# Log-domain interior-point methods for convex quadratic programming 

Frank Permenter

December 6, 2022


#### Abstract

Applying an interior-point method to the central-path conditions is a widely used approach for solving quadratic programs. Reformulating these conditions in the log-domain is a natural variation on this approach that to our knowledge is previously unstudied. In this paper, we analyze log-domain interior-point methods and prove their polynomial-time convergence. We also prove that they are approximated by classical barrier methods in a precise sense and provide simple computational experiments illustrating their superior performance.


## 1 Introduction

Interior-point methods (IPMs) are widely used numerical algorithms for solving convex quadratic programs (QPs) of the form

$$
\begin{array}{ll}
\operatorname{minimize} & \frac{1}{2} x^{T} W x+c^{T} x  \tag{1}\\
\text { subject to } & A x+b \geq 0
\end{array}
$$

where $x \in \mathbb{R}^{n}$ is the decision variable, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m}$ define linear inequality constraints, and $W \in \mathbb{R}^{n \times n}$ is a symmetric, positive semidefinite matrix that, together with $c \in \mathbb{R}^{n}$, defines a convex, quadratic objective. IPMs solve (1) by tracking the solution $(x, s, \lambda) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}^{m}$ to the central-path conditions

$$
\begin{equation*}
A^{T} \lambda=W x+c, \quad s=A x+b, \quad \lambda \geq 0, s \geq 0, \quad s_{i} \lambda_{i}=\mu \forall i \in\{1,2, \ldots, m\} \tag{2}
\end{equation*}
$$

for a decreasing sequence of $\mu>0$. When $\mu=0$, these are precisely the Karush-Kuhn-Tucker (KKT) optimality conditions for (1). Hence, by gradually reducing $\mu$ to zero, IPMs produce an optimal solution $x$ to (1) along with an optimal constraint slack $s$ and corresponding vector $\lambda$ of Lagrange multipliers. IPMs are efficient in practice and have several high quality implementations $[15,13]$. They are also efficient in theory, requiring just $\mathcal{O}(\sqrt{m})$ iterations to solve the QP (1) to fixed accuracy, where the per-iteration cost is the solution of an $n \times n$ linear system $[19,18,1]$.

Success of IPMs requires existence and uniqueness of the central path, i.e., of solutions $(x, s, \lambda)$ to (2) for all $\mu>0$. Using standard arguments (e.g., [8, Theorem 1]), this holds by further assuming the QP (1) satisfies the following conditions.

Assumption 1. The following conditions hold:

- There exist $x \in \mathbb{R}^{n}$ and $s \in \mathbb{R}^{m}$ with $s>0$ satisfying $s=A x+b$.
- For all $\beta \in \mathbb{R}$, the sublevel set $\left\{x \in \mathbb{R}^{n}: \frac{1}{2} x^{T} W x+c^{T} x \leq \beta, A x+b \geq 0\right\}$ is bounded.
- $A^{T} A+W \succ 0$, i.e., $A^{T} A+W$ is positive definite.

We assume that these conditions hold throughout. Note that these conditions impose no direct constraint on the shape of $A \in \mathbb{R}^{m \times n}$, i.e., we may have that $m=n, m<n$, or $m>n$. Further, if $W=0$, then the condition $A^{T} A \succ 0$ is the usual assumption for linear programming that the constraint matrix of $A x+b \geq 0$ is full column rank; see, e.g., [2].

### 1.1 Log-domain interior-point methods

The set of nonnegative $(s, \lambda)$ satisfying $s_{i} \lambda_{i}=\mu$ for $i \in\{1,2, \ldots, m\}$ is easily parameterized in the log-domain: letting $e^{v} \in \mathbb{R}^{m}$ denote elementwise exponentiation, this condition holds if and only if $\lambda=\sqrt{\mu} e^{v}$ and $s=\sqrt{\mu} e^{-v}$ for some $v \in \mathbb{R}^{m}$. This $v$-parametrization of $s$ and $\lambda$ yields the following log-domain reformulation of the central-path conditions (2)

$$
\begin{equation*}
\sqrt{\mu} A^{T} e^{v}=W x+c, \quad \sqrt{\mu} e^{-v}=A x+b \tag{3}
\end{equation*}
$$

and a template log-domain interior-point method for solving the QP (1):

- Update $(v, x)$ by applying Newton's method to (3) for fixed $\mu$.
- Reduce $\mu$ and repeat.

This paper studies this template algorithm, which, to our knowledge, has not previously appeared in the QP literature. As we show, there exist concrete instantiations that are both practically and theoretically efficient. In particular, we provide a short-step algorithm and prove the typical $\mathcal{O}(\sqrt{m})$ iteration bound. We also provide a long-step version and illustrate its practical performance.

### 1.2 Prior work

The literature on quadratic programming is vast and we will not attempt to cite it completely We do note that IPMs with polynomial-time complexity trace to $[10,9]$ and IPMs with $\mathcal{O}(\sqrt{m})$ iteration bounds include $[12,6,7,11]$. One can also obtain a $\mathcal{O}(\sqrt{m})$ bound by invoking the selfconcordance of a suitable barrier function; see [14]. For linear objectives $(W=0)$, our algorithms are special cases of geodesic interior-point methods [16], recent techniques for minimizing a linear function subject to symmetric cone inequalities. Indeed, our main analysis task is showing that key convergence results of [16] still hold when a quadratic objective term $x^{T} W x$ is included.

Linear updates of the $\log$ parameter $v$ are of course multiplicative updates of $s=\sqrt{\mu} e^{-v}$ and $\lambda=\sqrt{\mu} e^{v}$. Algorithms based on multiplicative updates have been developed for restricted families of QPs, e.g., linear programs $(W=0)$, nonnegative least-squares $(A=I, b=0)$, and model-predictive control; see, e.g., $[3,17,4]$. We emphasize that in each of these algorithms, the updates are distinct from ours and are designed in different ways. In particular, they are only applied to one variable, $\lambda$ or $s$, and are not based on the log-transformation (3) of the central-path conditions.

### 1.3 Outline of contributions

The contributions of this paper are organized as follows. Section 2 analyzes the application of Newton's method to the log-domain central-path equations (3), establishing a globallyconvergent step-size rule and a local region of quadratic convergence. Building on this analysis, Section 3 provides two algorithms for solving the QP (1) based on two different $\mu$-update rules. The first is a short-step algorithm that reduces $\mu$ at a fixed rate and terminates after at most $\mathcal{O}(\sqrt{m})$ Newton iterations. The second is a long-step algorithm that employs more aggressive $\mu$-updates via line-search. Section 4 provides theoretical and computational comparisons with barrier methods. In particular, we show that these methods are, in a precise sense, approximations of the presented log-domain IPMs.

## 2 Newton's method

Applying Newton's method to the log-domain central-path equations (3) proceeds by Taylorapproximating the exponential functions $e^{v}$ and $e^{-v}$. Letting $x \circ y$ denote elementwise multiplication of $x, y \in \mathbb{R}^{m}$, these approximations take the form

$$
e^{v+d} \approx e^{v}+e^{v} \circ d, \quad e^{-(v+d)} \approx e^{-v}-e^{-v} \circ d
$$

The Newton direction $d(v, \mu) \in \mathbb{R}^{m}$ and an associated $x(v, \mu) \in \mathbb{R}^{n}$ are then defined by substituting these approximations into (3).
Definition 2.1. For fixed $\mu>0$ and $v \in \mathbb{R}^{m}$, the Newton direction $d(v, \mu)$ and associated $x(v, \mu)$ are the $d \in \mathbb{R}^{m}$ and $x \in \mathbb{R}^{n}$ satisfying

$$
\sqrt{\mu} A^{T}\left(e^{v}+e^{v} \circ d\right)=W x+c, \quad \sqrt{\mu}\left(e^{-v}-e^{-v} \circ d\right)=A x+b
$$

Note that when $d(v, \mu)=0$, we obtain a point $(x, s, \lambda)$ on the central-path by taking $x=$ $x(v, \mu), s=\sqrt{\mu} e^{-v}$, and $\lambda=\sqrt{\mu} e^{v}$. Further, $x(v, \mu)$ is a "good" approximate solution of the QP (1) when $\mu$ and $\|d(v, \mu)\|$ are sufficiently small. (Exact error bounds will be given in Section 3.2.) It remains to prove that $d(v, \mu)$ and $x(v, \mu)$ are well-defined for all $\mu>0$ and $v \in \mathbb{R}^{m}$. For this, we next show that $d(v, \mu)$ is a function of $x(v, \mu)$ and that $x(v, \mu)$ is the unique solution of a consistent linear system. In particular, we show that this linear system is of the form $S x=f$ for $S \succ 0$, i.e., for $S$ symmetric and positive definite.
Theorem 2.1. For all $v \in \mathbb{R}^{m}$ and $\mu>0$, the Newton direction $d(v, \mu)$ and point $x(v, \mu)$ satisfy

$$
d=\mathbf{1}-\frac{1}{\sqrt{\mu}} e^{v} \circ(A x+b)
$$

where $\mathbf{1} \in \mathbb{R}^{m}$ denotes the vector of all ones. Moreover, $x(v, \mu)$ is the unique solution of

$$
\left(A^{T} Q(v) A+W\right) x=2 \sqrt{\mu} A^{T} e^{v}-\left(c+A^{T} Q(v) b\right)
$$

where $Q(v) \in \mathbb{R}^{m \times m}$ is the diagonal matrix with $[Q(v)]_{i i}=e^{2 v_{i}}$. Further, $A^{T} Q(v) A+W \succ 0$.
Proof. Rearranging $\sqrt{\mu}\left(e^{-v}-e^{-v} \circ d\right)=A x+b$, we conclude that

$$
d=\mathbf{1}-\frac{1}{\sqrt{\mu}} e^{v} \circ(A x+b) .
$$

Substituting into $\sqrt{\mu} A^{T}\left(e^{v}+e^{v} \circ d\right)=W x+c$ yields

$$
\begin{aligned}
W x+c & =\sqrt{\mu} A^{T} e^{v} \circ(\mathbf{1}+d) \\
& =\sqrt{\mu} A^{T} e^{v} \circ\left(\mathbf{1}+\mathbf{1}-\frac{1}{\sqrt{\mu}} e^{v} \circ(A x+b)\right) .
\end{aligned}
$$

Rearranging and using $Q(v) A x=e^{v} \circ\left(e^{v} \circ A x\right)$ shows that

$$
\left(A^{T} Q(v) A+W\right) x=2 \sqrt{\mu} A^{T} e^{v}-\left(c+A^{T} Q(v) b\right)
$$

Uniqueness of $x$ follows because $A^{T} Q(v) A+W$ is positive definite under our assumption that $A^{T} A+W$ is positive definite (Assumption 1). To see this, suppose that $\left(A^{T} Q(v) A+W\right) z=0$ for nonzero $z$. Then, $W z=0$ and $A^{T} Q(v) A z=0$. But this implies that $Q(v)^{1 / 2} A z=0$, which, in turn means that $A z=0$ since $Q(v)^{1 / 2}$ is invertible. We conclude that $\left(A^{T} A+W\right) z=0$, a contradiction.

The remainder of this section studies convergence of the Newton iterations

$$
v_{i+1}=v_{i}+\frac{1}{\alpha_{i}} d\left(v_{i}, \mu\right)
$$

under a simple step-size rule for choosing $\alpha_{i} \in \mathbb{R}$. We will show global convergence to centered points.

Definition 2.2 (Centered points). For $\mu>0$, the centered point $\hat{v}(\mu)$ is the $v \in \mathbb{R}^{m}$ that, for some $x \in \mathbb{R}^{n}$, solves the log-domain central-path equations (3).
Following [16], we will measure the distance of an iterate $v_{i}$ to $\hat{v}(\mu)$ using divergence.
Definition 2.3 (Divergence [16]). The divergence $h(u, v)$ of $(u, v) \in \mathbb{R}^{m} \times \mathbb{R}^{m}$ is

$$
h(u, v):=\left\langle e^{u}, e^{-v}\right\rangle+\left\langle e^{-u}, e^{v}\right\rangle-2 m .
$$

For fixed $\mu>0$, the function $h_{\mu}: \mathbb{R}^{m} \rightarrow \mathbb{R}$ denotes the map $v \mapsto h(\hat{v}(\mu), v)$.
While divergence is not a metric, it does have a set of properties useful for convergence analysis.
Lemma 2.1. The following properties hold for all $u, v \in \mathbb{R}^{m}$ and $\mu>0$.
(a) $h(u, v)=h(v, u)$ and $h(u, v) \geq 0$.
(b) $h(u, v)=0$ if and only if $u=v$. In particular, $h(u, v)=-2 m+\sum_{i=1}^{m} 2 \cosh \left(v_{i}-u_{i}\right)$.
(c) $h_{\mu}: \mathbb{R}^{m} \rightarrow \mathbb{R}$ is strongly convex. In particular, $\frac{1}{2} \nabla^{2} h_{\mu}(v) \succeq I$.

Leveraging these properties, Section 2.1 provides a step-size rule for which $h_{\mu}\left(v_{i}\right)<h_{\mu}\left(v_{i-1}\right)$ for all iterations $i$. Building on this, Section 2.2 shows that the sequence $v_{0}, v_{1}, v_{2}, \ldots$ converges to the centered point $\hat{v}(\mu)$ from an arbitrary initial $v_{0} \in \mathbb{R}^{m}$. Finally, Section 2.3 shows quadratic convergence when $h_{\mu}\left(v_{0}\right) \leq \frac{1}{2}$. As we will point out, some statements generalize previous results for linear optimization [16] to the quadratic program (1).

### 2.1 Step-size rule

Our step-size rule arises from bounds on the directional derivatives of divergence $h_{\mu}(v)$. Towards stating them, fix $v \in \mathbb{R}^{m}$ and $\mu>0$ and for brevity let $d \in \mathbb{R}^{m}$ denote the Newton direction $d(v, \mu)$. Assume that $d \neq 0$ or, equivalently, that $v \neq \hat{v}(\mu)$. Finally, let $f: \mathbb{R} \rightarrow \mathbb{R}$ denote the restriction of $h_{\mu}(v)$ to the line induced by $v$ and $d$, i.e.,

$$
f(t):=h_{\mu}(v+t d)
$$

The next lemma provides bounds on $f^{\prime}(0)$ and $f^{\prime \prime}(t)$ and generalizes [16, Lemma 3.4] and [16, Lemma 3.6].
Lemma 2.2. The following statements hold.

- $f^{\prime}(0) \leq-\left(f(0)+\|d\|^{2}\right)$.
- $f^{\prime \prime}(t) \leq\|d\|_{\infty}^{2} f(t)+2\|d\|^{2}$.
- For all intervals $[a, b] \subset \mathbb{R}$, we have $\sup _{\zeta \in[a, b]} f^{\prime \prime}(\zeta) \leq \max _{\zeta \in\{a, b\}}\left(\|d\|_{\infty}^{2} f(\zeta)+2\|d\|^{2}\right)$.

Proof. For brevity, let $w=e^{v}, \hat{w}=e^{\hat{v}(\mu)}$ and $k=\sqrt{\mu}$. Letting $z^{-1}$ denote elementwise inversion, define $p:=\left\langle w^{-1} \circ(\mathbf{1}-d)-\hat{w}^{-1}, w \circ(\mathbf{1}+d)-\hat{w}\right\rangle$. Expanding, we conclude that

$$
\begin{aligned}
p & =\langle(\mathbf{1}-d),(\mathbf{1}+d)\rangle-\left\langle w^{-1} \circ \hat{w},(\mathbf{1}-d)\right\rangle-\left\langle w \circ \hat{w}^{-1},(\mathbf{1}+d)\right\rangle+m \\
& =m-\|d\|^{2}-\left\langle w^{-1} \circ \hat{w},(\mathbf{1}-d)\right\rangle-\left\langle w \circ \hat{w}^{-1},(\mathbf{1}+d)\right\rangle+m \\
& =-\left(\left\langle\hat{w}, w^{-1}\right\rangle+\left\langle\hat{w}^{-1}, w\right\rangle-2 m\right)-\|d\|^{2}-\left\langle w \circ \hat{w}^{-1}-w^{-1} \circ \hat{w}, d\right\rangle \\
& =-f(0)-\|d\|^{2}-f^{\prime}(0)
\end{aligned}
$$

We now show that $p \geq 0$ for the Newton direction $d$, which will prove the first statement. Since $\hat{w}$ solves (2), we have for some $\hat{x}$, that $b=k \hat{w}^{-1}-A \hat{x}$ and $c=k A^{T} \hat{w}-W \hat{x}$. Substituting these expressions for $b$ and $c$ into the definition of $d$, we have, for some $x$, that

$$
k w^{-1} \circ(\mathbf{1}-d)-k \hat{w}^{-1}=A(x-\hat{x}), \quad k A^{T}(w \circ(\mathbf{1}+d)-\hat{w})=W(x-\hat{x})
$$

Using the first equation, we conclude that $p=\left\langle\frac{1}{k} A(x-\hat{x}), w \circ(\mathbf{1}+d)-\hat{w}\right\rangle$. Combining with the second yields $p=\left\langle\frac{1}{k}(x-\hat{x}), \frac{1}{k} W(x-\hat{x})\right\rangle \geq 0$, which proves the first statement. Proof of the second statement is identical to [16, Lemma 3.4.c]. The third statement follows from the second and convexity of $f(t)$.

Combining this lemma with the inequality

$$
\begin{equation*}
f(t) \leq f(0)+f^{\prime}(0) t+\frac{1}{2} \sup _{\zeta \in[0, t]} f^{\prime \prime}(\zeta) t^{2} \tag{4}
\end{equation*}
$$

yields a piecewise step-size rule for selecting $t$ such that $f(t)<f(0)$. This rule is parameterized by $0<\beta<1$ which, along with $\|d\|_{\infty}^{2}$, controls the transition from full to damped steps.
Theorem 2.2. For $\beta \in(0,1)$, let $\alpha=\max \left(1, \frac{1}{2 \beta}\|d\|_{\infty}^{2}\right)$. The following statements hold.
(a) $f\left(\frac{1}{\alpha}\right)<f(0)$
(b) If $\alpha=1$, then $f(1) \leq \frac{1}{2}\|d\|_{\infty}^{2} f(0) \leq \beta f(0)$.

Proof. Let $\hat{t} \geq 0$ denote the smallest $t$ for which $f(t)=f(0)$. By strong convexity (Lemma 2.1), we have that $f(t)<f(0)$ for all $t \in(0, \hat{t})$ since $d$ is a descent direction (Lemma 2.2). Further $\hat{t}>0$. Towards bounding $\hat{t}$, we first note that the combination of (4) with Lemma 2.2 implies that for all $t$,

$$
\begin{equation*}
f(t) \leq f(0)-t\left(f(0)+\|d\|^{2}\right)+\frac{1}{2}\left(\|d\|_{\infty}^{2} \max (f(0), f(t))+2\|d\|^{2}\right) t^{2} \tag{5}
\end{equation*}
$$

Substituting $t=\hat{t}$ and using $f(0)=f(\hat{t})$, we conclude that

$$
\hat{t}\left(\frac{\|d\|_{\infty}^{2}}{2} f(0)+\|d\|^{2}\right) \geq f(0)+\|d\|^{2}
$$

Hence, $t<\hat{t}$ if $t\left(\frac{\|d\|_{\infty}^{2}}{2} f(0)+\|d\|^{2}\right)<f(0)+\|d\|^{2}$, which holds if $t=\min \left(1, \frac{2 \beta}{\|d\|_{\infty}^{2}}\right)$, proving item (a). Item (b) follows by substituting $t=1$ and $\max (f(0), f(1))=f(0)$ into (5).

### 2.2 Global convergence

Newton iterations strictly decrease the divergence $h_{\mu}(v)$ under the step-size rule of Theorem 2.2. Combined with the strong convexity of $h_{\mu}$, this implies convergence to the centered point $\hat{v}(\mu)$.
Theorem 2.3. Fix $0<\beta<1$ and $\mu>0$. For all $v_{0} \in \mathbb{R}^{m}$, the Newton iterations $v_{i+1}=$ $v_{i}+\frac{1}{\alpha_{i}} d\left(v_{i}, \mu\right)$ with step-size rule $\alpha_{i}=\max \left\{1, \frac{1}{2 \beta}\left\|d\left(v_{i}, \mu\right)\right\|_{\infty}^{2}\right\}$ converge to the centered point $\hat{v}(\mu)$.

Proof. By choice of $\alpha_{i}$ and Theorem 2.2-(a), we have that $h_{\mu}\left(v_{i}\right)$ strictly decreases. In addition, $h_{\mu}\left(v_{i}\right) \geq 0$ for all $v_{i}$; hence, it converges to some nonnegative number $\delta$. We will show that $\delta=0$, which implies that $v_{i}$ converges to $\hat{v}(\mu)$ by Lemma 2.1-(b). To begin, note that all iterations $v_{i}$ are contained in the set $\Omega:=\left\{v \in \mathbb{R}^{m}: \delta \leq h_{\mu}(v) \leq h_{\mu}\left(v_{0}\right)\right\}$. But $\Omega$ is compact since $h_{\mu}$ is strongly convex (Lemma 2.1). Letting $\alpha(v):=\max \left\{1, \frac{1}{2 \beta}\|d(v, \mu)\|_{\infty}^{2}\right\}$, we conclude that the continuous function $D(v):=h_{\mu}(v)-h_{\mu}\left(v+\frac{1}{\alpha(v)} d(v, \mu)\right)$ obtains its infimum $D^{*} \in \mathbb{R}$ on $\Omega$. But if $\delta>0$, then $D^{*}>0$, which implies that $h_{\mu}\left(v_{m}\right) \leq h_{\mu}\left(v_{0}\right)-m D^{*}<0$ for all $m>h_{\mu}\left(v_{0}\right) / D^{*}$, a contradiction since $h_{\mu}\left(v_{m}\right) \geq 0$. Hence, $\delta=0$.

### 2.3 Local quadratic convergence

Theorem 2.2 states that a full Newton-step $v_{i+1}=v_{i}+d\left(v_{i}, \mu\right)$ decreases the divergence $h_{\mu}\left(v_{i}\right)$ by a factor of at least $\frac{1}{2}\left\|d\left(v_{i}, \mu\right)\right\|_{\infty}^{2}$. The next lemma, which generalizes [16, Corollary 3.1], shows that we can also upper-bound $\left\|d\left(v_{i}, \mu\right)\right\|^{2}$, and hence $\left\|d\left(v_{i}, \mu\right)\right\|_{\infty}^{2}$, using $h_{\mu}\left(v_{i}\right)$.

Lemma 2.3. For all $v \in \mathbb{R}^{m}$ and $\mu>0$, it holds that $\|d(v, \mu)\|^{2} \leq h_{\mu}(v)(1+\|d(v, \mu)\|)$.

Proof. For brevity, let $d$ denote $d(v, \mu)$ and let $a=e^{v-\hat{v}(\mu)}$ and $g=a-a^{-1}$. As in Section 2.2, let $f(t)=h_{\mu}(v+t d)$ such that $f(0)=h_{\mu}(v)$. Observing that $g$ is the gradient of $h_{\mu}(v)$ with respect to $v$, we have, by Lemma 2.2 and Cauchy-Schwartz, that

$$
f(0)+\|d\|^{2} \leq\left|f^{\prime}(0)\right|=|\langle g, d\rangle| \leq\|g\|\|d\|
$$

We also have that $\|g\|^{2} \leq f(0)^{2}+4 f(0)$ given that
$f(0)=\left\|a+a^{-1}-2 \mathbf{1}\right\|_{1} \geq\left\|a+a^{-1}-2 \mathbf{1}\right\|_{2}=\sqrt{\left\|a-a^{-1}\right\|^{2}-4\left\langle\mathbf{1}, a+a^{-1}-2 \mathbf{1}\right\rangle}=\sqrt{\|g\|^{2}-4 f(0)}$.
We conclude that $f(0)+\|d\|^{2} \leq\|d\| \sqrt{f(0)^{2}+4 f(0)}$. Squaring each side and rearranging yields

$$
0 \leq\|d\|^{2}\left(f(0)^{2}+4 f(0)\right)-\left(f(0)+\|d\|^{2}\right)^{2}=\left(-\|d\|^{2}+f(0)(1+\|d\|)\right)\left(\|d\|^{2}+f(0)(\|d\|-1)\right)
$$

This shows that if $f(0)(1+\|d\|)<\|d\|^{2}$, then $\|d\|^{2} \leq f(0)(1-\|d\|)$, which in turn implies that

$$
f(0)(1+\|d\|)<f(0)(1-\|d\|)
$$

which is impossible. Hence, $f(0)(1+\|d\|) \geq\|d\|^{2}$, as desired.

Combining this with Theorem 2.2 yields the following quadratic convergence result, which generalizes [16, Theorem 3.4].
Theorem 2.4. For $\mu>0$ and $v_{0} \in \mathbb{R}^{m}$, let $v_{i+1}=v_{i}+d\left(v_{i}, \mu\right)$. If $h_{\mu}\left(v_{0}\right) \leq \theta \leq \frac{1}{2}$, then $h_{\mu}\left(v_{i}\right) \leq \theta^{2^{i}}$.

Proof. Let $h_{i}=h_{\mu}\left(v_{i}\right)$ and $d_{i}=d\left(v_{i}, \mu\right)$. Make the inductive hypothesis that $h_{i} \leq \frac{1}{2}$. Then Lemma 2.3 implies that $\left\|d_{i}\right\| \leq 1$. Hence,

$$
h_{i+1} \leq \frac{1}{2} h_{i}\left\|d_{i}\right\|_{\infty}^{2} \leq \frac{1}{2} h_{i}\left\|d_{i}\right\|^{2} \leq \frac{1}{2} h_{i}\left(\left\|d_{i}\right\|+1\right) h_{i}
$$

where the first inequality is Theorem 2.2 (b) and the last is Lemma 2.3. Since $\left\|d_{i}\right\| \leq 1$, we further conclude that $h_{i+1} \leq h_{i}^{2}$, and that $h_{i+1}<\frac{1}{2}$. By induction, $h_{i+1} \leq h_{i}^{2}$ must hold for all $i$, which implies that $h_{i} \leq\left(h_{0}\right)^{2^{i}}$ as claimed.

Using the results of this section, we can now concretely instantiate the template log-domain interior-point method (Section 1.1). The next section will state two algorithms.

## 3 Algorithms

The analysis of Newton's method (Section 2) yields two concrete IPMs (Figure 1) for solving the QP (1). The first is a short-step algorithm: it conservatively updates $\mu$, takes full Newton steps, and never leaves the quadratic-convergence region of Newton's method. The next is a long-step algorithm: it aggressively updates $\mu$ via line-search and takes potentially damped steps. The first algorithm, shortstep, has an $\mathcal{O}(\sqrt{m})$ iteration bound, which is typical for interior-point methods. The second algorithm, longstep, is intended for practical implementation.

### 3.1 Short-step algorithm

The algorithm shortstep reduces the centering parameter $\mu$ by a fixed-factor $k$ after every $N$ Newton steps. With proper selection of $k$ and $N$, it updates a given centered-point $\hat{v}\left(\mu_{0}\right)$ to $\hat{v}\left(\mu_{f}\right)$ using at most $C \sqrt{m} \log \left(\mu_{0} \mu_{f}^{-1}\right)$ iterations, where $C$ is an explicit constant. If the quadratic objective term is zero $(W=0)$, it reduces to the short-step algorithm of $[16$, Section 2]. Its analysis is also identical once we show that a key divergence bound still holds for non-zero $W$.

To begin, let $q(t):=2(\cosh (t)-1)$. The next lemma establishes the aforementioned bound and reduces to $[16$, Theorem 3.1] when $W=0$.

```
Procedure shortstep( }\mp@subsup{v}{0}{},\mp@subsup{\mu}{0}{},\mp@subsup{\mu}{f}{}
    v}\leftarrow\mp@subsup{v}{0}{},\mu\leftarrow\mp@subsup{\mu}{0}{
    while }\mu>\mp@subsup{\mu}{f}{}\mathrm{ do
        \mu\leftarrow\frac{1}{k}\mu
        for i=1,2,\ldots.N do
            v\leftarrowv+d(v,\mu)
        end
    end
    return (v,\mu)
```

Procedure longstep ( $v_{0}, \mu_{0}, \mu_{f}$ )
$v \leftarrow v_{0}, \mu \leftarrow \mu_{0}$
while $\mu>\mu_{f}$ or $\|d(v, \mu)\|_{\infty}>1$ do
$\mu \leftarrow \min \left(\mu, \inf \left\{\mu>0:\|d(v, \mu)\|_{\infty} \leq 1\right\}\right)$
$\alpha \leftarrow \max \left(1, \frac{1}{2 \beta}\|d(v, \mu)\|_{\infty}^{2}\right)$
$v \leftarrow v+\frac{1}{\alpha} d(v, \mu)$
end
return $(v, \mu)$

Figure 1: Algorithms for finding an approximate solution $x(v, \mu)$ to the QP (1). For $(k, N)$ and ( $v_{0}, \mu_{0}$ ) specified by Theorem 3.1, the algorithm shortstep stays within the quadratic-convergence region of Newton's method (Theorem 2.4) and terminates in $\mathcal{O}\left(\sqrt{m} \log \left(\mu_{0} \mu_{f}^{-1}\right)\right)$ Newton steps. For any step-size parameter $\beta \in\left[\frac{1}{2}, 1\right)$, the algorithm longstep terminates given arbitrary initialization points (Theorem 3.2), and allows for construction of a $\mu_{f}$-sub-optimal feasible-point $x(v, \mu)$.

Lemma 3.1. For all $\mu_{1}, \mu_{2}>0$, the centered points $\hat{v}\left(\mu_{1}\right)$ and $\hat{v}\left(\mu_{2}\right)$ satisfy

$$
\frac{1}{m} h\left(\hat{v}\left(\mu_{1}\right), \hat{v}\left(\mu_{2}\right)\right) \leq q\left(\frac{1}{2} \log \frac{\mu_{1}}{\mu_{2}}\right)
$$

Proof. Let $w_{1}=e^{\hat{v}\left(\mu_{1}\right)}$ and $w_{2}=e^{\hat{\imath}\left(\mu_{2}\right)}$ and let $k_{1}=\sqrt{\mu_{1}}$ and $k_{2}=\sqrt{\mu_{2}}$. By primal-dual feasibility,

$$
b=k_{1} w_{1}^{-1}-A x_{1}=k_{2} w_{2}^{-1}-A x_{2} .
$$

Taking inner-products with $w_{1}$ and $w_{2}$ gives:

$$
\frac{k_{1} m+w_{1}^{T} A\left(x_{2}-x_{1}\right)}{k_{2}}=w_{1}^{T} w_{2}^{-1}, \quad \frac{k_{2} m+w_{2}^{T} A\left(x_{1}-x_{2}\right)}{k_{1}}=w_{2}^{T} w_{1}^{-1}
$$

Adding and simplifying yields:

$$
\begin{aligned}
h\left(\hat{v}\left(\mu_{1}\right), \hat{v}\left(\mu_{2}\right)\right)+2 m & :=w_{1}^{T} w_{2}^{-1}+w_{2}^{T} w_{1}^{-1} \\
& =\frac{\left(k_{2}^{2}+k_{1}^{2}\right) m+\left(k_{1} w_{1}^{T}-k_{2} w_{2}^{T}\right) A\left(x_{2}-x_{1}\right)}{k_{1} k_{2}} \\
& =\frac{\left(k_{2}^{2}+k_{1}^{2}\right) m+\left(c+W x_{1}-\left(c+W x_{2}\right)\right)^{T}\left(x_{2}-x_{1}\right)}{k_{1} k_{2}} \\
& =\frac{\left(k_{2}^{2}+k_{1}^{2}\right) m+\left(W\left(x_{1}-x_{2}\right)\right)^{T}\left(x_{2}-x_{1}\right)}{k_{1} k_{2}} \\
& =\frac{\left(k_{2}^{2}+k_{1}^{2}\right) m-\left(x_{1}-x_{2}\right)^{T} W\left(x_{2}-x_{1}\right)}{k_{1} k_{2}} \\
& \leq \frac{\left(k_{2}^{2}+k_{1}^{2}\right) m}{k_{1} k_{2}}=m\left(\frac{k_{2}}{k_{1}}+\frac{k_{1}}{k_{2}}\right)=2 m \cosh \left(\log \frac{k_{1}}{k_{2}}\right)=2 m \cosh \left(\log \sqrt{\frac{\mu_{1}}{\mu_{2}}}\right)
\end{aligned}
$$

Subtracting $2 m$ proves the claim.
We also need a lemma from [16] that relates divergence to Euclidean distance.
Lemma 3.2 (Lemma 3.2 of [16]). For all $v_{1}, v_{2} \in \mathbb{R}^{m}$ it holds that $\left\|v_{1}-v_{2}\right\|^{2} \leq h\left(v_{1}, v_{2}\right) \leq$ $q\left(\left\|v_{1}-v_{2}\right\|\right)$.
Using these lemmas, the $\mu$-update factor $k$ and the inner-iteration count $N$ are selected such that the divergence $h_{\mu}(v)$ remains bounded at each iteration by a specified $\theta \in\left(0, \frac{1}{2}\right]$. This in
turn implies that each Newton step is quadratically convergent (Theorem 2.4). To ensure that $h_{\frac{1}{k} \mu}(v) \leq \theta$ just before $\mu$ updates, we use the following upper-bound

$$
\begin{equation*}
h_{\frac{1}{k} \mu}(v) \leq q\left(\|v-\hat{v}(\mu)\|+\left\|\hat{v}(\mu)-\hat{v}\left(\frac{1}{k} \mu\right)\right\|\right) \tag{6}
\end{equation*}
$$

which follows from Lemma 3.2 and the triangle inequality. For a specified $\epsilon$, the parameter $N$ is chosen to ensure that $\|v-\hat{v}(\mu)\| \leq \epsilon$ using Theorem 2.4 and Lemma 3.2. Using Lemma 3.1 and Lemma 3.2, the parameter $k$ is then chosen to ensure that $\left\|\hat{v}(\mu)-\hat{v}\left(\frac{1}{k} \mu\right)\right\| \leq q^{-1}(\theta)-\epsilon$, where $q^{-1}: \mathbb{R} \rightarrow \mathbb{R}_{+}$denotes the nonnegative inverse of $q$. Together with (6), this implies that $h_{\frac{1}{k} \mu}(v) \leq \theta$.

A formal statement of the $(k, N)$-selection criteria and the convergence guarantees of shortstep follow. We omit proof, as it is identical to [16, Theorem 2.1].

Theorem 3.1. Let shortstep (Figure 1) have parameters ( $k, N$ ) that satisfy, for some $\frac{1}{2} \geq$ $\theta>0$ and $q^{-1}(\theta)>\epsilon>0$, the conditions

$$
\begin{equation*}
\theta^{2^{N}} \leq \epsilon^{2}, \quad \frac{1}{2} \log k=q^{-1}\left(\frac{1}{m} \zeta^{2}\right) \tag{7}
\end{equation*}
$$

where $\zeta:=q^{-1}(\theta)-\epsilon$. The following statements hold for shortstep given input $\left(\hat{v}\left(\mu_{0}\right), \mu_{0}, \mu_{f}\right)$ :
(a) At most $N\left\lceil c^{-1} \sqrt{m} \log \frac{\mu_{0}}{\mu_{f}}\right\rceil$ Newton steps execute, where $c:=2 q^{-1}\left(\zeta^{2}\right)$.
(b) The output $(v, \mu)$ satisfies $\|v-\hat{v}(\mu)\| \leq \epsilon$ and $\mu \leq \mu_{f}$.

Observe that shortstep takes only full Newton-steps and that its convergence guarantees assume a centered initialization point, i.e., that $v_{0}=\hat{v}\left(\mu_{0}\right)$. The next algorithm longstep will support arbitrary initialization through the use of damped Newton steps.

### 3.2 Long-step algorithm

The procedure longstep (Figure 1) supports arbitrary initialization and performs more aggressive updates of the centering parameter $\mu$. At each iteration, it finds the smallest $\mu$ for which the Newton direction $d(v, \mu)$ satisfies $\|d(v, \mu)\|_{\infty} \leq 1$, if such a $\mu$ exists. It terminates once both $\|d(v, \mu)\|_{\infty} \leq 1$ and $\mu \leq \mu_{f}$. The condition $\|d(v, \mu)\|_{\infty} \leq 1$ is motivated by the following lemma, which shows that under this condition, the approximate solution $x(v, \mu)$ associated with the Newton direction (Definition 2.1) is feasible and has bounded sub-optimality.
Lemma 3.3. For $\mu>0$ and $v \in \mathbb{R}^{m}$, let $d=d(v, \mu)$ and $x=x(v, \mu)$. Let $\lambda=\sqrt{\mu}\left(e^{v}+e^{v} \circ d\right)$ and $s=\sqrt{\mu}\left(e^{-v}-e^{-v} \circ d\right)$. If $\|d\|_{\infty} \leq 1$, then $(x, s, \lambda)$ satisfies the primal-dual feasibility conditions

$$
A x+b=s, \quad A^{T} \lambda=W x+c, \quad \lambda \geq 0, \quad s \geq 0
$$

Further, $\|s \circ \lambda\|_{1}=\mu\left(m-\|d\|^{2}\right)$.
Proof. The equality constraints hold by definition of $d(v, \mu)$. Nonnegativity of both $s$ and $\lambda$ hold by their definition and the fact that $\|d\|_{\infty} \leq 1$. Finally, since $s, \lambda \geq 0$, we have that $\|\lambda \circ s\|_{1}=\langle s, \lambda\rangle$. Expanding $\langle s, \lambda\rangle$ using the definition of $s$ and $\lambda$ proves the claim.

It remains to show that the algorithm will actually terminate. To prove this, we note that the algorithm, by design, monotonically decreases $\mu$ and always applies an update $v \leftarrow v+\frac{1}{\alpha} d(v, \mu)$ with $\|d(v, \mu)\|_{\infty} \geq 1$. We will show that infinitely many iterations, and the resulting convergence of $\mu$, contradicts $\|d(v, \mu)\|_{\infty} \geq 1$. Our analysis also selects the step-size parameter $\beta$ to ensure global convergence (Theorem 2.3) and full Newton steps when $\|d(v, \mu)\|_{\infty} \leq 1$.

Theorem 3.2. For any step-size parameter $\beta \in\left[\frac{1}{2}, 1\right)$ and input $\left(v_{0}, \mu_{0}, \mu_{f}\right) \in \mathbb{R}^{m} \times \mathbb{R} \times \mathbb{R}$ with $\mu_{0}>\mu_{f}>0$, the algorithm longstep terminates and returns $(v, \mu)$ with $\mu \leq \mu_{f}$. Further, letting $x$ denote $x(v, \mu)$, we have that

$$
A x+b \geq 0, \quad \frac{1}{2} x^{T} W x+c^{T} x \leq V^{*}+\mu m
$$

where $V^{*}$ denotes the optimal value of $Q P(1)$.
Proof. Let $v_{i}, \mu_{i}$ and $\alpha_{i}$ denote the sequences generated by longstep indexed by $i$ such that $v_{i+1}=v_{i}+\frac{1}{\alpha_{i}} d\left(v_{i}, \mu_{i}\right)$. Now suppose the algorithm does not terminate. We first consider the case where $\mu$ updates only finitely many times. In this case, there exists an $M$ such that $\mu_{i}=\mu_{M}$ for all iterations $i>M$, which implies that $v_{i}$ converges to $\hat{v}\left(\mu_{M}\right)$ by Theorem 2.3. But by selection of $\mu_{i}$, we also have that $\left\|d\left(v_{i}, \mu_{i}\right)\right\|_{\infty} \geq 1$ for all $i$, a contradiction. Hence, in this case, the algorithm must terminate.

Now suppose that $\mu$ updates infinitely many times and let $\sigma_{k}$ denote the subsequence of iterations $i$ where $\mu$ changes, i.e., $\mu_{\sigma_{k}}<\mu_{\sigma_{k-1}}$ and $\mu_{i}=\mu_{\sigma_{k-1}}$ for $\sigma_{k}>i \geq \sigma_{k-1}$. For brevity, let $h_{k}: \mathbb{R}^{m} \rightarrow \mathbb{R}$ denote the divergence $h\left(v, \hat{v}\left(\mu_{\sigma_{k}}\right)\right)$ as a function of $v$. We note that

$$
\begin{equation*}
\frac{1}{2} h_{k}\left(v_{\sigma_{k}}\right) \geq h_{k}\left(v_{1+\sigma_{k}}\right) \geq h_{k}\left(v_{\sigma_{k+1}}\right) \tag{8}
\end{equation*}
$$

where the first inequality holds by Theorem 2.2 since $\left\|d\left(v_{i}, \mu_{i}\right)\right\|_{\infty}=1$ for $i=\sigma_{k}$ and the second because Newton iterations decrease $h_{\mu}\left(v_{i}\right)$ for fixed $\mu$.

Since $\mu_{\sigma_{k}}$ is bounded below and monotonically decreasing, it converges. Since the map $\mu \mapsto \hat{v}(\mu)$ is continuous, the sequence $\hat{v}\left(\mu_{\sigma_{k+1}}\right)$ also converges. Hence, for any $\epsilon>0$, there exists an $N$ such that for all $k>N$,

$$
\left\|\hat{v}\left(\mu_{\sigma_{k}}\right)-\hat{v}\left(\mu_{\sigma_{k+1}}\right)\right\|<\epsilon
$$

This shows that for $k>N$,

$$
\begin{aligned}
h_{k+1}\left(v_{\sigma_{k+1}}\right)+2 m & =2 \sum_{j=1}^{m} \cosh \left(\left[v_{\sigma_{k+1}}-\hat{v}\left(\mu_{\sigma_{k+1}}\right)\right]_{j}\right) \\
& \leq 2 \sum_{j=1}^{m} \cosh \left(\left|\left[v_{\sigma_{k+1}}-\hat{v}\left(\mu_{\sigma_{k}}\right)\right]_{j}\right|+\epsilon\right) \\
& \leq 2 \sum_{j=1}^{m} \cosh \left(\left[v_{\sigma_{k+1}}-\hat{v}\left(\mu_{\sigma_{k}}\right)\right]_{j}\right)(\cosh (\epsilon)+\sinh (\epsilon)) \\
& =\left(h_{k}\left(v_{\sigma_{k+1}}\right)+2 m\right)(\cosh (\epsilon)+\sinh (\epsilon)) \\
& \leq\left(\frac{1}{2} h_{k}\left(v_{\sigma_{k}}\right)+2 m\right)(\cosh (\epsilon)+\sinh (\epsilon))
\end{aligned}
$$

where the third line uses the inequality $\cosh (x+y) \leq \cosh (x)(\cosh (y)+\sinh (|y|))$ and the last uses (8). Noting that $e^{\epsilon}=\cosh (\epsilon)+\sinh (\epsilon)$, we rearrange the last line to conclude that

$$
h_{k+1}\left(v_{\sigma_{k+1}}\right) \leq \frac{e^{\epsilon}}{2} h_{k}\left(v_{\sigma_{k}}\right)+2 m\left(e^{\epsilon}-1\right) .
$$

This shows that $h_{k}\left(v_{\sigma_{k}}\right)$ is upper-bounded by a sequence of the form $a_{k+1}=c_{1}(\epsilon) a_{k}+c_{2}(\epsilon)$, which, if $\left|c_{1}\right|<1$, converges to $L=\frac{c_{2}}{1-c_{1}}$. For any $\delta>0$, we can pick $\epsilon$ small enough to show that $L<\delta$. This shows that $h_{k}\left(v_{\sigma_{k}}\right)$ converges to zero, contradicting $\left\|d\left(v_{\sigma_{k}}, \mu_{\sigma_{k}}\right)\right\| \geq 1$ by Lemma 2.3. Hence, the algorithm must terminate. Finally, the feasibility and suboptimality guarantees for $x(v, \mu)$ follow from Lemma 3.3 and weak duality.

We conclude with three topics related to selection of $\mu$.

### 3.2.1 Computing the infimum

Fix $v \in \mathbb{R}^{m}$ and let $d(\mu)$ denote $d(v, \mu)$. In this notation, each iteration of longstep requires computation of $\inf \left\{\mu>0:\|d(\mu)\|_{\infty} \leq 1\right\}$. This is straight-forward upon recognition that $d(\mu)$ is an affine function of $(\sqrt{\mu})^{-1}$, i.e., it decomposes as $d(\mu)=d_{0}+(\sqrt{\mu})^{-1} d_{1}$ for some fixed $d_{0} \in \mathbb{R}^{m}$ and $d_{1} \in \mathbb{R}^{m}$. We give a constructive proof of this fact that demonstrates how to build this decomposition.
Proposition 3.1. There exists $d_{0}, d_{1} \in \mathbb{R}^{m}$ satisfying $d(\mu)=d_{0}+\frac{1}{\sqrt{\mu}} d_{1}$ for all $\mu>0$.
Proof. Fix $\mu_{1}>0$ and $\mu_{2}>0$ and $v \in \mathbb{R}^{m}$. Let $\hat{d}_{i}=d\left(v, \mu_{i}\right)$ and $k_{i}=\frac{1}{\sqrt{\mu_{i}}}$ for $i \in 1,2$. By Definition 2.1, there exists $\hat{x}_{1}$ and $\hat{x}_{2}$ satisfying

$$
A^{T}\left(e^{v}+e^{v} \circ \hat{d}_{1}\right)=W \hat{x}_{1}+k_{1} c, \quad A^{T}\left(e^{v}+e^{v} \circ \hat{d}_{2}\right)=W \hat{x}_{2}+k_{2} c
$$

Multiplying these equations by $t$ and $(1-t)$, respectively, and adding yields

$$
A^{T}\left(e^{v}+e^{v} \circ\left(t \hat{d}_{1}+(1-t) \hat{d}_{2}\right)\right)=W\left(t \hat{x}_{1}+(1-t) \hat{x}_{2}\right)+\left(t k_{1}+(1-t) k_{2}\right) c
$$

By similar argument,

$$
e^{-v}-e^{-v} \circ\left(t \hat{d}_{1}+(1-t) \hat{d}_{2}\right)=A\left(t \hat{x}_{1}+(1-t) \hat{x}_{2}\right)+\left(t k_{1}+(1-t) k_{2}\right) b
$$

By Definition 2.1, we conclude that $t \hat{d}_{1}+(1-t) \hat{d}_{2}=d(v, \mu)$ for $\frac{1}{\sqrt{\mu}}=t k_{1}+(1-t) k_{2}$. Solving for $t$ shows that $t=c_{1} \frac{1}{\sqrt{\mu}}+c_{0}$ for $c_{1}=\left(k_{1}-k_{2}\right)^{-1}$ and $c_{0}=-k_{2}\left(k_{1}-k_{2}\right)^{-1}$. Substituting into $t \hat{d}_{1}+(1-t) \hat{d}_{2}$, we deduce that

$$
d(v, \mu)=\left(c_{1} \frac{1}{\sqrt{\mu}}+c_{0}\right) \hat{d}_{1}+\left(1-c_{1} \frac{1}{\sqrt{\mu}}-c_{0}\right) \hat{d}_{2}
$$

Hence, the claim follows for $d_{0}=c_{0} \hat{d}_{1}+\left(1-c_{0}\right) \hat{d}_{2}$ and $d_{1}=c_{1}\left(\hat{d}_{1}-\hat{d}_{2}\right)$.
This decomposition in turn allows us to characterize the condition $\|d(\mu)\|_{\infty} \leq 1$ using a system of linear inequalities immediate from the definition of $\|\cdot\|_{\infty}$.
Proposition 3.2. For $d_{0}, d_{1} \in \mathbb{R}^{m}$, we have that $\left\|d_{0}+\frac{1}{\sqrt{\mu}} d_{1}\right\|_{\infty} \leq 1$ if and only if

$$
\begin{equation*}
-\mathbf{1} \leq d_{0}+\frac{1}{\sqrt{\mu}} d_{1} \leq \mathbf{1} \tag{9}
\end{equation*}
$$

where $\mathbf{1} \in \mathbb{R}^{m}$ denotes the vector of all ones.
Note that minimizing $\mu$ subject to these inequalities can be done in $\mathcal{O}(m)$ time simply by iterating over the components of $d_{0}, d_{1} \in \mathbb{R}^{m}$.

### 3.2.2 Reuse of factorizations

The constructive proof of Proposition 3.1 builds the decomposition $d_{0}+(\sqrt{\mu})^{-1} d_{1}$ from two Newton directions $d\left(v, \mu_{1}\right)$ and $d\left(v, \mu_{2}\right)$. Since $v$ is fixed, these directions are found by solving two Newton systems (Definition 2.1) with the same positive definite coefficient matrix $W+A^{T} Q(v) A$. Hence, one can find both directions using the same Cholesky factorization of $W+A^{T} Q(v) A$.

### 3.2.3 Least-squares $\mu$

The decomposition $d(\mu)=d_{0}+(\sqrt{\mu})^{-1} d_{1}$ from Proposition 3.1 also enables easy computation of the least-squares $\mu$, i.e., the $\mu$ that minimizes $\|d(\mu)\|^{2}$. Assuming $d_{0}^{T} d_{1}<0$, this $\mu$ is the solution of $-d_{0}^{T} d_{1} \sqrt{\mu}=\left\|d_{1}\right\|^{2}$ and provides a natural heuristic choice for the initial $\mu_{0}$ passed to longstep.

## 4 Comparison with barrier methods

The log-domain update $v \leftarrow v+d(v, \mu)$ induces multiplicative updates $s \leftarrow s \circ e^{-d}$ and $\lambda \leftarrow \lambda \circ e^{d}$ of the slack variable $s:=\sqrt{\mu} e^{-v}$ and the Lagrange multiplier $\lambda:=\sqrt{\mu} e^{v}$. Taylor expanding $e^{-d}$ and $e^{d}$ yields approximations of these updates:

$$
\begin{equation*}
s \circ e^{-d} \approx s \circ(\mathbf{1}-d), \quad \lambda \circ e^{d} \approx \lambda \circ(\mathbf{1}+d) \tag{10}
\end{equation*}
$$

In this section, we interpret these approximations in the context of barrier methods. Specifically, we show that $s \leftarrow s \circ(\mathbf{1}-d)$ is equivalent to an iteration of the primal barrier method applied to the QP (1). Similarly, we show that $\lambda \leftarrow \lambda \circ(\mathbf{1}+d)$ is equivalent to an iteration of the dual barrier method applied to the dual QP [5], which takes the form

$$
\underset{u, \lambda}{\operatorname{maximize}}-\left(\frac{1}{2} u^{T} W u+b^{T} \lambda\right) \text { subject to } A^{T} \lambda=W u+c, \lambda \geq 0
$$

Computational experiments illustrate that replacing the log-domain update with either of these approximations increases the number of longstep iterations needed to solve random QPs to fixed accuracy, illustrating in effect that barrier methods are less efficient than log-domain IPMs.

### 4.1 Primal barrier method

For $\mu>0$, the primal barrier method applies Newton's method to the optimality conditions of

$$
\underset{x, s}{\operatorname{minimize}} \frac{1}{2} x^{T} W x+c^{T} x-\mu \sum_{i=1}^{m} \log s_{i} \text { subject to } s=A x+b
$$

These conditions read $A^{T} \lambda=W x+c, s=A x+b$ and $\mu s^{-1}=\lambda$, where $\lambda$ is a Lagrange multiplier for the equality constraints. Letting $z:=(x, s, \lambda)$, Newton iterations take the form $z \leftarrow z+\Delta z$, where $\Delta z:=(\Delta x, \Delta s, \Delta \lambda)$ solves

$$
\begin{array}{r}
A^{T}(\lambda+\Delta \lambda)=W(x+\Delta x)+c, s+\Delta s=A(x+\Delta x)+b \\
\mu\left(s^{-1}-s^{-2} \circ \Delta s\right)=\lambda+\Delta \lambda \tag{11}
\end{array}
$$

The following shows that the Newton update of $s$ is precisely equivalent to the primal linearization (10) of the log-domain Newton step, i.e., it is equivalent to replacing $s \circ e^{-d}$ with its first-order approximation $s \circ e^{-d} \approx s \circ(\mathbf{1}-d)$.
Proposition 4.1. Consider $(x, s, \lambda) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}^{m}$ with $s>0$. Let $(\Delta x, \Delta s, \Delta \lambda)$ solve (11). Finally, let $v \in \mathbb{R}^{m}$ and $\mu>0$ satisfy $s=\sqrt{\mu} e^{-v}$. Then $s+\Delta s=s \circ(\mathbf{1}-d(v, \mu))$, where $d(v, \mu)$ is the log-domain Newton direction (Definition 2.1).

Proof. Letting $\hat{x}=x+\Delta x$, the conditions (11) simplify to

$$
\mu A^{T}\left(s^{-1}-s^{-2} \circ \Delta s\right)=W \hat{x}+c, \quad s+\Delta s=A \hat{x}+b
$$

Letting $\hat{d}:=-s^{-1} \circ \Delta s$, we conclude that

$$
\mu A^{T}\left[s^{-1} \circ(\mathbf{1}+\hat{d})\right]=W \hat{x}+c, s \circ(\mathbf{1}-\hat{d})=A x+b
$$

Substituting $s^{-1}=\sqrt{\mu}{ }^{-1} e^{v}$ and $s=\sqrt{\mu} e^{-v}$ yields

$$
\sqrt{\mu} A^{T}\left[e^{v} \circ(\mathbf{1}+\hat{d})\right]=W \hat{x}+c, \quad \sqrt{\mu} e^{-v} \circ(\mathbf{1}-\hat{d})=A \hat{x}+b,
$$

which is precisely the definition of $d(v, \mu)$. Hence, $d(v, \mu)=\hat{d}$ since $d(v, \mu)$ is unique by Theorem 2.1. By definition of $\hat{d}$, we also have that $\Delta s=-s \circ \hat{d}$, proving the claim.

We next show an analogous interpretation holds for the dual linearization $\lambda \circ e^{d} \approx \lambda \circ(\mathbf{1}+d)$.

### 4.2 Dual barrier method

Given $\mu>0$, the dual barrier method applies Newton's method to the optimality conditions of

$$
\underset{u, \lambda}{\operatorname{minimize}} \frac{1}{2} u^{T} W u+b^{T} \lambda-\mu \log \lambda \quad \text { subject to } A^{T} \lambda=W u+c .
$$

These optimality conditions read $W u=W \gamma, A^{T} \lambda=W u+c$ and $A \gamma+b=\mu \lambda^{-1}$, where $\gamma$ is a Lagrange multiplier for the equality constraints. Letting $z:=(\gamma, u, \lambda)$, Newton iterations take the form $z \leftarrow z+\Delta z$, where $\Delta z:=(\Delta \gamma, \Delta u, \Delta \lambda)$ solves

$$
\begin{array}{r}
W(u+\Delta u)=W(\gamma+\Delta \gamma), \quad A^{T}(\lambda+\Delta \lambda)=W(u+\Delta u)+c \\
A(\gamma+\Delta \gamma)+b=\mu\left(\lambda^{-1}-\lambda^{-2} \circ \Delta \lambda\right) \tag{12}
\end{array}
$$

The following shows that the Newton update of $\lambda$ is precisely equivalent to the dual linearization (10) of the log-domain Newton step, i.e., it is equivalent to replacing $\lambda \circ e^{d}$ with its first-order approximation $\lambda \circ e^{d} \approx \lambda \circ(\mathbf{1}+d)$.
Proposition 4.2. Consider $(\gamma, u, \lambda) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{m}$ with $\lambda>0$. Let $\Delta z:=(\Delta \gamma, \Delta u, \Delta \lambda)$ solve (12). Finally, let $v \in \mathbb{R}^{m}$ and $\mu>0$ satisfy $\lambda=\sqrt{\mu} e^{v}$. Then $\lambda+\Delta \lambda=\lambda \circ(\mathbf{1}+d(v, \mu))$, where $d(v, \mu)$ is the log-domain Newton direction (Definition 2.1).

Proof. Letting $\hat{\gamma}=\gamma+\Delta \gamma$, the conditions (12) simplify to

$$
A \hat{\gamma}+b=\mu\left(\lambda^{-1}-\lambda^{-2} \circ \Delta \lambda\right), \quad A^{T}(\lambda+\Delta \lambda)=W \hat{\gamma}+c
$$

Substituting $\lambda=\sqrt{\mu} e^{v}$ and letting $\hat{d}:=\lambda^{-1} \circ \Delta \lambda$, we obtain

$$
A \hat{\gamma}+b=\sqrt{\mu} e^{-v} \circ(\mathbf{1}-\hat{d}), \quad \sqrt{\mu} A^{T}\left[e^{v} \circ(\mathbf{1}+\hat{d})\right]=W \hat{\gamma}+c
$$

which is precisely the definition of $d(v, \mu)$. Hence, $d(v, \mu)=\hat{d}$ since $d(v, \mu)$ is unique by Theorem 2.1. By definition of $\hat{d}$, we have that $\Delta \lambda=\lambda \circ \hat{d}$, proving the claim.

### 4.3 Computational comparison

We next give simple computational experiments that compare barrier methods with longstep. These experiments show that barrier methods require more iterations to reach a target duality gap when initialized at identical starting points. Code for reproducing these experiments is located at

```
https://github.com/frankpermenter/LDIPMComputationalResults
```

Barrier method implementations We invoke Proposition 4.1 and implement the primal barrier method by modifying a single line of longstep. Precisely, we replace $v \leftarrow v+\alpha^{-1} d$ with the approximation $v \leftarrow-\log \left[e^{-v} \circ\left(\mathbf{1}-\alpha^{-1} d\right)\right]$, which is equivalent to taking $s \leftarrow s+\alpha^{-1} \Delta s$, with $(s, \Delta s)$ as defined in Proposition 4.1. We similarly implement the dual barrier method using Proposition 4.2. That is, we replace $v \leftarrow v+\alpha^{-1} d$ with $v \leftarrow \log \left[e^{v} \circ\left(\mathbf{1}+\alpha^{-1} d\right)\right]$, which is equivalent to taking $\lambda \leftarrow \lambda+\alpha^{-1} \Delta \lambda$, with $(\lambda, \Delta \lambda)$ as defined in Proposition 4.2. We also slightly modify selection of $\mu$, replacing the $\|d(v, \mu)\|_{\infty} \leq 1$ bound with $\|d(v, \mu)\|_{\infty} \leq 1-\epsilon$ : since $\alpha \geq 1$, this ensures that the argument to the $\log$ function is always positive.

Instances Our comparison uses randomly generated QPs that satisfy the regularity conditions of Assumption 1. The inequality matrix $A$ has entries drawn from a normal distribution with zero mean and unit variance. Each row is then rescaled to have unit norm. The cost matrix $W$ is constructed as $W=R^{T} R$ with $R \in \mathbb{R}^{r \times n}$ sampled and normalized the same way as $A$. Finally $c$ and $b$ are chosen by sampling $x, \lambda>0$ and $s>0$ and setting $b=s-A x$, $c=A^{T} \lambda-W x$. The vector $x$ is drawn from a normal distribution. The vector $s$ is chosen as $\mathbf{1}+\frac{1}{10}|w|$, where $w$ is also sampled from a normal distribution. Finally, $\lambda$ is chosen the same way as $s$ using a different random $w$.

| $m$ | Iterations |  |  |  |  |  | Iterations |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | rank $W$ | LS | DB | PB | $m$ | $\operatorname{rank} W$ | LS | DB | PB |
| 200 | 0 | 8.9 | 9.5 | 9.5 | 2000 | 0 | 10.8 | 11.6 | 11.7 |
| 200 | 50 | 7.1 | 8.3 | 7.6 | 2000 | 500 | 8.1 | 9.8 | 8.8 |
| 200 | 100 | 6.5 | 7.9 | 7.1 | 2000 | 1000 | 7.5 | 9.0 | 8.0 |
| 100 | 50 | 6.3 | 7.1 | 7.3 | 1000 | 500 | 7.3 | 8.0 | 8.7 |
| 150 | 50 | 6.8 | 7.7 | 7.6 | 1500 | 500 | 7.9 | 9.0 | 8.8 |
| 200 | 50 | 7.1 | 8.3 | 7.6 | 2000 | 500 | 8.1 | 9.8 | 8.8 |

Table 4.3.1: Average iterations on 30 random QPs with $m$ inequalities in $n=100$ variables (left) and $n=1000$ variables (right). Iterations needed to reach a target duality-gap are shown for longstep (LS), the dual barrier method (DB), and the primal barrier method (PB).

Results Table 4.3 .1 shows superior performance of longstep on a set of instances. In this set, we vary either the rank of $W \in \mathbb{R}^{n \times n}$ or the number of inequalities $m$, i.e., the number of rows of $A \in \mathbb{R}^{m \times n}$. The rank of $W$ is controlled by the number of columns $r$ of $R \in \mathbb{R}^{r \times n}$, recalling that $W=R^{T} R$. The algorithms are initialized with $v_{0}=0, \mu_{f}=10^{-3}$, and $\mu_{0}=\mu_{l s}$, where $\mu_{l s}$ denotes the least-squares $\mu$ (Section 3.2.3).

## References

[1] M. Achache. A new primal-dual path-following method for convex quadratic programming. Computational \& Applied Mathematics, 25:97-110, 2006.
[2] E. D. Andersen, C. Roos, and T. Terlaky. On implementing a primal-dual interior-point method for conic quadratic optimization. Mathematical Programming, 95(2):249-277, 2003.
[3] S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: a metaalgorithm and applications. Theory of Computing, 8(1):121-164, 2012.
[4] S. Di Cairano and M. Brand. On a multiplicative update dual optimization algorithm for constrained linear mpc. In 52nd IEEE Conference on Decision and Control, pages 1696-1701. IEEE, 2013.
[5] W. S. Dorn. Duality in quadratic programming. Quarterly of applied mathematics, 18(2): 155-162, 1960.
[6] D. Goldfarb and S. Liu. An $\mathcal{O}\left(n^{3} L\right)$ primal interior point algorithm for convex quadratic programming. Mathematical programming, 49(1):325-340, 1990.
[7] D. Goldfarb and S. Liu. An $\mathcal{O}\left(n^{3} L\right)$ primal—dual potential reduction algorithm for solving convex quadratic programs. Mathematical Programming, 61(1):161-170, 1993.
[8] L. Graña Drummond, A. N. Iusem, and B. F. Svaiter. On the central path for nonlinear semidefinite programming. RAIRO-Operations Research-Recherche Opérationnelle, 34(3): 331-345, 2000.
[9] S. Kapoor and P. M. Vaidya. Fast algorithms for convex quadratic programming and multicommodity flows. In Proceedings of the eighteenth annual ACM symposium on Theory of computing, pages 147-159, 1986.
[10] N. Karmarkar. A new polynomial-time algorithm for linear programming. In Proceedings of the sixteenth annual ACM symposium on Theory of computing, pages 302-311, 1984.
[11] M. Kojima, S. Mizuno, and A. Yoshise. An $O(\sqrt{n} L)$ iteration potential reduction algorithm for linear complementarity problems. Mathematical programming, 50(1):331-342, 1991.
[12] R. D. Monteiro and I. Adler. Interior path following primal-dual algorithms. Part II: Convex quadratic programming. Mathematical Programming, 44(1):43-66, 1989.
[13] Mosek APS. The MOSEK optimization software. Online at http://www.mosek.com.
[14] Y. Nesterov, A. Nemirovskii, and Y. Ye. Interior-point polynomial algorithms in convex programming, volume 13. SIAM, 1994.
[15] G. Optimization. Gurobi optimizer reference manual. Online at http://www.gurobi.com.
[16] F. Permenter. A geodesic interior-point method for linear optimization over symmetric cones. 2020.
[17] F. Sha, Y. Lin, L. K. Saul, and D. D. Lee. Multiplicative updates for nonnegative quadratic programming. Neural computation, 19(8):2004-2031, 2007.
[18] T. Terlaky. Interior point methods of mathematical programming, volume 5. Springer Science \& Business Media, 2013.
[19] S. J. Wright. Primal-dual interior-point methods. SIAM, 1997.

