfies condition  $\sharp$  at  $\lambda(\bar{X})$ , then the pair  $(\lambda^{-1}(\mathcal{X}), \lambda^{-1}(\mathcal{Y}))$  satisfies condition  $\sharp$  at  $\bar{X}$ , where  $\sharp$  can stand for (a), (b), or strong (a). Moreover, if  $\mathcal{Y}$  is  $C^3$ -smooth and  $\mathcal{X}$  is prox-regular, then we may set  $\sharp$  to be the strong (b) condition.*

## 2.6 Regularity of functions along manifolds

The previous sections developed basic examples and calculus rules for the four basic regularity conditions. In this section we interpret these results for functions through their epigraphs. We begin with the following lemma, which is the direct analogue of Propositions 2.3.1, 2.3.2, and 2.3.4.

**Lemma 2.6.1** (Basic examples). *Consider a function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$ , a set  $\mathcal{M} \subset \text{dom } f$ , and a point  $\bar{x} \in \mathcal{M}$ . The following statements are true.*

1. *If  $f$  is a sublinear function and  $\mathcal{M} = \{x : f(x) = -f(-x)\}$  is its lineality space, then  $f$  is both strongly (a) and strongly (b) regular along  $\mathcal{M}$  at  $\bar{x}$ .*
2. *If  $f$  is convex,  $\mathcal{M}$  is locally affine near  $\bar{x}$ , and  $f$  restricted to  $\mathcal{M}$  is a linear function near  $\bar{x}$ , then  $f$  is strongly (a)-regular along  $\mathcal{M}$  near  $\bar{x}$ .*
3. *If  $f$  is weakly convex and locally Lipschitz near  $\bar{x}$  and the subdifferential map  $x \mapsto \partial f(x)$  is inner-semicontinuous on  $\mathcal{M}$ , then  $f$  is (b)-regular along  $\mathcal{M}$  at  $\bar{x}$ .*

*Proof.* Let  $f$  be a sublinear function. Then  $\text{epi } f$  is a convex cone whose lineality space is precisely  $\text{gph } f|_{\mathcal{M}}$ . Applying proposition 2.3.1 we deduce that  $f$  is both strongly (a) and strongly (b) regular along  $\mathcal{M}$  at  $\bar{x}$ , as claimed. The second and third claims are immediate from Propositions 2.3.2 and 2.3.4.  $\square$

Theorem 2.4.1 takes the form of the following chain rule.

**Theorem 2.6.2** (Chain rule). *Consider a  $C^p$ -smooth map  $c: \mathbf{Y} \rightarrow \mathbf{E}$  and a closed function  $h: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  that is finite at some point  $c(\bar{x})$ . Define the composite function  $f(x) = h(c(x))$ , fix a set  $\mathcal{M} \subset \mathbf{E}$ , and define  $\mathcal{L} := c^{-1}(\mathcal{M})$ . Suppose the following:*

1.  *$h$  is (a)-regular (respectively (b)-regular) along  $\mathcal{M}$  at  $c(\bar{x})$ ,*
2.  *$\mathcal{M}$  is a  $C^p$  manifold and the restriction  $h|_{\mathcal{M}}$  is  $C^p$ -smooth near  $\bar{x}$ ,*
3. *the transversality condition holds:*

$$N_{\mathcal{M}}(c(\bar{x})) \cap \text{Null}(\nabla c(\bar{x})^*) = \{0\}. \quad (2.6.1)$$

Then  $\mathcal{L}$  is a  $C^p$  manifold around  $\bar{x}$ , the restriction of  $f$  to  $\mathcal{L}$  is  $C^p$ -smooth, and  $f$  is (a)-regular (respectively (b)-regular) along  $\mathcal{L}$  at  $\bar{x}$ . Analogous statements hold for strong (a) and strong (b) conditions, provided  $c$  is in addition  $C^2$ -smooth.

*Proof.* First, the transversality condition (2.6.2) classically guarantees that  $\mathcal{L}$  is a  $C^p$  manifold around  $\bar{x}$ . Moreover, for any  $x \in \mathcal{L}$ , we may write  $f(x) = h(c(x)) = (h|_{\mathcal{M}} \circ c)(x)$ . Therefore the restriction of  $f$  to  $\mathcal{L}$  is indeed  $C^p$ -smooth.

Next, observe that we may write  $\text{epi } f = \{(x, r) : (c(x), r) \in \text{epi } h\}$ . Thus in the notation of Theorem 2.4.1, setting  $\mathcal{X} = \text{epi } h$ ,  $\mathcal{Y} = \text{gph } h|_{\mathcal{M}}$ , and  $F(x, r) = (c(x), r)$ , we may write

$$\text{epi } f = F^{-1}(\mathcal{X}) \quad \text{and} \quad \text{gph } f|_{\mathcal{L}} = F^{-1}(\mathcal{Y}).$$

An application of Theorem 2.4.1 will then complete the proof as soon as we verify the required transversality condition (2.4.1). To this end, consider a vector  $(v, s) \in N_{\mathcal{Y}}(F(\bar{x})) \cap \text{Null}(\nabla F(\bar{x})^*)$ . We compute

$$(0, 0) = \nabla F(\bar{x})^*(v, s) = (\nabla c(\bar{x})^*v, s).$$

Therefore, the inclusion  $(v, 0) \in N_{\mathcal{Y}}(F(\bar{x}))$  holds and we have  $v \in \text{Null } \nabla(c(\bar{x})^*)$ . The former immediately implies that  $v$  lies in  $N_{\mathcal{M}}(c(\bar{x}))$  and therefore (2.6.2) implies  $v = 0$ , as claimed.  $\square$

In particular, decomposable functions of [82] are regular along their active manifolds.

**Corollary 2.6.3** (Decomposable functions are regular). *Suppose that a function  $f$  is properly  $C^1$  decomposable as  $h \circ c$  around  $\bar{x}$ . Then  $f$  is (b)-regular along  $c^{-1}(\text{Lin}(h))$  at  $\bar{x}$ . If  $f$  is  $C^2$ -smooth, then  $f$  is strongly (a) and strongly (b) regular along  $c^{-1}(\text{Lin}(h))$  at  $\bar{x}$ .*

As usual, the chain rule immediately implies a sum rule—the final result of the section.

**Theorem 2.6.4** (Sum rule). *Consider closed functions  $f_i: \mathbf{R} \rightarrow \mathbf{R} \cup \{\infty\}$  indexed by a finite set  $i \in I$ , and which are finite at a point  $\bar{x}$ . Define the function  $f(x) = \sum_{i \in I} f_i(x)$ , fix some set  $\mathcal{M}_i$  for each  $i \in I$ , and define the intersection  $\mathcal{M} = \cap_{i \in I} \mathcal{M}_i$ . Suppose that the following hold for each  $i \in I$ :*

1.  $f_i$  is (a)-regular (respectively (b)-regular) along  $\mathcal{M}_i$  at  $\bar{x}$ ,
2.  $\mathcal{M}_i$  is a  $C^p$  manifold and the restriction of  $f_i$  to  $\mathcal{M}_i$  is  $C^p$ -smooth near  $\bar{x}$ ,
3. the implication holds:

$$\sum_{i \in I} v_i = 0 \quad \text{and} \quad v_i \in N_{\mathcal{M}_i}(\bar{x}) \quad \forall i \in I \quad \implies \quad v_i = 0 \quad \forall i \in I. \quad (2.6.2)$$

Then  $\mathcal{M}$  is a  $C^p$  manifold around  $\bar{x}$ , the restriction of  $f$  to  $\mathcal{M}$  is  $C^p$ -smooth, and  $f$  is (a)-regular (respectively (b)-regular) along  $\mathcal{L}$  at  $\bar{x}$ . Analogous statements hold for strong (a) and strong (b) conditions, provided  $f_i$  are in addition  $C^2$ -smooth for each  $i \in I$ .

*Proof.* Enumerate the indices as  $I = \{1, \dots, k\}$ . The result follows from Theorem 2.6.2 by setting  $c(x) = (x, \dots, x)$  and  $h(x_1, \dots, x_k) = x_1 + \dots + x_k$ .  $\square$

## 2.7 Generic regularity along active manifolds

How can one justify the use of a particular regularity condition? One approach, highlighted in the previous sections, is to verify the conditions for certain basic examples and then show that they are preserved under transverse smooth deformations. Stratification theory adapts another viewpoint, wherein a regularity condition between two manifolds is considered acceptable if reasonable sets (e.g. semi-algebraic, subanalytic or definable) can always be partitioned into finitely many smooth manifolds so that the regularity condition holds along any two “adjacent” manifolds. See the survey [88] for an extensive discussion.

To formalize this viewpoint, we begin with a definition of a stratification.

**Definition 2.7.1** (Stratification). A  $C^p$ -stratification ( $p \geq 1$ ) of a set  $Q \subset \mathbf{E}$  is a partition of  $Q$  into finitely many  $C^p$  manifolds, called *strata*, such that any two strata  $\mathcal{X}$  and  $\mathcal{Y}$  satisfy the implication:

$$\mathcal{Y} \cap \text{cl } \mathcal{X} \neq \emptyset \implies \mathcal{Y} \subset \text{cl } \mathcal{X}.$$

A stratum  $\mathcal{Y}$  is said to be *adjacent* to a stratum  $\mathcal{X}$  if the inclusion  $\mathcal{Y} \subset \text{cl } \mathcal{X}$  holds. If each stratum is a definable manifold, the stratification is called *definable*.

Thus a stratification of  $Q$  is simply a partition of  $Q$  into smooth manifolds so that the closure of any stratum is a union of strata. Stratifications such that any pair of adjacent strata are strongly (a)-regular are called Verdier stratifications.

**Definition 2.7.2.** A  $C^p$  Verdier stratification ( $p \geq 1$ ) of a set  $Q \subset \mathbf{E}$  is a  $C^p$  stratification of  $Q$  such that whenever a stratum  $\mathcal{Y}$  is adjacent to a stratum  $\mathcal{X}$ , the pair  $(\mathcal{X}, \mathcal{Y})$  is strongly (a)-regular at any point  $x \in \mathcal{Y}$ .

It is often useful to refine stratifications. To this end, a stratification is *compatible* with a collection of sets  $Q_1, \dots, Q_k$  if for every index  $i$ , every stratum  $\mathcal{M}$  is either contained in  $Q_i$  or is disjoint from it. The following theorem, due to Ta Le Loi [48], shows that definable sets admit a Verdier stratification, which is compatible with any finite collection of definable sets.

**Theorem 2.7.3** (Verdier stratification). *For any  $p \geq 1$ , any definable set  $Q \subset \mathbf{E}$  admits a definable  $C^p$  Verdier stratification. Moreover, given finitely many definable subsets  $Q_1, \dots, Q_k$ , we may ensure that the Verdier stratification of  $Q$  is compatible with  $Q_1, \dots, Q_k$ .*

The analogous theorem for condition (b) (and therefore condition (a)) was proved earlier; see the discussion in [89]. The strong (b) condition does not satisfy such decomposition properties. It can fail even relative to a single point of a definable set in  $\mathbf{R}^2$ , as Example 2.7.1 shows. Nonetheless, as we have seen in previous sections, it does hold in a number of interesting settings in optimization (e.g. for cone reducible sets along the active manifold).

**Example 2.7.1** (Strong (b) is not generic). Define the curve  $\gamma(t) = (t, t^{3/2})$  in  $\mathbf{R}^2$ . Let  $\mathcal{X}$  be the graph of  $\gamma$  and let  $\mathcal{Y}$  be the origin in  $\mathbf{R}^2$ . Then a quick computation shows that a unit normal  $u(t) \in N_{\mathcal{X}}(\gamma(t))$  is given by  $(-\frac{2}{3}\sqrt{t}, 1)/\sqrt{1 + \frac{4}{9}t}$  and therefore

$$\left\langle u(t), \frac{\gamma(t)}{\|\gamma(t)\|^2} \right\rangle = \frac{t^{3/2}}{3(t^2 + t^3)\sqrt{1 + \frac{4}{9}t}} \rightarrow \infty \quad \text{as } t \rightarrow 0.$$

Therefore, the strong condition (b) fails for the pair  $(X, Y)$  at the origin.

Applying Theorem 2.7.3, to epigraphs immediately yields the following.

**Theorem 2.7.4** (Verdier stratification of a function). *Consider a definable function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  that is continuous on its domain. Then for any  $p > 0$ , there exists a partition of  $\text{dom } f$  into finitely many  $C^p$ -smooth manifolds such that  $f$  is strongly (a) regular and (b)-regular along any manifold  $\mathcal{M}$  at any point  $x \in \mathcal{M}$ .*

*Proof.* We first form a nonvertical stratification  $\{\mathcal{M}_i\}$  of  $\text{gph } f$ , guaranteed to exist by [8]. Choose any integer  $p \geq 2$ . Restratifying using Theorem 2.7.3 yields a nonvertical  $C^p$ -Verdier stratification  $\{\mathcal{K}_j\}$  of  $\text{gph } f$ . Let  $\mathcal{X}_j$  denote the image of  $\mathcal{K}_j$  under the canonical projection  $(x, r) \mapsto x$ . As explained in [8], each set  $\mathcal{X}_j$  is a  $C^p$ -smooth manifolds, the function  $f$  restricted to  $\mathcal{X}_j$  is  $C^p$ -smooth, and equality  $\text{gph } f|_{\mathcal{X}_j} = \mathcal{K}_j$  holds.

Consider now an arbitrary point  $\bar{X} \in \mathcal{K}_j$ . It remains to verify that the pair  $(\text{epi } f, \mathcal{K}_j)$  is strongly (a)-regular at  $\bar{X}$ . To this end, since  $\mathcal{K}_j$  is a  $C^p$  manifold, there exists  $C > 0$  satisfying

$$\Delta(N_{\mathcal{K}_j}(X), N_{\mathcal{K}_j}(Y)) \leq C\|X - Y\|$$

for all  $X, Y \in \mathcal{K}_j$  near  $\bar{X}$ . Moreover since there are only finitely many strata and using the definition of the Verdier stratification, shrinking  $C > 0$  we may be sure the estimate

$$\Delta(N_{\mathcal{K}_l}(X), N_{\mathcal{K}_j}(Y)) \leq C\|X - Y\|$$

holds for all  $Y \in \mathcal{K}_j$  near  $\bar{X}$  and for all  $X$  near  $\bar{X}$  lying in any stratum  $\mathcal{K}_l$  that is adjacent to  $\mathcal{K}_j$ . The projection formula [8, Proposition 4] applied to the indicator function of the epigraph implies the inclusion  $N_{\text{epi } f}^c(X) \subset N_{\mathcal{K}_l}(X)$  for any index  $l$  and any point  $X \in \mathcal{K}_l$ . We therefore immediately deduce that the pair  $(\text{epi } f, \mathcal{K}_j)$  is strongly (a)-regular at  $\bar{X}$ , as claimed.  $\square$

In this work, we will be interested in sets that are regular along a particular manifold—the active one. Theorem 2.7.3 quickly implies that critical points of “generic” definable functions lie on an active manifold along which the objective function is strongly (a) regular.

**Theorem 2.7.5** (Regularity at critical points of generic functions). *Consider a closed definable function  $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$ . Then for almost every direction  $v \in \mathbf{E}$  in the sense of Lebesgue measure, the perturbed function  $f_v := f(x) - \langle v, x \rangle$  has at most finitely many limiting critical points, each lying on a unique  $C^p$ -smooth active manifold and along which the function  $f_v$  is strongly (a)-regular.*

This theorem is a special case of a more general result that applies to structured problems of the form

$$\min_x g(x) + h(x) \tag{2.7.1}$$

for definable functions  $g$  and  $h$ . Algorithms that utilize this structure, such as the proximal subgradient method, generate a sequence that may convergence to *composite Clarke critical points*  $\bar{x}$ , meaning those satisfying

$$0 \in \partial_c g(\bar{x}) + \partial_c h(\bar{x}).$$

This condition is typically weaker than  $0 \in \partial_c(g + h)(\bar{x})$ . Points  $\bar{x}$  satisfying the stronger inclusion  $0 \in \partial g(\bar{x}) + \partial h(\bar{x})$  will be called *composite limiting critical*.

The following theorem shows that under a reasonably rich class of perturbations, the problem (2.7.1) admits no extraneous composite limiting critical points. Moreover each of the functions involved admits an active manifold along which the function is strongly (a)-regular. The proof is a small modification of [28, Theorem 5.2].

**Theorem 2.7.6** (Regularity at critical points of generic functions). *Consider closed definable functions  $g: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  and  $h: \mathbf{E} \rightarrow \mathbf{R} \cup \{\infty\}$  and define the parametric family of problems*

$$\min_x f_{y,v}(x) = g(x) - \langle v, x \rangle + h(x + y) \quad (2.7.2)$$

*Define the tilted function  $g_v(x) = g(x) - \langle v, x \rangle$ . Then there exists an integer  $N > 0$  such that for almost all parameters  $(v, y)$  in the sense of Lebesgue measure, the problem (2.7.2) has at most  $N$  composite Clarke critical points. Moreover, for any limiting composite critical point  $\bar{x}$ , there exists a unique vector*

$$\bar{\lambda} \in \partial h(\bar{x} + y) \quad \text{satisfying} \quad -\bar{\lambda} \in \partial g_v(\bar{x}),$$

*and the following properties are true.*

1. *The inclusions  $\bar{\lambda} \in \hat{\partial} h(\bar{x} + y)$  and  $-\bar{\lambda} \in \hat{\partial} g_v(\bar{x})$  hold.*
2.  *$g_v$  admits a  $C^p$  active manifold  $\mathcal{M}$  at  $\bar{x}$  for  $-\bar{\lambda}$  and  $h$  admits a  $C^p$  active manifold  $\mathcal{K}$  at  $\bar{x} + y$  for  $\bar{\lambda}$ , and the two manifolds intersect transversally:*

$$N_{\mathcal{K}}(\bar{x}) \cap N_{\mathcal{M}}(\bar{x}) = \{0\}.$$

3.  *$\bar{x}$  is either a local minimizer of  $f_{y,v}$  or a  $C^p$  strict active saddle point of  $\varphi_{y,v}$ .*
4.  *$g_v$  is strongly (a)-regular along  $\mathcal{M}$  at  $\bar{x}$  and  $h$  is strongly (a)-regular along  $\mathcal{K}$  at  $\bar{x} + y$ .*

*Proof.* All the claims, except for 3 and 4, are proved in [28]; note, that in that work, active manifolds are defined using the limiting subdifferential, but exactly the same arguments apply under the more restrictive Definition 1.4.1. Claim 3 is proved in [24, Theorem 5.2]<sup>4</sup>; it is a direct consequence of the classical Sard's theorem and existence of stratifications. Claim 4 follows from a small modification to the proof of [28]. Namely, the first-bullet point in the proof may be replaced by “ $g$  is  $C^p$ -smooth and strongly (a) regular on  $X_i^j(\widehat{U}_i)$  and  $h$  is  $C^p$ -smooth and strongly (a)-regular on  $F_i^j(\widehat{U}_i)$ ”.  $\square$

---

<sup>4</sup>weak convexity is invoked in the theorem statement but is not necessary for the result.

# Chapter 3

## Algorithmic Consequences: local rates of convergence, asymptotic normality, and saddle point avoidance

### 3.1 Algorithm and main assumptions

In this chapter, we introduce our main algorithmic consequences of the strong (a) and (b) regularity properties developed in Chapter 2. Setting the stage, throughout we consider a minimization problem

$$\min_{x \in \mathbb{R}^d} f(x), \tag{3.1.1}$$

where  $f: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is a closed function. The function  $f$  may enforce constraints or regularization; it may also be the population loss of a stochastic optimization problem. In order to simultaneously model algorithms which exploit such structure, we take a fairly abstract approach, assuming access to a *generalized gradient mapping* for  $f$ :

$$G: \mathbb{R}_{++} \times \text{dom } f \times \mathbb{R}^d \rightarrow \mathbb{R}^d$$

We then consider the following stochastic method: given  $x_0 \in \mathbb{R}^d$ , we iterate

$$x_{k+1} = x_k - \alpha_k G_{\alpha_k}(x_k, \nu_k), \tag{3.1.2}$$

where  $\alpha_k$  is a control sequence and  $\nu_k$  is stochastic noise, satisfying Assumption E, which we defer to Section 3.3. A first classical example of (3.1.2), valid for locally Lipschitz  $f$ , is the stochastic subgradient method:

$$x_{k+1} = x_k - \alpha_k(w_k + \nu_k) \quad \text{where } w_k \in \partial_c f(x_k),$$

In this case, mapping  $G$  satisfies

$$G_\alpha(x, \nu) \in \partial_c f(x) + \nu \quad \text{for all } x, \nu \in \mathbb{R}^d \text{ and } \alpha > 0. \tag{3.1.3}$$



More generally,  $G$  may represent a stochastic projected gradient method or a stochastic proximal gradient method—two algorithms we examine in detail in Section 3.2.

The purpose of this chapter is to understand how iteration (3.1.2) is affected by the existence of “active manifolds”  $\mathcal{M}$  contained within the domain of  $f$ . For this, we posit a tight interaction between  $G$  and the active manifold, which we first motivate by the subgradient mapping (3.1.3).

Indeed, let us suppose that  $f$  is locally Lipschitz and has a critical point  $\bar{x}$  contained in an active manifold  $\mathcal{M}$ . Then we posit the following two properties that describe how  $f$  and  $\mathcal{M}$  interact: we assume there exists  $C, \mu > 0$  such that for all  $x$  near  $\bar{x}$  and  $v \in \partial_c f(x)$ ,

$$\begin{aligned} \textbf{Strong (a):} \quad & \|P_{T_{\mathcal{M}}(y)}(v - \nabla_{\mathcal{M}} f(y))\| \leq C\|x - y\|; \\ \textbf{Proximal aiming:} \quad & \langle v, x - P_{\mathcal{M}}(x) \rangle \geq \mu \cdot \text{dist}(x, \mathcal{M}). \end{aligned}$$

Recalling the terminology and results of Chapter 2, we find that the first condition is simply strong (a) regularity along  $\mathcal{M}$  and  $\bar{x}$ , while the proximal aiming condition is an immediate consequence of (b) regularity of  $f$  along  $\mathcal{M}$  at  $\bar{x}$  (see Corollary 2.1.5). Moreover, these properties have natural algorithmic consequences: we will later show that strong (a) regularity ensures that the shadow sequence

$$y_k = P_{\mathcal{M}}(x_k)$$

is locally an inexact stochastic Riemannian gradient sequence with implicit retraction:

$$\textbf{Shadow Iteration:} \quad y_{k+1} = y_k - \alpha_k(\nabla_{\mathcal{M}} f(y_k) + P_{T_{\mathcal{M}}(y_k)}(\nu_k)) + \alpha_k E_k,$$

with error sequence  $E_k$  that decays as  $x_k$  approaches the manifold. On the other hand, the proximal aiming condition ensures that the subgradient method aims towards the manifold. As we will later show, this property implies that  $\text{dist}(x_k, \mathcal{M})$  and the error sequence  $E_k$  tend to zero at a controlled rate. These two properties in turn are sufficient to establish all of our claimed results for the subgradient method: local rates of convergence, asymptotic normality, and saddle-point avoidance.

Motivated by the strong (a) and proximal aiming conditions, we propose the following analogous conditions for generalized gradient mapping  $G$ , which models the tight interaction between  $f$ ,  $G$ , and an “active manifold”  $\mathcal{M}$  containing a given point  $\bar{x} \in \mathcal{M}$ .

**Assumption A** (Strong (a) and Aiming at  $\bar{x}$ ). Fix  $\bar{x} \in \text{dom } f$ . Then there exists constants  $C, \mu > 0$ , a neighborhood  $\mathcal{U}$  of  $\bar{x}$ , and a  $C^3$  manifold  $\mathcal{M} \subseteq \text{dom } f$  containing  $\bar{x}$  such that the following hold on  $\mathcal{U}_f := \mathcal{U} \cap \text{dom } f$ : for all  $\nu \in \mathbb{R}^d$  and  $\alpha > 0$ ,

(A1) **(Local Boundedness)** We have

$$\sup_{x \in \mathcal{U}_f} \|G_{\alpha}(x, \nu)\| \leq C(1 + \|\nu\|).$$

(A2) **(Strong (a))** The function  $f$  is  $C^2$  on  $\mathcal{M}$  and for all  $x \in \mathcal{U}_f$ , we have

$$\|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(G_{\alpha}(x, \nu) - \nabla_{\mathcal{M}} f(P_{\mathcal{M}}(x)) - \nu)\| \leq C(1 + \|\nu\|)^2(\text{dist}(x, \mathcal{M}) + \alpha).$$

(A3) **(Proximal Aiming)** For  $x \in \mathcal{U}_f$  tending to  $\bar{x}$ , we have

$$\langle G_\alpha(x, \nu) - \nu, x - P_{\mathcal{M}}(x) \rangle \geq \mu \cdot \text{dist}(x, \mathcal{M}) - (1 + \|\nu\|)^2(o(\text{dist}(x, \mathcal{M})) + C\alpha).$$

Some comments are in order. Assumption (A1) is similar to classical Lipschitz assumptions and ensures the steplength can only scale linearly in  $\|\nu\|$ . Assumption (A2) is the natural analogue of strong (a) regularity for the operator  $G_\alpha(x, \nu)$ . It ensures that the shadow sequence  $y_k = P_{\mathcal{M}}(x_k)$  locally remains an inexact stochastic Riemannian gradient sequence with implicit retraction. Assumption (A3) ensures that after subtracting the noise from  $G_{\alpha_k}(x_k, \nu_k)$ , the update direction  $x_k - x_{k+1}$  locally points towards the manifold  $\mathcal{M}$ . We will later show that this ensures the iterates  $x_k$  approach the manifold  $\mathcal{M}$  at a controlled rate. Finally we note in passing that the power of  $(1 + \|\nu\|)$  in the above expressions must be at least 2 for common iterative algorithms to satisfy Assumption A; one may also take higher powers, but this requires higher moment bounds on  $\|\nu_k\|$ . Before making these results precise in Section 3.3, we first formalize our statements about the subgradient method and introduce several examples.

## 3.2 Example algorithms satisfying Assumption A

The most immediate example of operator  $G$  arises from the subgradient method applied to a locally Lipschitz function  $f$ . In this setting, any selection  $s: \mathbb{R}^d \rightarrow \mathbb{R}$  of  $\partial_c f(x)$  gives rise to a mapping

$$G_\alpha(x, \nu) = s(x) + \nu, \tag{3.2.1}$$

which is independent of  $\alpha$ . Then Algorithm (3.1.2) gives rise to the classical stochastic subgradient method:

$$x_{k+1} = x_k - \alpha_k(s(x_k) + \nu_k). \tag{3.2.2}$$

For this mapping  $G$ , Assumption A holds under the following conditions:

**Assumption B** (Assumptions for the subgradient mapping). Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  be a function. Fix  $\bar{x} \in \mathbb{R}^d$  and let  $\mathcal{M} \subseteq \mathcal{X}$  be a  $C^3$  manifold containing  $\bar{x}$  and suppose that  $f$  is  $C^2$  on  $\mathcal{M}$  near  $\bar{x}$ .

(B1) **(Local Lipschitz)** The function  $f$  is locally Lipschitz.

(B2) **(Strong (a))** The function  $f$  is strong (a)-regular along  $\mathcal{M}$  at  $\bar{x}$ .

(B3) **(Proximal aiming)** There exists  $\mu > 0$  such that the inequality holds

$$\langle v, x - P_{\mathcal{M}}(x) \rangle \geq \mu \cdot \text{dist}(x, \mathcal{M}) \quad \text{for all } x \text{ near } \bar{x} \text{ and } v \in \partial_c f(x). \tag{3.2.3}$$

The following proposition is then immediate.

**Proposition 3.2.1** (Subgradient method). *If assumption B holds at  $\bar{x} \in \mathbb{R}^d$ , then  $f$  and  $G$  satisfy Assumption A at  $\bar{x}$ .*

Thus, all three properties arise from reasonable assumptions on the function  $f$ , as we have already seen in Chapter 1. Moreover, for definable functions, they are in fact automatic, as the following corollary shows:

**Corollary 3.2.2.** *Suppose that  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz and semialgebraic. Then there exists a finite  $N$  such that for a generic set of  $v \in \mathbb{R}^d$  the tilted function  $f_v(x) := f(x) - \langle v, x \rangle$  has at most  $N$  Clarke critical points  $\bar{x}$ . Moreover, each limiting critical point  $\bar{x}$  is in fact Fréchet critical and satisfies*

1. *The function  $f$  and the subgradient mapping (3.2.2) satisfy Assumption A at  $\bar{x}$  with respect to an active manifold  $\mathcal{M}$ .*
2. *Critical point  $\bar{x}$  is either a local minimizer or an active strict saddle point of  $f$ .*

The proof of this Corollary follows from a more general result on the projected subgradient method. This is the topic of the next section.

### 3.2.1 Projected subgradient method

Throughout this section let  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  be a locally Lipschitz function and let  $\mathcal{X}$  be a closed set and consider the constrained minimization problem

$$\min f(x) := g(x) + \delta_{\mathcal{X}}.$$

A classical algorithm for solving this problem is known as the stochastic projected subgradient method. Each iteration of the method updates

$$x_{k+1} \in P_{\mathcal{X}}(x_k - \alpha_k(v_k + \nu_k)) \quad \text{where } v_k \in \partial_c g(x_k) \quad (3.2.4)$$

This algorithm can be reformulated as an instance of (3.1.2). Indeed, let  $s_{\mathcal{X}}: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a measurable selection of  $P_{\mathcal{X}}$ , let  $s_g: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a measurable selection of  $\partial_c g$ , and define the generalized gradient mapping

$$G_{\alpha}(x, \nu) := \frac{x - s_{\mathcal{X}}(x - \alpha(s_g(x) + \nu))}{\alpha} \quad \text{for all } x \in \mathbb{R}^d, \nu \in \mathbb{R}^d, \alpha > 0. \quad (3.2.5)$$

Evidently, the update rule (3.1.2) becomes

$$x_{k+1} = x_k - \alpha_k G_{\alpha_k}(x_k, \nu_k) \in P_{\mathcal{X}}(x_k - \alpha_k(s_g(x_k) + \nu_k)),$$

an instance of the classical stochastic projected subgradient method.

To understand Assumption A for the stochastic projected subgradient method, we introduce the following assumptions on  $g$  and  $\mathcal{X}$ .

**Assumption C** (Assumptions for the projected gradient mapping). Let  $f := g + \delta_{\mathcal{X}}$ , where  $\mathcal{X}$  is a closed set and let  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  be a function. Fix  $\bar{x} \in \mathbb{R}^d$  and let  $\mathcal{M} \subseteq \mathcal{X}$  be a  $C^3$  manifold containing  $\bar{x}$  and suppose that  $f$  is  $C^2$  on  $\mathcal{M}$  near  $\bar{x}$ .

(C1) **(Local Lipschitz)** The function  $g$  is locally Lipschitz.

(C2) **(Strong (a))** The function  $g$  and set  $\mathcal{X}$  are strong (a)-regular along  $\mathcal{M}$  at  $\bar{x}$ .

(C3) **(Proximal aiming)** There exists  $\mu > 0$  such that the inequality holds

$$\langle v, x - P_{\mathcal{M}}(x) \rangle \geq \mu \cdot \text{dist}(x, \mathcal{M}) \quad \text{for all } x \in \mathcal{X} \text{ near } \bar{x} \text{ and } v \in \partial_c g(x). \quad (3.2.6)$$

(C4) **(Prox regularity/condition (b))** The set  $\mathcal{X}$  is either prox-regular at  $\bar{x}$  or is (b)-regular along  $\mathcal{M}$  at  $\bar{x}$ .

The following proposition shows that Assumption C is sufficient to ensure Assumption A. The proof of the Proposition appears in Appendix 3.11.1.

**Proposition 3.2.3** (Projected subgradient method). *If assumption C holds at  $\bar{x} \in \text{dom } f$ , then  $f$  and  $G$  satisfy Assumption A at  $\bar{x}$ .*

Given this proposition, an immediate question is whether Assumption C is automatically satisfied for some reasonably large class of functions. The following corollary, which is an immediate consequence of Proposition 3.2.3, Theorem 2.7.6 and Corollary 2.1.5, shows that the answer is yes.

**Corollary 3.2.4.** *Suppose that  $f = g + \delta_{\mathcal{X}}$ , where  $\mathcal{X} \subseteq \mathbb{R}^d$  semialgebraic and closed, and  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz and semialgebraic. Then there exists a finite  $N$  such that for a generic set of  $v, w \in \mathbb{R}^d$  the tilted function  $f_{v,w}(x) := g(x + w) + \delta_{\mathcal{X}}(x) - \langle v, x \rangle$  has at most  $N$  composite Clarke critical points  $\bar{x}$ . Moreover, each composite limiting critical point  $\bar{x}$  is in fact Fréchet critical and satisfies*

1. *The function  $f$  and the projected subgradient mapping  $G$  (3.2.5) satisfy Assumption A at  $\bar{x}$  with respect to an active manifold  $\mathcal{M}$ .*
2. *Critical point  $\bar{x}$  is either a local minimizer or an active strict saddle point of  $f$ .*

In the above corollary, the qualification *composite critical points*, as defined in Theorem 2.7.6, is important, since the projected subgradient method is only known to converge to such points.

### 3.2.2 Proximal gradient method

Throughout this section let  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^1$  function and let  $h: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  be a closed function. We then consider the minimization problem

$$\min_{x \in \mathbb{R}^d} f(x) := g(x) + h(x).$$

A classical algorithm for solving this problem is known as the stochastic proximal gradient method. Each iteration of the method solves the proximal problem:

$$x_{k+1} \in \operatorname{argmin}_{x \in \mathbb{R}^d} \left\{ h(x) + \langle \nabla g(x_k) + \nu_k, x - x_k \rangle + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}.$$

This algorithm can be reformulated as an instance of (3.1.2). Indeed, let  $s: \mathbb{R}_{++} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a measurable selection of the proximal problem and consider the mapping  $G$  defined by

$$G_\alpha(x, \nu) = \frac{x - s_\alpha(x - \alpha(\nabla g(x) + \nu))}{\alpha} \quad \text{for all } x \in \mathbb{R}^d, \nu \in \mathbb{R}^d \text{ and } \alpha > 0. \quad (3.2.7)$$

Evidently, the update rule (3.1.2) becomes

$$x_{k+1} = x_k - \alpha_k G_{\alpha_k}(x_k, \nu_k) = s_{\alpha_k}(x_k - \alpha(\nabla g(x) + \nu)),$$

an instance of the stochastic proximal gradient method.

To understand Assumption A for the stochastic proximal gradient method, we introduce the following assumptions on  $g$  and  $h$ .

**Assumption D** (Assumptions for the proximal gradient mapping). Let  $f := g + h$ , where  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is  $C^1$  and  $h: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is closed. Denote  $\mathcal{X} := \text{dom } h$  and let  $\mathcal{M} \subseteq \mathcal{X}$  be a  $C^3$  manifold containing some point  $\bar{x}$  and suppose that  $f$  is  $C^2$  on  $\mathcal{M}$  near  $\bar{x}$ .

(D1) **(Lipschitz gradient/boundedness)** The gradient  $\nabla g|_{\mathcal{X}}$  is Lipschitz near  $\bar{x}$ . Moreover, there exists  $C > 0$  such that  $\|\nabla g(x)\| \leq C(1 + \|x\|)$  for all  $x \in \mathcal{X}$ .

(D2) **(Lipschitz proximal term)** The function  $h$  is Lipschitz on  $\mathcal{X}$ .

(D3) **(Strong (a))** The function  $f$  is strong (a)-regular along  $\mathcal{M}$  at  $\bar{x}$ .

(D4) **(Proximal Aiming)** There exists  $\mu > 0$  such that the inequality holds

$$\langle \nabla g(x) + v, x - P_{\mathcal{M}}(x) \rangle \geq \mu \cdot \text{dist}(x, \mathcal{M}) - (1 + \|v\|)o(\text{dist}(x, \mathcal{M})) \quad (3.2.8)$$

as  $x \in \text{dom } h$  tends to  $\bar{x}$  and  $v \in \hat{\partial} h(x)$ .

The following proposition shows that Assumption D is sufficient to ensure Assumption A. The proof of the Proposition appears in Appendix 3.11.2

**Proposition 3.2.5** (Proximal gradient method). *If assumption D holds at  $\bar{x} \in \text{dom } f$ , then  $f$  and  $G$  satisfy Assumption A at  $\bar{x}$ .*

The following corollary, which is an immediate consequence of Proposition 3.2.5 and Theorem 2.7.5, shows that assumption D is automatically true for generic semialgebraic functions.

**Corollary 3.2.6.** *Suppose that  $f = g + h_0 + \delta_{\mathcal{X}}$ , where  $\mathcal{X} \subseteq \mathbb{R}^d$ ,  $g$  is a  $C^1$  function with Lipschitz gradient on  $\mathcal{X}$ , the function  $h_0: \mathbb{R}^d \rightarrow \mathbb{R}$  is Lipschitz on  $\mathcal{X}$ , and we define  $h := h_0 + \delta_{\mathcal{X}}$ . Suppose that  $g, h_0$ , and  $\mathcal{X}$  are semialgebraic. Then there exists a finite  $N$  such that for a full measure set of  $v, w \in \mathbb{R}^d$ , the tilted function  $f_{v,w} := g(x+w) + h_0(x+w) + \delta(x) - \langle v, x \rangle$  has at most  $N$  composite Clarke critical points  $\bar{x}$ . Moreover, each composite limiting critical point  $\bar{x}$  is in fact composite Fréchet critical and satisfies*

1. *The function  $f$  and the proximal gradient mapping (3.2.7) satisfy Assumption A at  $\bar{x}$  with respect to an active manifold  $\mathcal{M}$ .*

2. Critical point  $\bar{x}$  is either a local minimizer or an active strict saddle point of  $f$ .

Thus, we find that Assumption A is satisfied for common iterative mappings, under reasonable assumptions, and is even automatic for certain generic classes of functions. In the next several sections, we turn our attention to the statements of the main results of this chapter.

### 3.3 The two pillars

Assumption A at a point  $\bar{x}$  guarantees two useful behaviors, provided the iterates  $\{x_k\}$  of iteration (3.1.2) remain in a small ball around  $\bar{x}$ . First  $x_k$  must approach the manifold  $\mathcal{M}$  containing  $\bar{x}$  at a controlled rate, a consequence of the proximal aiming condition. Second the shadow  $y_k = P_{\mathcal{M}}(x_k)$  of the iterates along the manifold form an approximate Riemannian stochastic gradient sequence with an implicit retraction. Moreover, the approximation error of the sequence decays with  $\text{dist}(x_k, \mathcal{M})$  and  $\alpha_k$ , quantities that quickly tend to zero. These two properties together are sufficient to establish the results outlined in Section 1.2.

The statements of the results in this section are necessarily highly technical and can be skipped on first reading. However, the following technical notation and assumptions cannot be skipped, since they will appear in our results on local rates of convergence. First, the formal statements of our results crucially require local arguments and frequently refer to the following stopping time: given an index  $k \geq 1$  and a  $\delta > 0$ , define

$$\tau_{k,\delta} := \inf\{j \geq k : x_j \notin B_\delta(\bar{x})\}. \quad (3.3.1)$$

Note that the stopping time implicitly depends on  $\bar{x}$ , a point at which Assumption A is satisfied. In the statements of our result, the point  $\bar{x}$  will always be clear from the context. Second, we make the following standing assumption on the stepsizes  $\alpha_k$  and  $\nu_k$ . We assume they are in force throughout the rest of the chapter.

**Assumption E** (Standing assumptions).

(E1) We assume that  $G$  is measurable.

(E2) There exists constants  $c_1, c_2 > 0$  and  $\gamma \in (1/2, 1]$  such that

$$\frac{c_1}{k^\gamma} \leq \alpha_k \leq \frac{c_2}{k^\gamma}.$$

(E3)  $\{\nu_k\}$  is a martingale difference sequence w.r.t. to the increasing sequence of  $\sigma$ -fields

$$\mathcal{F}_k = \sigma(x_j : j \leq k \text{ and } \nu_j : j < k)$$

That is, there exists a function  $q : \mathbb{R}^d \rightarrow \mathbb{R}_+$  that is bounded on bounded sets with

$$\mathbb{E}[\nu_k \mid \mathcal{F}_k] = 0 \quad \text{and} \quad \mathbb{E}[\|\nu_k\|^4 \mid \mathcal{F}_k] < q(x_k).$$

We let  $\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_k]$  denote the conditional expectation.

(E4) The inclusion  $x_k \in \text{dom } f$  for all  $k \geq 1$ .

All items in Assumption E are standard in the literature on stochastic approximation methods and mirror those found in [26, Assumption C]. The only exception is the fourth moment bound on  $\|\nu_k\|$ , which stipulates that  $\nu_k$  has slightly lighter tails. This bound appears to be necessary for the setting we consider. We now turn to the first pillar.

### 3.3.1 Pillar I: Aiming towards the manifold

The following proposition ensures the sequence  $x_k$  approaches the manifold. Our results are separated into two cases: small  $\nu_k$  and arbitrary  $\nu_k$ . When  $\nu_k$  is small, we can establish deterministic bounds with faster  $O(k^{-\gamma})$  convergence rates. Obtaining this rate is crucial for fast local rates of convergence near strong local minimizers of  $f$ . When  $\nu_k$  is arbitrary, the corresponding rates of convergence degrade, but are sufficient to establish all of our other results, including asymptotic normality and saddle point avoidance. The proof appears in Section 3.8.1.

**Proposition 3.3.1.** *Suppose that  $f$  satisfies Assumption A at  $\bar{x}$ . Let  $\gamma \in (1/2, 1]$  and assume  $c_1 \geq 32/\mu$  if  $\gamma = 1$ . Then for all  $k_0 \geq 1$  and sufficiently small  $\delta > 0$ , there exists a constant  $C$ , such that the following hold with stopping time  $\tau_{k_0, \delta}$  defined in (3.3.1):*

1. *Then there exists a random variable  $V_{k_0, \delta}$  such that*

(a) *The limit holds:*

$$\frac{k^{2\gamma-1}}{\log(k+1)^2} \text{dist}^2(x_k, \mathcal{M}) 1_{\tau_{k_0, \delta} > k} \xrightarrow{a.s.} V_{k_0, \delta}$$

(b) *The sum is almost surely finite:*

$$\sum_{k=1}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \text{dist}(x_k, \mathcal{M}) 1_{\tau_{k_0, \delta} > k} < +\infty$$

2. *We have*

(a) *The expected squared distance satisfies:*

$$\mathbb{E}[\text{dist}^2(x_k, \mathcal{M}) 1_{\tau_{k_0, \delta} > k}] \leq C\alpha_k \quad \text{for all } k \geq 1.$$

(b) *The tail sum is bounded:*

$$\mathbb{E} \left[ \sum_{i=k}^{\infty} \alpha_i \text{dist}(x_i, \mathcal{M}) 1_{\tau_{k_0, \delta} > i} \right] \leq C \sum_{i=k}^{\infty} \alpha_i^2 \quad \text{for all } k \geq 1.$$

3. *If  $\|\nu_k\| \leq \mu/4$  for all  $k \geq 1$ , we have*

$$\text{dist}(x_k, \mathcal{M}) 1_{\tau_{k_0, \delta} > k} \leq C\alpha_k \quad \text{for all } k \geq k_0.$$

In Section 3.4 we will comment further on how to interpret these conditional convergence results. For now we remark that Part 1b of the proposition holds not only almost surely, but also in expectation, which is a stronger statement in general. Now we turn our attention to Pillar II: the shadow iteration.

### 3.3.2 Pillar II: The shadow iteration

Next we study the evolution of the shadow  $y_k = P_{\mathcal{M}}(x_k)$  along the manifold, showing that  $y_k$  is locally an inexact Riemannian stochastic gradient sequence with error that asymptotically decays as  $x_k$  approaches the manifold. Consequently, we may control the error using Proposition 3.3.1. As before, we prove that the error decays at a rate  $O(k^{-\gamma})$  when  $\nu_k$  is small, but degrades when  $\nu_k$  is arbitrary. The proof appears in Section 3.8.2

**Proposition 3.3.2.** *Suppose that  $f$  satisfies Assumption A at  $\bar{x}$ . Then for all  $k_0 \geq 1$  and sufficiently small  $\delta > 0$ , there exists a constant  $C$ , such that the following hold with stopping time  $\tau_{k_0, \delta}$  defined in (3.3.1): there exists a sequence  $\mathcal{F}_{k+1}$  measurable random vectors  $E_k \in \mathbb{R}^d$  such that*

1. *The shadow sequence*

$$y_k = \begin{cases} P_{\mathcal{M}}(x_k) & \text{if } x_k \in B_{2\delta}(\bar{x}) \\ \bar{x} & \text{otherwise.} \end{cases}$$

*satisfies  $y_k \in B_{4\delta}(\bar{x}) \cap \mathcal{M}$  for all  $k$  and the recursion holds:*

$$\boxed{y_{k+1} = y_k - \alpha_k \nabla_{\mathcal{M}} f(y_k) - \alpha_k P_{T_{\mathcal{M}}(y_k)}(\nu_k) + \alpha_k E_k \quad \text{for all } k \geq 1.} \quad (3.3.2)$$

*Moreover, for such  $k$ , we have  $\mathbb{E}_k[P_{T_{\mathcal{M}}(y_k)}(\nu_k)] = 0$ .*

2. *Let  $\gamma \in (1/2, 1]$  and assume that  $c_1 \geq 32/\mu$  if  $\gamma = 1$ .*

- (a) *We have the following bounds for  $k_0 \leq k \leq \tau_{k_0, \delta} - 1$ :*

- i.  $\|E_k\| 1_{\tau_{k_0, \delta} > k} \leq C(1 + \|\nu_k\|)^2(\text{dist}(x_k, \mathcal{M}) + \alpha_k) 1_{\tau_{k_0, \delta} > k}$*

- ii.  $\max\{\mathbb{E}_k[\|E_k\|] 1_{\tau_{k_0, \delta} > k}, \mathbb{E}_k[\|E_k\|^2] 1_{\tau_{k_0, \delta} > k}\} \leq C$ .*

- iii.  $\mathbb{E}[\|E_k\|^2] 1_{\tau_{k_0, \delta} > k} \leq C\alpha_k$*

- (b) *The following sums are finite*

- i.  $\sum_{k=1}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \max\{\|E_k\| 1_{\tau_{k_0, \delta} > k}, \mathbb{E}_k[\|E_k\|] 1_{\tau_{k_0, \delta} > k}\} < +\infty$*

- ii.  $\sum_{k=1}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \max\{\|E_k\|^2 1_{\tau_{k_0, \delta} > k}, \mathbb{E}_k[\|E_k\|^2] 1_{\tau_{k_0, \delta} > k}\} < +\infty$*

- (c) *The tail sum is bounded*

$$\mathbb{E} \left[ 1_{\tau_{k_0, \delta} = \infty} \sum_{i=k}^{\infty} \alpha_i \|E_i\| \right] \leq C \sum_{i=k}^{\infty} \alpha_i^2 \quad \text{for all } k \geq 1.$$

- (d) *If  $\|\nu_k\| \leq \mu/2$  for all  $k_0 \leq k \leq \tau_{k_0, \delta} - 1$ , we have*

$$\|E_k\| 1_{\tau_{k_0, \delta} > k} \leq C\alpha_k \quad \text{for all } k \geq 1.$$

With the two pillars we separate our study of the sequence  $x_k$  into two orthogonal components: In the tangent/smooth directions, we study the sequence  $y_k$ , which arises from an inexact gradient method with rapidly decaying errors and is amenable to the techniques of smooth optimization. In the normal/nonsmooth directions, we steadily approach the manifold, allowing us to infer strong properties of  $x_k$  from corresponding properties for  $y_k$ , e.g., asymptotic normality and saddle point avoidance. In the following three sections, we formally state our main results: local rates of convergence, asymptotic normality, and saddle point avoidance.



### 3.4 Local convergence guarantees

Our first application of the two pillars states: if  $\{x_k\}$  remains in  $B_\delta(\bar{x})$  for some small  $\delta$  and all sufficiently large  $k$ , then  $\text{dist}(x_k, \mathcal{M})$  and  $\|\nabla_{\mathcal{M}}f(P_{\mathcal{M}}(x_k))\|$  decay at a controlled rate. Given the properties of the shadow sequence, the reason for this behavior is fairly straightforward: the function  $f_{\mathcal{M}} := f \circ P_{\mathcal{M}}$  is  $C^2$  near  $\bar{x}$ , its gradient agrees with  $\nabla_{\mathcal{M}}f$  along the manifold, and the shadow iteration is an inexact gradient descent sequence for  $f_{\mathcal{M}}$ , yielding the conditional controlled decrease:

$$\mathbb{E}_k[f_{\mathcal{M}}(y_{k+1})] \leq f_{\mathcal{M}}(y_k) - C\|\nabla_{\mathcal{M}}f(y_k)\|^2 + O(\alpha_k \mathbb{E}_k\|E_k\| + \alpha_k^2(\mathbb{E}_k\|E_k\|^2 + \mathbb{E}_k\|\nu_k\|^2)).$$

Then, the summability properties of  $\|E_k\|$  outlined in Proposition 3.3.2 yield the following proposition. We place the proof in Section 3.9.1.

**Theorem 3.4.1** (Local rates near critical points). *Suppose that  $f$  satisfies Assumption A at  $\bar{x}$ . Let  $\gamma \in (1/2, 1]$  and assume that  $c_1 \geq 32/\mu$  if  $\gamma = 1$ . Then for all  $k_0 \geq 1$  and sufficiently small  $\delta > 0$ , the following hold with stopping time  $\tau_{k_0, \delta}$  defined in (3.3.1):*

$$\sum_{k=1}^{\infty} \alpha_k \|\nabla_{\mathcal{M}}f(y_k)\|^2 1_{\tau_{k_0, \delta} > k} < \infty. \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \text{dist}(x_k, \mathcal{M}) 1_{\tau_{k_0, \delta} > k} < \infty.$$

If moreover,  $\|\nu_k\| \leq \mu/2$  for all  $k \geq 1$ , we have

$$\sup_k \left\{ \frac{\text{dist}(x_k, \mathcal{M})}{\alpha_k} 1_{\tau_{k_0, \delta} > k} \right\} < +\infty.$$

An immediate consequence of Theorem 3.4.1 is following: on the event  $\{\tau_{k_0, \delta} = \infty\}$ , there exists a finite nonnegative random variable  $W_{k_0, \delta}$  such that for any  $\epsilon > 0$ , and any  $k \geq 1$

$$\min_{i=1, \dots, k} \text{dist}(x_i, \mathcal{M}) \leq \frac{W_{k_0, \delta}}{k^{\gamma-\epsilon}} \quad \text{and} \quad \min_{i=1, \dots, k} \|\nabla_{\mathcal{M}}f(P_{\mathcal{M}}(x_i))\| \leq \frac{W_{k_0, \delta}}{k^{1/2-\gamma/2}}. \quad (3.4.1)$$

Additionally, when  $\|\nu_k\| \leq \mu/2$ , the rates of convergence for  $\text{dist}(x_k, \mathcal{M})$  improve. Indeed, in the event  $\{\tau_{k_0, \delta} = \infty\}$  there exists a finite nonnegative random variable  $W'_{k_0, \delta}$  such that

$$\text{dist}(x_k, \mathcal{M}) \leq \frac{W'_{k_0, \delta}}{k^\gamma} \quad \text{for all } k \geq 1.$$

How should we interpret these results? This question is most easily understood when the covariant gradient is Hölder metrically subregular to a subset  $\mathcal{Y} \subseteq \mathcal{M}$  of the manifold, a property connected to the Kurdyka-Łojasiewicz inequality [8, 47, 63]:

$$\|\nabla_{\mathcal{M}}f(y)\| \geq \hat{\mu} \cdot \text{dist}^\eta(y, \mathcal{Y}) \quad \text{for all } y \in B_{4\delta}(\bar{x}) \cap \mathcal{M},$$

where  $\hat{\mu} > 0, \eta > 0$ . In this case, we find that

$$\text{dist}(x_k, \mathcal{Y}) \leq \text{dist}(x_k, \mathcal{M}) + \text{dist}(y_k, \mathcal{Y}) \leq \text{dist}(x_k, \mathcal{M}) + \hat{\mu}^{-1/\eta} \|\nabla_{\mathcal{M}}f(y_k)\|^{1/\eta}$$

Recalling (3.4.1) immediately yields local  $O(k^{-\min\{\gamma-\epsilon, 1/2\eta-\gamma/2\eta\}})$  rates of convergence for how quickly  $x_k$  approaches  $\mathcal{Y}$ .

A natural question is whether we observe improved behavior when  $f$  satisfies further regularity conditions. Our next theorem shows that  $x_k$  converges to  $\bar{x}$  if  $\bar{x}$  is critical for the covariant gradient and covariant Hessian is positive definite on tangent space at  $\bar{x}$ . We place the proof in Section 3.9.2

**Theorem 3.4.2** (Local rates near strong local minimizers). *Suppose that  $f$  satisfies Assumption A at  $\bar{x}$ . Suppose that  $\nabla_{\mathcal{M}}f(\bar{x}) = 0$  and  $\nabla_{\mathcal{M}}^2f(\bar{x})$  is  $\sigma$ -positive definite on  $T_{\mathcal{M}}(\bar{x})$ . Finally, suppose that  $c_1 \geq \max\{16/\mu, 64/\sigma\}$  if  $\gamma = 1$ . Then for all  $k_0 \geq 1$  and sufficiently small  $\delta > 0$ , the following hold with stopping time  $\tau_{k_0, \delta}$  defined in (3.3.1):*

1. *There exists a finite random variable  $V$  such that the following holds:*

$$\limsup_{k \geq 1} \left\{ \frac{k^{2\gamma-1}}{\log(k+1)^2} \|x_k - \bar{x}\|^2 1_{\tau_{k_0, \delta} > k} \right\} \leq V \quad (3.4.2)$$

*Moreover, we have  $\sum_{k=1}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \|x_k - \bar{x}\|^2 1_{\tau_{k_0, \delta} > k} < +\infty$ .*

2. *If  $\nu_k = 0$  for all  $k \geq 1$ , then the following holds:*

$$\|x_k - \bar{x}\| 1_{\tau_{k_0, \delta} > k} \leq \frac{C}{k^\gamma}, \quad \forall k \geq k_0.$$

The most interesting setting of this theorem is the case  $\gamma = 1$ . In this case, on the event  $\{\tau_{k_0, \delta} = \infty\}$ , the following bound holds:

$$\|x_k - \bar{x}\| = O\left(\frac{\sqrt{V} \log(1+k)}{\sqrt{k}}\right).$$

This rate nearly matches that of the stochastic gradient method for smooth and strongly convex problems, which is known to be optimal [71]. In the next section, we prove that positive definiteness of  $\nabla_{\mathcal{M}}^2f(\bar{x})$  on  $T_{\mathcal{M}}(\bar{x})$  not only guarantees improves rates of convergence, but leads to asymptotic normality of the iterate sequence.

## 3.5 Asymptotic normality

Polyak and Juditsky [78] famously showed that the stochastic gradient method for minimizing smooth and strongly convex functions enjoys a central limit theorem: the error sequence  $\sqrt{k}(\bar{x}_k - \bar{x})$ , where  $x_k := \frac{1}{k} \sum_{i=1}^k x_i$ , converges in distribution to a normal random vector with a problem-dependent covariance matrix. In this section we ask whether a similar property is available in nonsmooth optimization. To provide context for our result, for the moment assume that

$$\min f(x) = \mathbb{E}_{z \sim P}[f(x; z)],$$

where  $P$  is a fixed, unknown distribution, the map  $x \mapsto f(x; z)$  is  $C^2$ , and  $f$  is strongly convex. For this class of problems, a popular algorithm is the stochastic gradient method

$$\begin{aligned} \text{Sample: } z_k &\sim P \\ x_{k+1} &= x_k - \alpha_k \nabla f(x_k; z_k), \end{aligned}$$

for an iid sequence of samples  $z_k$ . Then the result of [78] states that

$$\sqrt{k}(\bar{x}_k - \bar{x}) \xrightarrow{d} N(0, \nabla^2 f(\bar{x})^{-1} \text{Cov}(\nabla f(\bar{x}, z)) \nabla^2 f(\bar{x})^{-1}),$$

where  $\bar{x}$  is the unique minimizer of the  $f$ . Hence, the covariance structure of the error depends on second-order smoothness properties of  $f$ . In the nonsmooth setting, we will shortly prove a similar result, showing that the optimal covariance depends instead on the covariant Hessian  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  at the solution.

To prove our results, we must make the following more restrictive assumption on the noise sequence  $\nu_k$ , which appears in [33, Assumption D']. The assumption is modeled on stochastic gradient noise, which in the setting of Section 1.2.2 decomposes into stationary  $\nu_k^{(1)}$  and “small” noise  $\nu_k^{(2)}(x)$  components, as follows:

$$\nu_k = \nabla f(x_k; z_k) - \nabla f(x_k) = \underbrace{\nabla f(\bar{x}; z_k)}_{=:\nu_k^{(1)}} + \underbrace{(\nabla f(x_k; z_k) - \nabla f(\bar{x}; z_k)) + (\nabla f(\bar{x}) - \nabla f(x_k))}_{=:\nu_k^{(2)}(x)}.$$

**Assumption F.** Fix a point  $\bar{x} \in \text{dom } f$  at which Assumption A holds and let  $U$  be a matrix whose column vectors form an orthogonal basis of  $T_{\mathcal{M}}(\bar{x})$ . We suppose the noise sequence has decomposable structure  $\nu_k = \nu_k^{(1)} + \nu_k^{(2)}(x)$ , where  $\nu_k^{(2)}: \text{dom } f \rightarrow \mathbb{R}^d$  is a random function satisfying

$$\mathbb{E}_k[\|U^\top \nu_k^{(2)}(x)\|^2] \leq C\|x - \bar{x}\|^2 \quad \text{for all } x \in \text{dom } f \text{ near } \bar{x},$$

and some  $C > 0$ . In addition, we suppose that for all  $x \in \text{dom } f$ , we have  $\mathbb{E}_k[\nu_k^{(1)}] = \mathbb{E}_k[\nu_k^{(2)}(x)] = 0$  and the following limit holds:

$$\frac{1}{\sqrt{k}} \sum_{i=1}^k U^\top \nu_i^{(1)} \xrightarrow{d} N(0, \Sigma).$$

for some symmetric positive semidefinite matrix  $\Sigma$ .

Based on this assumption, we establish the following asymptotic normality guarantee, which is reminiscent of the classical result of Polyak and Juditsky for smooth problems [78]. The proof appears in Section 3.9.3.

**Theorem 3.5.1** (Asymptotic Normality). *Suppose that  $f$  satisfies Assumption A at  $\bar{x}$  and  $\nu_k$  satisfies assumption F. Suppose that  $\gamma \in (\frac{1}{2}, 1)$  and  $x_k$  converges to  $\bar{x}$  with probability one. Suppose  $\nabla_{\mathcal{M}} f(\bar{x}) = 0$  and that  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  is positive definite on  $T_{\mathcal{M}}(\bar{x})$ . Then the average iterate  $\bar{x}_k = \frac{1}{k} \sum_{i=1}^k x_i$  satisfies*

$$\sqrt{k}(\bar{x}_k - \bar{x}) \xrightarrow{d} N(0, \nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger \Sigma \nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger).$$

The conclusion of this theorem is surprising: although the sequence  $x_k$  never reaches the manifold, the distribution of  $\sqrt{k}(\bar{x}_k - \bar{x})$  is supported on the tangent space  $T_{\mathcal{M}}(\bar{x})$ . Thus asymptotically, the “directions of nonsmoothness,” which are normal to the  $\mathcal{M}$ , are quickly “averaged out.” When  $\|G_{\alpha_k}(x_k, \nu_k)\|$  is bounded away from 0 for all  $k$ , this means that  $x_k$  must oscillate across the manifold, instead of approaching it from one direction.

The proof of the theorem works as follows. First, we show that it suffices to understand the limit distribution of  $y_k$ , since  $\text{dist}(x_k, \mathcal{M})$  decreases sufficiently rapidly. While the shadow iteration (3.3.2) looks like a standard stochastic gradient sequence with quickly decreasing errors, it does not live within the ambient space  $\bar{x} + T_{\mathcal{M}}(\bar{x})$ , where the distribution is supported. Thus, we prove that it suffices to study the sequence  $z_k = P_{\bar{x} + T_{\mathcal{M}}(\bar{x})}(y_k)$ , which due to the smoothness of  $\mathcal{M}$ , closely approximates  $y_k$ . Then we find that  $z_k$  is amenable to existing proof techniques for stochastic gradient methods with errors, e.g., [1, Theorem 2].

### 3.5.1 Asymptotic normality in nonlinear programming

To illustrate this result, we consider the classical nonlinear programming problem

$$\begin{aligned} \min_x \quad & f_0(x) := \mathbb{E}_{z \sim P}[f_0(x; z)] \\ \text{subject to: } & x \in \mathcal{X} := \{x \in \mathbb{R}^d : f_i(x) = 0, i \in I_1; f_i(x) \leq 0, i \in I_2\}. \end{aligned} \quad (3.5.1)$$

where  $I_1, I_2 \subseteq \mathbb{N}$  are finite index sets, each  $f_i: \mathbb{R}^d \rightarrow \mathbb{R}$  is smooth,  $P$  is a fixed, unknown probability distribution, and for each  $z$  the function  $x \mapsto f(x; z)$  is  $C^1$ . Consider the following stochastic projected gradient method for solving (3.5.1):

$$\begin{aligned} \text{Sample: } & z_k \sim P \\ \text{Update: } & x_{k+1} \in P_{\mathcal{X}}(x_k - \alpha_k \nabla f_0(x_k; z_k)). \end{aligned} \quad (3.5.2)$$

Now fix a point  $\bar{x} \in \mathbb{R}^d$  and suppose the following conditions hold: For a given point  $\bar{x} \in \mathbb{R}^d$ , we assume that

(G1) **(Local Smoothness)** The functions  $f_0$  and  $f_i$  are  $C^3$  smooth near  $\bar{x}$  for all  $i \in I_1 \cup I_2$ .

(G2) **(Constraint Qualification/Activity)** The vector  $\nabla f(\bar{x})$  satisfies

$$-\nabla f_0(\bar{x}) \in \text{relint } N_{\mathcal{X}}(\bar{x}) \quad (3.5.3)$$

Moreover, let  $I = \{i \in I_1 \cup I_2 : f_i(\bar{x}) = 0\}$  denote the active constraints. Then the gradients  $\{\nabla_i f(\bar{x}) : i \in I\}$  are linearly independent.

Two consequences of (G2) follow: First there exists unique  $\bar{\lambda}_i \in \mathbb{R}_{++}$  ( $i \in I$ ) such that

$$\nabla f_0(\bar{x}) + \sum_{i \in I} \bar{\lambda}_i \nabla f_i(\bar{x}) = 0.$$

Second, the set

$$\mathcal{M} = \{x \in \mathbb{R}^d : f_i(x) = 0 \ \forall i \in I\},$$

is a  $C^3$  manifold with tangent space  $T_{\mathcal{M}}(\bar{x}) = \{w: \mathbb{R}^d : \nabla f_i(\bar{x})^\top w = 0 \ \forall i \in I\}$ . Moreover, due to local smoothness,  $f_0$  is  $C^3$  along  $\mathcal{M}$  near  $\bar{x}$ . Consequently, we may examine the second order optimality conditions through the covariant Hessian:

(G3) **(Second Order Sufficient Condition)** There exists  $\sigma > 0$  such that

$$w^\top \left[ \nabla^2 f_0(\bar{x}) + \sum_{i \in I} \bar{\lambda}_i \nabla^2 f_i(\bar{x}) \right] w \geq \sigma \|w\|^2 \quad \text{for all } w \in T_{\mathcal{M}}(\bar{x}).$$

Note that this condition is simply the requirement that the covariant Hessian of  $f := f_0 + \delta_{\mathcal{X}}$ , computed in [68],

$$\nabla_{\mathcal{M}}^2 f(\bar{x}) = P_{T_{\mathcal{M}}(\bar{x})} \left[ \nabla^2 f_0(\bar{x}) + \sum_{i \in I} \bar{\lambda}_i \nabla^2 f_i(\bar{x}) \right] P_{T_{\mathcal{M}}(\bar{x})}$$

is positive definite on  $T_{\mathcal{M}}(\bar{x})$ .

Finally, to ensure our noise sequence

$$\begin{aligned} \nu_k &= \nabla f_0(x_k; z_k) - \nabla f_0(x_k) \\ &= \underbrace{\nabla f_0(\bar{x}; z_k) - \nabla f_0(\bar{x})}_{=: \nu_k^{(1)}} + \underbrace{(\nabla f_0(x_k; z_k) - \nabla f_0(\bar{x}; z_k) + \nabla f_0(\bar{x}) - \nabla f_0(x_k))}_{=: \nu_k^{(2)}(x_k)}, \end{aligned}$$

satisfies Assumption F, we assume the stochasticity is sufficiently well-behaved:

(G5) **(Stochastic Gradients)** As a function of  $x$ , the fourth moment

$$x \in \mathcal{X} \mapsto \mathbb{E}_{z \sim P} [\|\nabla f_0(x; z) - \nabla f_0(x)\|^4]$$

is bounded on bounded sets. Moreover, there exists  $C > 0$  such that

$$\mathbb{E}_{z \sim P} [\|\nabla f_0(x; z) - \nabla f_0(\bar{x}; z)\|^2] \leq C \|x - \bar{x}\|^2 \quad \text{for all } x \in \mathcal{X}.$$

Finally, the gradients  $P_{T_{\mathcal{M}}(\bar{x})} \nabla f_0(\bar{x}; z)$  have finite covariance  $\Sigma = \text{Cov}(P_{T_{\mathcal{M}}(\bar{x})} \nabla f_0(\bar{x}; z))$ .

With these assumptions in hand, we have the following asymptotic normality result for nonlinear programming. The proof appears in Appendix 3.11.3.

**Corollary 3.5.2** (Asymptotic normality in nonlinear programming). *Suppose that Assumptions (G1)-(G5) hold and denote  $f = f_0 + \delta_{\mathcal{X}}$ . Suppose that  $\gamma \in (\frac{1}{2}, 1)$  and consider the iterates  $x_k$  generated by the stochastic projected gradient method (3.5.2). Then if  $x_k$  converges to  $\bar{x}$  with probability 1, the average iterate  $\bar{x}_k = \frac{1}{k} \sum_{i=1}^k x_i$  satisfies*

$$\sqrt{k}(\bar{x}_k - \bar{x}) \xrightarrow{d} N(0, \nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger \text{Cov}(P_{T_{\mathcal{M}}(\bar{x})} \nabla f_0(\bar{x}; z)) \nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger).$$

Moreover, if  $\bar{x}$  is the unique critical point of (3.5.1) and  $x_k$  remains bounded with probability one, then  $x_k$  converges to  $\bar{x}$  with probability 1.

As stated in introduction (Section 1.2.2), this appears to be the first asymptotic normality guarantee for the standard stochastic projected gradient method in general nonlinear programming problems with  $C^3$  data, even in the convex setting. Moreover, as shown in [33, Theorem 1] the covariance matrix  $\nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger \Sigma \nabla_{\mathcal{M}}^2 f(\bar{x})^\dagger$  is in a well-defined sense optimal, matching the estimation quality of classical empirical risk minimization methods [81, Theorem 3.3]. Finally we note that even for simple optimization problems dual averaging procedures can achieve suboptimal convergence [33]. This is surprising since such methods identify the active manifold [52] (also see [33, Section 4.1]), while projected stochastic gradient methods do not.

### 3.6 Avoiding saddle points

In this section, we ask whether  $x_k$  can converge to points  $\bar{x}$  at which  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  has at least one strictly negative eigenvalue. We call such points *strict saddle points*, and when  $\mathcal{M}$  is in addition an active manifold for  $f$ , then we call such points *active strict saddle points*, following [24]. As outlined in the introduction (Section 1.2.3), the recent work [24] proved that almost surely several proximal methods cannot converge to active strict saddles of weakly convex functions, when randomly initialization. Since the subgradient method does not identify the active manifold, we cannot apply the same strategy. Instead we resort to another technique, well-known in the stochastic approximation literature: random perturbations [2, 3, 13, 74].

Let us briefly describe this technique. Fix a point  $p \in \mathbb{R}^d$ , consider a  $C^2$  mapping  $F_p: \mathbb{R}^d \rightarrow \mathbb{R}^d$  with “unstable zero” at  $p$ , meaning  $\nabla F_p(p)$  has an eigenvalue with positive real part. Then a well-known result of Pemantle [74] states that, with probability 1, the following perturbed iteration cannot converge to  $p$ :

$$\left\{ \begin{array}{l} \text{Sample } \xi_k \sim \text{Unif}(B_1(0)) \\ \text{Set } Y_{k+1} = Y_k + \alpha_k F_p(Y_k) + \alpha_k \xi_k \end{array} \right\}. \quad (3.6.1)$$

As stated, the result of [74] does not shed light on the iteration (1.2.2). Nevertheless, the shadow iteration  $y_k$  does satisfy an iteration similar to (3.6.1) with mapping

$$F_p(y) = -\nabla(f \circ P_{\mathcal{M}})(P_{\mathcal{M}}(y)),$$

which under reasonable assumptions is locally  $C^2$  near  $p$  and satisfies  $F_p(y) = -\nabla_{\mathcal{M}} f(y)$  and  $\nabla F_p(y) = -\nabla_{\mathcal{M}}^2 f(y)$  for all  $y \in \mathcal{M}$  near  $p$ . Moreover, if  $p$  is an active strict saddle of  $f$ , then  $\nabla_{\mathcal{M}}^2 f(p)$  has a strictly negative eigenvalue, so  $p$  is an “unstable zero” of  $F_p$ . Thus, we might reasonably expect  $y_k$  to converge to  $p$  only with probability zero. If this is the case, we can then lift the argument to  $x_k$ , showing that if  $x_k$  converges to  $p$ , then so does  $y_k$ —a probability zero event. This is the strategy we will apply in what follows, taking into account the additional error term  $E_k$  in the shadow iteration (1.0.2), a key technical issue that we have so far ignored.

In order to formalize the above plan, we prove the following extension of the main result of [74] which models the relationship between  $x_k$  and  $y_k$  described above. The proof, which we defer Section 3.9.4 draws on and the techniques of [2, 3, 12, 13, 74].

**Theorem 3.6.1** (Nonconvergence). *Fix  $c_1, c_2 > 0$  and let  $S \subseteq \mathbb{R}^d$ . Suppose for any  $p \in S$ , there exists a ball  $B_{\epsilon_p}(p)$  centered at  $p$  and a  $C^2$  mapping  $F_p: B_{\epsilon_p}(p) \rightarrow \mathbb{R}^d$  that vanishes at  $p$  and has a symmetric Jacobian  $\nabla F_p(p)$  that has at least one positive eigenvalue. Suppose  $\{X_k\}_{k=1}^{\infty}$  is a stochastic process and for any  $k_0, p \in S$ , and  $\delta > 0$  define the stopping time:*

$$\tau_{k_0, \delta}(p) = \inf \{k \geq k_0: X_k \notin B_{\delta}(p)\}.$$

*Suppose that for any  $p \in S$ ,  $k_0 \geq 1$ , and all sufficiently small  $\delta_p \leq \epsilon_p$  the following hold: there exists  $c_3, c_4 > 0$  possibly depending on  $p$ , but not on  $\delta_p$  and  $\epsilon_p$ , such that on the event  $\Omega_0 = \{\tau_{k_0, \delta_p}(p) = \infty\}$ , we have*

1. (**Local iteration.**) There exists a process  $\{Y_k: k \geq k_0\} \subseteq B_{\epsilon_p/2}(p)$  satisfying

$$Y_{k+1} = Y_k + \alpha_k F_p(Y_k) + \alpha_k \xi_k + \alpha_k E_k \quad (3.6.2)$$

for error sequence  $\{E_k\}$ , noise sequence  $\{\xi_k\}$ , and deterministic stepsize sequence  $\{\alpha_k\}$  that is square summable, but not summable.

2. (**Noise Conditions.**) Let  $\mathcal{F}_k$  be the sigma algebra generated by  $X_{k_0}, \dots, X_k$  and  $Y_{k_0}, \dots, Y_k$ . Define  $W_p$  to be the subspace of eigenvectors of  $\nabla F_p(p)$  with positive eigenvalues. Then we have

$$(a) \mathbb{E}[\xi_k \mid \mathcal{F}_k] = 0.$$

$$(b) \limsup_k \mathbb{E}[\|\xi_k\|^4 \mid \mathcal{F}_k] \leq c_3.$$

$$(c) \mathbb{E}[\langle \xi_k, w \rangle \mid \mathcal{F}_k] \geq c_4 \quad \text{for } k \geq k_0 \text{ and all unit norm } w \in W_p.$$

3. (**Error Conditions.**)

$$(a) \text{ We have } \limsup_k \mathbb{E}[\|E_k\|^4 \mid \mathcal{F}_k] < \infty.$$

$$(b) \text{ For all } n \geq k_0, \text{ we have } \mathbb{E}[1_{\Omega_0} \sum_{k=n}^{\infty} \alpha_k \|E_k\|] = O_{k_0}(\sum_{k=n}^{\infty} \alpha_k^2).$$

Then  $P(\lim_{k \rightarrow \infty} X_k \in S) = 0$ .

Looking at the theorem, recursion condition (3.6.2) is clearly modeled on the shadow sequence of Proposition 3.3.2. Moreover, the error condition 3b on  $E_k$  precisely matches 2c. Finally, the noise  $\xi_k$  is modeled on  $P_{T_{\mathcal{M}}(y_k)}(\nu_k)$  in the shadow iteration, which is mean zero and has bounded fourth moment. Condition 2c is not automatic for all noise distributions and requires that  $\nu_k$  has nontrivial mass in all directions of negative curvature for  $f$ .

Given Theorem 3.6.1, we now ask: can  $x_k$  converge to critical points  $\bar{x}$  at which  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  has a strict negative eigenvalue? In the following theorem we show that the answer is no, provided that we choose the noise  $\nu_k$  according to the following assumption:

**Assumption G** (Uniform noise). There exists  $r > 0$  such that  $\nu_k \sim \text{Unif}(B_r(0))$  for all  $k$ .

The proof of the theorem appears in Section 3.9.5.

**Theorem 3.6.2** (Nonconvergence to strict saddle point). *Let  $S \subseteq \mathbb{R}^d$  and suppose that Assumption A holds at each point  $\bar{x} \in S$ , where each manifold is  $C^4$ . Let  $\mathcal{M}$  be the manifold associated to  $\bar{x}$  and suppose the  $\nabla_{\mathcal{M}}^2 f(\bar{x})$  has a strictly negative eigenvalue. Suppose that  $\nu_k$  satisfies Assumption G. In addition, suppose that  $\gamma \in (\frac{1}{2}, 1)$ . Then*

$$P\left(\lim_{k \rightarrow \infty} x_k \in S\right) = 0. \quad (3.6.3)$$

Some comments are in order. First note that the theorem applies to arbitrary sets  $S$ , making no assumptions on countability/isolatedness. Second the result does not preclude the limit points of  $x_k$  from lying in  $S$ . Thus, the result is useful only when  $x_k$  is known to converge. Third, the noise sequence  $\nu_k$  may be chosen from different distributions. The main requirement is that it be sufficiently well-spread in all directions of negative curvature for  $f$  and that it have a bounded eighth moment.

We now examine two applications of the above theorem. The following corollary provides sufficient conditions for the projected subgradient method to avoid active strict saddle points. We place the proof in Appendix 3.11.4.

**Corollary 3.6.3** (Projected subgradient methods). *Suppose that  $f = g + \delta_{\mathcal{X}}$ , where  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz and  $\mathcal{X} \subseteq \mathbb{R}^d$  is closed. Let  $S$  be a set of Fréchet  $C^4$  active strict saddle points of  $f$ . Suppose the following hold for all  $x \in S$  with associated active manifold  $\mathcal{M}_x$ :*

1. *The function  $g$  and the set  $\mathcal{X}$  are strong (a)-regular along  $\mathcal{M}_x$  at  $x$ .*
2. *The function  $g$  is prox-regular at  $x$  or (b)-regular along  $\mathcal{M}_x$  at  $x$ .*
3. *The set  $\mathcal{X}$  is prox-regular at  $x$  or (b)-regular along  $\mathcal{M}_x$  at  $x$ .*

*Suppose that  $\nu_k$  satisfies Assumption G. Then the iterates of the stochastic projected subgradient method (3.2.4) satisfy*

$$P \left( \lim_{k \rightarrow \infty} x_k \in S \right) = 0.$$

Next we analyze the proximal gradient method. Recall that the paper [24] showed that randomly initialized proximal gradient methods avoid active strict saddles of weakly convex functions. The following Corollary shows that the same behavior holds for perturbed proximal gradient methods beyond the weakly convex class. We place the proof in Appendix 3.11.5.

**Corollary 3.6.4** (Proximal gradient methods). *Suppose that  $f = g + h$ , where  $h: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  is closed that is Lipschitz on its domain  $\mathcal{X} := \text{dom } h$  and  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is  $C^1$  with Lipschitz continuous gradient on  $\mathcal{X}$ . Let  $S$  be a set of Fréchet  $C^4$  active strict saddle points of  $f$ . Suppose that for all  $x \in S$  with associated active manifold  $\mathcal{M}_x$ , the function  $f$  is strong (a)-regular and (b)-regular along  $\mathcal{M}_x$  at  $x$ . Suppose that  $\nu_k$  satisfies Assumption G. Then the iterates of the stochastic proximal gradient method (3.2.4) satisfy*

$$P \left( \lim_{k \rightarrow \infty} x_k \in S \right) = 0.$$

### 3.6.1 Consequences for generic semialgebraic functions

The above results show that the perturbed projected subgradient and the proximal gradient method cannot converge to Fréchet active strict saddle points, provided that  $x_k$  converges and various regularity properties hold. Although the convergence of  $x_k$  and the required regularity properties may seem stringent, they are in a precise sense generic. Indeed, the genericity of the regularity properties was already addressed in Chapter 2 and Section 3.2. Convergence also holds generically: it is known that all limit points of the stochastic subgradient method, the stochastic projected subgradient method, and the stochastic proximal method are (composite) Clarke critical points, as long as  $f$  is a semialgebraic function [26, Corollary 6.4.]. Thus, since generic semialgebraic functions have only finitely many (composite) Clarke critical points and one can show (with small effort) that the set of limit points of each algorithm is connected, it follows that the entire sequence  $x_k$  must converge on generic problems (if the sequence remains bounded). Thus we have the following three corollaries, whose proofs we place in Appendix 3.11.6.



**Corollary 3.6.5** (Subgradient method on generic semialgebraic functions). *Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  be a locally Lipschitz semialgebraic function. Then for a full measure set of  $v$  the following is true for the tilted function  $f_v(x) := f(x) - \langle v, x \rangle$ : Let  $\{x_k\}_{k \in \mathbb{N}}$  be generated by the subgradient method 3.2.2 on  $f_v$ . Suppose that  $\nu_k$  satisfies Assumption G. Then on the event  $\{x_k\}_{k \in \mathbb{N}}$  is bounded, almost surely we have only two possibilities*

1.  $x_k$  converges to a local minimizer  $\bar{x}$  of  $f_v$ .
2.  $x_k$  converges to a Clarke critical point of  $f_v$ .

*Thus, if  $f$  is Clarke regular, the sequence  $x_k$  must converge to a local minimizer of  $f_v$ .*

**Corollary 3.6.6** (Projected subgradient method on generic semialgebraic functions). *Let  $f = g + \delta_{\mathcal{X}}$ , where  $\mathcal{X} \subseteq \mathbb{R}^d$  semialgebraic and closed and  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz and semialgebraic. Then for a full measure set of  $v, w \in \mathbb{R}^d$  the following is true for the tilted function  $f_{v,w}(x) := g(x + w) + \delta_{\mathcal{X}}(x) - \langle v, x \rangle$ . Let  $\{x_k\}_{k \in \mathbb{N}}$  be generated by the projected subgradient method 3.2.5. Suppose that  $\nu_k$  satisfies Assumption G. Then on the event  $\{x_k\}_{k \in \mathbb{N}}$  is bounded, almost surely we have only two possibilities*

1.  $x_k$  converges to a local minimizer  $\bar{x}$  of  $f_{v,w}$ .
2.  $x_k$  converges to a composite Clarke critical point of  $f_{v,w}$ .

*Thus, if  $g$  and  $\mathcal{X}$  are Clarke regular, the sequence  $x_k$  converges to a local minimizer of  $f_{v,w}$ .*

**Corollary 3.6.7** (Proximal gradient method on generic semialgebraic functions). *Suppose that  $f = g + h_0 + \delta_{\mathcal{X}}$ , where  $\mathcal{X} \subseteq \mathbb{R}^d$ ,  $g$  is a  $C^1$  function with Lipschitz gradient on  $\mathcal{X}$ , the function  $h_0: \mathbb{R}^d \rightarrow \mathbb{R}$  is Lipschitz on  $\mathcal{X}$ , and we define  $h := h_0 + \delta_{\mathcal{X}}$ . Then for a full measure set of  $v, w \in \mathbb{R}^d$  the following is true for the tilted function  $f_{v,w} := g(x + w) + h_0(x + w) + \delta(x) - \langle v, x \rangle$ . Let  $\{x_k\}_{k \in \mathbb{N}}$  be generated by the proximal gradient method 3.2.7. Suppose that  $\nu_k$  satisfies Assumption G. Then on the event  $\{x_k\}_{k \in \mathbb{N}}$  is bounded, almost surely we have only two possibilities*

1.  $x_k$  converges to a local minimizer  $\bar{x}$  of  $f_{v,w}$ .
2.  $x_k$  converges to a composite Clarke critical point of  $f_{v,w}$ .

*Thus, if  $h_0$  and  $\mathcal{X}$  are Clarke regular, the sequence  $x_k$  converges to a local minimizer of  $f_{v,w}$ .*

In short, the main conclusion of the above three theorems is

On generic regular semialgebraic functions, perturbed subgradient/proximal  
methods converge only to local minimizers

We note in passing that the results hold verbatim if one replaces the word “semialgebraic” with “definable in an  $\mathcal{o}$ -minimal structure,” throughout.

## 3.7 Outline

This concludes our statement of main results for Chapter 3. The remaining portion of this paper consists of proofs. In Section 3.8 we prove the two pillars. This section forms the core of the arguments for the remainder of the paper. In Section 3.9, we prove the remaining theorems: local rates of converges (Section 3.9.1 and 3.9.2), asymptotic normality (Section 3.9.3), and saddle point avoidance (Sections 3.9.4 and 3.9.5). Finally the appendix of the paper includes several secondary results: proofs that Assumption A holds for subgradient, projected subgradient (Section 3.11.1), and proximal gradient (Section 3.11.2) algorithms; asymptotic normality in nonlinear programming (Section 3.11.3); saddle avoidance for projected subgradient (Section 3.11.4) and proximal gradient methods (Section 3.11.4); saddle avoidance for generic semialgebraic problems (Section 3.11.6); and a small appendix on sequences and stochastic processes 3.11.7.

## 3.8 Proofs of the two pillars

Throughout this work we let  $\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_k]$  denote the conditional expectation. We now present the proofs of the two pillars.

### 3.8.1 Proof of Proposition 3.3.1: aiming towards the manifold

Throughout the proof we let  $C$  denote a constant depending on  $k_0$  and  $\delta$ , which may change from line to line. Choose  $\delta \leq \min\{1, \frac{c_1\mu}{12\gamma}\}$ , satisfying  $B_\delta(\bar{x}) \subseteq U$  where  $U$  is the neighborhood in which Assumption A holds. Define  $Q := \max\{\sup_{x \in B_\delta} q(x), 1\}$ . By shrinking  $\delta$  slightly, we can assume that the little  $o$  term in (A3) satisfies

$$o(\text{dist}(x, \mathcal{M})) \leq \frac{\mu}{4(1+Q)} \text{dist}(x, \mathcal{M}) \quad \text{for all } x \in B_\delta(\bar{x}).$$

Now define:  $D_k := \text{dist}(x_k, \mathcal{M})$  for all  $k \geq 0$ . We prove a recurrence relation satisfied by the sequence  $D_k$ . To that end, denote  $v_k = G_{\alpha_k}(x_k, \nu_k)$  and observe that in the event  $A_k := \{\tau_{k_0, \delta} > k\}$ , we have

$$\begin{aligned} D_{k+1}^2 &\leq \|x_{k+1} - P_{\mathcal{M}}(x_k)\|^2 \\ &= \|x_k - \alpha_k v_k - P_{\mathcal{M}}(x_k)\|^2 \\ &= \|x_k - P_{\mathcal{M}}(x_k)\|^2 - 2\alpha_k \langle v_k, x_k - P_{\mathcal{M}}(x_k) \rangle + \alpha_k^2 \|v_k\|^2 \\ &\leq D_k^2 - 2\alpha_k \mu D_k + 2\alpha_k (1 + \|\nu_k\|)^2 o(D_k) \\ &\quad - 2\alpha_k \langle \nu_k, x_k - P_{\mathcal{M}}(x_k) \rangle + \underbrace{C(1 + \|\nu_k\|)^2}_{:= B_k} \alpha_k^2, \end{aligned} \tag{3.8.1}$$

where the second inequality follows from the proximal aiming and local boundedness properties of  $G$ ; see Assumption A. This inequality will allow us to prove all parts of the result.

Indeed, let us prove Part 1. To that end, first note that the bound  $\mathbb{E}_k[\|\nu_k\|^2]1_{A_k} \leq q(x_k)1_{A_k} \leq Q$  implies that there exists  $C > 0$  such that

$$\mathbb{E}_k[B_k]1_{A_k} \leq C,$$

meaning the conditional expectation is bounded for all  $k$ . Moreover, by our choice of  $\delta$ ,

$$\mathbb{E}_k[(1 + \|\nu\|)^2 o(D_k) 1_{A_k}] \leq \frac{\mu}{2} D_k 1_{A_k}.$$

Thus, for each  $k$ , we have

$$\begin{aligned} \mathbb{E}_k[D_{k+1}^2 1_{A_{k+1}}] &\leq \mathbb{E}_k[D_{k+1}^2 1_{A_k}] \\ &\leq D_k^2 1_{A_k} - \alpha_k \mu D_k 1_{A_k} + \mathbb{E}_k[B_k] 1_{A_k} \alpha_k^2 - 2\alpha_k \langle \mathbb{E}_k[\nu_k], x_k - P_{\mathcal{M}}(x_k) \rangle 1_{A_k} \\ &\leq D_k^2 1_{A_k} - \alpha_k \mu D_k 1_{A_k} + C\alpha_k^2 \\ &\leq (1 - (\alpha_k/2)\mu) D_k^2 1_{A_k} - (\alpha_k/2)\mu D_k 1_{A_k} + C\alpha_k^2, \end{aligned} \quad (3.8.2)$$

where the first inequality follows from  $1_{A_{k+1}} \leq 1_{A_k}$ ; the second inequality follows from  $\mathcal{F}_k$ -measurability of  $A_k$ ; and the fourth inequality follows since  $D_k 1_{A_k} \geq D_k^2 1_{A_k}$  (recall  $\delta \leq 1$ ). Now apply Lemma 3.11.6 with the sequences  $X_k := D_k^2 1_{A_k}$ ,  $Y_k := \alpha_k \mu D_k 1_{A_k}$ , and  $Z_k := C\alpha_k^2$  and deduce that  $(k^{2\gamma-1}/\log(k+1)^2) D_k^2$  almost surely converges to a finite valued random variable and the following sum is finite:

$$\sum_{k=1}^{\infty} \frac{k^{2\gamma-1} \alpha_k}{\log(k+1)^2} \mu D_k 1_{A_k} < +\infty.$$

Recalling that  $\alpha_k \geq c_1/k^\gamma$ , we get the claimed summability result.

Next we prove Part 2. To that end, take expectation of (3.8.2) and use the law of total expectation to deduce that for some  $C > 0$ , we have

$$\begin{aligned} \mathbb{E}[D_{k+1}^2 1_{A_k}] &\leq (1 - \mu\alpha_k/2) \mathbb{E}[D_k^2 1_{A_k}] - (\alpha_k/2)\mu \mathbb{E}[D_k 1_{A_k}] + C\alpha_k^2 \\ &\leq (1 - \mu c_1 k^{-\gamma}/2) \mathbb{E}[D_k^2 1_{A_k}] - (\alpha_k/2)\mu \mathbb{E}[D_k 1_{A_k}] + Ck^{-2\gamma} \end{aligned}$$

To prove part 2a, simply apply Lemma 3.11.8 applied with sequence  $s_k = \mathbb{E}[D_k^2 1_{A_k}]$  and constants  $c = \mu c_1/2$  and  $C$ . To prove part 2b, sum the above inequality from  $n$  to infinity to get

$$\sum_{k=n}^{\infty} (\alpha_k/2)\mu \mathbb{E}[D_k 1_{A_k}] \leq \mathbb{E}[D_n^2 1_{A_n}] + C \sum_{k=n}^{\infty} \alpha_k^2 \leq Cn^{-\gamma} + C \sum_{k=n}^{\infty} \alpha_k^2,$$

where the second inequality follows from Part 2a. Noting that  $n^{-\gamma} = O(\sum_{k=n}^{\infty} \alpha_k^2)$  proves the result.

Now we prove Part 3. We first slightly shrink  $\delta$  so that

$$o(\text{dist}(x, \mathcal{M})) \leq \frac{\mu}{4(1 + \mu/4)} \text{dist}(x, \mathcal{M}) \quad \text{for all } x \in B_\delta(\bar{x}).$$

Next, based on our assumptions on  $\|\nu_k\|$ , we have

$$|\langle \nu_k, x_k - P_{\mathcal{M}}(x_k) \rangle| \leq (\mu/4) D_k.$$

In addition, we have that for some constant  $C > 0$ , that  $\|B_k\| \leq C$ . Consequently, inequality (3.8.1) shows that in the event  $A_k$ , we have

$$\begin{aligned} D_{k+1}^2 &\leq D_k^2 - 2\alpha_k \mu D_k - 2\alpha_k (1 + \|\nu_k\|) o(D_k) + B_k \alpha_k^2 - 2\alpha_k \langle \nu_k, x_k - P_{\mathcal{M}}(x_k) \rangle \\ &\leq D_k^2 - \alpha_k \mu D_k + C\alpha_k^2. \end{aligned}$$

Therefore, the result follows from Lemma 3.11.7 applied with the sequence  $s_k = D_k 1_{A_k}$  and constants  $c = \mu c_1$ , and  $C$ , as desired.

### 3.8.2 Proof of Proposition 3.3.2: the shadow iteration

Throughout the proof we let  $C$  denote a constant depending on  $k_0$  and  $\delta$ , but not on  $k$ , which may change from line to line. We assume  $\delta$  is small enough that the conclusions of Proposition 3.3.1 hold; that  $B_{4\delta}(\bar{x}) \subseteq U$  where  $U$  is the neighborhood in which Assumption A holds; and that  $P_{\mathcal{M}}$  and  $\nabla P_{\mathcal{M}}$  are Lipschitz continuous on  $B_{4\delta}(\bar{x})$ . Write  $\tau = \tau_{k_0, \delta}$  and fix index  $k \geq 1$ . Finally, recall that  $P_{\mathcal{M}}$  is  $C^2$  on  $U$  and  $\nabla P_{\mathcal{M}}(x) = P_{T_{\mathcal{M}}(x)}$  for all  $x \in \mathcal{M}$ .

Let us first prove that  $y_k \in B_{4\delta}(\bar{x})$ . Clearly, we need only consider the case  $x \in B_{2\delta}(\bar{x})$ . In this case,

$$\|y_k - \bar{x}\| \leq \|y_k - x_k\| + \|x_k - \bar{x}\| \leq 2\|x_k - \bar{x}\| \leq 4\delta,$$

where the final inequality follows since  $\bar{x} \in \mathcal{M}$ . Therefore, we always have  $\|y_k - \bar{x}\| \leq 4\delta$ .

Next, let us define the error sequence  $E_k$  in the shadow iteration. To that end, denote  $T_k := T_{\mathcal{M}}(y_k)$  and

$$w_k := y_k - \alpha_k \nabla_{\mathcal{M}} f(y_k) - \alpha_k P_{T_k}(\nu_k)$$

Then with error sequence  $E_k := (y_{k+1} - w_k)/\alpha_k$ , the claimed recursion is trivially true. Thus, in the remainder of the proof, we bound  $E_k$ .

Turning to the bound, we first note that throughout the proof, we must separate the analysis into two cases:  $x_{k+1} \in B_{2\delta}(\bar{x})$  and  $x_{k+1} \notin B_{2\delta}(\bar{x})$ . In the second case, the following preliminary observation will be useful:

*Claim 1.* Suppose that in the event  $\{\tau > k\}$  it holds that  $x_{k+1} \notin B_{2\delta}(\bar{x})$ . Then there exists  $C > 0$  such that

$$\|y_{k+1} - y_k\| \leq 4\delta \leq C\|x_{k+1} - x_k\|. \quad (3.8.3)$$

*Proof.* First notice that

$$\|x_{k+1} - x_k\| \geq \|x_{k+1} - \bar{x}\| - \|x_k - \bar{x}\| \geq 2\delta - \delta \geq \delta.$$

Therefore, the result trivially holds since  $\|y_{k+1} - y_k\| \leq 4\delta$ .  $\square$

With the preliminaries set, we now bound  $\|E_k\|$ . To that end, in what follows we assume we are in the event  $\{\tau > k\}$  where  $k \geq k_0$ . In this event, our strategy will be to bound the terms  $R_1$  and  $R_2$  in the following decomposition:

$$\begin{aligned} \|E_k\| &= \|(y_{k+1} - w_k)/\alpha_k\| \\ &\leq \underbrace{\|y_{k+1} - y_k - P_{T_k}(y_{k+1} - y_k)\|/\alpha_k}_{:=R_1} + \underbrace{\|P_{T_k}(y_{k+1} - y_k)/\alpha_k + \nabla f_{\mathcal{M}}(y_k) + P_{T_k}(\nu_k)\|}_{:=R_2}. \end{aligned} \quad (3.8.4)$$

In our bounds of these terms, we frequently use the following bound: there exists  $C > 0$  such that

$$\|x_{k+1} - x_k\| \leq \alpha_k \|G_{\alpha_k}(x_k, \nu_k)\| \leq C(1 + \|\nu_k\|)\alpha_k. \quad (3.8.5)$$

We now bound  $R_1$  and  $R_2$  separately.

The following claim bounds  $R_1$ .

*Claim 2.* There exists  $C > 0$  such that

$$R_1 1_{\tau > k} \leq C(1 + \|\nu_k\|)^2 \alpha_k 1_{\tau > k}. \quad (3.8.6)$$

*Proof.* We consider two cases. First suppose  $x_{k+1} \in B_{2\delta}(\bar{x})$ . Let  $C > 0$  be a local Lipschitz constant of  $\nabla P_M$  and  $P_M$ . Then it follows that vector  $y_{k+1} - y_k = P_M(x_{k+1}) - P_M(x_k)$  is nearly tangent to the manifold at  $y_k$ :

$$\|y_{k+1} - y_k - P_{T_k}(y_{k+1} - y_k)\| \leq C\|y_{k+1} - y_k\|^2 \leq C^2\|x_{k+1} - x_k\|^2.$$

Thus, taking into account (3.8.5), we have for some  $C > 0$ , the bound:

$$R_1 \leq C(1 + \|\nu_k\|)^2 \alpha_k,$$

as desired.

Now suppose that  $x_{k+1} \notin B_{2\delta}(\bar{x})$ . Therefore, there exists  $C > 0$  such that

$$\|y_{k+1} - y_k - P_{T_k}(y_{k+1} - y_k)\| \leq 2\|y_{k+1} - y_k\| \leq C\|x_{k+1} - x_k\| \leq \frac{C^2}{\delta}\|x_{k+1} - x_k\|^2,$$

where the first inequality follows since  $\|P_{T_k}\| \leq 1$  and the second and third inequalities follow from Claim 1. Thus taking into account (3.8.5), we again have for some  $C > 0$ , the bound:

$$R_1 \leq C(1 + \|\nu_k\|)^2 \alpha_k,$$

Thus, putting together both bounds on  $R_1$ , the result follows.  $\square$

The following claim bounds  $R_2$ .

*Claim 3.* There exists  $C > 0$  such that

$$R_2 1_{\tau > k} \leq C(1 + \|\nu_k\|)^2 (\text{dist}(x_k, \mathcal{M}) + \alpha_k) 1_{\tau > k}. \quad (3.8.7)$$

*Proof.* To bound  $R_2$ , we first simplify:

$$\begin{aligned} R_2 &= \|P_{T_k}(y_{k+1} - y_k)/\alpha_k + \nabla_{\mathcal{M}} f(y_k) + P_{T_k}(\nu_k)\| \\ &\leq \|P_{T_k}(y_{k+1} - x_{k+1})/\alpha_k\| + \|P_{T_k}(x_k - y_k)/\alpha_k\| + \|P_{T_k}(x_{k+1} - x_k)/\alpha_k + \nabla_{\mathcal{M}} f(y_k) + P_{T_k}(\nu_k)\| \\ &\leq \|P_{T_k}(y_{k+1} - x_{k+1})/\alpha_k\| + C(1 + \|\nu_k\|)^2 (\text{dist}(x_k, \mathcal{M}) + \alpha), \end{aligned} \quad (3.8.8)$$

where the second equality follows from by Assumption A and the inclusion  $x_k - y_k \in N_{\mathcal{M}}(y_k)$ , which implies that  $P_{T_k}(x_k - y_k) = 0$ . We now bound the term  $\|P_{T_k}(y_{k+1} - x_{k+1})/\alpha_k\|$ .

First suppose that  $x_{k+1} \in B_{2\delta}(\bar{x})$  and note that  $y_{k+1} \in B_{4\delta}(\bar{x}) \cap \mathcal{M} \subseteq U \cap \mathcal{M}$ . Let  $C' > 0$  be a local Lipschitz constant of  $\nabla P_M$  and  $P_M$ . Then for some  $C > 0$  larger than  $C'$ , we have

$$\begin{aligned} \|P_{T_k}(y_{k+1} - x_{k+1})/\alpha_k\| &\leq \|(P_{T_{k+1}} - P_{T_k})(y_{k+1} - x_{k+1})/\alpha_k\| \\ &\leq C'\|y_{k+1} - y_k\| \text{dist}(x_{k+1}, \mathcal{M})/\alpha_k \\ &\leq (C')^2\|x_{k+1} - x_k\| (\text{dist}(x_k, \mathcal{M}) + \|x_{k+1} - x_k\|)/\alpha_k \\ &\leq C^3(1 + \|\nu_k\|) \text{dist}(x_k, \mathcal{M}) + C^4(1 + \|\nu_k\|)^2 \alpha_k, \end{aligned}$$

where the first inequality follows from  $x_{k+1} - y_{k+1} \in N_{\mathcal{M}}(y_{k+1})$ , which implies  $P_{T_{k+1}}(y_{k+1} - x_{k+1}) = 0$ ; the second inequality follows from Lipschitz continuity of  $\nabla P_{\mathcal{M}}(y) = P_{T_{\mathcal{M}}(y)}$  in  $y$ ; the third inequality follows from Lipschitz continuity of  $P_{\mathcal{M}}$  and Lipschitz continuity of  $\text{dist}(\cdot, \mathcal{M})$ ; and the fourth inequality follows from (3.8.5). Plugging this bound into (3.8.8), yields that for some  $C > 0$ , we have

$$R_2 \leq C(1 + \|\nu_k\|)^2(\text{dist}(x_k, \mathcal{M}) + \alpha_k),$$

as desired.

Now suppose that  $x_{k+1} \notin B_{2\delta}(\bar{x})$ . Then, there exists  $C > 0$  such that

$$\begin{aligned} \|P_{T_k}(y_{k+1} - x_{k+1})/\alpha_k\| &\leq \|P_{T_k}(y_{k+1} - x_k)\|/\alpha_k + \|P_{T_k}(x_k - x_{k+1})\|/\alpha_k \\ &\leq 2\delta/\alpha_k + \|x_k - x_{k+1}\|/\alpha_k \\ &\leq (1 + C)\|x_k - x_{k+1}\|/\alpha_k \\ &\leq \frac{(1 + C)C}{\delta\alpha_k}\|x_k - x_{k+1}\|^2 \\ &\leq \frac{(1 + C)C^3}{\delta}(1 + \|\nu_k\|)^2\alpha_k \end{aligned}$$

where first inequality follows from the triangle inequality; the second inequality follows since  $x_k \in B_{2\delta}(\bar{x})$  and  $y_{k+1} = \bar{x}$ ; the third and fourth third inequalities follow from Claim 1; and the fifth follows from (3.8.5). Thus, in this case, we find that there exists  $C > 0$  with

$$R_2 \leq C(1 + \|\nu_k\|)^2(\text{dist}(x_k, \mathcal{M}) + \alpha_k).$$

Therefore, putting together both bounds on  $R_2$ , the result follows.  $\square$

Now we prove Part 2a. Beginning with subpart 2(a)i, we find that by Claim 2 and 3, we have that for some  $C > 0$ , the bound

$$\|E_k\|1_{\tau > k} \leq R_1 1_{\tau > k} + R_2 1_{\tau > k} \leq C(1 + \|\nu_k\|)^2(\text{dist}(x_k, \mathcal{M}) + \alpha_k)1_{\tau > k}, \quad (3.8.9)$$

as desired. Turning to Part 2(a)ii, first note that that  $\text{dist}(x_k, \mathcal{M})1_{\tau > k} \leq \delta$ . Thus, the bound will follow if the conditional expectation of  $(1 + \|\nu_k\|)^4$  is bounded whenever  $x_k \in B_{\delta}(\bar{x})$ . This holds by assumption, since

$$\mathbb{E}_k[\|\nu_k\|^4]1_{\tau > k} \leq \sup_{x \in B_{\delta}(\bar{x})} q(x) < \infty.$$

Finally, we prove Part subpart 2(a)iii. Again using the boundedness of the conditional fourth moment of  $\|\nu_k\|1_{\tau > k}$ , we find that there exists a  $C > 0$  such that

$$\mathbb{E}_k[\|E_k\|^2 1_{\tau > k}] \leq C \text{dist}^2(x_k, \mathcal{M})1_{\tau > k} + C\alpha_k^2 1_{\tau > k}, \quad (3.8.10)$$

where the first inequality follows from Jensen's inequality and the second inequality follows from (3.8.9). Consequently, there exists  $C' > 0$  such that

$$\mathbb{E}[\|E_k\|^2 1_{\tau > k}] = \mathbb{E}[\mathbb{E}_k[\|E_k\|^2 1_{\tau > k}]] \leq C\mathbb{E}[\text{dist}^2(x_k, \mathcal{M})1_{\tau > k}] + C\alpha_k^2 \leq C'\alpha_k,$$

where the third inequality follows from Part 3 of Proposition 3.3.1. This prove Part 2a.

Now we prove Part 2b, beginning with Part 2(b)i. To that end, define  $F_k = \frac{k^{\gamma-1}}{\log(k+1)^2} \|E_k\| 1_{\tau > k}$ . Recall that by the conditional Borel-Cantelli theorem (Lemma 3.11.2), the sequence  $F_k$  is summable whenever  $\mathbb{E}_k[F_k]$  is summable. Thus, we first upper bound  $\mathbb{E}_k[F_k]$  by a summable sequence: there exists  $C > 0$  such that

$$\begin{aligned} \mathbb{E}_k[F_k] &\leq C \frac{k^{\gamma-1}}{\log(k+1)^2} (\text{dist}(x_k, \mathcal{M}) + \alpha_k) 1_{\tau > k} \\ &\leq C \frac{k^{\gamma-1}}{\log(k+1)^2} \text{dist}(x_k, \mathcal{M}) 1_{\tau > k} + C \frac{c_2}{k \log(k+1)^2}, \end{aligned}$$

where the first inequality follows from (3.8.10) and the second inequality follows by definition of  $\alpha_k$ . By Part 1 of Proposition 3.3.1, it follows that we have upper bounded  $\mathbb{E}_k[F_k]$  by a summable sequence. Therefore, it follows that  $F_k$  is summable, as desired. This proves part 2(b)i.

Now we prove part 2(b)ii. The conditional expectation is summable by Part 2(a)iii, since

$$\sum_{k=k_0}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \mathbb{E}[\|E_k\|^2 1_{\tau > k}] \leq C \sum_{k=k_0}^{\infty} \frac{k^{-1}}{\log(k+1)^2} < +\infty.$$

By conditional Borel-Cantelli theorem (Lemma 3.11.2), we also have that

$$\sum_{k=k_0}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \|E_k\|^2 1_{\tau > k} < +\infty,$$

as desired.

Now we prove Part 2c. To that end, note that there exists  $C > 0$  such that

$$\mathbb{E}[\alpha_k \|E_k\| 1_{\tau > k}] = \mathbb{E}[\alpha_k \mathbb{E}_k[\|E_k\| 1_{\tau > k}]] \leq C \mathbb{E}[\alpha_k \text{dist}(x_k, \mathcal{M}) 1_{\tau > k} + \alpha_k^2 1_{\tau > k}].$$

where the inequality from (3.8.10). Thus, the result follows by Part 2b of Proposition 3.3.1.

Finally, we prove Part 2d. To that end, Part 3 of Proposition 3.3.1 ensures that for all  $k \geq k_0$ , we have

$$\text{dist}(x_k, \mathcal{M}) 1_{\tau > k} \leq C k^{-\gamma},$$

for some  $C > 0$ . Therefore, since  $\|\nu_k\| \leq \mu/2$ , it follows from 2(a)i that there exists  $C' > 0$  with

$$\|E_k\| 1_{\tau > k} \leq C' k^{-\gamma},$$

as desired.

## 3.9 Proofs of the main theorems

In this section, we prove the remaining theorems stated above: local rates of convergence, asymptotic normality, and saddle avoidance. We now outline common notation and conventions used in all of the sections. Common to all proofs below is that Assumption A holds at

a point  $\bar{x}$  with associated neighborhood  $\mathcal{U}$ . When this assumption holds, there exists  $\epsilon > 0$  such that the function  $f_{\mathcal{M}}: B_{2\epsilon}(\bar{x}) \rightarrow \mathbb{R}$ , defined as the composition

$$f_{\mathcal{M}} := f \circ P_{\mathcal{M}} \quad (3.9.1)$$

is  $C^2$  and satisfies

$$\nabla f_{\mathcal{M}}(x) = \nabla_{\mathcal{M}} f(x) \quad \text{and} \quad \nabla^2 f_{\mathcal{M}}(x) = \nabla_{\mathcal{M}}^2 f(x)$$

for all  $x \in B_{2\epsilon}(\bar{x}) \cap \mathcal{M}$ . Moreover, we may also assume that the projection map  $P_{\mathcal{M}}: B_{2\epsilon}(\bar{x}) \rightarrow \mathbb{R}^d$  is  $C^2$ , in particular, Lipschitz with Lipschitz Jacobian. Throughout the proofs, we assume that  $\delta \leq \epsilon/4$  is small enough that conclusions of Propositions 3.3.1 and 3.3.2 are valid; we shrink  $\delta$  several further times throughout the proofs. In addition, we let  $C$  denote a constant depending on  $k_0$  and  $\delta$ , which may change from line to line.

Now, denote stopping time (3.3.1) by  $\tau := \tau_{k_0, \delta}$  and the noise bound by  $Q := \sup_{x \in B_{\delta}(\bar{x})} q(x)$ . Observe that by Proposition 3.3.2, the shadow sequence  $y_k$  satisfies  $y_k \in B_{4\delta}(x_k) \cap \mathcal{M} \subseteq B_{\epsilon}(\bar{x}) \cap \mathcal{M}$  and recursion (3.3.2) holds. Due to the identity  $\nabla f_{\mathcal{M}}(y_k) = \nabla_{\mathcal{M}} f(y_k)$ , we use the equivalent recursion throughout:

$$y_{k+1} = y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \alpha_k P_{T_{\mathcal{M}}(y_k)}(\nu_k) + \alpha_k E_k.$$

In addition, defining

$$f^* := \inf_{x \in B_{\epsilon}(\bar{x})} f_{\mathcal{M}},$$

we have the bound  $f^* 1_{\tau > k} \leq f(y_k) 1_{\tau > k}$  for all  $k$ . We now turn to the proofs.

### 3.9.1 Proof of Theorem 3.4.1: general rates

We first note that the summability and claimed bounds on the distance immediately follow from Proposition 3.3.1. Thus, the remainder of the proof proves the desired summability properties of the gradient.

To that end, since  $\nabla f_{\mathcal{M}}$  is Lipschitz on  $B_{4\delta}(\bar{x})$ , there exists  $C > 0$  such that

$$\begin{aligned} & \mathbb{E}_k[(f_{\mathcal{M}}(y_{k+1}) - f^*) 1_{\tau > k}] \\ & \leq (f_{\mathcal{M}}(y_k) - f^*) 1_{\tau > k} + \langle \nabla f_{\mathcal{M}}(y_k), \mathbb{E}_k[y_{k+1} - y_k] \rangle 1_{\tau > k} + C \mathbb{E}_k \|y_{k+1} - y_k\|^2 1_{\tau > k} \\ & \leq (f_{\mathcal{M}}(y_k) - f^*) 1_{\tau > k} - \alpha_k \|\nabla f_{\mathcal{M}}(y_k)\|^2 1_{\tau > k} \\ & \quad + \underbrace{\alpha_k \mathbb{E}_k[\langle \nabla f_{\mathcal{M}}(y_k), E_k \rangle] 1_{\tau > k}}_{=: Z_{k,1}} + \underbrace{C \mathbb{E}_k \|y_{k+1} - y_k\|^2 1_{\tau > k}}_{=: Z_{k,2}}. \end{aligned} \quad (3.9.2)$$

We now prove that  $Z_{k,1}$  and  $Z_{k,2}$  are summable.

Let us begin with  $Z_{k,1}$ . To that end, we first observe that  $\alpha_k \leq C \frac{k^{\gamma-1}}{\log(k+1)^2}$  for some  $C > 0$ . Consequently, Parts 2(b)i and 2d of Proposition 3.3.2 both imply the sequence  $\alpha_k \mathbb{E}_k \|E_k\| 1_{\tau > k}$  is summable. Thus, since  $\|\nabla f_{\mathcal{M}}(y)\|$  is bounded in  $B_{4\delta}(\bar{x})$  by continuity, there exists  $C > 0$  such that

$$|Z_{k,1}| \leq C \alpha_k \mathbb{E}_k[\|E_k\|] 1_{\tau > k}$$

Thus,  $Z_{k,1}$  is summable.



We now turn to  $Z_{k,2}$ . To that end, we first note that Parts 2(b)i and 2d of Proposition 3.3.2 both imply that the sequence  $\alpha_k^2 \mathbb{E}_k[\|E_k\|^2] 1_{\tau > k}$  is summable. Moreover, there exists  $C > 0$  such that

$$|Z_{k,2}| \leq \mathbb{E}_k(3\alpha_k^2 \|\nabla f_{\mathcal{M}}(y_k)\|^2 + 3\alpha_k^2 \|E_k\|^2 + 3\alpha_k^2 \|\nu_k\|^2) 1_{\tau > k} \leq C(1 + \mathbb{E}_k[\|E_k\|^2] 1_{\tau > k} + Q)\alpha_k^2.$$

Therefore it follows that  $Z_{k,2}$  is also summable as desired.

Now define nonnegative  $\mathcal{F}_k$  adapted random variables  $X_k := (f_{\mathcal{M}}(y_k) - f^*) 1_{\tau > k}$ ,  $Y_k := \alpha_k \|\nabla f_{\mathcal{M}}(y_k)\|^2$ , and  $Z_k := Z_{k,1} + Z_{k,2}$ . It follows from (3.9.2) and the bound  $1_{\tau > k+1} \leq 1_{\tau > k}$ , that

$$\mathbb{E}_k[X_{k+1}] \leq X_k - Y_k + Z_k.$$

Thus, by Robbins-Siegmund Lemma (Lemma 3.11.1), we find that  $Y_k$  is almost surely summable, as claimed.

### 3.9.2 Proof of Theorem 3.4.2: rates near strong local minimizers

We first begin with a Lemma ensuring  $\nabla^2 f_{\mathcal{M}}(x)$  has sufficient curvature in  $B_{2\delta}(\bar{x})$ .

**Lemma 3.9.1.** *For all sufficiently small  $\delta$  and all  $x \in B_{2\delta}(\bar{x})$ , the following hold. First, the Hessian satisfies:*

$$(y - \bar{x})^\top \nabla^2 f_{\mathcal{M}}(x)(y - \bar{x}) \geq \frac{\sigma}{4} \|y - \bar{x}\|^2, \quad \text{for all } x \in B_{2\delta}(\bar{x}), y \in B_{4\delta}(\bar{x}) \cap \mathcal{M}.$$

*Second, the negative gradient points towards  $\bar{x}$ :*

$$\langle \nabla f_{\mathcal{M}}(y), y - \bar{x} \rangle \geq \frac{\sigma}{4} \|y - \bar{x}\|^2 \quad \text{for all } y \in B_{4\delta}(\bar{x}) \cap \mathcal{M}.$$

*Proof.* Observe that by continuity of  $\nabla^2 f_{\mathcal{M}}(x)$  near  $\bar{x}$ , the following holds for all small  $\delta$ :

$$u^\top \nabla^2 f_{\mathcal{M}}(x) u \geq \sigma/2, \quad \text{for all } u \in \mathbb{S}^{d-1} \cap T_{\mathcal{M}}(\bar{x}) \text{ and } x \in B_{2\delta}(\bar{x}).$$

In addition, for all  $y \in \mathcal{M}$  near  $\bar{x}$ , there exists some constant  $C$  that

$$\|y - \bar{x} - P_{T_{\mathcal{M}}(\bar{x})}(y - \bar{x})\| \leq C \|y - \bar{x}\|^2.$$

Hence, when  $y \neq \bar{x}$ ,  $\frac{y - \bar{x}}{\|y - \bar{x}\|}$  is in the tangent space up to error at the order of  $\|y - \bar{x}\|$ , which yields the second statement of the theorem, possibly after shrinking  $\delta$ . To complete the proof, note that by Newton-Leibniz rule,

$$\begin{aligned} \langle \nabla f_{\mathcal{M}}(y), y - \bar{x} \rangle &= \langle \nabla f_{\mathcal{M}}(y) - \nabla f_{\mathcal{M}}(\bar{x}), y - \bar{x} \rangle \\ &= \int_0^1 (y - \bar{x})^\top \nabla^2 f_{\mathcal{M}}(\bar{x} + t(y - \bar{x}))(y - \bar{x}) dt \\ &\geq \frac{\sigma}{4} \|y - \bar{x}\|^2, \end{aligned}$$

as desired. □

Turning to the proof, we begin a preliminary bound, which will also be used in our proof of asymptotic normality.

**Lemma 3.9.2.** *For all sufficiently small  $\delta$ , there exists a constant  $C$  such that for any  $k \geq k_0$ , we have*

$$\begin{aligned} \mathbb{E}_k[\|y_{k+1} - \bar{x}\|^2 1_{\tau > k}] &\leq (1 - \alpha_k \sigma / 2) \|y_k - \bar{x}\|^2 1_{\tau > k} + C\alpha_k^2 + \alpha_k^2 \mathbb{E}_k[\|E_k\|^2 1_{\tau > k}] \\ &\quad + 2\alpha_k \mathbb{E}_k[\langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x}, E_k \rangle 1_{\tau > k}]. \end{aligned} \quad (3.9.3)$$

*Proof.* Expanding  $\|y_{k+1} - \bar{x}\|^2$ , we obtain

$$\begin{aligned} &\|y_{k+1} - \bar{x}\|^2 1_{\tau > k} \\ &= \|y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \alpha_k P_{T_{\mathcal{M}}(y_k)}(\nu_k) + \alpha_k E_k - \bar{x}\|^2 1_{\tau > k} \\ &= \|y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) + \alpha_k E_k - \bar{x}\|^2 1_{\tau > k} + \alpha_k^2 \|P_{T_{\mathcal{M}}(y_k)}(\nu_k)\|^2 1_{\tau > k} \\ &\quad + 2\alpha_k \langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) + \alpha_k E_k - \bar{x}, P_{T_{\mathcal{M}}(y_k)}(\nu_k) \rangle 1_{\tau > k} \\ &= \|y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x}\|^2 1_{\tau > k} + \alpha_k^2 \|E_k\|^2 1_{\tau > k} + 2\alpha_k \langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x}, E_k \rangle 1_{\tau > k} \\ &\quad + \alpha_k^2 \|P_{T_{\mathcal{M}}(y_k)}(\nu_k)\|^2 1_{\tau > k} + 2\alpha_k \langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) + \alpha_k E_k - \bar{x}, P_{T_{\mathcal{M}}(y_k)}(\nu_k) \rangle 1_{\tau > k}. \end{aligned} \quad (3.9.4)$$

In addition, Lemma 3.9.1, ensures that for some  $C > 0$ , we have

$$\begin{aligned} \|y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x}\|^2 1_{\tau > k} &= (\|y_k - \bar{x}\|^2 - 2\alpha_k \langle \nabla f_{\mathcal{M}}(y_k), y_k - \bar{x} \rangle + \alpha_k^2 \|\nabla f_{\mathcal{M}}(y_k)\|^2) 1_{\tau > k} \\ &\leq ((1 - \alpha_k \sigma / 2) \|y_k - \bar{x}\|^2 + C\alpha_k^2) 1_{\tau > k} \end{aligned}$$

To complete the proof, plug the above bound into (3.9.4), take the conditional expectation of both sides, recall that  $\mathbb{E}_k[\|P_{T_{\mathcal{M}}(y_k)}(\nu_k)\|^2] \leq Q^{1/2}$ , and apply Cauchy-Schwarz to the dot product:

$$\begin{aligned} 2\alpha_k \mathbb{E}_k[\langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) + \alpha_k E_k - \bar{x}, P_{T_{\mathcal{M}}(y_k)}(\nu_k) \rangle 1_{\tau > k}] &= 2\alpha_k^2 \mathbb{E}_k[\langle E_k, P_{T_{\mathcal{M}}(y_k)}(\nu_k) \rangle] \\ &\leq \alpha_k^2 \mathbb{E}_k[\|E_k\|^2 1_{\tau > k}] + Q^{1/2} \alpha_k^2, \end{aligned}$$

as desired.  $\square$

We now prove Part 1. To that end, Define  $X_k := \|y_k - \bar{x}\|^2 1_{\tau > k}$  and  $Y_k = \frac{\alpha_k \sigma}{4} X_k$ . Observe that  $C > 0$ , satisfying  $C > \|(y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x})\| 1_{\tau > k}$  for all  $k \geq k_0$ . Thus, by this bound and Lemma 3.9.2, there exists  $C > 0$  such that

$$\begin{aligned} \mathbb{E}_k[X_{k+1}] &\leq (1 - \sigma \alpha_k / 4) X_k - Y_k \\ &\quad + C\alpha_k^2 + \alpha_k^2 \mathbb{E}_k[\|E_k\|^2 1_{\tau > k}] + 2\alpha_k \mathbb{E}_k[\langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x}, E_k \rangle 1_{\tau > k}]. \\ &\leq (1 - \sigma \alpha_k / 4) X_k - Y_k + C\alpha_k^2 + \alpha_k^2 \mathbb{E}_k[\|E_k\|^2 1_{\tau > k}] + C\alpha_k \mathbb{E}_k[\|E_k\| 1_{\tau > k}], \end{aligned}$$

where the second inequality follows from Cauchy-Schwarz. Now define

$$Z_k := C\alpha_k^2 + \alpha_k^2 \mathbb{E}_k[\|E_k\|^2 1_{\tau > k}] + C\alpha_k \mathbb{E}_k[\|E_k\| 1_{\tau > k}]$$

and note that by Proposition 3.3.2, we have

$$\sum_{k=1}^{\infty} \frac{(k+1)^{2\gamma-1}}{\log(k+2)^2} Z_k < +\infty.$$

Moreover, we have

$$\mathbb{E}_k X_{k+1} \leq \left(1 - \frac{\sigma c_1}{4k^\gamma}\right) X_k - Y_k + Z_k.$$

Applying Lemma 3.11.6, we therefore deduce that there exists a finite random variable  $V_y$

$$\frac{k^{2\gamma-1}}{\log^2(k+1)} \|y_k - \bar{x}\|^2 1_{\tau > k} \xrightarrow{\text{a.s.}} V_y \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \|y_k - \bar{x}\|^2 1_{\tau > k} < +\infty \quad (3.9.5)$$

Similarly by Proposition 3.3.1 there exists a finite random variable  $V_x$  such that

$$\frac{k^{2\gamma-1}}{\log^2(k+1)} \text{dist}^2(x_k, \mathcal{M}) 1_{\tau > k} \xrightarrow{\text{a.s.}} V_x \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{k^{\gamma-1}}{\log(k+1)^2} \text{dist}^2(x_k, \mathcal{M}) 1_{\tau > k} < +\infty \quad (3.9.6)$$

To complete the proof, note the decomposition

$$\|x_k - \bar{x}\|^2 \leq 2\text{dist}^2(x_k, \mathcal{M}) + 2\|y_k - \bar{x}\|^2$$

and apply simply apply (3.9.5) and (3.9.6). This completes the proof of Part 1

Now we prove Part 2: Applying Proposition 3.3.2 to the case when  $\nu_k = 0$ , we find that  $y_k$  satisfies

$$y_{k+1} = y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) + \alpha_k E_k \quad \text{for } k \geq k_0,$$

where the error sequence  $\{E_k\}$  satisfies

$$\|E_k\| 1_{\tau > k} \leq Ck^{-\gamma} \quad \text{for } k \geq k_0.$$

By Newton-Leibniz rule and Lemma 3.9.1, we obtain in the event  $\{\tau > k\}$ , that

$$\begin{aligned} \|y_{k+1} - \bar{x}\| &= \|(y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k)) - (\bar{x} - \alpha_k \nabla f_{\mathcal{M}}(\bar{x})) + \alpha_k E_k\| \\ &\leq \left\| \int_0^1 (I - \alpha_k \nabla^2 f_{\mathcal{M}}(\bar{x} + t(y_k - \bar{x}))) (y_k - \bar{x}) dt \right\| + \alpha_k \|E_k\| \\ &\leq \left(1 - \frac{\alpha_k \sigma}{4}\right) \|y_k - \bar{x}\| + C\alpha_k k^{-\gamma} \\ &\leq \left(1 - \frac{c_1 \sigma}{4} k^{-\gamma}\right) \|y_k - \bar{x}\| + Ck^{-2\gamma} \end{aligned}$$

Applying Lemma 3.11.8 to the sequence  $s_k = \|y_k - \bar{x}\| 1_{\tau > k}$ , it follows that there exists a constant  $C$  such that

$$\|y_k - \bar{x}\| 1_{\tau > k} \leq Ck^{-\gamma} \quad \text{for } k \geq k_0.$$

The result therefore follows by (i) bounding  $\|x_k - \bar{x}\| \leq \text{dist}(x_k, \mathcal{M}) + \|y_k - \bar{x}\|$  and (ii) applying Proposition 3.3.1, which ensures that for some  $C > 0$ , we have  $\text{dist}(x_k, \mathcal{M}) \leq Ck^{-\gamma}$  for all  $k \geq k_0$ .

### 3.9.3 Proof of Theorem 3.5.1: asymptotic normality

Throughout we adopt the same conventions that were outlined at the start of Section 3.9, except that we write  $\tau_{k_0} = \tau_{k_0, \delta}$ , since we will consider several values of  $k_0$ . Under these conventions, we have the following Proposition, which will be useful in ensuring summability of certain sequences.

**Lemma 3.9.3.** *Under the condition of Theorem 3.5.1, there exists  $C > 0$  such that*

1.  $\mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}] \leq C/k^\gamma$  for all  $k \geq 1$ .
2.  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|y_k - \bar{x}\|^2 < \infty$  almost surely.
3.  $\frac{1}{\sqrt{n}} \sum_{k=1}^n \|y_k - \bar{x}\|^2 \rightarrow 0$  almost surely.
4.  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|E_k\| < +\infty$ .

*Proof.* We first prove Part 1. Recall by Proposition 3.3.2 that there exists  $C > 0$  such that  $\mathbb{E}[\|E_k\|^2 1_{\tau_{k_0} > k}] \leq C\alpha_k$  for all  $k \geq 1$ . In addition, we may also assume by enlarging  $C$  that  $\|\nabla f(y_k)\| 1_{\tau_{k_0} > k} \leq C$  for all  $k \geq 1$ . Therefore, there exists  $C' > 0$  such that

$$\begin{aligned}
& \mathbb{E}[\langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x}, E_k \rangle 1_{\tau_{k_0} > k}] \\
& \leq C\alpha_k \mathbb{E}[\|E_k\|] + \left( \mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}] \right)^{\frac{1}{2}} \left( \mathbb{E}[\|E_k\|^2 1_{\tau_{k_0} > k}] \right)^{\frac{1}{2}} \\
& \leq C^{3/2} \alpha_k^{3/2} + C^{1/2} \left( \mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}] \right)^{\frac{1}{2}} \alpha_k^{1/2} \\
& \leq C^{3/2} \alpha_k^{3/2} + \frac{\sigma}{4} \mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}] + \frac{4C\alpha_k}{\sigma} \\
& \leq C' \alpha_k + \frac{\sigma}{4} \mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}], \tag{3.9.7}
\end{aligned}$$

where the first inequality follows by Cauchy-Schwarz and the third inequality follows by Young's inequality. Therefore, by inequality (3.9.3) of Lemma 3.9.2, we obtain there exists  $C'' > 0$  such that

$$\begin{aligned}
\mathbb{E}[\|y_{k+1} - \bar{x}\|^2 1_{\tau_{k_0} > k+1}] & \leq \mathbb{E}[\|y_{k+1} - \bar{x}\|^2 1_{\tau_{k_0} > k}] \\
& = \mathbb{E}[\mathbb{E}_k[\|y_{k+1} - \bar{x}\|^2 1_{\tau_{k_0} > k}]] \\
& \leq (1 - \alpha_k \sigma / 2) \mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}] + C\alpha_k^2 + \alpha_k^2 \mathbb{E}[\|E_k\|^2 1_{\tau_{k_0} > k}] \\
& \quad + 2\alpha_k \mathbb{E}[\langle y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \bar{x}, E_k \rangle 1_{\tau_{k_0} > k}] \\
& \leq (1 - \alpha_k \sigma / 4) \mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}] + C'' \alpha_k^2,
\end{aligned}$$

where the first inequality follows from Proposition 3.3.2 and the final inequality follows from (3.9.7). To complete the proof apply Lemma 3.11.8 to the sequence  $s_k = \mathbb{E}[\|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}]$ .

We now prove Part 2. By Part 1, we have

$$\mathbb{E} \left[ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k} \right] \leq \sum_{k=1}^{\infty} \frac{C}{k^{\gamma + \frac{1}{2}}} < \infty.$$

Therefore,  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}$  is finite almost surely. Since  $x_k \rightarrow \bar{x}$  almost surely, for almost every sample path, we can find a  $k_0$  such that  $\tau_{k_0} = \infty$ . Therefore,  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|y_k - \bar{x}\|^2$  is finite almost surely. Note that Part 3 now immediately follows from Kronecker lemma 3.11.5

Finally, we prove Part 4. By Proposition 3.3.2, we know that the error sequence  $E_k$  almost surely satisfies

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|E_k\| 1_{\tau_{k_0} > k} < +\infty.$$

Since  $x_k \rightarrow \bar{x}$  almost surely, for almost every sample path, we can find a  $k_0$  such that  $\tau_{k_0} = \infty$ . Therefore, almost surely we have  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|E_k\| < +\infty$ , as desired.  $\square$

Turning to the proof, we introduce an additional shadow sequence

$$z_k = P_{\bar{x} + T_{\mathcal{M}}(\bar{x})}(y_k). \quad (3.9.8)$$

Evidently, for all  $\delta$  sufficiently small,  $z_k$  closely approximates  $y_k$ . Indeed, due to the smoothness of  $\mathcal{M}$ , there exists  $C > 0$  such that

$$\|y_k - z_k\| 1_{\tau_{k_0} > k} \leq C \|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}. \quad (3.9.9)$$

Our first result states that it suffices to study the distribution of  $\frac{1}{\sqrt{n}} \sum_{k=1}^n (z_k - \bar{x})$ .

**Lemma 3.9.4.** *If  $\frac{1}{\sqrt{n}} \sum_{k=1}^n (z_k - \bar{x})$  converges in distribution to some distribution  $\mathcal{D}$ , then  $\frac{1}{\sqrt{n}} \sum_{k=1}^n (x_k - \bar{x})$  converges in distribution to  $\mathcal{D}$ .*

*Proof.* Note that

$$\left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n (x_k - \bar{x}) \right\| = \frac{1}{\sqrt{n}} \sum_{k=1}^n (z_k - \bar{x}) + \frac{1}{\sqrt{n}} \sum_{k=1}^n (x_k - y_k) + \frac{1}{\sqrt{n}} \sum_{k=1}^n (y_k - z_k).$$

By the [34, Exercise 3.2.13], the result will follow if the following two limits hold:

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n (x_k - y_k) \rightarrow 0 \quad \text{and} \quad \frac{1}{\sqrt{n}} \sum_{k=1}^n (y_k - z_k) \rightarrow 0.$$

almost surely. To that end, we recall that Proposition 3.3.1 guarantees that almost surely we have

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|x_k - y_k\| 1_{\tau_{k_0} > k} < +\infty$$

Since  $x_k \rightarrow \bar{x}$  almost surely, for almost every sample path, we can find a  $k_0$  such that  $\tau_{k_0} = \infty$ . Therefore, almost surely we have  $\sum_{k=1}^{\infty} \frac{\|x_k - y_k\|}{\sqrt{k}} < \infty$ . Applying Kronecker lemma 3.11.5, almost surely we have

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n \|x_k - y_k\| \rightarrow 0,$$

which implies  $\frac{1}{\sqrt{n}} \sum_{k=1}^n (x_k - y_k) \rightarrow 0$ . On the other hand, we have by Lemma 3.9.3 and inequality (3.9.9), that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|y_k - z_k\| 1_{\tau_{k_0} > k} \leq \sum_{k=1}^{\infty} \frac{C}{\sqrt{k}} \|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k} < +\infty$$

Again since for almost every sample path we may find  $k_0$  such that  $\tau_{k_0} = \infty$ , we have that  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|y_k - z_k\| < +\infty$ , as desired.  $\square$

In the remainder of the proof, we study the limit distribution of  $\frac{1}{\sqrt{n}} \sum_{k=1}^n (z_k - \bar{x})$ . In the following Lemma, notice that we state the covariance matrix in a different, equivalent form that is more convenient for computation.

**Lemma 3.9.5.** *Under the conditions of Theorem 3.5.1,  $\frac{1}{\sqrt{n}} \sum_{k=1}^n (z_k - \bar{x})$  converges in distribution to*

$$N(0, U(U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})U)^{-1} \Sigma (U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})U)^{-1} U^\top),$$

where that  $U$  is a matrix whose column vectors form an orthogonal basis of  $T_{\mathcal{M}}(\bar{x})$

*Proof.* Clearly,  $z_k = \bar{x} + UU^\top(y_k - \bar{x})$ . Moreover, multiplying both sides of (3.3.2) by  $U^\top$ , we have

$$\begin{aligned} U^\top(y_{k+1} - \bar{x}) &= U^\top(y_k - \bar{x}) - \alpha_k U^\top \nabla f_{\mathcal{M}}(y_k) - \alpha_k U^\top P_{T_{\mathcal{M}}(y_k)}(\nu_k) + \alpha_k U^\top E_k \\ &= U^\top(y_k - \bar{x}) - \alpha_k U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(y_k - \bar{x}) - \alpha_k (U^\top \nabla f_{\mathcal{M}}(y_k) - U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(y_k - \bar{x})) \\ &\quad - \alpha_k U^\top P_{T_{\mathcal{M}}(\bar{x})}(\nu_k) - \alpha_k (U^\top P_{T_{\mathcal{M}}(y_k)}(\nu_k) - U^\top P_{T_{\mathcal{M}}(\bar{x})}(\nu_k)) + \alpha_k U^\top E_k \\ &= U^\top(y_k - \bar{x}) - \alpha_k U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})UU^\top(y_k - \bar{x}) - \alpha_k U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(I - UU^\top)(y_k - \bar{x}) \\ &\quad - \alpha_k (U^\top \nabla f_{\mathcal{M}}(y_k) - U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(y_k - \bar{x})) - \alpha_k U^\top P_{T_{\mathcal{M}}(\bar{x})}(\nu_k) \\ &\quad - \alpha_k (U^\top P_{T_{\mathcal{M}}(y_k)}(\nu_k) - U^\top P_{T_{\mathcal{M}}(\bar{x})}(\nu_k)) + \alpha_k U^\top E_k \end{aligned}$$

Define  $\Delta_k = U^\top(y_k - \bar{x})$ ,  $H = U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})U$ ,  $\zeta_k = U^\top P_{T_{\mathcal{M}}(y_k)}(\nu_k) - U^\top P_{T_{\mathcal{M}}(\bar{x})}(\nu_k)$ , and

$$R(y) = U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(I - UU^\top)(y - \bar{x}) + U^\top \nabla f_{\mathcal{M}}(y_k) - U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(y - \bar{x}).$$

By our assumption,  $H$  is positive definite and  $U^\top P_{T_{\mathcal{M}}(\bar{x})}(\nu_k) = U^\top UU^\top \nu_k = U^\top \nu_k$ . Thus we can rewrite the update of  $\Delta_k$  as

$$\Delta_{k+1} = \Delta_k - \alpha_k H \Delta_k - \alpha_k U^\top \nu_k - \alpha_k (R(y_k) + \zeta_k - U^\top E_k).$$

In the remainder of the proof, we prove that  $\frac{1}{\sqrt{n}} \sum_{k=1}^n \Delta_k$  converges in distribution to  $N(0, (U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})U)^{-1} \Sigma (U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})U)^{-1})$ . This implies that result since  $\frac{1}{\sqrt{n}}(z_k - \bar{x}) = \frac{1}{\sqrt{n}} \sum_{k=1}^n U \Delta_k$ . We note that our proof closely mirrors [1, Theorem 2].

To prove this claim, define matrices

$$B_k^n = \alpha_k \sum_{i=k}^n \prod_{j=k+1}^i (I - \alpha_j H) \text{ and } A_k^n = B_k^n - H^{-1}.$$

Polyak and Juditsky [78, Lemma 2] show that  $\bar{\Delta}_n = \frac{1}{n} \sum_{k=1}^n \Delta_k$  satisfies the equality

$$\begin{aligned} \sqrt{n}\bar{\Delta}_n &= \frac{1}{\sqrt{n}} \sum_{k=1}^n H^{-1} U^\top \nu_k \\ &\quad + \frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k + \frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n [R(y_k) + \zeta_k - U^\top E_k] + O\left(\frac{1}{\sqrt{n}}\right) \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^n H^{-1} U^\top \nu_k^{(1)} \\ &\quad + \frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k^{(1)} + \frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n [R(y_k) + \zeta_k - U^\top E_k + \nu_k^{(2)}(x_k)] + O\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

where  $\sup_{k,n} \max\{\|B_k^n\|, \|A_k^n\|\} < +\infty$  and  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \|A_k^n\| = 0$ . Notice that by assumption, the sum  $\frac{1}{\sqrt{n}} \sum_{k=1}^n H^{-1} U^\top \nu_k^{(1)}$  converges in distribution to

$$N\left(0, (U^\top \nabla^2 f_{\mathcal{M}}(\bar{x}) U)^{-1} \Sigma (U^\top \nabla^2 f_{\mathcal{M}}(\bar{x}) U)^{-1}\right).$$

Thus the claim holds if we can show that the other sums in our expression for  $\sqrt{n}\bar{\Delta}_n$  converge to 0 almost surely. To complete the proof, we now prove these limits.

*Claim 4.* We have that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k^{(1)} \xrightarrow{\text{a.s.}} 0.$$

*Proof.* Observe that  $\|A_k^n\|$  is bounded, so

$$\begin{aligned} \mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k \right\|^2 1_{\tau_{k_0} > k} \right] &= \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \left\| A_k^n U^\top \nu_k^{(1)} 1_{\tau_{k_0} > k} \right\|^2 \right] \\ &\leq \frac{C}{n} \sum_{k=1}^n \|A_k^n\| \\ &\rightarrow 0. \end{aligned}$$

As a result,  $\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k 1_{\tau_{k_0} > k}$  is a  $L^2$ -bounded martingale. By [34, Theorem 4.4.6], we know that  $\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k 1_{\tau_{k_0} > k} \xrightarrow{L^2} 0$ . On the other hand, by [34, Theorem 4.2.11],  $\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k 1_{\tau_{k_0} > k}$  converges almost surely. Therefore, since for almost every sample path there exists  $k_0$  such that  $\tau_{k_0} = \infty$ , we have  $\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n U^\top \nu_k \xrightarrow{\text{a.s.}} 0$ , as desired.  $\square$

*Claim 5.* We have that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n U^\top R(y_k) \xrightarrow{\text{a.s.}} 0.$$

*Proof.* Recall that

$$R(y) = U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(I - UU^\top)(y - \bar{x}) + U^\top \nabla f_{\mathcal{M}}(y_k) - U^\top \nabla^2 f_{\mathcal{M}}(\bar{x})(y - \bar{x}).$$

By smoothness of  $\mathcal{M}$  and the fact that  $f_{\mathcal{M}}$  is  $C^2$ , for  $y$  near  $\bar{x}$ , we have  $\|R(y)\| = O(\|y - \bar{x}\|^2)$ . In addition, by our assumption that  $x_k \xrightarrow{\text{a.s.}} \bar{x}$ , we have  $y_k \xrightarrow{\text{a.s.}} \bar{x}$ . Consequently, there exists a constant  $C$  depending on sample path such that  $\|R(y_k)\| \leq C\|y_k - \bar{x}\|^2$  almost surely. By Lemma 3.9.3, we know that  $\frac{1}{\sqrt{n}}\|R(y_k)\| \xrightarrow{\text{a.s.}} 0$ . Therefore,  $\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n U^\top R(y_k) \xrightarrow{\text{a.s.}} 0$ .  $\square$

*Claim 6.* We have that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n \zeta_k \xrightarrow{\text{a.s.}} 0.$$

*Proof.* For  $k \geq 1$ , define truncated variables  $\zeta_k^{(k_0)} = \zeta_k 1_{\tau_{k_0} > k}$ . Note that suffices to show that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n \zeta_k^{(k_0)} \xrightarrow{\text{a.s.}} 0,$$

since on every sample path there exists a  $k_0$  such that  $\tau_{k_0} = \infty$ . Thus, we will work with these truncated variables throughout.

Turning to the proof, we first show that  $\frac{1}{\sqrt{n}} \sum_{k=1}^n \zeta_k^{(k_0)} \xrightarrow{P} 0$  and  $\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n \zeta_k^{(k_0)} \xrightarrow{P} 0$ . Recall that  $\zeta_k = U^\top P_{T_{\mathcal{M}}(y_k)}(\nu_k) - U^\top P_{T_{\mathcal{M}}(\bar{x})}(\nu_k)$ , so we have

$$\mathbb{E} \left[ \zeta_k^{(k_0)} \mid \mathcal{F}_k \right] = \mathbb{E} [\zeta_k \mid \mathcal{F}_k] 1_{\tau_{k_0} > k} = 0.$$

Since  $x \mapsto P_{T_{\mathcal{M}}(x)}$  is locally Lipschitz on a neighborhood of  $\bar{x}$  in  $\mathcal{M}$ , we have the following bound for some  $C > 0$  and all sufficiently small  $\delta$ :

$$\left\| \zeta_k^{(k_0)} \right\| \leq C \|y_k - \bar{x}\| 1_{\tau_{k_0} > k},$$

In particular, it holds that

$$\mathbb{E} \left[ \left\| \zeta_k^{(k_0)} \right\|^2 \mid \mathcal{F}_k \right] \leq C^2 \|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k}.$$

Combining with Lemma 3.9.3, we know that  $\zeta_k^{(k_0)}$  is a martingale difference sequence and almost surely,

$$\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{E} \left[ \left\| \zeta_k^{(k_0)} \right\|^2 \mid \mathcal{F}_k \right] \leq C^2 \sum_{k=1}^{\infty} \frac{1}{k} \|y_k - \bar{x}\|^2 < \infty.$$

Therefore, by Lemma 3.11.4, we have

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n \zeta_k^{(k_0)} \xrightarrow{\text{a.s.}} 0.$$

In particular, it holds that  $\frac{1}{\sqrt{n}} \sum_{k=1}^n \zeta_k^{(k_0)} \xrightarrow{P} 0$ .



Next we show that for any  $k_0 < \infty$ , we have  $n^{-1/2} \sum_{k=1}^n A_k^n \zeta_k^{(k_0)} \xrightarrow{P} 0$ . To see this, note that by Lemma 3.9.3, there exists  $C' > 0$  such that

$$\mathbb{E} \left[ \left\| \zeta_k^{(k_0)} \right\|^2 \right] \leq C' \mathbb{E} \left[ \|y_k - \bar{x}\|^2 1_{\tau_{k_0} > k} \right] \leq C' \alpha_k. \quad (3.9.10)$$

Hence, the following limit holds

$$\mathbb{E} \left[ \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n \zeta_k^{(k_0)} \right\|^2 \right] = \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \left\| A_k^n \zeta_k^{(k_0)} \right\|^2 \right] \leq \frac{C' \alpha_k}{n} \sum_{k=1}^n \|A_k^n\|^2 \leq \frac{C' \alpha_k \sup_{k,n} \|A_k^n\|}{n} \sum_{k=1}^n \|A_k^n\| \rightarrow 0,$$

where the first equality follows from the martingale difference property, the second inequality follows from the boundedness of moments of  $\zeta_k^{(k_0)}$ , and the limit holds since  $\|A_k^n\|$  is bounded. Consequently, we have shown that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n \zeta_k^{(k_0)} \xrightarrow{L^2} 0,$$

which implies that  $\frac{1}{\sqrt{n}} \sum_{k=1}^n A_k^n \zeta_k^{(k_0)} \xrightarrow{P} 0$ .

We have therefore proved that  $\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n \zeta_k^{(k_0)} \xrightarrow{P} 0$ . We now show that  $\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n \zeta_k^{(k_0)}$  converges almost surely. Since the almost sure limits and limits in probability agree when both exist, this will complete the proof.

To this end, define the sequence

$$Z_{n,k_0} = \sum_{k=1}^n B_k^n \zeta_k^{(k_0)}.$$

The result follows if we can prove that for any finite  $k_0$ , the sequence  $n^{-1/2} Z_{n,k_0}$  almost surely converges. To that end, note that  $B_k^{n+1} - B_k^n = \alpha_k \prod_{i=k+1}^n (I - \alpha_i H)$ . Thus, defining

$$W_k^n = \prod_{i=k}^n (I - \alpha_i H), \quad V_{n,k_0} = \sum_{k=1}^n \alpha_k W_{k+1}^{n+1} \zeta_k^{(k_0)},$$

we deduce that  $V_{n,k_0}$  is  $\mathcal{F}_{n+1}$  measurable and  $Z_{n,k_0}$  admits the decomposition:

$$Z_{n,k_0} = Z_{n-1,k_0} + V_{n-1,k_0} + \alpha_n \zeta_n^{(k_0)} = \sum_{k=1}^{n-1} V_{k,k_0} + \sum_{k=1}^n \alpha_k \zeta_k^{(k_0)}.$$

Note that the sum  $\sum_{k=1}^n \alpha_k \zeta_k^{(k_0)}$  is a square-integrable martingale with summable squared increments, so it converges almost surely [34, Theorem 4.2.11]. As a result, we have the following limit  $n^{-1/2} \sum_{k=1}^n \alpha_k \zeta_k^{(k_0)} \xrightarrow{\text{a.s.}} 0$ . It thus suffices to show that  $n^{-1/2} \sum_{k=1}^{n-1} V_{k,k_0}$  converges almost surely.

To that end, let  $\lambda$  denote the smallest eigenvalue of  $H$ . Then we have

$$\mathbb{E} [\|V_{n,k_0}\|^2] = \sum_{k=1}^n \alpha_k^2 \|W_{k+1}^{n+1}\|^2 \mathbb{E} \left[ \left\| \zeta_k^{(k_0)} \right\|^2 \right] \leq C' \sum_{k=1}^n \alpha_k^3 \prod_{i=k+1}^{n+1} |1 - \lambda \alpha_i|,$$

where the inequality follows from the bound  $\mathbb{E} \left[ \left\| \zeta_k^{(k_0)} \right\|^2 \right] \leq C' \alpha_k$  (see Equation (3.9.10)) By [1, Lemma A.7], there thus exists some constant  $C$  such that

$$\mathbb{E} [\|V_{n,k_0}\|^2] \leq \frac{C \log n}{n^{2\gamma}}.$$

Hence, for any  $\epsilon > 0$ , we can find some  $C$  such that

$$\mathbb{E} [\|V_{n,k_0}\|^2] \leq \frac{C}{n^{2\gamma-\epsilon}}.$$

Now define  $T_{n,k_0} = \frac{1}{\sqrt{n}} \sum_{k=1}^n V_{k,k_0}$ . We claim that  $T_{n,k_0}$  almost surely has finite length. Indeed, for any  $\epsilon > 0$  there exists  $C, C' > 0$  such that

$$\begin{aligned} \mathbb{E} [\|T_{n,k_0} - T_{n+1,k_0}\|] &\leq \left| \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} \right| \sum_{k=1}^n \mathbb{E} [\|V_{k,k_0}\|] + \frac{1}{\sqrt{n+1}} \mathbb{E} [\|V_{n+1,k_0}\|] \\ &\leq \frac{C}{n^{\frac{3}{2}}} \sum_{k=1}^n \frac{1}{k^{\gamma-\epsilon}} + \frac{1}{\sqrt{n}} \frac{1}{n^{\gamma-\epsilon}} \\ &\leq \frac{C'}{n^{\gamma+1/2-\epsilon}}. \end{aligned}$$

Since  $\gamma \in (\frac{1}{2}, 1)$ , we therefore have  $\sum_n \mathbb{E} [\|T_{n,k_0} - T_{n+1,k_0}\|] < \infty$ . Consequently, the sum is finite almost surely:  $\sum_n \|T_{n,k_0} - T_{n+1,k_0}\| < +\infty$ . This implies that  $T_{n,k_0} = n^{-1/2} \sum_{k=1}^n V_{k,k_0}$  converges almost surely. Recalling the definition of  $V_{k,k_0}$ , we find that  $n^{-1/2} Z_{n,k_0}$  almost surely converges, which completes the proof.  $\square$

*Claim 7.* We have that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n \nu_k^{(2)}(x_k) \xrightarrow{\text{a.s.}} 0.$$

*Proof.* This may be proved by argument that mirrors Claim 6. Indeed, observe that the sequence  $\xi_k = \nu_k^{(2)}(x_k) 1_{\tau > k_0}$  is a martingale difference sequence, the bounds hold for some  $C > 0$

$$\mathbb{E}_k [\|\xi_k\|^2] \leq C \|x_k - \bar{x}\|^2 1_{\tau > k_0} \quad \text{and} \quad \mathbb{E} [\|\xi_k\|^2] \leq C \alpha_k,$$

and  $\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{E}_k [\|\xi_k\|^2] \leq \sum_{k=1}^{\infty} \frac{1}{k} \|x_k - \bar{x}\|^2 1_{\tau > k_0} < +\infty$ . Only these facts for  $\zeta_k^{(k_0)}$  were used to prove Claim 6.  $\square$

*Claim 8.* We have that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n U^\top E_k \xrightarrow{\text{a.s.}} 0.$$

*Proof.* Recall that  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|E_k\| < \infty$  almost surely and that  $\sup_{k,n} \|B_k^n\| < \infty$ . Therefore, there exists  $C' > 0$  such that almost surely we have

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|B_k^n U^\top E_k\| \leq C' \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|E_k\| < \infty.$$

Therefore, by Kronecker lemma, we have

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n B_k^n U^\top E_k \xrightarrow{\text{a.s.}} 0,$$

as desired.  $\square$

Taking these claims into account, the proof is complete.  $\square$

### 3.9.4 Proof of Theorem 3.6.1: nonconvergence of stochastic process

We begin by recalling and slightly reframing Proposition 3 in [74]. This result provides a Lyapunov function, which we will use to show that each local process  $Y_k$  escapes a local neighborhood of each  $p \in S$ .

**Proposition 3.9.6** (Lyapunov Function). *Fix  $p \in \mathbb{R}^d$  and suppose  $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a  $C^2$  mapping that vanishes at  $p$  and has symmetric Jacobian  $\nabla F(p)$ . Suppose that  $\nabla F(p)$  has at least one positive eigenvalue and let  $W$  denote the subspace of eigenvectors of  $\nabla F(p)$  with positive eigenvalues. Then, there exists a matrix  $A \in \mathbb{R}^{d \times d}$  with  $\text{range}(A^\top) = W$ , a ball  $\mathcal{B}$  centered at  $p$ , and a  $C^2$  mapping  $\Phi: \mathcal{B} \rightarrow \mathbb{R}^d$  with  $\Phi(p) = p$  and  $\nabla \Phi(p) = I_d$  such that the weakly convex function  $\eta: \mathcal{B} \rightarrow \mathbb{R}$  defined as*

$$\eta(v) = \|A(\Phi(v) - p)\|_2$$

*satisfies the following condition: There exists  $c, c' > 0$  such that*

$$\eta(v + \epsilon F(v)) \geq (1 + c\epsilon)\eta(v) - c'\epsilon^2 \quad \text{for } v \text{ in } \mathcal{B} \text{ and all sufficiently small } \epsilon.$$

*In particular, we have*

$$\eta'(v; F(v)) \geq c\eta(v) \quad \text{for all } v \in \mathcal{B}.$$

Turning to the proof of Theorem 3.6.1, we begin with a covering argument: For any  $p \in S$ , choose  $\epsilon_p$  small enough that both the conditions of Theorem 3.6.1 and Proposition 3.9.6 hold in  $B_{\epsilon_p}(p)$ . Let  $\delta_p \leq \epsilon_p$  and  $c_3, c_4, c_5 > 0$  be the associated constants. Clearly, the union  $\cup_{p \in S} B_{\delta_p}(p)$  is an open cover of set  $S$ , so by second countability of  $\mathbb{R}^d$ , there exists a countable index set  $\Lambda$  such that  $S \subset \cup_{p \in \Lambda} B_{\delta_p}(p)$ . Therefore, to prove Theorem 3.6.1, it suffices to show that

$$P(X_k \in B_{\delta_p}(p), \forall k \geq k_0) = 0 \quad \text{for all } k_0 \geq K_p. \quad (3.9.11)$$

To this end, fix  $p \in \Lambda$  and  $k_0 \geq K_p$ . Let  $F = F_p$  denote the local mapping in Condition 1 of Theorem 3.6.1. In addition, let  $\eta = \eta_p$ , denote the mapping associated to  $F$ , guaranteed to exist by Theorem 3.9.6.<sup>1</sup> Furthermore, recall the stopping time  $\tau_{k_0} = \tau_{k_0, \delta_p}(p)$ , defined as  $\tau_{k_0, \delta_p}(p) = \inf\{k \geq k_0: X_k \notin B_{\delta_p}(p)\}$ . Note that (3.9.11) holds if  $P(\tau_{k_0} = \infty) = 0$ .

Our strategy is as follows. We first prove that on the event that on the event  $\{\tau_{k_0} = \infty\}$ , we have  $\eta(Y_k) \rightarrow 0$  almost surely. Then we show that  $P(\{\tau_{k_0} = \infty\} \cap \{\eta(Y_k) \rightarrow 0\}) = 0$ . This will imply that  $P(\{\tau_{k_0} = \infty\}) = 0$  and the proof will be complete. These two claims are subjects of the following two subsections.

---

<sup>1</sup>Note that strictly speaking we should extend  $F$  to all  $\mathbb{R}^d$ , for example, by a partition of unity [53, Lemma 2.26]. Since the argument that follows is local, we omit this discussion for simplicity.

**Claim:** On the event  $\{\tau_{k_0} = \infty\}$ , we have  $\eta(Y_k) \rightarrow 0$

To prove this claim, note that the following hold for almost all sample paths in the event  $\{\tau_{k_0} = \infty\}$ :

1. The sequence  $Y_k$  is bounded.
2. Define  $\beta_k = \sum_{i=0}^{k-1} \alpha_i$ . Then for each  $T > 0$ , the limit holds:

$$\lim_{n \rightarrow \infty} \left( \sup_{k: 0 \leq \beta_k - \beta_n \leq T} \left\| \sum_{i=n}^{k-1} \alpha_i \cdot (\xi_i + E_k) \right\| \right) = 0. \quad (3.9.12)$$

Indeed, note that by Condition 3b of the Theorem, it suffices to show  $M_k = \sum_{i=0}^k \alpha_i \xi_i$  converges almost surely, since then it is a Cauchy sequence. To prove that  $M_k$  converges, note that  $\sum_i \alpha_i^2 < \infty$  and  $\limsup \mathbb{E}[\|\xi_i\|^2 \mid \mathcal{F}_k] < \infty$ , so  $M_k$  is a martingale. Moreover,

$$\sup_{k \geq 0} \mathbb{E}[\|M_k\|^2] \leq \sup_{k \geq 0} \mathbb{E}[\|M_k\|^2] \leq c_4^2 \sum_{i \geq 0} \alpha_i^2 < \infty. \quad (3.9.13)$$

Standard martingale theory then shows that  $M_k$  converges almost surely (Theorem 4.2.11 in [34]). Therefore, (3.9.12) holds almost surely.

These conditions match those of [2, Theorem 1.2]. Consequently, by this result it holds that the set of limit points of  $Y_k$  is almost surely invariant under the mapping  $\Theta_t: B_{\epsilon_p/2}(p) \rightarrow \mathbb{R}^d$ , defined as the time- $t$  map of the ODE  $\dot{\gamma}(t) = F(\gamma(t))$ . Thus, for any  $x'$  in limit set of  $Y_k$ , we have  $\Theta_t(x') \in \overline{B_{\epsilon_p/2}(p)}$  for all  $t \geq 0$ . Consequently, by Proposition 3.9.6, we have

$$\eta'(\Theta_t(x'); F(\Theta_t(x'))) \geq c\eta(\Theta_t(x')) \quad \text{for all } t \geq 0. \quad (3.9.14)$$

Therefore, by integrating  $\eta'$  with respect to  $t$ , we have for all  $t \geq 0$ , the bound

$$\eta(\Theta_t(x')) = \eta(\Theta_0(x')) + \int_0^t \eta'(\Theta_s(x'); F(\Theta_s(x'))) ds \geq \eta(\Theta_0(x')) + \int_0^t c\eta(\Theta_s(x')) ds.$$

Thus, by Gronwall's inequality [40] it holds that

$$\eta(\Theta_t(x')) \geq e^{ct} \eta(\Theta_0(x')) = e^{ct} \eta(x') \quad \text{for all } t \geq 0.$$

Now observe that since  $\Theta_t(x') \in \overline{B_{\epsilon_p/2}(p)}$ , the quantity  $\eta(\Theta_t(x'))$  is bounded for all  $t \geq 0$ . Consequently, we must have  $\eta(x') = 0$ . Thus, we have shown that for all limits points  $x'$  of  $Y_k$ , we have  $\eta(x') = 0$ . Since  $\eta$  is continuous in  $\overline{B_{\epsilon_p/2}(p)}$ , we must therefore have  $\eta(Y_k) \rightarrow 0$ .

**Claim:** We have  $P(\{\tau_{k_0} = \infty\} \cap \{\eta(Y_k) \rightarrow 0\}) = 0$ .

We begin by stating the following straightforward extension of [12, Theorem 4.1].

**Lemma 3.9.1.** *Let  $\{\zeta_k\}_k$  be a nonnegative sequence of random variables adapted to a filtration  $\{\mathcal{F}_k\}$  satisfying the following recurrence almost surely on an  $\mathcal{F}_\infty$ -measurable set  $\Omega_0$ :*

$$\zeta_{k+1} \geq \zeta_k + \alpha_k(e_{k+1} + r_{k+1} + \hat{r}_{k+1}) \quad \text{for all } k \geq k_0.$$

*where  $\{\alpha_k\}$  is a square-summable, but not summable sequence. Assume that  $\{e_k\}_k$ ,  $\{r_k\}$ , and  $\{\hat{r}_k\}_k$  are  $\mathcal{F}_k$  measurable and satisfy*

$$\begin{aligned} \mathbb{E}[e_{k+1} \mid \mathcal{F}_k] &= 0; & \sum_{k=1}^{\infty} r_k^2 &< +\infty \\ \limsup_k \mathbb{E}[e_{k+1}^2 \mid \mathcal{F}_k] &< \infty & \liminf_k \mathbb{E}[|e_{k+1}| \mid \mathcal{F}_k] &> 0, \end{aligned}$$

*almost surely on  $\Omega_0$ . Assume that for  $n \geq k_0$ , we have*

$$\mathbb{E} \left[ 1_{\Omega_0} \sum_{k=n}^{\infty} \alpha_k |\hat{r}_{k+1}| \right] = O \left( \sum_{k=n}^{\infty} \alpha_k^2 \right).$$

*Then we have  $P(\Omega_0 \cap \{\zeta_k \rightarrow 0\}) = 0$ .*

*Proof.* Without loss of generality we may assume  $k_0 = 0$ . Following [12, Theorem 4.1] (itself based on [13, Page 401]) it suffices to work in the case where there exist fixed constants  $\mu$  and  $C > 0$  such that almost surely on the whole probability space, we have

$$\begin{aligned} \mathbb{E}[e_{k+1} \mid \mathcal{F}_k] &= 0 \quad \text{and} \quad \limsup_k \mathbb{E}[e_{k+1}^2 \mid \mathcal{F}_k] < C \\ \liminf_k \mathbb{E}[|e_{k+1}| \mid \mathcal{F}_k] &> \mu > 0 \quad \text{and} \quad \sum_{k=1}^{\infty} r_k^2 < C. \end{aligned}$$

Now define the nonnegative residual sequence:

$$\alpha_k U_{k+1} = \zeta_{k+1} - \zeta_k - \alpha_k(e_{k+1} + r_{k+1} + \hat{r}_{k+1})$$

Notice that for all  $k \geq 0$ , we have

$$\zeta_k = \left[ \zeta_0 + \sum_{j=0}^k \alpha_j (e_{j+1} + r_{j+1} + \hat{r}_{j+1} + U_{j+1}) \right] \quad \text{on } G := \Omega_0 \cap \{\zeta_k \rightarrow 0\}.$$

Therefore, on  $G$ , we have

$$-\zeta_0 = \left[ \sum_{j=0}^{\infty} \alpha_j (e_{j+1} + r_{j+1} + \hat{r}_{j+1} + U_{j+1}) \right].$$

Then as argued the proof of [12, Theorem 4.1] it suffices by Theorem A of [13] (included as Lemma 3.11.3 in the Appendix) to show that

$$\mathbb{E} \left[ 1_G \sum_{k=n}^{\infty} \alpha_k |U_{k+1} + \hat{r}_{j+1}| \right] = o \left( \left( \sum_{k=n}^{\infty} \alpha_k^2 \right)^{1/2} \right),$$

Clearly, it suffices to bound the series  $\mathbb{E} \left[ 1_G \sum_{j=K}^{\infty} \alpha_j U_{j+1} \right]$  (which consists of nonnegative terms), since by assumption, we have

$$\mathbb{E} \left[ 1_G \sum_{k=n}^{\infty} \alpha_k |\hat{r}_{j+1}| \right] = O \left( \sum_{j=n}^{\infty} \alpha_j^2 \right) = o \left( \sum_{j=n}^{\infty} \alpha_j^2 \right)^{1/2}.$$

To that end, note that for all  $k, n \geq 0$ , we have

$$\zeta_{n+k} = \left[ \zeta_n + \sum_{j=n}^{n+k} \alpha_j (e_{j+1} + r_{j+1} + \hat{r}_{j+1} + U_{j+1}) \right]$$

Hence on  $G$ , we may let  $k$  tend to infinity, yielding:

$$-\zeta_n = \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1} + \hat{r}_{j+1} + U_{j+1}).$$

Thus, on the event  $G$ , we have

$$\sum_{j=n}^{\infty} \alpha_j U_{j+1} = -\zeta_n - \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1} + \hat{r}_{j+1})$$

Therefore, we find that

$$\begin{aligned} \mathbb{E} \left[ 1_G \sum_{j=n}^{\infty} \alpha_j U_{j+1} \right] &\leq -\zeta_n - \mathbb{E} \left[ 1_G \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1} + \hat{r}_{j+1}) \right] \\ &\leq \left| \mathbb{E} \left[ 1_G \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1}) \right] \right| + o \left( \sum_{j=n}^{\infty} \alpha_j^2 \right)^{1/2}. \end{aligned}$$

where the second inequality follows from nonnegativity of  $\zeta_n$  and our assumptions on  $\hat{r}_{j+1}$ . Thus, to complete the bound of  $\mathbb{E}[1_G \sum_{j=K}^{\infty} \alpha_j U_{j+1}]$  we must show that

$$\left| \mathbb{E} \left[ 1_G \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1}) \right] \right| = o \left( \sum_{j=n}^{\infty} \alpha_j^2 \right)^{1/2}.$$

The above bound follows by the exact same argument as [12, Theorem 4.1], which we reproduce for completeness: First let  $G_n = \mathbb{E}[1_G \mid \mathcal{F}_n]$ , recall that  $G$  is  $\mathcal{F}_{\infty}$  measurable and that  $G_n$  converges to  $1_G$  almost surely in  $L^p$  for every  $p \geq 1$ , e.g.,  $\mathbb{E}[(G_n - 1_G)^2] \rightarrow 0$ . Turning

to the bound, we have

$$\begin{aligned}
& \left| \mathbb{E} \left[ 1_G \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1}) \right] \right| \\
& \left| \mathbb{E} \left[ (1_G - G_n) \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1}) \right] \right| + \left| \mathbb{E} \left[ G_n \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1}) \right] \right| \\
& \leq \mathbb{E}[(1_G - G_n)^2]^{1/2} \underbrace{\left( \mathbb{E} \left[ \left( \sum_{j=n}^{\infty} \alpha_j (e_{j+1} + r_{j+1}) \right)^2 \right] \right)^{1/2}}_{=:R_1} + \underbrace{\mathbb{E} \left[ \sum_{j=n}^{\infty} \alpha_j |r_{j+1}| \right]}_{=:R_2}.
\end{aligned}$$

The proof will be complete if  $R_1 = O\left(\sum_{j=n}^{\infty} \alpha_j^2\right)^{1/2}$  and  $R_2 = o\left(\sum_{j=n}^{\infty} \alpha_j^2\right)^{1/2}$ . Let us first bound  $R_2$ :

$$R_2 \leq \left( \sum_{j=n}^{\infty} \alpha_j^2 \right)^{1/2} \mathbb{E} \left[ \sum_{j=n}^{\infty} r_{j+1}^2 \right]^{1/2} = o \left( \left( \sum_{j=n}^{\infty} \alpha_j^2 \right)^{1/2} \right),$$

where the last inequality follows from the bound  $\sum_{k=1}^{\infty} r_{k+1}^2 < C$ . Now we bound  $R_1$ :

$$\begin{aligned}
R_1 & \leq \left( \mathbb{E} \left[ \left( \sum_{j=n}^{\infty} \alpha_j e_{j+1} \right)^2 \right] \right)^{1/2} + \left( \mathbb{E} \left[ \left( \sum_{j=n}^{\infty} \alpha_j r_{j+1} \right)^2 \right] \right)^{1/2} \\
& \leq \left( \mathbb{E} \left[ \sum_{j=n}^{\infty} \alpha_j^2 \mathbb{E}[e_{j+1}^2 \mid \mathcal{F}_k] \right] \right)^{1/2} + \left( \sum_{j=n}^{\infty} \alpha_j^2 \right)^{1/2} \left( \mathbb{E} \left[ \sum_{j=n}^{\infty} r_{j+1}^2 \right] \right)^{1/2} = O \left( \sum_{j=n}^{\infty} \alpha_j^2 \right)^{1/2}.
\end{aligned}$$

Therefore, the proof is complete.  $\square$

Now we apply the above Lemma. To that end, we state a few simplifications and facts to be used below. First, throughout the proof, we let  $C$  be a positive constant that changes from line to line. Second, we simplify notation and let  $\tau$  denote  $\tau_{k_0, \delta}$ . Third, we recall the bound  $\frac{c_1}{k^\gamma} \leq \alpha_k \leq \frac{c_2}{k^\gamma}$ . Fourth, the function  $\eta$  is weakly convex and Lipschitz continuous on  $B_{\epsilon_p}(p)$ . Fifth, the Jacobian  $\nabla \Phi$  is  $\text{Lip}_{\nabla \Phi}$ -Lipschitz in  $B_{\epsilon_p}(p)$ . Sixth, we note that for sufficiently large  $k$ , we have the following on  $\{\tau = \infty\}$ :  $Y_k + \alpha_k F(Y_k) \in B_{\epsilon_p}(p)$ . We may assume without loss of generality that these assertions hold for all  $k \geq 1$ . Finally, we note that by shrinking  $\epsilon_p$ , if necessary, we can assume that on the event  $\{\tau = \infty\}$ , we have

$$\begin{aligned}
s_{\min}(A) \liminf_k \inf_{w \in W \cap \mathbb{S}^{d-1}} \mathbb{E}[\langle w, \xi_k \rangle \mid \mathcal{F}_k] - \epsilon_p \limsup_k \mathbb{E}[\|\xi_k\| \mid \mathcal{F}_k] \|A\| \text{Lip}_{\nabla \Phi} \\
\geq c_4 s_{\min}(A) - \epsilon_p c_3^{1/4} \|A\| \text{Lip}_{\nabla \Phi} > 0
\end{aligned} \tag{3.9.15}$$

where  $c_4$  and  $c_3$  are independent of  $\epsilon_p$  and  $\delta_p$ ,  $A$  is defined in Proposition 3.9.6, and  $s_{\min}(A)$  denotes the minimal nonzero singular value of  $A$ .

Now let  $s: B_{\epsilon_p}(p) \rightarrow \mathbb{R}^d$  be a selection of  $\partial \eta$  defined as follows: for all  $y \in B_{\epsilon_p}(p)$ ,

- If  $\eta(y) \neq 0$ , then  $\eta$  is differentiable at  $Y$ , so set  $s(y) = \nabla\eta(y)$ .
- If  $\eta(y) = 0$ , then  $\eta$  is nondifferentiable, so we choose subgradient

$$s(Y) = \nabla\Phi(y)^\top A^\top u \in \partial\eta(y)$$

where  $u \in \mathbb{S}^{d-1}$  satisfies  $\|A^\top u\| = \|A\| > 0$ .

Next, consider the event  $\Omega_0 = \{\tau = \infty\}$ . Then by the boundedness of  $s(Y_k + \alpha_k F(Y_k))$  and the weak convexity of  $\eta$  on  $B_{\epsilon_p}(p)$ , there exists  $C > 0$  such that

$$\begin{aligned} \eta(Y_{k+1}) &\geq \eta(Y_k + \alpha_k F(Y_k)) + \langle s(Y_k + \alpha_k F(Y_k)), \alpha_k E_k + \alpha_k \xi_k \rangle - C\|\alpha_k E_k + \alpha_k \xi_k\|^2 \\ &\geq \eta(Y_k + \alpha_k F(Y_k)) + \langle s(Y_k + \alpha_k F(Y_k)), \alpha_k \xi_k \rangle - C\|\alpha_k E_k + \alpha_k \xi_k\|^2 - C\alpha_k \|E_k\| \\ &\geq (1 + c\alpha_k)\eta(Y_k) + \langle s(Y_k + \alpha_k F(Y_k)), \alpha_k \xi_k \rangle - C\|\alpha_k E_k + \alpha_k \xi_k\|^2 - C\alpha_k \|E_k\| - C\alpha_k^2. \end{aligned} \quad (3.9.16)$$

Now define four sequences:

$$\zeta_k := \eta(Y_k); \quad e_{k+1} := \langle s(Y_k + \alpha_k F(Y_k)), \xi_k \rangle; \quad r_{k+1} := -C\alpha_k (1 + \|E_k + \xi_k\|^2); \quad \hat{r}_{k+1} := -C\|E_k\|$$

and observe that on  $\Omega_0$ , we have

$$\zeta_{k+1} \geq \zeta_k + \alpha_k(e_{k+1} + r_{k+1} + \hat{r}_{k+1}).$$

Now we must verify the assumptions of the Lemma. We begin with  $\hat{r}_{k+1}$ . To that end, observe that

$$\mathbb{E} \left[ 1_{\Omega_0} \sum_{k=n}^{\infty} \alpha_k \hat{r}_{k+1} \right] = O \left( \sum_{k=n}^{\infty} \alpha_k^2 \right),$$

by our assumption on  $\|E_k\|$ . Next we prove square summability of  $r_{k+1}$  on  $\Omega_0$ : Indeed, observe

$$r_{k+1}^2 \leq C\alpha_k^2(\|\xi_k\|^4 + \|E_k\|^4 + 1).$$

Moreover both  $\limsup_k \mathbb{E}_k[\|\xi_k\|^4 \mid \mathcal{F}_k] < \infty$  and  $\limsup_k \mathbb{E}_k[\|E_k\|^4 \mid \mathcal{F}_k] < \infty$  are bounded on  $\Omega_0$ . Therefore, by conditional Borel-Cantelli Lemma 3.11.2, we have

$$\sum_{k=1}^{\infty} r_{k+1}^2 < +\infty.$$

almost surely on  $\Omega_0$ .

Finally we prove that  $e_k$  has the desired properties. First note that we have

$$\mathbb{E}[e_{k+1} \mid \mathcal{F}_k] = 0 \quad \text{and} \quad \limsup_k \mathbb{E}[e_{k+1}^2 \mid \mathcal{F}_k] < \infty.$$

on  $\Omega_0$ . Indeed, this follows since  $\limsup_k \mathbb{E}[\|\xi_k\|^4 \mid \mathcal{F}_k] < \infty$  almost surely and  $Y_k + \alpha_k F(Y_k) \in B_{\epsilon_p}(p)$  on  $\Omega_0$ . Next, since  $\eta$  is globally Lipschitz on  $B_{\epsilon_p}(p)$ , we have that  $s(Y_k + \alpha_k F(Y_k))$  is uniformly bounded. Thus,

$$\limsup_k \mathbb{E}[e_{k+1}^2 \mid \mathcal{F}_k] \leq \limsup_k \mathbb{E}[\|s(Y_k + \alpha_k F(Y_k))\|^2 \|\xi_k\|^2 \mid \mathcal{F}_k] < \infty,$$



on  $\Omega_0$ , as desired.

Now we prove that  $\liminf \mathbb{E}[|e_{k+1}| \mid \mathcal{F}_k]$  is positive on  $\Omega_0$ . To that end, recall that the mapping  $\Phi$  satisfies  $\nabla\Phi(p) = I_d$ . Turning to the proof, there are two cases to consider. First suppose that  $\eta(Y_k + \alpha_k F(Y_k)) \neq 0$ . Then  $\eta$  is differentiable at  $Y_k + \alpha_k F(Y_k)$ . Now define  $u_k := \frac{A(\Phi(Y_k + \alpha_k F(Y_k)) - p)}{\|A(\Phi(Y_k + \alpha_k F(Y_k)) - p)\|}$  and note that

$$\begin{aligned} s(Y_k + \alpha_k F(Y_k)) &= \nabla\eta(Y_k + \alpha_k F(Y_k)) = \nabla\Phi(Y_k + \alpha_k F(Y_k))^\top A^\top u_k \\ &= A^\top u_k + (\nabla\Phi(Y_k + \alpha_k F(Y_k)) - \nabla\Phi(p))^\top A^\top u_k \\ &\in A^\top u_k + \epsilon_p \|A\| \text{Lip}_{\nabla\Phi} B_1(0), \end{aligned}$$

where the inclusion follows since  $Y_k + \alpha_k F(Y_k) \in B_{\epsilon_p}(p)$ . Let  $s_{\min}(A)$  denote the minimal nonzero singular value of  $A$  and notice that since  $u_k \in \mathbb{S}^{d-1} \cap \text{range}(A)$ , we have that  $w_k := A^\top u_k$  satisfies and

$$w_k \in W \quad \text{and} \quad \|w_k\| \geq s_{\min}(A) > 0.$$

Therefore, it follows that on the event  $\Omega_0$ , we have

$$\begin{aligned} \mathbb{E}[|e_{k+1}| \mid \mathcal{F}_k] &= \mathbb{E}[|\langle s(Y_k + \alpha_k F(Y_k)), \xi_k \rangle| \mid \mathcal{F}_k] \\ &\geq \mathbb{E}[|\langle w_k, \xi_k \rangle| \mid \mathcal{F}_k] - \epsilon_p \mathbb{E}[\|\xi_k\| \mid \mathcal{F}_k] \|A\| \text{Lip}_{\nabla\Phi} \\ &\geq s_{\min}(A) \inf_{w \in W \cap \mathbb{S}^{d-1}} \mathbb{E}[|\langle w, \xi_k \rangle| \mid \mathcal{F}_k] - \epsilon_p \mathbb{E}[\|\xi_k\| \mid \mathcal{F}_k] \|A\| \text{Lip}_{\nabla\Phi} \end{aligned}$$

We now consider the case  $\eta(Y_k + \alpha_k F(Y_k)) = 0$ . In this case, there exists  $u_k \in \mathbb{S}^{d-1}$  such that  $\|A^\top u_k\| = \|A\|$  and

$$s(Y_k + \alpha_k F(Y_k)) = \nabla\Phi(Y_k + \alpha_k F(Y_k))^\top A^\top u_k \in A^\top u_k + \epsilon_p \|A\| \text{Lip}_{\nabla\Phi} B_1(0),$$

Recall  $\text{range}(A^\top) = W$ . Thus, we have that the vector  $w_k := A^\top u_k$  is in  $W$  and  $\|w_k\| = \|A\| > 0$ . Thus, for all  $v \in \mathbb{R}^d$ , we have

$$|\langle s(Y_k + \alpha_k F(Y_k)), v \rangle| = \langle \nabla\Phi(Y_k + \alpha_k F(Y_k))^\top A^\top u_k, v \rangle \geq \langle w_k, v \rangle - \epsilon_p \text{Lip}_{\nabla\Phi} \|A\| \|v\|.$$

Taking  $v = \xi_k$ , we obtain

$$\mathbb{E}[|\langle s(Y_k + \alpha_k F(Y_k)), \xi_k \rangle| \mid \mathcal{F}_k] \geq \|A\| \inf_{w \in W \cap \mathbb{S}^{d-1}} \mathbb{E}[|\langle w, \xi_k \rangle| \mid \mathcal{F}_k] - \epsilon_p \mathbb{E}[\|\xi_k\| \mid \mathcal{F}_k] \|A\| \text{Lip}_{\nabla\Phi}$$

Thus, putting both cases together, we find that on the event  $\Omega_0$ , we have

$$\liminf_k \mathbb{E}[|e_{k+1}| \mid \mathcal{F}_k] \geq s_{\min}(A) \liminf_k \inf_{w \in W \cap \mathbb{S}^{d-1}} \mathbb{E}[|\langle w, \xi_k \rangle| \mid \mathcal{F}_k] - \epsilon_p \limsup_k \mathbb{E}[\|\xi_k\| \mid \mathcal{F}_k] \|A\| \text{Lip}_{\nabla\Phi} > 0,$$

where the last inequality follows from (3.9.15).

Therefore, we have verified that all the conditions of Lemma 3.9.1. It follows that

$$P(\{\tau = \infty\} \cap \{\eta(Y_k) \rightarrow 0\}) = 0,$$

as desired.

### 3.9.5 Proof of Theorem 3.6.2: nonconvergence to saddle points

In this section, prove Theorem 3.6.2 by verifying that the iterates  $\{x_k\}_{k \in \mathbb{N}}$  satisfy the conditions of Theorem 3.6.1. To that end, fix a point  $p \in S$  with associated manifold  $\mathcal{M}$  and neighborhood  $\mathcal{U}$ . Let  $\epsilon_p$  be small enough that  $B_{\epsilon_p}(\bar{x}) \subseteq \mathcal{U}$  and define the  $C^2$  mapping  $F_p: B_{\epsilon_p}(p) \rightarrow \mathbb{R}^d$  by:

$$F_p(y) = -\nabla f_{\mathcal{M}}(y),$$

where  $f_{\mathcal{M}} := f \circ P_{\mathcal{M}}$ . Note that the mapping  $F$  is indeed  $C^2$ , since  $\mathcal{M}$  is a  $C^4$  manifold, and hence,  $f_{\mathcal{M}}$  is  $C^3$ . Moreover, since  $\nabla F(p) = -\nabla_{\mathcal{M}}^2 f(p)$ , the mapping  $F_p$  has at least one eigenvector with positive eigenvalue. In addition, the subspace  $W_p$  spanned by such eigenvectors is contained in  $T_{\mathcal{M}}(p)$ .

Turning to the proof, define  $X_k = x_k$  for all  $k \geq 1$ . We now construct the sequences  $Y_k, \xi_k$ , and  $E_k$  and show they satisfy the assumptions of the theorem. Beginning with  $Y_k$ , recall that by Proposition 3.3.2, for all  $k \geq 1$  and all sufficiently small  $\delta > 0$ , the sequence

$$Y_k := \begin{cases} P_{\mathcal{M}}(X_k) & \text{if } x_k \in B_{2\delta}(\bar{x}) \\ p & \text{otherwise.} \end{cases}, \quad (3.9.17)$$

satisfies  $Y_k \in B_{4\delta}(\bar{x}) \cap \mathcal{M}$  and the recursion

$$Y_{k+1} = Y_k - \alpha_k \nabla f_{\mathcal{M}}(y_k) - \alpha_k \xi_k + \alpha_k E_k \quad \text{for all } k \geq 1.$$

where  $\xi_k := P_{T_{\mathcal{M}}(Y_k)}(\nu_k)$  and  $E_k$  is an error sequence. Moving to  $E_k$ , let us show that the error sequence satisfies the assumptions of theorem. To that end, Proposition 3.3.2 shows that for  $\delta$  sufficiently small, there exists  $C > 0$  such that for all  $n \geq k_0$ , we have

$$\mathbb{E} \left[ 1_{\tau_{k_0, \delta} = \infty} \sum_{k=n}^{\infty} \alpha_k \|E_k\| \right] \leq C \sum_{k=n}^{\infty} \alpha_k^2.$$

Moreover, by the inequality from Proposition 3.3.2, the sequence  $\|E_k\| 1_{\tau_{k_0, \delta} > k}$  is bounded above by a bounded sequence that almost surely converges to zero:

$$\|E_k\| 1_{\tau_{k_0, \delta} > k} \leq C(1 + \|\nu_k\|)^2 (\text{dist}(x_k, \mathcal{M}) + \alpha_k) 1_{\tau_{k_0, \delta} > k} \leq C(1 + r)^2 (\delta + \alpha_k),$$

Thus, on the event  $\{\tau_{k_0, \delta} = \infty\}$ , we have

$$\limsup_k 1_{\Omega_0} \mathbb{E}[\|E_k\|^4 \mid \mathcal{F}_k] \leq \limsup_k \mathbb{E}[\|E_k\|^4 1_{\tau_{k_0, \delta} > k} \mid \mathcal{F}_k] \leq C(1 + r)^2 (\delta + \alpha_k).$$

Therefore,  $Y_k$  and  $E_k$  satisfy the conditions 1 and 3 of Theorem 3.6.1 for all sufficiently small  $\delta_p$  satisfying  $\delta_p \leq \epsilon_p/8$ .

To conclude the proof, we now show that Condition 2 of Theorem 3.6.1 is satisfied. To that end, clearly  $\|\xi_k\| = \|P_{T_k}(\nu_k)\| \leq r =: c_3$  for all  $k \geq k_0$ . In addition, we have that

$$\mathbb{E}[\xi_k \mid \mathcal{F}_k] = P_{T_k}(\mathbb{E}[\nu_k \mid X_{k_0}, \dots, X_k]) = 0.$$

Indeed, this follows from two facts: first  $Y_k$  is a measurable function of  $X_k$ ; and second the noise sequence  $\nu_k$  is mean zero and independent of  $X_{k_0}, \dots, X_k$ . Finally, we must show that  $\xi_k$  has positive correlation with the unstable subspace  $W_p$ .

To prove correlation with the unstable subspace, recall that there exists  $C' > 0$  such that the mapping  $x \mapsto P_{T_{\mathcal{M}}(x)}$  is  $C'$ -Lipschitz mapping on  $\mathcal{M} \cap B_{\epsilon_p}(p)$ . In addition, we have that  $W_p \subseteq T_{\mathcal{M}}(p)$ . Therefore, since  $Y_k \in \mathcal{M} \cap B_{\epsilon_p}(p)$  for all  $k \geq k_0$ , we have the following bound for all  $w \in W \cap \mathbb{S}^{d-1}$ :

$$\begin{aligned} \mathbb{E}[\langle \xi_k, w \rangle \mid \mathcal{F}_k] &= \mathbb{E}[\langle \nu_k, P_{T_{\mathcal{M}}(Y_k)} w \rangle \mid \mathcal{F}_k] \\ &\geq \mathbb{E}[\langle \nu_k, w \rangle \mid \mathcal{F}_k] - r \|(P_{T_{\mathcal{M}}(Y_k)} - P_{T_{\mathcal{M}}(p)})w\| \\ &\geq rc_d - rC'\|Y_k - p\|, \end{aligned}$$

where  $c_d$  is a constant dependent only on  $d$  since  $\nu_k \sim \text{Unif}(B_r(0))$ . By slightly shrinking  $\epsilon_p$  if needed, we can ensure that  $\inf_{x \in B_{\epsilon_p}(p)} \{rc_d - rC'\|x - p\|\} > (1/2)rc_d =: c_4$ , as desired.

# Bibliography

- [1] Hilal Asi and John C Duchi. Stochastic (approximate) proximal point methods: Convergence, optimality, and adaptivity. *SIAM Journal on Optimization*, 29(3):2257–2290, 2019.
- [2] Michel Benaïm. A dynamical system approach to stochastic approximations. *SIAM Journal on Control and Optimization*, 34(2):437–472, 1996.
- [3] Michel Benaïm. Dynamics of stochastic approximation algorithms. In *Seminaire de probabilites XXXIII*, pages 1–68. Springer, 1999.
- [4] S. Bhojanapalli, B. Neyshabur, and N. Srebro. Global optimality of local search for low rank matrix recovery. In *Advances in Neural Information Processing Systems*, pages 3873–3881, 2016.
- [5] Pascal Bianchi, Walid Hachem, and Sholom Schechtman. Stochastic subgradient descent escapes active strict saddles. *arXiv preprint arXiv:2108.02072*, 2021.
- [6] Jérôme Bolte and Edouard Pauwels. Conservative set valued fields, automatic differentiation, stochastic gradient methods and deep learning. *Mathematical Programming*, 188(1):19–51, 2021.
- [7] Jérôme Bolte, Shoham Sabach, and Marc Teboulle. Proximal alternating linearized minimization for nonconvex and nonsmooth problems. *Mathematical Programming*, 146(1):459–494, 2014.
- [8] Jérôme Bolte, Aris Daniilidis, Adrian Lewis, and Masahiro Shiota. Clarke subgradients of stratifiable functions. *SIAM Journal on Optimization*, 18(2):556–572, 2007.
- [9] J Frédéric Bonnans and Alexander Shapiro. *Perturbation analysis of optimization problems*. Springer Science & Business Media, 2013.
- [10] Jonathan Borwein and Adrian S Lewis. *Convex analysis and nonlinear optimization: theory and examples*. Springer Science & Business Media, 2010.
- [11] Nicolas Boumal. An introduction to optimization on smooth manifolds. *Available online*, Aug, 2020.
- [12] Odile Brandiere. Some pathological traps for stochastic approximation. *SIAM journal on control and optimization*, 36(4):1293–1314, 1998.

- [13] Odile Brandière and Marie Duflo. Les algorithmes stochastiques contournent-ils les pièges ? *Annales de l'I.H.P. Probabilités et statistiques*, 32(3):395–427, 1996.
- [14] Louis H. Y. Chen. A Short Note on the Conditional Borel-Cantelli Lemma. *The Annals of Probability*, 6(4):699 – 700, 1978.
- [15] Francis H Clarke, Yuri S Ledyaev, Ronald J Stern, and Peter R Wolenski. *Nonsmooth analysis and control theory*, volume 178. Springer Science & Business Media, 2008.
- [16] Francis H Clarke, RJ Stern, and PR Wolenski. Proximal smoothness and the lower-c2 property. *J. Convex Anal*, 2(1-2):117–144, 1995.
- [17] C. Criscitiello and N. Boumal. Efficiently escaping saddle points on manifolds. *arXiv preprint arXiv:1906.04321*, 2019.
- [18] Aris Daniilidis, Dmitriy Drusvyatskiy, and Adrian S Lewis. Orthogonal invariance and identifiability. *SIAM Journal on Matrix Analysis and Applications*, 35(2):580–598, 2014.
- [19] Aris Daniilidis, Adrian Lewis, Jérôme Malick, and Hristo Sendov. Prox-regularity of spectral functions and spectral sets. *Journal of Convex Analysis*, 15(3):547–560, 2008.
- [20] Chandler Davis. All convex invariant functions of hermitian matrices. *Archiv der Mathematik*, 8(4):276–278, 1957.
- [21] Damek Davis, Mateo Díaz, and Dmitriy Drusvyatskiy. Escaping strict saddle points of the moreau envelope in nonsmooth optimization. *arXiv preprint arXiv:2106.09815*, 2021.
- [22] Damek Davis and Dmitriy Drusvyatskiy. Active strict saddles in nonsmooth optimization. *arXiv preprint arXiv:1912.07146*, 2019.
- [23] Damek Davis and Dmitriy Drusvyatskiy. Stochastic model-based minimization of weakly convex functions. *SIAM Journal on Optimization*, 29(1):207–239, 2019.
- [24] Damek Davis and Dmitriy Drusvyatskiy. Proximal methods avoid active strict saddles of weakly convex functions. *Foundations of Computational Mathematics*, pages 1–46, 2021.
- [25] Damek Davis, Dmitriy Drusvyatskiy, and Vasileios Charisopoulos. Stochastic algorithms with geometric step decay converge linearly on sharp functions. *arXiv preprint arXiv:1907.09547*, 2019.
- [26] Damek Davis, Dmitriy Drusvyatskiy, Sham Kakade, and Jason D Lee. Stochastic sub-gradient method converges on tame functions. *Foundations of computational mathematics*, 20(1):119–154, 2020.
- [27] A Dembo. Lecture notes on probability theory: Stanford statistics 310. *Accessed October*, 1:2016, 2016.

- [28] D. Drusvyatskiy, A. D. Ioffe, and A. S. Lewis. Generic minimizing behavior in semialgebraic optimization. *SIAM J. Optim.*, 26(1):513–534, 2016.
- [29] D. Drusvyatskiy and C. Paquette. Variational analysis of spectral functions simplified. *J. Convex Anal.*, 25(1):119–134, 2018.
- [30] Dmitriy Drusvyatskiy. The proximal point method revisited. *arXiv preprint arXiv:1712.06038*, 2017.
- [31] Dmitriy Drusvyatskiy, Alexander D Ioffe, and Adrian S Lewis. Curves of descent. *SIAM Journal on Control and Optimization*, 53(1):114–138, 2015.
- [32] Dmitriy Drusvyatskiy and Adrian S Lewis. Optimality, identifiability, and sensitivity. *Mathematical Programming*, 147(1):467–498, 2014.
- [33] John C Duchi and Feng Ruan. Asymptotic optimality in stochastic optimization. *The Annals of Statistics*, 49(1):21–48, 2021.
- [34] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.
- [35] Herbert Federer. Curvature measures. *Transactions of the American Mathematical Society*, 93(3):418–491, 1959.
- [36] R. Ge, F. Huang, C. Jin, and Y. Yuan. Escaping from saddle pointsonline stochastic gradient for tensor decomposition. In *Conference on Learning Theory*, pages 797–842, 2015.
- [37] R. Ge, C. Jin, and Y. Zheng. No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 1233–1242. JMLR. org, 2017.
- [38] R. Ge, J.D. Lee, and T. Ma. Matrix completion has no spurious local minimum. In *Advances in Neural Information Processing Systems*, pages 2973–2981, 2016.
- [39] H. Gfrerer and V. Outrata. On a semismooth\* newton method for solving generalized equations. *SIAM Journal on Optimization*, 31(1):489–517, 2021.
- [40] Thomas Hakon Gronwall. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Annals of Mathematics*, pages 292–296, 1919.
- [41] Nadav Hallak and Marc Teboulle. Finding second-order stationary points in constrained minimization: A feasible direction approach. *Journal of Optimization Theory and Applications*, 186(2):480–503, 2020.
- [42] Warren L Hare and Adrian S Lewis. Identifying active constraints via partial smoothness and prox-regularity. *Journal of Convex Analysis*, 11(2):251–266, 2004.

- [43] Morris W Hirsch, Charles Chapman Pugh, and Michael Shub. *Invariant manifolds*, volume 583. Springer, 2006.
- [44] Minhui Huang. Escaping saddle points for nonsmooth weakly convex functions via perturbed proximal algorithms. *arXiv preprint arXiv:2102.02837*, 2021.
- [45] T-C Kuo. The ratio test for analytic whitney stratifications. In *Proceedings of Liverpool Singularities Symposium I*, pages 141–149. Springer, 1971.
- [46] Tzee-Char Kuo. Characterizations of v-sufficiency of jets. *Topology*, 11(1):115–131, 1972.
- [47] Krzysztof Kurdyka. On gradients of functions definable in o-minimal structures. In *Annales de l’institut Fourier*, volume 48, pages 769–783, 1998.
- [48] Ta Lê Loi. Verdier and strict thom stratifications in o-minimal structures. *Illinois Journal of Mathematics*, 42(2):347–356, 1998.
- [49] J.D. Lee, I. Panageas, G. Piliouras, M. Simchowitz, M.I. Jordan, and B. Recht. First-order methods almost always avoid strict saddle points. *Math. Program.*, 176(1-2):311–337, July 2019.
- [50] J.D. Lee, M. Simchowitz, M.I. Jordan, and B. Recht. Gradient descent only converges to minimizers. In *Conference on learning theory*, pages 1246–1257, 2016a.
- [51] John M Lee. Smooth manifolds. In *Introduction to Smooth Manifolds*, pages 1–31. Springer, 2013.
- [52] Sangkyun Lee and Stephen J. Wright. Manifold identification in dual averaging for regularized stochastic online learning. *Journal of Machine Learning Research*, 13(55):1705–1744, 2012.
- [53] Sangkyun Lee, Stephen J Wright, and Léon Bottou. Manifold identification in dual averaging for regularized stochastic online learning. *Journal of Machine Learning Research*, 13(6), 2012.
- [54] Claude Lemaréchal, François Oustry, and Claudia Sagastizábal. The  $\mathcal{U}$ -lagrangian of a convex function. *Transactions of the American mathematical Society*, 352(2):711–729, 2000.
- [55] Adrian S Lewis. Convex analysis on the hermitian matrices. *SIAM Journal on Optimization*, 6(1):164–177, 1996.
- [56] Adrian S Lewis. Nonsmooth analysis of eigenvalues. *Mathematical Programming*, 84(1):1–24, 1999.
- [57] Adrian S Lewis. Active sets, nonsmoothness, and sensitivity. *SIAM Journal on Optimization*, 13(3):702–725, 2002.

- [58] Adrian S Lewis and Jingwei Liang. Partial smoothness and constant rank. *arXiv preprint arXiv:1807.03134*, 2018.
- [59] Adrian S Lewis and Hristo S Sendov. Nonsmooth analysis of singular values. part i: Theory. *Set-Valued Analysis*, 13(3):213–241, 2005.
- [60] Adrian S Lewis and Stephen J Wright. A proximal method for composite minimization. *Mathematical Programming*, 158(1):501–546, 2016.
- [61] Adrian Stephen Lewis. Derivatives of spectral functions. *Mathematics of Operations Research*, 21(3):576–588, 1996.
- [62] Jingwei Liang, Jalal Fadili, Gabriel Peyré, and Russell Luke. Activity identification and local linear convergence of douglas-rachford/admm under partial smoothness. In *International Conference on Scale Space and Variational Methods in Computer Vision*, pages 642–653. Springer, 2015.
- [63] Stanislaw Lojasiewicz. Une propriété topologique des sous-ensembles analytiques réels. *Les équations aux dérivées partielles*, 117:87–89, 1963.
- [64] Szymon Majewski, Błażej Miasojedow, and Eric Moulines. Analysis of nonsmooth stochastic approximation: the differential inclusion approach. *arXiv preprint arXiv:1805.01916*, 2018.
- [65] John Mather. *Notes on topological stability*. Citeseer, 1970.
- [66] Robert Mifflin. Semismooth and semiconvex functions in constrained optimization. *SIAM Journal on Control and Optimization*, 15(6):959–972, 1977.
- [67] Robert Mifflin and Claudia Sagastizábal. A VU-algorithm for convex minimization. *Mathematical programming*, 104(2):583–608, 2005.
- [68] Scott A Miller and Jérôme Malick. Newton methods for nonsmooth convex minimization: connections among-lagrangian, riemannian newton and sqp methods. *Mathematical programming*, 104(2):609–633, 2005.
- [69] A. Mokhtari, A. Ozdaglar, and A. Jadbabaie. Escaping saddle points in constrained optimization. In *Advances in Neural Information Processing Systems*, pages 3629–3639, 2018.
- [70] Boris S Mordukhovich. *Variational analysis and generalized differentiation I: Basic theory*, volume 330. Springer Science & Business Media, 2006.
- [71] Arkadij Semenovič Nemirovskij and David Borisovich Yudin. Problem complexity and method efficiency in optimization. 1983.
- [72] Yurii Nesterov. Primal-dual subgradient methods for convex problems. *Mathematical Programming*, 120(1):221–259, 2009.



- [73] Maher Nouiehed, Jason D Lee, and Meisam Razaviyayn. Convergence to second-order stationarity for constrained non-convex optimization. *arXiv preprint arXiv:1810.02024*, 2018.
- [74] Robin Pemantle et al. Nonconvergence to unstable points in urn models and stochastic approximations. *The Annals of Probability*, 18(2):698–712, 1990.
- [75] Jean-Paul Penot. *Calculus without derivatives*, volume 266. Springer Science & Business Media, 2012.
- [76] René Poliquin, R Rockafellar, and Lionel Thibault. Local differentiability of distance functions. *Transactions of the American mathematical Society*, 352(11):5231–5249, 2000.
- [77] René A Poliquin and Ralph Tyrell Rockafellar. A calculus of prox-regularity. *J. Convex Anal*, 17(1):203–210, 2010.
- [78] Boris T Polyak and Anatoli B Juditsky. Acceleration of stochastic approximation by averaging. *SIAM journal on control and optimization*, 30(4):838–855, 1992.
- [79] Herbert Robbins and David Siegmund. A convergence theorem for non negative almost supermartingales and some applications. In *Optimizing methods in statistics*, pages 233–257. Elsevier, 1971.
- [80] R Tyrrell Rockafellar and Roger J-B Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.
- [81] Alexander Shapiro. Asymptotic Properties of Statistical Estimators in Stochastic Programming. *The Annals of Statistics*, 17(2):841 – 858, 1989.
- [82] Alexander Shapiro. On a class of nonsmooth composite functions. *Mathematics of Operations Research*, 28(4):677–692, 2003.
- [83] Michael Shub. *Global stability of dynamical systems*. Springer Science & Business Media, 2013.
- [84] Stephen Smale. Differentiable dynamical systems. *Bulletin of the American mathematical Society*, 73(6):747–817, 1967.
- [85] J. Sun, Q. Qu, and J. Wright. When are nonconvex problems not scary? *arXiv preprint arXiv:1510.06096*, 2015.
- [86] J. Sun, Q. Qu, and J. Wright. A geometric analysis of phase retrieval. *Foundations of Computational Mathematics*, 18(5):1131–1198, 2018.
- [87] Y. Sun, N. Flammarion, and M. Fazel. Escaping from saddle points on riemannian manifolds. *arXiv preprint arXiv:1906.07355*, 2019.
- [88] David Trotman. Stratification theory. In *Handbook of Geometry and Topology of Singularities I*, pages 243–273. Springer, 2020.

- [89] Lou van den Dries and Chris Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 84(2):497–540, 1996.
- [90] Jean-Louis Verdier. Stratifications de whitney et théoreme de bertini-sard. *Inventiones mathematicae*, 36(1):295–312, 1976.
- [91] John Wells. Invariant manifolds on non-linear operators. *Pacific Journal of Mathematics*, 62(1):285–293, 1976.
- [92] Hassler Whitney. Elementary structure of real algebraic varieties. In *Hassler Whitney Collected Papers*, pages 456–467. Springer, 1992.
- [93] Hassler Whitney. Local properties of analytic varieties. In *Hassler Whitney Collected Papers*, pages 497–536. Springer, 1992.
- [94] Hassler Whitney. Tangents to an analytic variety. In *Hassler Whitney Collected Papers*, pages 537–590. Springer, 1992.
- [95] Stephen J Wright. Identifiable surfaces in constrained optimization. *SIAM Journal on Control and Optimization*, 31(4):1063–1079, 1993.

## 3.10 Appendix to Part 1

### 3.10.1 Proof of Proposition 2.3.3

Since  $X$  is a  $C^3$  manifold, the projection  $P_Y$  is  $C^2$ -smooth. Therefore, there exist constants  $\epsilon, L > 0$  satisfying

$$\|P_Y(y + h) - P_Y(y) - \nabla P_Y(y)h\| \leq L\|h\|^2 \quad (3.10.1)$$

for all  $y \in B_\epsilon(\bar{x})$  and  $h \in \epsilon\mathbb{B}$ . Fix now two points  $x \in X$  and  $y \in Y$  and a unit vector  $v \in N_X(x)$ . Clearly, we may suppose  $v \notin N_Y(y)$ , since otherwise the claim is trivially true. Define the normalized vector  $w := -\frac{P_{T_Y(y)}(v)}{\|P_{T_Y(y)}(v)\|}$ . Noting the equality  $\nabla P_Y(y) = P_{T_Y(y)}$  and appealing to (3.10.1), we deduce the estimate

$$\|P_Y(y - \alpha w) - (y - \alpha w)\| \leq L\|\alpha w\|^2 = L\alpha^2,$$

for all  $y \in B_\epsilon(\bar{x})$  and  $\alpha \in (0, \epsilon)$ . Shrinking  $\epsilon > 0$ , prox-regularity yields the estimate

$$\langle v, P_Y(y - \alpha w) - x \rangle \leq \frac{\rho}{2}\|x - P_Y(y - \alpha w)\|^2,$$

for some constant  $\rho > 0$ . Therefore, we conclude

$$\begin{aligned} \alpha\|P_{T_Y(y)}v\| &= -\alpha\langle v, w \rangle = \langle v, x - y \rangle + \langle v, P_Y(y - \alpha w) - x \rangle + \langle v, (y - \alpha w) - P_Y(y - \alpha w) \rangle \\ &\leq \|x - y\| + \frac{\rho}{2}\|x - P_Y(y - \alpha w)\|^2 + L\alpha^2. \end{aligned}$$

Note that the middle term is small:

$$\|P_Y(y - \alpha w) - x\|^2 \leq 2\|P_Y(y - \alpha w) - (y - \alpha w)\|^2 + 2\|y - \alpha w - x\|^2 \leq 2L^2\alpha^4 + 4\|y - x\|^2 + 4\alpha^2.$$

Thus, we have

$$\alpha \|P_{T_Y(y)} v\| \leq \|x - y\| + \rho L^2 \alpha^4 + 2\rho \|x - y\|^2 + 2\rho \alpha^2 + L\alpha^2.$$

Dividing both sides by  $\alpha$  and setting  $\alpha = \sqrt{\|x - y\|}$  completes the proof of (2.3.1).

## 3.11 Appendix to Part 2

### 3.11.1 Proof of Proposition 3.2.3: the projected gradient method

Let  $\epsilon > 0$  be a neighborhood small enough that the following hold for all  $x \in B_\epsilon(\bar{x}) \cap \mathcal{X}$ : first (3.2.6) holds; second we require that for some  $C > 0$ , we have

$$\begin{aligned} \|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(s_g(x) - \nabla_{\mathcal{M}} f(P_{\mathcal{M}}(x)))\| &\leq C \text{dist}(x, \mathcal{M}); \\ \|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(w)\| &\leq C \text{dist}(x, \mathcal{M}), \end{aligned}$$

for all  $w \in N_{\mathcal{X}}(x)$  of unit norm, a consequence of strong (a);<sup>2</sup> third we require that

$$\langle w, x - P_{\mathcal{M}}(x) \rangle \geq -\frac{\mu}{2} \text{dist}(x, \mathcal{M})$$

for all  $w \in N_{\mathcal{X}}(x)$  of unit norm, a consequence of prox-regularity or (b). Finally, we assume that  $P_{\mathcal{M}}$  is  $C^3$ , in particular,  $C$  Lipschitz with  $C$ -Lipschitz Jacobian.

#### Verifying Assumption (A1)

Turning to the proof, throughout we let  $C$  be a constant that varies from line to line. We also fix  $x \in B_{\epsilon/2}(\bar{x}) \cap \mathcal{X}$ ,  $\alpha > 0$  and  $\nu \in \mathbb{R}^d$ . Define

$$w = G_\alpha(x, \nu) - s_g(x) - \nu \quad \text{and} \quad x_+ = s_{\mathcal{X}}(x - \alpha(s_g(x) + \nu)).$$

We begin with the following bound, which verifies Assumption (A1):

*Claim 9.* We have  $w \in N_{\mathcal{X}}(x_+)$  and there exists  $C > 0$  such that the following bounds hold:

$$\|w\| \leq C + \|\nu\|; \quad \|G_\alpha(x, \nu)\| \leq 2(C + \|\nu\|); \quad \|x_+ - x\| \leq 2(C + \|\nu\|)\alpha$$

*Proof.* Beginning with the inclusion, first-order optimality conditions imply that:

$$w = \frac{x - \alpha(s_g(x) + \nu) - x_+}{\alpha} \in N_{\mathcal{X}}(x_+),$$

as desired. Next we bound  $\|w\|$ : there exists  $C > 0$  such that

$$\|w\| = \|x - \alpha(s_g(x) + \nu) - x_+\|/\alpha \leq \|x - \alpha(s_g(x) + \nu) - x\|/\alpha \leq C + \|\nu\|,$$

---

<sup>2</sup>The strong (a) regularity property for  $g$  is a consequence of Equation (2.1.3), which is stated with respect to the limiting subdifferential. However, for locally Lipschitz functions, the relationship extends to Clarke subdifferentials through convexification:  $\partial_c g(x) = \text{Conv } \partial g(x)$ .

where the first inequality follows from the definition of the projection and the second inequality follows from our local Lipschitz assumptions on  $g$ . Next, we bound  $\|x_+ - x\|$ :

$$\|x_+ - x\| \leq \|x_+ - (x - \alpha(s_g(x) + \nu))\| + \alpha\|s_g(x) + \nu\| = \alpha\|w\| + \alpha\|s_g(x) + \nu\| \leq 2\alpha(C + \|\nu\|),$$

as desired. Finally, we bound  $\|G_\alpha(x, \nu)\|$ :

$$\|G_\alpha(x, \nu)\| = \|x_+ - x\|/\alpha \leq 2(C + \|\nu\|),$$

as desired.  $\square$

In the remainder of the proof, we will make use of the following claim:

*Claim 10.* Suppose that  $y \in B_{\epsilon/2}(\bar{x})$  and  $z \notin B_\epsilon(\bar{x})$ . Then

$$\text{dist}(y, \mathcal{M}) \leq \frac{1}{2}\epsilon \leq \|z - y\|$$

*Proof.* We have

$$\|z - y\| \geq \|z - \bar{x}\| - \|y - \bar{x}\| \geq \frac{1}{2}\epsilon \geq \|y - \bar{x}\| \geq \text{dist}(y, \mathcal{M})$$

where the final inequality follows from the inclusion  $\bar{x} \in \mathcal{M}$ .  $\square$

### Verifying Assumption (A3)

Now we verify Assumption (A3). Observe the following bound:

$$\begin{aligned} \langle G_\alpha(x, \nu) - \nu, x - P_{\mathcal{M}}(x) \rangle &= \langle w + s_g(x), x - P_{\mathcal{M}}(x) \rangle \\ &\geq \langle w, x - P_{\mathcal{M}}(x) \rangle + \mu \text{dist}(x, \mathcal{M}) \\ &= \mu \text{dist}(x, \mathcal{M}) \\ &\quad + \underbrace{\langle w, x_+ - P_{\mathcal{M}}(x_+) \rangle}_{=: R_1} - \underbrace{\|w\| \|(x - P_{\mathcal{M}}(x)) - (x_+ - P_{\mathcal{M}}(x_+))\|}_{=: R_2}. \end{aligned} \tag{3.11.1}$$

In what follows we will bound  $R_1$  and  $R_2$ .

*Claim 11.* There exists  $C > 0$  such that

$$R_1 \geq -\frac{\mu}{2} \text{dist}(x, \mathcal{M}) - \mu(C + \|\nu\|)\alpha - 4(C + \|\nu\|)^2\alpha$$

*Proof.* First suppose that  $x_+ \in B_\epsilon(\bar{x})$ . Then

$$\begin{aligned} R_1(x) &= \langle w, x_+(\alpha, \nu) - P_{\mathcal{M}}(x_+(\alpha, \nu)) \rangle \\ &\geq -\frac{\mu}{2} \text{dist}(x_+(\alpha, \nu), \mathcal{M}) \\ &\geq -\frac{\mu}{2} \text{dist}(x, \mathcal{M}) - \frac{\mu}{2} \|x_+ - x\| \\ &\geq -\frac{\mu}{2} \text{dist}(x, \mathcal{M}) - \frac{\mu}{2} 2(C + \|\nu\|)\alpha, \end{aligned}$$

as desired. Now suppose that  $x_+ \notin B_\epsilon(\bar{x})$ . Then by Claim 10, we have  $\text{dist}(x, \mathcal{M}) \leq \|x - x_+\|$ . Consequently,

$$\begin{aligned} R_1 &\geq -\|w\|\text{dist}(x_+, \mathcal{M}) \\ &\geq -\|w\|\text{dist}(x, \mathcal{M}) - \|w\|\|x_+ - x\| \\ &\geq -2\|w\|\|x_+ - x\| \\ &\geq -4(C + \|\nu\|)^2\alpha \end{aligned}$$

□

*Claim 12.* There exists  $C > 0$  such that

$$R_2(x) \leq 6(C + \|\nu\|)^2\alpha$$

*Proof.* First suppose that  $x_+ \in B_\epsilon(\bar{x})$ . Then the result follows from the bound:

$$\begin{aligned} \|w\|\|x - P_{\mathcal{M}}(x) - (x_+ - P_{\mathcal{M}}(x_+))\| &\leq (1 + C)\|w\|\|x_+ - x\| \\ &\leq 2(C + \|\nu\|)^2\alpha, \end{aligned}$$

as desired. Next suppose that  $x_+ \notin B_\epsilon(\bar{x})$ . Then by Claim 10, we have  $\text{dist}(x, \mathcal{M}) \leq \|x - x_+\|$ . Consequently,

$$\|x - P_{\mathcal{M}}(x) - (x_+ - P_{\mathcal{M}}(x_+))\| \leq \text{dist}(x, \mathcal{M}) + \text{dist}(x_+, \mathcal{M}) \leq 3\|x_+ - x\|.$$

Therefore, we have

$$\|w\|\|x - P_{\mathcal{M}}(x) - (x_+ - P_{\mathcal{M}}(x_+))\| \leq 3\|w\|\|x_+ - x\| \leq 6(C + \|\nu\|)^2\alpha,$$

as desired. □

Therefore, plugging these bounds in (3.11.1), it follows that there exists  $C' > 0$  such that

$$\langle G_\alpha(x, \nu) - \nu, x - P_{\mathcal{M}}(x) \rangle \geq \frac{\mu}{2}\text{dist}(x, \mathcal{M}) - C'(1 + \|\nu\|)^2\alpha,$$

as desired.

### Verifying Assumption (A2)

Now we verify Assumption (A2). To that end, suppose that  $x_+ \in B_\epsilon(\bar{x})$ . Then

$$\begin{aligned} \|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(G_\alpha(x, \nu) - s_g(x) - \nu)\| &= \|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}w\| \\ &\leq C\|w\|\|x_+ - P_{\mathcal{M}}(x)\| \\ &\leq C\|w\|(\|x_+ - x\| + \text{dist}(x, \mathcal{M})) \\ &\leq C(2(C + \|\nu\|)^2\alpha + (C + \|\nu\|)\text{dist}(x, \mathcal{M})), \end{aligned} \quad (3.11.2)$$

where the first inequality follows from strong (a) regularity of  $\mathcal{X}$  along  $\mathcal{M}$ . Now suppose that  $x_+ \notin B_\epsilon(\bar{x})$ . Then by Claim 10, we have

$$\|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(G_\alpha(x, \nu) - s_g(x) - \nu)\| \leq \|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}w\| \leq \|w\| \leq \frac{2\|w\|}{\epsilon}\|x_+ - x\| \leq \frac{4}{\epsilon}(C + \|\nu\|)^2\alpha \quad (3.11.3)$$

Therefore, putting together (3.11.2) and (3.11.2), we find that there exists  $C' > 0$  such that

$$\|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(G_\alpha(x, \nu) - s_g(x) - \nu)\| \leq C'(1 + \|\nu\|)^2(\alpha + \text{dist}(x, \mathcal{M})),$$

as desired.

### 3.11.2 Proof of Proposition 3.2.5: the proximal gradient method

Let  $\epsilon > 0$  be a neighborhood small enough that the following hold for all  $x \in B_\epsilon(\bar{x}) \cap \mathcal{X}$ : first (3.2.8) holds; second we require that for some  $C > 0$ , we have

$$\|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(\nabla g(x) + u - \nabla_{\mathcal{M}}f(P_{\mathcal{M}}(x)))\| \leq C\sqrt{1 + \|u\|^2}\text{dist}(x, \mathcal{M})$$

for all  $u \in \partial h(x)$ , a consequence of strong (a);<sup>3</sup> third, we assume that  $\nabla_{\mathcal{M}}f$  is  $C$ -Lipschitz on  $B_\epsilon(\bar{x}) \cap \mathcal{M}$ ; fourth, we assume that  $\nabla g$  is  $C$ -Lipschitz on  $B_\epsilon(\bar{x}) \cap \mathcal{X}$ ; Finally, we assume that  $P_{\mathcal{M}}$  is  $C^3$ , in particular,  $C$ -Lipschitz with  $C$ -Lipschitz Jacobian on  $B_\epsilon(\bar{x})$ .

Turning to the proof, fix  $x \in B_{\epsilon/2}(\bar{x})$  and  $\nu \in \mathbb{R}^d$ . We also define

$$w = G_\alpha(x, \nu) - \nabla g(x) - \nu \quad \text{and} \quad x_+ = s_\alpha(x - \alpha(\nabla g(x) + \nu)).$$

Finally, we let  $C$  be a constant independent of  $x, \alpha$  and  $\nu$ , which changes from line to line.

#### Verifying Assumption (A1)

We begin with the following bound, which verifies Assumption (A1).

*Claim 13.* We have  $w \in \hat{\partial}h(x_+)$  and there exists a constant  $C$  independent of  $x, \nu, \alpha$ , such that the following bounds hold:

$$\max\{\|G_\alpha(x, \nu)\|, \|w\|\} \leq C(1 + \|\nu\|); \quad \text{and} \quad \|x_+ - x\| \leq C(1 + \|\nu\|)\alpha.$$

*Proof.* Beginning with the inclusion, first-order optimality conditions imply that  $w$  is a Fréchet subgradient:

$$w = \frac{x - \alpha(\nabla g(x) + \nu) - x_+}{\alpha} \in \hat{\partial}h(x_+),$$

as desired. First, we bound  $\|x_+ - x\|$ : Let  $v = \nabla g(x) + \nu$  and observe that there exists  $C > 0$  such that

$$\begin{aligned} \frac{1}{2\alpha}\|x_+ - x\|^2 &\leq h(x) - h(x_+) - \langle v, x_+ - x \rangle \\ &\leq C\|x_+ - x\| + \|v\|\|x_+ - x\|. \end{aligned}$$

---

<sup>3</sup>The strong (a) regularity property for  $f$  is a consequence of Equation (2.1.3) and the fact that  $\nabla g(x)$  is bounded near  $\bar{x}$ .

Consequently, we have  $\|x_+ - x\| \leq (2C + 2\|\nu\|)\alpha \leq 2(2C + \|\nu\|)\alpha$ , as desired. Second, the bound  $G_\alpha(x, \nu)$  follows trivially

$$\|G_\alpha(x, \nu)\| = \|x_+ - x\|/\alpha \leq 2(2C + \|\nu\|),$$

as desired. Finally, we bound  $\|w\|$ :

$$\|w\| = \|x - x_+\|/\alpha + \|\nabla g(x) + \nu\| \leq 4(2C + \|\nu\|),$$

as desired.  $\square$

### Verifying Assumption (A3)

We now verify Assumption (A3). Let us first assume that  $x_+ \notin B_\epsilon(\bar{x})$ . Then using Claim 10, we find that there exists  $C > 0$  such that

$$\text{dist}(x, \mathcal{M}) \leq C\|x - x_+\|.$$

Consequently, there exists  $C' > 0$  such that

$$\begin{aligned} \langle G_\alpha(x, \nu) - \nu, x - P_{\mathcal{M}}(x) \rangle &\geq \text{dist}(x, \mathcal{M}) - \text{dist}(x, \mathcal{M})(1 + \|G_\alpha(x, \nu) - \nu\|) \\ &\geq \text{dist}(x, \mathcal{M}) - C\|x - x_+\|(1 + C(1 + \|\nu\|) + \|\nu\|) \\ &\geq \text{dist}(x, \mathcal{M}) - C^2(1 + \|\nu\|)(1 + C(1 + \|\nu\|) + \|\nu\|)\alpha \\ &\geq \text{dist}(x, \mathcal{M}) - C'(1 + \|\nu\|)^2\alpha, \end{aligned}$$

as desired.

Now let us consider the case where  $x_+ \in B_\epsilon(\bar{x})$ . To that end, let  $v = \nabla g(x) + w$  and observe that

$$\begin{aligned} &\langle G_\alpha(x, \nu) - \nu, x - P_{\mathcal{M}}(x) \rangle \\ &= \langle v, x - P_{\mathcal{M}}(x) \rangle \\ &\geq \langle v, x_+ - P_{\mathcal{M}}(x_+) \rangle - \|v\| \|(x - P_{\mathcal{M}}(x)) - (x_+ - P_{\mathcal{M}}(x_+))\| \\ &\geq \langle \nabla g(x_+) + w, x_+ - P_{\mathcal{M}}(x_+) \rangle \\ &\quad - \underbrace{\|\nabla g(x) - \nabla g(x_+)\| \|x_+ - P_{\mathcal{M}}(x_+)\|}_{=:R_1(x)} - \underbrace{\|v\| \|(x - P_{\mathcal{M}}(x)) - (x_+ - P_{\mathcal{M}}(x_+))\|}_{=:R_2(x)} \quad (3.11.4) \end{aligned}$$

In what follows we will bound  $R_1$  and  $R_2$ .

*Claim 14.* There exists a constant  $C$  independent of  $x, \nu$ , and  $\alpha$  such that

$$R_1(x) \leq C(1 + \|\nu\|)^2\alpha$$

*Proof.* By Lipschitz continuity, there exists  $C > 0$  with

$$\|\nabla g(x_+) - \nabla g(x)\| \leq C\|x - x_+\| \leq C^2(1 + \|\nu\|)\alpha.$$

Therefore, the result follows from the bound  $\|x_+ - P_{\mathcal{M}}(x_+)\| \leq \|x_+ - \bar{x}\| \leq \epsilon$ .  $\square$

*Claim 15.* There exists a constant  $C$  independent of  $x, \nu$ , and  $\alpha$  such that

$$R_2(x) \leq C(1 + \|\nu\|)^2 \alpha$$

*Proof.* By local Lipschitz continuity of  $P_{\mathcal{M}}$ , we have

$$\|(x - P_{\mathcal{M}}(x)) - (x_+ - P_{\mathcal{M}}(x_+))\| \leq (1 + C)\|x - x_+\| \leq (1 + C)C(1 + \|\nu\|)\alpha.$$

Therefore, to complete the proof, we must only note that there exists  $C > 0$  such that

$$\|v\| \leq \|\nabla g(x) + w\| \leq C + C(1 + \|\nu\|),$$

as desired.  $\square$

Now we lower bound the dot product in 3.11.4.

*Claim 16.* There there exists  $C > 0$  such that

$$\langle \nabla g(x_+) + w, x_+ - P_{\mathcal{M}}(x_+) \rangle \geq \mu \text{dist}(x, \mathcal{M}) - (1 + \|\nu\|)o(\text{dist}(x, \mathcal{M})) - C(1 + \|\nu\|)^2 \alpha.$$

*Proof.* Recall that by the proximal aiming (3.2.8) property, we have

$$\begin{aligned} \langle \nabla g(x_+) + w, x_+ - P_{\mathcal{M}}(x_+) \rangle &\geq \mu \text{dist}(x_+, \mathcal{M}) - (1 + \|w\|)o(\text{dist}(x_+, \mathcal{M})) \\ &\geq \mu \text{dist}(x, \mathcal{M}) - \mu\|x - x_+\| - (1 + \|w\|)o(\text{dist}(x_+, \mathcal{M})) \\ &\geq \mu \text{dist}(x, \mathcal{M}) - \mu C(1 + \|\nu\|)\alpha - (1 + \|w\|)o(\text{dist}(x_+, \mathcal{M})) \end{aligned}$$

To complete the proof, combine the bound  $(1 + \|w\|) \leq C(1 + \|\nu\|)$  and the following calculation:

$$\begin{aligned} (1 + \|\nu\|)o(\text{dist}(x_+, \mathcal{M})) &\leq (1 + \|\nu\|)o(\text{dist}(x, \mathcal{M})) + (1 + \|\nu\|)\|x - x_+\| \\ &\leq (1 + \|\nu\|)o(\text{dist}(x, \mathcal{M})) + (1 + \|\nu\|)^2 \alpha, \end{aligned}$$

which holds as long as  $\epsilon$  is sufficiently small.  $\square$

Thus, to complete the proof of assumption Assumption (A3), note that Equation (3.11.4), together with the claims implies the existence of a constant  $C > 0$  such that

$$\langle G_{\alpha}(x, \nu) - \nu, x - P_{\mathcal{M}}(x) \rangle \geq \mu \text{dist}(x, \mathcal{M}) - (1 + \|\nu\|)o(\text{dist}(x, \mathcal{M})) - C(1 + \|\nu\|)^2 \alpha,$$

as desired.

### Verifying Assumption (A2)

We now verify Assumption (A2). First suppose that  $x_+ \in B_{\epsilon}(\bar{x})$ . To that end, notice that

$$\begin{aligned} &P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x))) \\ &= \underbrace{(P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))} - P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x_+))})(G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x)))}_{=:R_3} + \\ &\quad + \underbrace{P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x_+))}(G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x)))}_{=:R_4}. \end{aligned}$$



Let us first bound  $R_3$ : there exists  $C, C' > 0$  such that

$$\begin{aligned}\|R_3\| &\leq C\|P_{\mathcal{M}}(x_+) - P_{\mathcal{M}}(x)\|\|G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x))\| \\ &\leq C^2\|x_+ - x\|\|G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x))\| \\ &\leq C'(1 + \|\nu\|)^2\alpha.\end{aligned}$$

where the first and second inequalities follow from Lipschitz continuity of  $P_{\mathcal{M}}$  and  $\nabla P_{\mathcal{M}}$ , and the third inequality follows from Claim 13 and the boundedness of  $\nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x))$  on  $B_{\epsilon}(\bar{x})$ . Turning to  $R_4$ , we first use the following decomposition, which is a consequence of the expression  $G_{\alpha}(x, \nu) - \nu = \nabla g(x) + w$ :

$$\begin{aligned}R_4 &= P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x_+))}(G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x))) \\ &= \underbrace{P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x_+))}(\nabla g(x_+) + w - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x_+)))}_{=:R_5} + \underbrace{P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x_+))}(\nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x_+)) - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x)))}_{=:R_6} \\ &+ \underbrace{P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x_+))}(\nabla g(x) - \nabla g(x_+))}_{=:R_7}.\end{aligned}$$

We now bound each term in turn, beginning with  $\|R_5\|$ : by the strong (a) regularity for  $f$  and the boundedness of  $w$ , There exists  $C > 0$  such that

$$\|R_5\| \leq C\text{dist}(x_+, \mathcal{M}) \leq C(\text{dist}(x, \mathcal{M}) + \|x_+ - x\|) \leq C\text{dist}(x, \mathcal{M}) + C^2(1 + \|\nu\|)\alpha.$$

Now we bound  $\|R_6\|$ : by local Lipschitz continuity of  $\nabla f_{\mathcal{M}}$  and  $P_{\mathcal{M}}$ , there exists  $C > 0$  such that

$$\|R_6\| \leq C\|P_{\mathcal{M}}(x_+) - P_{\mathcal{M}}(x)\| \leq C^2\|x - x_+\| \leq C^3(1 + \|\nu\|)\alpha.$$

Finally we bound  $\|R_7\|$ : by local Lipschitz continuity of  $\nabla g$ , there exists  $C > 0$  such that

$$\|R_7\| \leq \|\nabla g(x) - \nabla g(x_+)\| \leq C\|x - x_+\| \leq C^2(1 + \|\nu\|)\alpha.$$

Putting these bounds together, we find that there exists  $C > 0$  such that

$$\begin{aligned}\|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x)))\| &\leq \|R_3\| + \|R_5\| + \|R_6\| + \|R_5\| + \|R_7\| \\ &\leq C(1 + \|\nu\|)^2(\text{dist}(x, \mathcal{M}) + \alpha),\end{aligned}$$

as desired.

Now suppose that  $x_+ \notin B_{\epsilon}(\bar{x})$ . Then we have  $\|x_+ - x\| \geq \epsilon/2$  since  $x \in B_{\epsilon/2}(\bar{x})$ . Therefore, there exists  $C > 0$  such that

$$\begin{aligned}\|P_{T_{\mathcal{M}}(P_{\mathcal{M}}(x))}(G_{\alpha}(x, \nu) - \nu - \nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x)))\| &\leq \|G_{\alpha}(x, \nu)\| + \|\nu\| + \|\nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x))\| \\ &\leq \frac{2}{\epsilon}(\|G_{\alpha}(x, \nu)\| + \|\nu\| + \|\nabla f_{\mathcal{M}}(P_{\mathcal{M}}(x))\|)\|x - x_+\| \\ &\leq C(1 + \|\nu\|)^2\alpha,\end{aligned}$$

as desired.

### 3.11.3 Proof of Corollary 3.5.2: asymptotic normality in nonlinear programming

To prove the result, we verify that  $f := f_0(x) + \delta_{\mathcal{X}}$  and  $G_\alpha(x, \nu) = (x - P_{\mathcal{X}}(x - \alpha(\nabla f(x) + \nu)))/\alpha$  satisfies Assumption A at  $\bar{x}$ . By Proposition 3.2.3, this will follow if we can verify Assumption C. To that end, we first note that Assumption (C1) holds by assumption. Second, notice  $\mathcal{X}$  is prox-regular at  $\bar{x}$ , so Condition (C4) holds. Third,  $g := f_0$  satisfies strong (a) along  $\mathcal{M}$  at  $\bar{x}$ , since it is smooth along  $\mathcal{M}$ . Moreover, by Corollary 2.4.2,  $\mathcal{X}$  satisfies strong (a) along  $\mathcal{M}$  at  $\bar{x}$  since it is cone-reducible at  $\bar{x}$ . Finally, by Corollary 2.1.5, the aiming condition holds for  $f$ , since  $f$  is weakly convex. Then, there exists a constant  $c > 0$  such that for any  $\delta > 0$ , there exists  $\epsilon > 0$  satisfying

$$\langle v, x - P_{\mathcal{M}}(x) \rangle \geq (c - \delta\sqrt{1 + \|v\|^2}) \cdot \text{dist}(x, \mathcal{M}), \quad (3.11.5)$$

for all  $x \in B_\epsilon(\bar{x})$  and all  $v \in \partial f(x)$ . Then since  $f_0$  is  $C^1$  smooth near  $\bar{x}$ , we have  $\partial f(x) = \nabla f_0(x) + N_{\mathcal{X}}(x)$ . Consequently, plugging in  $v = \nabla f_0(x) \in \partial f(x)$  and using the local boundedness of  $\nabla f_0(x)$ , we may find a  $\mu > 0$  such that

$$\langle \nabla f_0(x), x - P_{\mathcal{M}}(x) \rangle \geq \mu \text{dist}(x, \mathcal{M}),$$

for all  $x \in \mathcal{X}$  near  $\bar{x}$ . Recalling that  $g := f_0$ , completes the proof of the aiming assumption.

Finally we deal with convergence in the case that  $\sup_k \|x_k\| < +\infty$  with probability 1. In this case, it is well-known that with probability 1, every limit point of the sequence is first-order stationary, i.e., equal to  $\bar{x}$  (see eg; [26, Corollary 6.4.]<sup>4</sup>). Consequently,  $x_k$  almost surely converges to  $\bar{x}$ .

### 3.11.4 Proof of Corollary 3.6.3: avoiding active strict saddle via projected subgradient method

By Proposition 3.2.3 we need only show that Assumption C holds. To that end, note that Assumptions (C1), (C2), and (C4) hold by assumption. Next we prove (C3). Note that if  $g$  satisfies (b) along  $\mathcal{M}$ , then (C3) holds by Corollary 2.1.5. Next, suppose that  $g$  is prox-regular at  $x$ . In this case, since each  $x \in S$  is Fréchet critical and  $\mathcal{M}_x$  is an active manifold, it follows by Proposition 1.4.2 that for some  $\mu > 0$ , we have

$$g(y) - g(P_{\mathcal{M}_x}(y)) \geq \mu \text{dist}(y, \mathcal{M}),$$

near  $x$ . Consequently, for all  $v \in \partial_c g(x)$ , we have

$$\langle v, y - P_{\mathcal{M}_x}(y) \rangle \geq g(y) - g(P_{\mathcal{M}_x}(y)) - O(\|x - y\|^2) \geq (\mu/2) \text{dist}(x, \mathcal{M}),$$

for all  $y$  near  $x$ , verifying (C3).

---

<sup>4</sup>Although this corollary is stated for definable functions, its conclusion holds as long as the problem  $\mathcal{X}$  “admits a chain rule.” This easily follows for nonlinear programming, due to Clarke regularity as outlined in [26, Lemma 5.4.].

### 3.11.5 Proof of Corollary 3.6.4: avoiding active strict saddle via proximal gradient method

By Proposition 3.2.5, we need only show that Assumption D holds. Note that (D1), (D2), and (D3) hold by assumption. Thus, we need only verify (D4), which is immediate from (b)-regularity and Corollary 2.1.5.

### 3.11.6 Proofs of Corollaries 3.6.5, 3.6.6, and 3.6.7: saddle point avoidance for generic semialgebraic problems.

We first claim that the collection of limit points for all three methods is a connected set of composite Clarke critical points. To that end, note that by [26, Theorem 6.2/Corollary 6.4], we know that for each method, on the event the sequence  $x_k$  is bounded, all limit points are composite Clarke critical. We claim that the set of limit points is in fact connected. Indeed, by [7, Lemma 5(iii)], this will follow if

$$\lim_{k \rightarrow 0} \|x_{k+1} - x_k\| = \lim_{k \rightarrow 0} \|\alpha_k G_{\alpha_k}(x_k, \nu_k)\| = 0.$$

This in turn follows from [26, Lemma A.4, A.5, and A.6], which shows that  $G_{\alpha_k}(x_k, \nu_k) = w_k + \xi_k$ , where  $w_k$  is bounded and  $\sum_{k=1}^{\infty} \alpha_k \xi_k$  exists almost surely. Consequently, we have  $\|\alpha_k G_{\alpha_k}(x_k, \nu_k)\| = \alpha_k \|w_k + \xi_k\| \rightarrow 0$  almost surely, as desired.

Next we claim that the sequence  $x_k$  converges for all three methods. Indeed, by Corollaries 3.2.2, 3.2.4, and 3.2.6, it follows that each of the set of composite Clarke critical points for all three problems is finite for generic semialgebraic problems. Therefore, since the set of limit points of  $x_k$  is connected and discrete, it follows that on the event the sequence  $x_k$  is bounded, it must converge to a composite Clarke critical point.

To wrap up the proof, suppose that  $x_k$  converges to a composite limiting critical point. Then by Corollaries 3.2.2, 3.2.4, and 3.2.6 for any of the three methods, every composite limiting critical point of  $f$  is a composite Fréchet critical point which is either a local minimizer or an active strict saddle point at which Assumption A holds along the active manifold. By Theorem 3.6.2, the sequence  $x_k$  can converge to the such active strict saddle points only with probability zero. Therefore, the limit point must be a local minimizer, as desired.

### 3.11.7 Sequences and Stochastic Processes

**Lemmas from other works.**

**Lemma 3.11.1** (Robbins-Siegmund [79]). *Let  $A_k, B_k, C_k, D_k \geq 0$  be non-negative random variables adapted to the filtration  $\mathcal{F}_k$  and satisfying*

$$\mathbb{E}_k[A_{k+1}] \leq (1 + B_k)A_k + C_k - D_k.$$

*Then on the event  $\{\sum_k B_k < \infty, \sum_k C_k < \infty\}$ , there is a random variable  $A_\infty < \infty$  such that  $A_k \xrightarrow{a.s.} A_\infty$  and  $\sum_k D_k < \infty$  almost surely.*

**Lemma 3.11.2** (Conditional Borel-Cantelli [14]). *Let  $\{X_n: n \geq 1\}$  be a sequence of non-negative random variables defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $\{\mathcal{F}_n: n \geq 0\}$  be a sequence of sub- $\sigma$ -algebras of  $\mathcal{F}$ . Let  $M_n = \mathbb{E}[X_n | \mathcal{F}_{n-1}]$  for  $n \geq 1$ . If  $\{\mathcal{F}_n: n \geq 0\}$  is nondecreasing, i.e., it is a filtration, then  $\sum_{n=1}^{\infty} X_n < \infty$  almost surely on  $\{\sum_{n=1}^{\infty} M_n < \infty\}$ .*

**Lemma 3.11.3** ([13, Theorem A]). *Let  $\{\mathcal{F}_k\}$  be a filtration and let  $\{\epsilon_k\}$  be a sequence of random variables adapted to  $\mathcal{F}$  satisfying for all  $k$  the bound*

$$\mathbb{E}(\epsilon_{k+1}^2 | \mathcal{F}_k) < \infty \quad \text{and} \quad \mathbb{E}[\epsilon_{k+1} | \mathcal{F}_k] = 0.$$

*Let  $\{\Phi_k\}_k$  be another sequence of random variables adapted to  $\{\mathcal{F}_k\}$ . Let  $\{c_k\}$  be a deterministic sequence that is square summable but not summable. Suppose that the following hold almost surely on an event  $H$ :*

- *We have that Marcinkiewick-Zygmund conditions:*

$$\limsup_k \mathbb{E}[\epsilon_{k+1}^2 | \mathcal{F}_k] < \infty \quad \text{and} \quad \liminf_k \mathbb{E}[|\epsilon_{k+1}| | \mathcal{F}_k] > 0.$$

- *There exists sequences of random variables  $\{r_k\}$  and  $\{R_k\}$ , adapted to  $\mathcal{F}_k$  such that  $\Phi_k = r_k + R_k$  and*

$$\sum_k \|r_i\|^2 < \infty \quad \text{and} \quad \mathbb{E} \left[ 1_H \sum_{k=K}^{\infty} c_k |R_k| \right] = o \left( \left( \sum_{k=K}^{\infty} c_k^2 \right)^{1/2} \right).$$

*Then on  $H$  the series  $\sum_{k=1}^{\infty} c_k(\Phi_k + \epsilon_k)$  converges almost surely to a finite random variable  $L$ . Moreover, for any  $p \in \mathbb{N}$  and any  $\mathcal{F}_p$ -measurable random variable  $Y$  we have*

$$P(H \cap (L = Y)) = 0.$$

**Lemma 3.11.4** ([27, Exercise 5.3.35]). *Let  $M_k$  be an  $L^2$  martingale adapted to a filtration  $\mathcal{F}_k$  and let  $b_k \uparrow \infty$  is a positive deterministic sequence. Then if*

$$\sum_{k \geq 1} b_k^{-2} \mathbb{E}[(M_k - M_{k-1})^2 | \mathcal{F}_{k-1}] < +\infty,$$

*we have  $b_n^{-1} M_n \xrightarrow{a.s.} 0$ .*

**Lemma 3.11.5** (Kronecker Lemma). *Suppose  $\{x_k\}_k$  is an infinite sequence of real number such that*

$$\sum_{k=1}^{\infty} x_k = s$$

*exists and finite. Then for any divergent positive nondecreasing sequence  $\{b_k\}$ , we have*

$$\lim_{K \rightarrow \infty} \frac{1}{b_K} \sum_{k=1}^K b_k x_k = 0.$$

### Lemmas proved in this work

We will use the following two Lemmas on sequences. The proof of the following Lemma may be found in Appendix 3.11.8.

**Lemma 3.11.6.** *Fix  $k_0 \in \mathbb{N}$ ,  $c > 0$ , and  $\gamma \in (1/2, 1]$ . Suppose that  $\{X_k\}$ ,  $\{Y_k\}$ , and  $\{Z_k\}$  are nonnegative random variables adapted to a filtration  $\mathcal{F}_k$ . Suppose the relationship holds:*

$$\mathbb{E}_k[X_{k+1}] \leq (1 - ck^{-\gamma})X_k - Y_k + Z_k \quad \text{for all } k \geq k_0.$$

*Assume furthermore that  $c \geq 6$  if  $\gamma = 1$ . Then on the event  $\{\sum_{k=1}^{\infty} \frac{(k+1)^{2\gamma-1}}{\log(k+2)^2} Z_k < +\infty\}$ , there exists a random variable  $V < \infty$  satisfying the following:*

1. *The limit holds*

$$\frac{k^{2\gamma-1}}{\log(k+1)^2} X_k \xrightarrow{\text{a.s.}} V.$$

2. *The sum is finite*

$$\sum_{k=1}^{\infty} \frac{(k+1)^{2\gamma-1}}{\log(k+2)^2} Y_k < +\infty.$$

The proof of the following Lemma may be found in Appendix 3.11.8.

**Lemma 3.11.7.** *Fix  $k_0 \in \mathbb{N}$ ,  $c, C > 0$ , and  $\gamma \in (1/2, 1]$ . Suppose that  $\{s_k\}_k$  is a nonnegative sequence satisfying*

$$s_k \leq \frac{c}{12^\gamma} \quad \text{and} \quad s_{k+1}^2 \leq s_k^2 - ck^{-\gamma}s_k + Ck^{2\gamma}, \quad \text{for all } k \geq k_0,$$

*Then, there exists a constant  $C_{ub}$  depending only on  $c, C, \gamma$  and  $k_0$  such that*

$$s_k \leq C_{ub}k^{-\gamma}, \quad \forall k \geq 1.$$

The proof of the following Lemma may be found in Appendix 3.11.8.

**Lemma 3.11.8.** *Fix  $k_0 \in \mathbb{N}$ ,  $c, C > 0$ , and  $\gamma \in (1/2, 1]$ . Suppose that  $\{s_k\}_k$  is a nonnegative sequence satisfying*

$$s_{k+1} \leq (1 - ck^{-\gamma})s_k + Ck^{-2\gamma}, \quad \text{for all } k \geq k_0,$$

*Assume furthermore that  $c \geq 16$  if  $\gamma = 1$ . Then, there exists a constant  $C_{ub}$  depending only on  $c, C, \gamma$  and  $k_0$  such that*

$$s_k \leq C_{ub}k^{-\gamma}, \quad \forall k \geq 1.$$

### 3.11.8 Proof of Lemma 3.11.6

*Proof.* For all  $k \geq 0$ , define  $z_k := \frac{k^{2\gamma-1}}{\log(k+1)^2}$  and observe that

$$\mathbb{E}_k[z_{k+1}X_{k+1}] \leq z_{k+1}(1 - ck^{-\gamma})X_k - z_{k+1}Y_k + z_{k+1}Z_k \quad \text{for all } k \geq k_0.$$

Thus, the result will follow from Robbins-Siegmund Lemma 3.11.1 if  $z_{k+1}(1 - ck^{-\gamma}) \leq z_k$  for all sufficiently large  $k$ . To that end, notice that for sufficiently large  $k$ , we have

$$\left(\frac{k+1}{k}\right)^{2\gamma-1} \leq 1 + \frac{2(2\gamma-1)}{k}.$$

Therefore,

$$\frac{z_{k+1}}{z_k} \leq 1 + \frac{2(2\gamma-1)}{k} \quad \text{for all sufficiently large } k.$$

Now we deal separately with the cases  $\gamma < 1$  and  $\gamma = 1$ . First suppose first that  $\gamma < 1$ . Then there exists a constant  $C' > 0$  such that

$$\frac{1}{1 - ck^{-\gamma}} \geq 1 + \frac{C'}{k^\gamma}, \quad \text{for all sufficiently large } k.$$

Consequently,  $z_{k+1}/z_k \leq (1 - ck^{-\gamma})^{-1}$  for all sufficiently large  $k$ , as desired.

Now assume that  $\gamma = 1$ .

$$\frac{1}{1 - ck^{-1}} \geq 1 + \frac{c}{2k^\gamma}, \quad \text{for all sufficiently large } k.$$

Consequently,  $z_{k+1}/z_k \leq (1 - ck^{-1})^{-1}$  for all large  $k$ , provided that whenever  $c \geq 6$ .  $\square$

### Proof of Lemma 3.11.7

*Proof.* It suffices to exhibit  $C'_{\text{ub}} > 0$  such that

$$s_k \leq C'_{\text{ub}} k^{-\gamma} \quad \text{for all sufficiently large } k \geq k_0.$$

To that end, choose  $k_1$  large enough that the following two bounds hold:

1.  $C'_{\text{ub}} := \max \left\{ \frac{ck_1^\gamma}{12\gamma}, 2\sqrt{C}, \frac{4C}{c} \right\} = \frac{ck_1^\gamma}{12\gamma}$
2.  $\left(\frac{k_1+1}{k_1}\right)^{2\gamma} \leq \min \left\{ 2, 1 + \frac{3\gamma}{k_1} \right\}.$

Then by assumption, we have

$$k^{2\gamma} s_{k+1}^2 \leq k^{2\gamma} s_k^2 - ck^\gamma s_k + C \quad \text{for all } k \geq k_1.$$

Denoting  $t_k := k^\gamma s_k$ , we obtain the following bound for all  $k \geq k_1$ :

$$t_{k+1}^2 \leq \left(\frac{k+1}{k}\right)^{2\gamma} (t_k^2 - ct_k + C) \leq \left(\frac{k_1+1}{k_1}\right)^{2\gamma} (t_k^2 - ct_k + C) \quad (3.11.6)$$

Thus the claim will follow if  $t_k \leq C'_{\text{ub}}$  for all  $k \geq k_1$ . We prove the claim by induction. First the case  $k = k_1$  holds by definition of  $C'_{\text{ub}}$ . Now suppose  $t_k \leq C'_{\text{ub}}$  for some  $k \geq k_1$  and consider two cases

First suppose  $t_k \in [0, \frac{1}{2}C'_{\text{ub}}]$ . By (3.11.6) and definition of  $C'_{\text{ub}}$ , we have

$$\begin{aligned} t_{k+1}^2 &\leq \left(\frac{k_1+1}{k_1}\right)^{2\gamma} (t_k^2 + C) \\ &\leq \left(\frac{k_1+1}{k_1}\right)^{2\gamma} \left(\frac{1}{4}C'^2_{\text{ub}} + \frac{1}{4}C'^2_{\text{ub}}\right) \\ &\leq C'^2_{\text{ub}}. \end{aligned}$$

Second, suppose  $t_k \in [\frac{1}{2}C'_{\text{ub}}, C'_{\text{ub}}]$ . By (3.11.6) and definition of  $C'_{\text{ub}}$ , we have

$$\begin{aligned} t_{k+1}^2 &\leq \left(\frac{k_1+1}{k_1}\right)^{2\gamma} (t_k^2 - ct_k + C) \\ &\leq \left(\frac{k_1+1}{k_1}\right)^{2\gamma} \left(C'^2_{\text{ub}} - \frac{cC'_{\text{ub}}}{2} + C\right) \\ &\leq \left(\frac{k_1+1}{k_1}\right)^{2\gamma} \left(C'^2_{\text{ub}} - \frac{cC'_{\text{ub}}}{4}\right) \\ &= C'_{\text{ub}} \left(\frac{k_1+1}{k_1}\right)^{2\gamma} \left(C'_{\text{ub}} - \frac{c}{4}\right) \end{aligned}$$

We claim that  $\left(\frac{k_1+1}{k_1}\right)^{2\gamma} (C'_{\text{ub}} - \frac{c}{4}) \leq C'_{\text{ub}}$ . Indeed, we have

$$\begin{aligned} &\left(\frac{k_1+1}{k_1}\right)^{2\gamma} \left(C'_{\text{ub}} - \frac{c}{4}\right) \\ &\leq \left(1 + \frac{3\gamma}{k_1}\right) \left(C'_{\text{ub}} - \frac{c}{4}\right) \\ &\leq C'_{\text{ub}} + \frac{3\gamma C'_{\text{ub}}}{k_1} - \frac{c}{4} \\ &\leq C'_{\text{ub}} + \frac{c}{4k_1^{1-\gamma}} - \frac{c}{4} \\ &\leq C'_{\text{ub}}, \end{aligned}$$

as desired. This completes the induction. □

### Proof of Lemma 3.11.8

*Proof.* It suffices to exhibit  $C_{\text{ub}} > 0$  such that

$$s_k \leq C_{\text{ub}} k^{-\gamma} \quad \text{for all sufficiently large } k \geq k_0.$$

To that end, choose  $k_1$  large enough that the following two bounds hold:

1.  $\left(\frac{k+1}{k}\right)^\gamma \leq 1 + \frac{2\gamma}{k} \leq 2$  for all  $k \geq k_1$ .
2.  $k_1^{1-\gamma} \geq \frac{16\gamma}{c}$  if  $\gamma \in (\frac{1}{2}, 1)$ .

Now let  $t_k = s_k k^\gamma$ , then we rewrite the above inequality as

$$t_{k+1} \leq \left(\frac{k+1}{k}\right)^\gamma \left[ (1 - ck^{-\gamma})t_k + \frac{C}{k^\gamma} \right], \quad \text{for all } k \geq k_0. \quad (3.11.7)$$

Let  $C_{\text{ub}} = \max\{s_{k_1} k_1^\gamma, 4C, \frac{8C}{c}\}$ . By definition of  $C_{\text{ub}}$ , we know that

$$t_{k_1} = s_{k_1} k_1^\gamma \leq C_{\text{ub}}.$$

For the induction step, we consider two cases.

First suppose  $t_k \in [0, \frac{1}{4}C_{\text{ub}}]$ . By (3.11.7) and definition of  $C_{\text{ub}}$ , we have

$$\begin{aligned} t_{k+1} &\leq \left(\frac{k_0+1}{k_0}\right)^\gamma (t_k + C) \\ &\leq \left(\frac{k_0+1}{k_0}\right)^\gamma \left(\frac{1}{4}C_{\text{ub}} + \frac{1}{4}C_{\text{ub}}\right) \\ &\leq C_{\text{ub}}. \end{aligned}$$

Second, suppose  $t_k \in [\frac{1}{4}C_{\text{ub}}, C_{\text{ub}}]$ . By (3.11.7) and definition of  $\tilde{C}_{\text{ub}, k_0, \bar{x}}$ , we have

$$\begin{aligned} t_{k+1} &\leq \left(\frac{k+1}{k}\right)^\gamma \left(t_k - \frac{ct_k}{k^\gamma} + \frac{C}{k^\gamma}\right) \\ &\leq \left(\frac{k+1}{k}\right)^\gamma \left(C_{\text{ub}} - \frac{cC_{\text{ub}}}{4k^\gamma} + \frac{C}{k^\gamma}\right) \\ &\leq \left(\frac{k+1}{k}\right)^\gamma \left(C_{\text{ub}} - \frac{cC_{\text{ub}}}{8k^\gamma}\right) \\ &= C_{\text{ub}} \left(\frac{k+1}{k}\right)^{2\gamma} \left(1 - \frac{c}{8k^\gamma}\right) \end{aligned}$$

We claim that  $\left(\frac{k+1}{k}\right)^\gamma \left(1 - \frac{c}{8k^\gamma}\right) \leq 1$ . Indeed, we have

$$\begin{aligned} &\left(\frac{k+1}{k}\right)^\gamma \left(1 - \frac{c}{8k^\gamma}\right) \\ &\leq \left(1 + \frac{2\gamma}{k}\right) \left(1 - \frac{c}{8k^\gamma}\right) \\ &\leq 1 + \frac{2\gamma}{k} - \frac{c}{8k^\gamma} \end{aligned}$$

When  $\gamma = 1$ ,  $1 + \frac{2\gamma}{k} - \frac{c}{8k^\gamma}$  by our assumption on  $c$ . When  $\gamma \in (\frac{1}{2}, 1)$ ,  $1 + \frac{2\gamma}{k} - \frac{c}{8k^\gamma} \leq 1$  by our choice of  $k_0$ . This completes the induction.  $\square$