Multi-Mode Capacitated Lot Sizing Problem with Periodic Carbon Emission Constraints

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Abstract

In this paper, we study the single item capacitated multi-mode lot sizing problem with periodic carbon emission constraints where the carbon emission constraints define an upper bound for the average emission per product produced in any period. The uncapacitated version of this problem was introduced in Absi et al. (2013) and solved in polynomial time. We show that this generalization of the problem is NP-Hard, in general, and present important properties for the optimal solutions of the problem. We propose algorithms to construct the piecewise linear total production cost functions for each period when the number of modes is fixed. This enables us to solve the problem using the dynamic programming algorithms developed for the lot sizing problem with piecewise concave production cost functions. We also consider an extension of the problem where at most two resources can be used at any period, and develop a polynomial time algorithm to solve it when the number of resources, the cost and emission parameters, and the capacities of the resources are time-invariant.

Keywords: Capacitated lot sizing, Periodic carbon emission constraints, Multi-mode, Complexity analysis, Dynamic programming

1 Introduction

The effect of climate change and global warming is growing fast, and this forces the authorities to control the greenhouse gases caused by different sectors through legal regulations. One of the major policies to control and reduce the carbon emitted by the operations of the companies is

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to introduce carbon emission constraints that should be respected during the planning stages in the companies. As a result of this development, the literature on the production planning and distribution problems with carbon emission constraints has grown recently.

The carbon emission constraints are considered in both static inventory models and in dynamic lot sizing models. Since our study is related to the second one, we refer the reader to a recent literature review by Christata and Daryanto (2020) for the EOQ problems with carbon emission constraints, and here, we focus on the studies on the dynamic lot sizing problems with carbon emission constraints.

Benjaafar et al. (2012) is one of the first studies that consider the carbon emission constraints in the planning phase, and investigate the impact of different emission regulations (carbon capacities, carbon tax, cap-and-trade policy, carbon offsets) based on a global carbon emission constraint where the total carbon emitted by the production processes over the whole planning horizon is limited. Helmrich et al. (2015) study the lot sizing problem with global carbon emission constraints, show that the problem is NP-Hard in most of the cases, and propose a heuristic, and also an approximation algorithm to find solutions for the problem. The same problem is considered by Akbalik and Rapine (2014) under the cap-and-trade policy where the company can buy and sell carbon units in the carbon market in case of need or surplus. They study the problem with different budget constraints and show that the problems can be reduced to the problem of Helmrich et al. (2015), hence NP-Hard in general. The bi-objective model of Romeijn et al. (2014) is a generalization of the uncapacitated lot sizing problem with periodic and global carbon emission constraints. The authors study the complexity of the problem with nonspeculative costs and under various settings of the carbon emission parameters. Different from these studies, Hong et al. (2016) studies the lot sizing problem with two uncapacitated resources where the resources are different from each other in terms of their costs and carbon emissions. They consider the problem under periodic emission constraints where a carbon capacity is enforced for each period, and propose a polynomial time algorithm for the problem when the cost and emission parameters of the resources are time-invariant.

The studies discussed above enforce a capacity limit on the carbon emissions, and they inherently impose an upper bound on the production quantities. In other words, the carbon emission constraints considered in these studies can be seen as different versions of the capacity constraints, and this might be the reason of the uncapacitated resource assumptions in these studies. Absi et al. (2013) study the uncapacitated lot sizing problem with a different type of carbon emission constraints where a limit for the average carbon emission for each unit produced is enforced, i.e. they do not limit the production quantities by themselves. The authors consider the periodic, global, cumulative and rolling carbon constraints, and discuss the complexity of the problem under each setting. For the problem with the periodic emission constraints, they show that the problem can be reduced to the lot sizing problem with M^2 uncapacitated modes where M is the number of resources, and therefore can be solved in polynomial time. A similar problem is considered by Absi et al. (2016) where the carbon emission constraints include the emissions due to setups of the resources, i.e. fixed carbon emissions. The problem becomes NP-Hard due to new components in the carbon emission constraints, and the authors propose dynamic programming algorithms to solve the problem in polynomial time when the number of resources is fixed.

In this paper, we consider the periodic carbon emission constraints proposed by Absi et al. (2013) in a capacitated multi-mode lot sizing problem. We consider a production system with resources (or modes) that are capable of producing a single product type. The resources are different from each other in terms of their production costs and carbon emissions which might be due to their technologies. We analyze how the resource capacities and the periodic emission constraints affect the problem complexity and the structure of the optimal solutions. To the best of our knowledge, this problem is not studied in the literature before.

Even though we focus on a production system and use production related terms, a similar problem might arise in different environments where demand for a single item should be satisfied by selecting different supply or transportation modes. Hence, the resources might represent machines, or different supplying modes such as plants, suppliers, transportation modes, etc. Note that assuming that each of these resources have a limited capacity makes the problem more realistic.

The remainder of this paper is organized as follows. In Section 2, we formally define the problem and present the mathematical formulation. In Section 3, we show that the problem is NP-Hard in general, and prove several optimal solution properties. Then, we consider the special cases of the problem with exactly two resources in Section 4, and without setup costs in Section 5 which allow us to determine the structure of the total production cost function for the general case in Section 6. We also study an interesting extension of the problem where at most two resources can be used at any period in Section 7. Finally, in Section 8 we present some concluding remarks.

2 Problem Definition and Formulation

We consider a multi-mode capacitated lot sizing problem with periodic carbon emission constraints (CLS-PC) where the demand for a single product d_t for period t should be satisfied with minimum cost over the planning horizon $t = 1, \ldots, T$. We assume that the company has M different resources with setup costs f_t^m, unit production costs p_t^m, and capacities C_t^m for $m = 1, \ldots, M$, $t = 1, \ldots, T$. As a result of the environmental regulations, the company has to ensure that the average carbon emission due to production does not exceed the maximum unitary environmental impact allowed E_t^{max} at any period t for $t = 1, \ldots, T$. We assume that the carbon emission of resource m due to producing one unit of the product is given by e_t^m for $m = 1, \ldots, M$, $t = 1, \ldots, T$. The unit inventory holding cost for period t is denoted by h_t for $t = 1, \ldots, T$. The main goal is to find a production plan for the next T periods to satisfy the demand of each period on time with minimum cost while respecting the resource capacities and the periodic carbon emission constraints.

To formulate the problem, we define the decision variables y_t^m which will be equal to one if resource m is used at period t, and x_t^m to represent the production quantity in resource m at period t for $m = 1, \dots, M$, $t = 1, \dots, T$. The inventory amount at the end of period t will be given by s_t for $t = 0, 1, \dots, T$.

We consider the following periodic carbon emission constraints introduced by Absi et al. (2013) which enforce an upper bound for the *average* carbon emission at each period:

$$\frac{\sum_{m=1}^{M} e_t^m x_t^m}{\sum_{m=1}^{M} x_t^m} \leq E_t^{max}, \qquad t = 1, \dots, T$$

$$\implies \sum_{m=1}^{M} (e_t^m - E_t^{max})\, x_t^m \leq 0, \qquad t = 1, \dots, T \tag{1}$$

Note that constraints (1) do not impose an upper bound on the quantities that can be produced in any period. In other words, if there exists a resource m with $e_t^m - E_t^{max} \leq 0$, then resource m can produce any amount if there is no additional capacity constraint for it. So, the periodic emission constraints (1) do not limit the production quantities in that case.

If the emission parameters are redefined as $\bar{e}_t^m = e_t^m - E_t^{max}$ for all m and t, then the emission constraints (1) can be written as $\sum_{m=1}^{M} \bar{e}_t^m x_t^m \leq 0$ for $t = 1, \dots, T$. We call a resource as "green" ("regular") if its unit carbon emission is less than or equal to (greater than) the unit carbon allowance, i.e. $\bar{e}_t^m \leq 0$ ($\bar{e}_t^m > 0$). We denote the number of green and regular resources with $M_g \geq 1$ and $M_r \geq 0$, respectively.

The mathematical formulation of CLS-PC is given below:

$$\min \quad \sum_{t=1}^{T} \sum_{m=1}^{M} (f_t^m y_t^m + p_t^m x_t^m) + \sum_{t=1}^{T} h_t s_t \tag{2a}$$

$$\text{s.t.} \quad s_{t-1} + \sum_{m=1}^{M} x_t^m = d_t + s_t, \qquad t = 1, \dots, T \tag{2b}$$

$$x_t^m \leq C_t^m y_t^m, \qquad m = 1, \dots, M,\ t = 1, \dots, T \tag{2c}$$

$$\sum_{m=1}^{M} \bar{e}_t^m x_t^m \leq 0, \qquad t = 1, \dots, T \tag{2d}$$

$$s_0 = 0 \tag{2e}$$

$$s_t \geq 0, \qquad t = 1, \dots, T \tag{2f}$$

$$x_t^m \geq 0,\ y_t^m \in \{0, 1\}, \qquad m = 1, \dots, M,\ t = 1, \dots, T \tag{2g}$$

The objective function (2a) minimizes the total setup, production and inventory holding cost over the planning horizon. Constraints (2b) are the inventory balance constraints which ensure that the inventory at the beginning of period t plus the amount produced in period t is sufficient to satisfy the demand of period t, and the remaining items will be kept in the inventory at the end of period t. Constraints (2c) are the capacity constraints for each resource at each period, and they also relate the continuous x variables with the binary y variables. Constraints (2d) are the carbon emission constraints for each period. We assume that the initial inventory is zero with (2e), and the remaining constraints (2f) and (2g) define the types of the decision variables.

CLS-PC is a generalization of the uncapacitated lot sizing problem with periodic carbon emission constraints (ULS-PC) which is studied by Absi et al. (2013) under the assumption that the resources are uncapacitated. It is also a generalization of the multi-mode lot sizing problem, called as MMLS.

We next discuss the complexity of CLS-PC, and prove several important optimal solution properties.

3 Complexity and Structural Properties of Optimal Solutions

In this section, we first show that CLS-PC is NP-Hard in general. Then, we prove several optimal solution properties for CLS-PC, and show that some of the important optimal solution properties for ULS-PC, which allow the problem to be solved in polynomial time, do not hold in CLS-PC in general.

In the next theorem, we prove that the problem is NP-Hard even if there exists a single period and the emission constraints (2d) are redundant.

Theorem 1. *Single period CLS-PC is NP-Hard.*

Proof. We will use the knapsack problem which is known to be NP-Complete (Garey and Johnson, 1990) to prove the statement. An instance of decision version of the knapsack problem (KDP) can be stated as follows: given N items with values b_i and weights w_i for $i = 1, \ldots, N$, capacity of the knapsack W and an integer B, does there exist a subset S of items such that $\sum_{i \in S} b_i \geq B$ and $\sum_{i \in S} w_i \leq W$?

Given an instance of KDP, the following instance of CLS-PC can be constructed: $T = 1$, $M = N$, $C^i = b_i$, $f^i = w_i$, $p^i = 0$, $e^i = 0$ for $i = 1, \ldots, M$, $h_1 = 0$, $d_1 = B$. The decision version of CLS-PC can be stated as follows: does there exist a subset of resources $S \subseteq \{1, \ldots, M\}$ such that the total capacity is sufficient to satisfy the demand of the single period, and the total cost is at most W, i.e. $\sum_{i \in S} C_1^i = \sum_{i \in S} b_i \geq d_1 = B$ and $\sum_{i \in S} f_1^i = \sum_{i \in S} w_i \leq W$? Consequently, the answer is "yes" if and only if the answer to KDP is also "yes". □

Theorem 1 shows that the single period multi-mode capacitated lot sizing problem, which is a special case of CLS-PC with $e_t^m = 0$ for all t and m, is also NP-Hard.

Note that due to the periodic carbon emission constraints (2d), the resources might not be fully utilized in any period. Indeed, the actual production capacities of periods depend both the capacities and the emissions of the resources. Moreover, as it will be discussed in the next section, the breakpoints of the total production cost function depend on the relation between the cost components of the resources. Therefore, we conclude that CLS-PC is NP-Hard, in general, if

- the capacities of the resources are time-dependent, or
- the emission parameters are time-dependent, or
- the cost parameters of the resources are time-dependent.

Hence, in the proceeding sections, we consider CLS-PC where the number of resources M is fixed, and the capacities and the emission parameters of the resources are time-invariant, i.e. $C_t^m = C^m$, $f_t^m = f^m$, $p_t^m = p^m$, $e_t^m = e^m$ for all t and m, and $E_t^{max} = \bar{e}$ for all t.

Next, we state several important results for the optimal solutions for CLS-PC. Due to the periodic carbon emission constraints (2d), the following optimal solution property can be stated.

Property 1. *In any feasible solution for CLS-PC, at least one green resource should be used in any production period.*

Proof. The proof follows from the periodic carbon emission constraints (2d). □

Property 2. *Zero inventory ordering (ZIO) policy of ULS-PC (Theorem 3 of Absi et al. (2013)) does not hold in CLS-PC, in general.*

Proof. In ZIO policy, the initial inventories of production periods should be zero, i.e. $s_{t-1} . \sum_{m=1}^{M} x_t^m = 0$ for $t = 1, \ldots, T$, which implies that demand of each period should be produced in a single period. Indeed, this policy mostly holds in the uncapacitated lot sizing problems where there is no restriction on the amount that can be produced in any period (see e.g. Wagner and Whitin (1958)).

Consider the following example to see that this property does not hold in CLS-PC:
$T = 2$, $M = 2$, $d = [5, 15]$; $h_t = 1$ for $t = 1, 2$; $f_t^1 = 50$, $p_t^1 = 5$, $C_t^1 = 10$, $e_t^1 = 0$ for $t = 1, 2$; $f_t^1 = 50$, $p_t^1 = 5$, $C_t^1 = 10$ for $t = 1, 2$; $f_t^2 = 100$, $p_t^2 = 5$, $C_t^2 = 10$, $e_t^2 = 0$ for $t = 1, 2$; $E_t^{max} = 0$ for $t = 1, 2$. The unique optimal solution for this problem is given by $y_1^1 = y_2^1 = 1$, $x_1^1 = x_2^1 = 10$, $s_1 = 5$, $y_1^2 = y_2^2 = 0$, $x_1^2 = x_2^2 = 0$.

Note that the problem instance given above is actually an instance of capacitated MMLS (without carbon emission constraints). In other words, here, the important aspect of the problem that rejects the ZIO policy is the resource capacities. □

Property 3. *More than two resources might be used in all optimal solutions for CLS-PC. In other words, Theorem 1 of Absi et al. (2013) does not hold for CLS-PC, i.e. there might not be an optimal solution for CLS-PC that uses at most two resources in a period.*

Proof. Consider the following example: $T = 1$,$M = 3$, $d_1 = 80$, $h_1 = 1$, $f_1 = [1000, 500, 100]$, $p_1 = [15, 10, 1]$, $C_1 = [50, 50, 25]$, $e_1 = [1, 2, 5]$, $E_1^{max} = 3$. The unique optimal solution for this problem instance is given by $y_1^m = 1$ for $m = 1, 2, 3$ and $x_1 = [5, 50, 25]$. Hence, more than two resources should be used in the optimal solution. □

Absi et al. (2013) uses (the reverse of) Property 3 to reduce an ULS-PC instance to an instance of uncapacitated MMLS with M^2 modes where there are no carbon emission constraints at any period. Due to the polynomial time dynamic programming algorithm of Wagelmans et al. (1992) that can be used to solve the latter one, the result for ULS-PC follows. As it is proved in Property 3, this property does not hold in CLS-PC. Even though all optimal solutions might require using more than two resources at any period, in the next theorem we show that at most two of these resources might be used at a level that is strictly less than its capacity. We call these resources as "fractional" resources. In other words, resource i is called as *fractional* if its production level is positive but not equal to the capacity, i.e. $0 < x_t^i < C^i$.

Theorem 2. *There exists an optimal solution for CLS-PC that uses at most two fractional resources at any period.*

Proof. Let X_t be the total amount that will be produced in period t: $X = \sum_m x_t^m$. Similar to Absi et al. (2013), we can decompose the problem using the Bender's decomposition approach into a master problem and a set of T independent subproblems:

$$\begin{aligned} \min\ & \sum_{t=1}^{T} \gamma_t\left(X_t\right) + \sum_{t=1}^{T} h_t s_t \\ \text{s.t. } & s_{t-1} + X_t = d_t + s_t, && t = 1, \ldots, T \\ & X_t \leq \bar{C}_t, && t = 1, \ldots, T \\ & s_0 = 0 \\ & X_t, s_t \geq 0, && t = 1, \ldots, T \end{aligned}$$

where $\bar{C}_t$ is the maximum amount that can be produced in period t, and $\gamma_t\left(X_t\right)$ is the minimum cost for producing X_t units in period t which is given by the optimal value of the following subproblem for period t

$$
\begin{aligned}
\min \ & \sum_{m=1}^{M} \left(f_t^m y_t^m + p_t^m x_t^m\right) \\
\text{s.t.} \ & \sum_{m=1}^{M} x_t^m = X_t, \\
& \sum_{m=1}^{M} \bar{e}_t^m x_t^m \leq 0, \\
& x_t^m \leq C_t^m y_t^m, & m = 1, \ldots, M \\
& x_t^m \geq 0, \ y_t^m \in \{0,1\}, & m = 1, \ldots, M
\end{aligned}
$$

Note that the master problem represents the production planning (lot sizing) problem while the subproblems are allocation problems. Moreover, the subproblem for period t is a single period problem, and since the total production quantity X_t is less than or equal to the actual capacity of period t which is denoted by $\bar{C}_t$, the subproblem is always feasible. Even though we assume that $\bar{C}_t$'s are known beforehand, they should be determined based on the capacities of the resources and the emission restrictions. But this does not change the result that will be proved here.

We first consider the case with no setup costs, i.e. the decision variables y_t^m are removed. In this case, the subproblem reduces to a linear program with M decision variables and $M+2$ constraints (and M non-negativity constraints). Due to the LP theory, it is known that M linearly independent constraints should be active at any basic solution. Hence, if the emission constraint is tight, then $M-2$ of the capacity and nonnegativity constraints should be tight, which implies that at most two resources might be fractional. Note that, if the emission constraint is not active, then $M-1$ of the capacity and nonnegativity constraints should be tight which implies that at most one resource might be fractional.

For the case with setup costs, let $(\hat{y}, \hat{x})$ be a feasible solution and define the subset of resources used in period t, i.e. $\hat{M}_t = \{m : \hat{y}_t^m = 1\}$. Consider the subproblem for period t where only the resources in $\hat{M}_t$ can be used, i.e. the setup variables are fixed to $y_t^m = \hat{y}_t^m$. Again, the subproblem reduces to an LP with $|\hat{M}_t|$ variables, and similar to the previous case, there exists at most two fractional resources at an optimal basic solution. Hence, $(\hat{y}, \hat{x})$ can be transformed into a solution with a lower (or the same) cost with the desired property, and the result follows. □

As it can be observed from the proof of Theorem 2, the minimum production cost $\gamma_t(X_t)$ for a given quantity X_t is the minimum of a set of linear cost functions. Thus, the total production cost function for any period is a piecewise linear function. This implies that if one can determine the total production cost function efficiently, then the problem can be solved in polynomial time when the number of breakpoints of the total production cost function is fixed due to the algorithms of Koca et al. (2014) or Ou (2017) which are developed for the lot sizing problem with piecewise concave production costs. Accordingly, we focus on the construction of the total production cost functions in Sections 4, 5 and 6.

4 CLS-PC with Two Resources

In this section, we consider a special case of CLS-PC with two resources since it is easier to see the structure of the total production cost function for any period in this case. Note that if both resources are regular, then the problem is infeasible due to Property 1. Hence, there are two possibilities: i) one of the resources is green and the other one is regular, or ii) both resources are green. These cases will be considered separately in the next two subsections.

4.1 One Green - One Regular Resource Case

Assuming that there are two resources with $e^1 < \bar{e} < e^2$, we will construct the total production cost function by considering possible relations between the unit production costs of the resources, p^1 and p^2.

Case 1: $p^1 \geq p^2$
If there were no emission constraints, then to produce a given quantity in any period, resource 2 would be used alone until its capacity is fully utilized, and if the capacity of resource 2 is not sufficient, resource 1 would be used in addition to resource 2. Hence, the total production capacity for any period would be $C^1 + C^2$ in this case. However, due to the carbon emission constraints (2d), and Property 1, it is not feasible to use resource 2 alone. Indeed, there are two possibilities: i) the green resource (resource 1 in our case) is used alone, or ii) both resources are used together such that the emission constraints are satisfied.

Note that if the setup costs are zero, i.e. $f^1 = f^2 = 0$, then the cheapest solution would be to produce in both resources proportional to their emissions. For example, if $e^1 = 1$, $e^2 = 5$, and $\bar{e} = 3$, then in the optimal solution, to ensure that the emission constraints are satisfied, for each unit produced in resource 2 another unit should be produced in resource 1 until at least one of the resources is fully utilized. If the emission parameters are $e^1 = 1$, $e^2 = 4$, then the strategy would be to produce one unit in resource 1 for every two units produced in resource 2. In general, these ratios can be determined by assuming that the emission constraint is tight in an optimal solution:

$$e^1 r_1 + e^2 r_2 = \bar{e}\,(r_1 + r_2)$$

where r_1 and r_2 represent the proportional production quantities for resources 1 and 2, respectively. Note that the above equation has infinitely many solutions due to the assumption $e^1 < \bar{e} < e^2$, but here the ratio of r_1 to r_2 is important, and it will be the same in all solutions. Consequently, we set $r_1 = e^2 - \bar{e}$ and $r_2 = \bar{e} - e^1$ to solve it. When there are no setup costs, a given quantity x should be produced in these resources as follows: $x^1 = x\frac{r_1}{r_1+r_2}$ and $x^2 = x\frac{r_2}{r_1+r_2}$

given that $x^1 \leq C^1$ and $x^2 \leq C^2$. This production strategy is used until one of the resources is fully utilized, so the maximum that can be produced with using both resources is given by $b_1 = \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}r_1 + \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}r_2 = \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}(r_1 + r_2)$. This will be the end point of the first segment of the total production cost function and its slope (unit production cost) is $\frac{p^1 r_1 + p^2 r_2}{r_1 + r_2}$.

If resource 1 is fully utilized at b_1, i.e. $\min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} = \frac{C^1}{r_1}$, b_1 will be the maximum that can be produced in any period, since resource 2 cannot be used alone. In this case, the total production cost function will have a single segment. Otherwise, if there is available capacity in resource 1, then it can be used until its capacity is fully used. So, the maximum that can be produced in any period is given by $b_2 = b_1 + C^1 - \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}r_1 = C^1 + \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}r_2$. In this case, the total production cost function will have two segments, and the slope and the endpoint of the second segment will be given by p^2 and b_2, respectively. Note that if there is no available capacity in resource 1, then we have $b_2 = b_1$.

When the setup costs are zero, it is cheaper to use both resources instead of using resource 1 alone. But, if the setup costs are positive - specifically for resource 2, then resource 1 should be used alone (since resource 2 cannot be used alone) until the setup cost of resource 2 is justified. Hence, there will be another segment before the first segment of the total production cost function explained above. The endpoint of this segment which is given by $\frac{f^2(r_1+r_2)}{(p^1-p^2)r_2}$ can be determined by solving the following equation

$$f^1 + p^1 x = f^1 + p^1 x \frac{r_1}{r_1 + r_2} + f^2 + p^2 x \frac{r_2}{r_1 + r_2}.$$

Consequently, the total production cost function for any period will have at most 3 segments and 3 breakpoints as shown in Figure 1(a). However, since the total production cost function is concave in the interval $[0, b_1]$, the point given by $\frac{f^2(r_1+r_2)}{(p^1-p^2)r_2}$ will not be considered as a breakpoint in the algorithm of Koca et al. (2014) since production at that level can occur at most once in any regeneration interval (see the proof of Theorem 3). As a result, the total production cost function has 2 breakpoints given by b_1 and b_2.

Case 2: $p^1 < p^2$
In this case, since resource 1 is both the green and the cheaper one, first resource 1 is used until its capacity is fully utilized. So, the first breakpoint of the total production cost function will be $b_1 = C^1$ with the slope (unit production cost) of p^1. Then, for producing larger amounts, resource 2 might be used as long as the carbon emission and capacity constraints are satisfied. Again, the proportional production quantities can be used to determine the maximum that can be produced in resource 2: $\min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}r_2$. Accordingly, the second segment has the endpoint of $b_2 = C^1 + \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}r_2$, and the unit production cost (slope) for this segment will be p^2. The total production cost function will have two breakpoints as it is depicted in Figure 1(b).

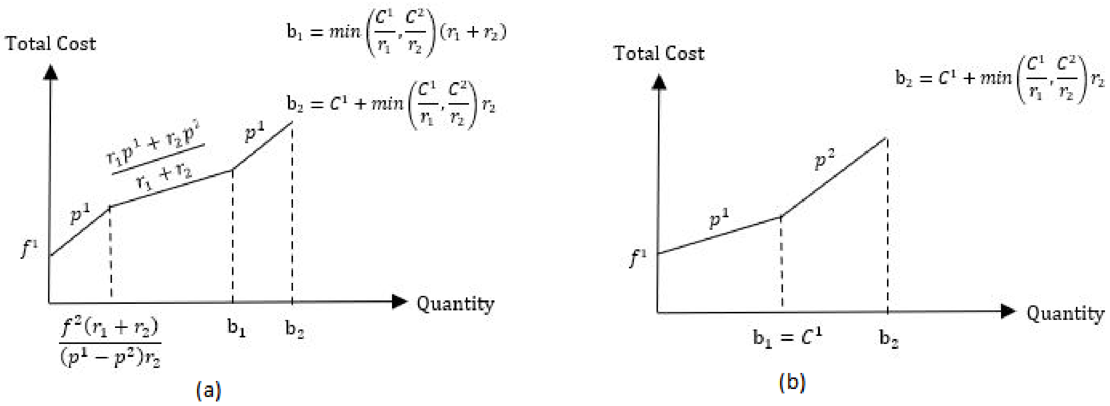

Figure 1: Total production cost functions for one green - one regular resource case

4.2 Two Green Resources

Even though the periodic emission constraints become irrelevant when both resources are green, and the problem reduces to the capacitated MMLS, since complexity of this problem is also open, and since green-green machine pairs should be considered in Sections 6 and 7, we discuss the possible cases briefly.

Case I: $f^1 \geq f^2$ **and** $p^1 \geq p^2$
In this case, resource 1 dominates resource 2, and it is cheaper to use resource 1 until its capacity is fully utilized. Then, in addition to resource 1, resource 2 will be used. Hence, the total production cost function includes two segments with the breakpoints $b_1 = C^2$, $b_2 = C^1 + C^2$, and the slopes p^2 and p^1, respectively.

Case II: $f^1 \geq f^2$ **and** $p^1 < p^2$
In this case, as the setup cost of resource 2 is smaller it will be used until it is cheaper to use resource 1, i.e. solve $f^2 + p^2x = f^1 + p^1x$ for x. After $x = \frac{f^1-f^2}{p^2-p^1}$ units, resource 1 is used alone until its capacity is fully used, i.e. $b_1 = C^1$, and then, in addition to resource 1, resource 2 will be used until its capacity is fulfilled, i.e. $b_2 = C^1 + C^2$. Similar to Case 1 of the previous part, here since the cost function is concave in the interval $[0, b_1]$, we will not consider the point given by x as a breakpoint. So, the cost function includes 2 breakpoints.

Case III: $f^1 < f^2$ **and** $p^1 \geq p^2$
This case is similar to the previous one. This time, as the setup cost of resource 1 is smaller, it will be used until it becomes cheaper to use resource 2 - but this threshold value will not be considered as a breakpoint as explained in the previous case. Then, resource 2 and then resource 1 will be used until their capacities are fully utilized. So the total production cost function will have two breakpoints: $b_1 = C^2$, $b_2 = C^1 + C^2$.

Case IV: $f^2 \geq f^1$ **and** $p^2 \geq p^1$
This case is the symmetric of Case I, where resource 2 dominates resource 1. Hence this case

has also 2 breakpoints.

The total production cost functions under each of these cases can be seen in Figure 2 in the same order, i.e. (a) is for Case I, etc. Note that, the total production cost function has two breakpoints at each case.

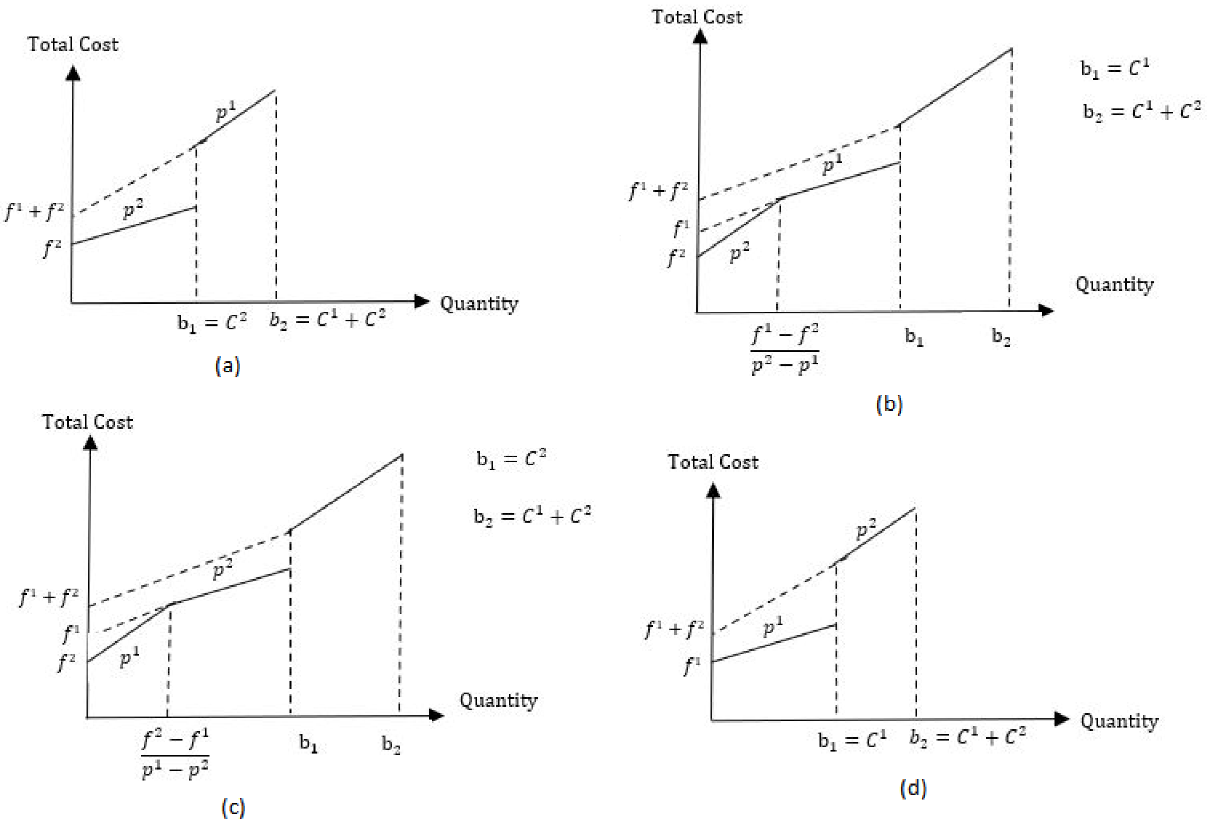

Figure 2: Total production cost functions for two green resources

Remark 1. *Note that the breakpoints of the total production cost function do not depend on the cost parameters in any case. However, the case that will apply depends on the relation between the cost parameters of the resources, and the breakpoints are different at different cases. This is the reason of our assumption for stationary cost parameters of the resources.*

Remark 2. *Since the number of breakpoints is 2 at each case, this special case of the problem can be solved in* $O\left(T^6\right)$ *time by using the dynamic programming algorithm of Koca et al. (2014).*

5 CLS-PC with No Setup Costs (CLS-PC0)

When the setup costs are zero, i.e. $f_t^m = 0$ for all m and t, the problem (2) reduces to a linear program which can be solved in polynomial time due to the interior point method of Karmarkar (1984). Still, we consider this special case since the algorithm presented here allows us to reformulate the problem as a minimum cost network flow problem with piecewise linear convex costs which can be solved by the special purpose algorithms of Orlin (1988), or Pinto and Shamir (1994). Moreover, the definitions given in this section will be also used in Section 6 to solve the general version of the problem.

Note that if there are no setup costs and no carbon emission constraints (MMLS without setup costs), then a resource with smaller unit production cost will be preferred to a resource with larger unit production cost until its capacity is fully utilized. Hence, the total production cost function for any period will be a piecewise linear convex function, and it can be determined easily by ordering the resources in nondecreasing order of their unit production costs, and use them incrementally. Consequently, the breakpoints and the slopes of the production cost function will be $0, C_1, C_1 + C_2, \ldots, C_1 + C_2 + \ldots + C_M$, and $p_1, p_2, \ldots, p_M$, respectively, if the resources are numbered in nondecreasing order of their unit production costs. Even though this procedure does not work in CLS-PC0 directly due to the carbon emission constraints, below we discuss how it can be applied to a set of artificial resources that will be obtained by a preprocessing step.

Due to Theorem 2, it is known that there exists a subset of resources to produce a given quantity with minimum cost where at most two of the resources are fractional. Moreover, since there are no fixed costs, the same subset of resources and the same fractional resources would give the smallest cost for larger quantities until it is not possible to produce more due to the capacity limitations. In other words, the same subset would be the cheapest option to use until one of the fractional resources hit the capacity. As a result, we arrive an important property of the breakpoints of the total production cost function that will allow us to construct it: *there will be at most one fractional resource at any breakpoint.* To this end, we see different combinations of the resources as the artificial resources, and construct the production cost function for a period by considering the unit production cost of each artificial resource.

We define two sets $\bar{S}^0$ and $\bar{S}^1$ to determine the possible subsets of resources with no fractional resources and with exactly one fractional resource, respectively. The subsets of resources that will produce in full capacity and satisfy the carbon emission constraints is given by $\bar{S}^0 = \{S \subseteq \{1, \ldots, M\} : \sum_{j \in S} \bar{e}^j C^j \leq 0\}$. For each $S \in \bar{S}^0$, we define the average unit production cost and the total quantity that can be produced as $\bar{p}_S = \frac{\sum_{j \in S} p^j C^j}{\sum_{j \in S} C^j}$ and $q_S = \sum_{j \in S} C^j$, respectively. Similarly, the subsets of resources that will produce in full capacity except one, and satisfy the carbon emission constraints can be determined by $\bar{S}^1 = \{(S, i) : 0 < x^{(S,i)} := \frac{-\sum_{j \in S \setminus \{i\}} \bar{e}^j C^j}{\bar{e}^i} < C^i, i \in S, S \subseteq \{1, \ldots, M\}\}$ where resource i is the fractional resource. For each $(S, i) \in \bar{S}^1$, the average unit production cost and the total quantity that can be produced are given by $\bar{p}_{S,i} = \frac{\sum_{j \in S \setminus \{i\}} p^j C^j + p^i x^{(S,i)}}{\sum_{j \in S \setminus \{i\}} C^j + x^{(S,i)}}$ and $q_{S,i} = \sum_{j \in S \setminus \{i\}} C^j + x^{(S,i)}$, respectively.

Note that $\bar{p}$ values do not give the actual unit production costs for the segments. Indeed, they represent the average unit production cost at the the potential breakpoints of the total production cost function. Since, the total production cost function is piecewise linear convex, and since there no setup costs, it is obvious that the unit, and consequently the average, production costs are nondecreasing in the segments of the function. We make use of this property to construct the total production cost function.

Let $\bar{S} = \bar{S}^0 \cup \bar{S}^1$. We construct an ordered list L of the elements in $\bar{S}$ where the elements are ordered in nondecreasing average production costs $\bar{p}$, and for the elements with the same $\bar{p}$ value, ones with smallest $|\bar{S}|$ and largest q values are considered first. Then, the following algorithm which is very similar to the one described above for the case with no carbon emission constraints gives the total production cost function for any period.

Algorithm 1: Algorithm to determine the total production cost function for CLS-PC0

Input: L, $\bar{S}$, $\bar{\mathbf{p}}$, $\mathbf{q}$

Output: The breakpoints $\mathbf{b}$ of the total production cost function for CLS-PC0

```
Let b_0 = 0 and i = 1.
while L ≠ ∅ do
    Select the first element, say π, in L
    Let b_i = q_π
    Remove all elements in L s.t. q_S ≤ b_i or q_{S,i} ≤ b_i
    Increase i by one
```

In Algorithm 1, we determine the breakpoints of the total production cost function. The slopes (unit production costs) can be determined easily by considering the difference between the total costs of two consecutive breakpoints.

Note that each element in $\bar{S}$ should include at least one green resource due to Property 1. Hence, the sizes of the sets $\bar{S}^0$ and $\bar{S}^1$ are given by $O\left(2^M - 2^{M_r}\right)$ and $O\left(M\left(2^M - 2^{M_r}\right)\right)$, respectively. Consequently, the size of the set $\bar{S}$ is $O\left((M+1)\left(2^M - 2^{M_r}\right)\right)$. Note that these values are exponential for an arbitrary M, but when the number of resources M is fixed, then the complexity of determining all these sets will be also fixed. Since each element in the ordered set L will be considered once in Algorithm 1, the dominant part in Algorithm 1 is constructing the list L, i.e. sorting the elements of the set $\bar{S}$, which can be done in $O\left(|\bar{S}|log|\bar{S}|\right)$, i.e. in fixed time.

Once the total production cost function is obtained, CLS-PC0 reduces to a minimum cost network flow problem with piecewise linear convex costs. This problem can be solved directly by using the special purpose algorithm of Pinto and Shamir (1994), or by using the algorithm of Orlin (1988) after converting the network by defining a separate arc from node 0 (dummy node to represent production) to node t (representing period t) for each segment of the total production cost function.

6 CLS-PC

When the setup costs are positive, the single fractional resource property of the breakpoints of CLS-PC0 might not hold any more. Actually, in addition to the breakpoints with this property (we call them as breakpoints due to capacity limitations), there might be additional breakpoints of the total production cost function with exactly two fractional resources due to the intersections of production cost functions of different resource combinations (we call them as breakpoints due to intersections). This can be observed from Figure 1(a) where the first breakpoint $\frac{f^2(r_1+r_2)}{(p_1-p_2)r_2}$ occurs due to the intersection of the production cost function of resource 1 with the production cost function for the case where resources 1 and 2 are used together.

Note that when the production cost functions of two different resource combinations intersect, since we select the option with minimum cost, the slope of the total production cost function decreases -see Figure 1(a). In other words, the total production cost function is concave between the breakpoints due to intersections. Therefore, one does not need to consider these new breakpoints as the "breakpoints" in the dynamic programming algorithm of Koca et al. (2014). On the other hand, we still have to determine these additional breakpoints and the slopes of these segments to correctly compute the total production cost while solving the problem. To do that, we make use of the definitions given in the previous section, i.e. $\bar{S}$, $\bar{S}^0$ and $\bar{S}^1$. But, this time, considering the points in the plane -as it is done in the previous section- is not sufficient to construct the total production cost function since it is neither convex nor concave, in general. Hence, we have to determine the line segments representing the production cost of each resource combination. For each resource combination in $\bar{S}$, we determine the minimum quantity that will be produced along with its total cost, the unit production cost and the maximum quantity that can be produced with this combination. Note that this information is sufficient to draw the production cost function for that combination. Below, we present a systematic way to do that.

For each $S \in \bar{S}^0$, $Q(S) = \sum_{j \in S} C^j$ units will be produced by using all the resources in the set S in full capacity with the cost of $F(S) = \sum_{j \in S} \left(f^j + p^j C^j\right)$. More units can be produced if a new resource $i \notin S$ is added to the combination S under the assumption that the carbon emission constraint is still satisfied with this new combination. The following scenarios should be considered for each $S \in \bar{S}^0$:

(a) A new resource $i \notin S$ might be included in the combination, and this new combination can be used until either i is fully utilized or the carbon emission constraint (2d) becomes tight.

(b) If the emission constraint is tight, then two new resources $i, j \notin S$ might be included in the combination, and this new combination can be used until either i or j is fully utilized.

(c) If the emission constraint is tight, then a new resource $i \notin S$ might be included in the combination, and production level of a resource $j \in S$ that was fully used in the combination might be decreased while the production level of i is increased until either i is fully used or j's production level reaches to zero.

Table 1: Possible scenarios for $S \in \bar{S}^0$

Condition	**Starting point**	**Slope**	**Additional quantity**
$i \notin S$, $\bar{e}^i \leq 0$ if $\sum_{j \in S} \bar{e}^j C^j = 0$	$\left(Q(S), F(S) + f^i\right)$	p^i	$\min\left\{\frac{-\sum_{j\in S} \bar{e}^j C^j}{\max\{\bar{e}^i, 0\}}, C^i\right\}$
$i, j \notin S$, $\bar{e}^i \bar{e}^j < 0$, $\sum_{k \in S} \bar{e}^k C^k = 0$	$\left(Q(S), F(S) + f^i + f^j\right)$	$\frac{p^i\|\bar{e}^j\| + p^j\|\bar{e}^i\|}{\|\bar{e}^i\| + \|\bar{e}^j\|}$	$\min\left\{\frac{C^i}{\|\bar{e}^j\|}, \frac{C^j}{\|\bar{e}^i\|}\right\}\left(\|\bar{e}^j\| + \|\bar{e}^i\|\right)$
$i \notin S$, $j \in S$, $\sum_{k \in S} \bar{e}^k C^k = 0$, $\bar{e}^i \bar{e}^j > 0$, $\|\bar{e}^j\| > \|\bar{e}^i\|$, $p^i\|\bar{e}^j\| > p^j\|\bar{e}^i\|$	$\left(Q(S), F(S) + f^i\right)$	$\frac{p^i\|\bar{e}^j\| - p^j\|\bar{e}^i\|}{\|\bar{e}^j\| - \|\bar{e}^i\|}$	$\min\left\{\frac{C^i}{\|\bar{e}^j\|}, \frac{C^j}{\|\bar{e}^i\|}\right\}\left(\|\bar{e}^j\| - \|\bar{e}^i\|\right)$

Similarly, for each $(S, i) \in \bar{S}^1$, $Q(S,i) = \sum_{j \in S} C^j + x^{(S,i)}$ units will be produced by using all the resources in the set S in full capacity except i with the cost of $F(S,i) = \sum_{j \in S} f^j + \sum_{j \in S \setminus \{i\}} p^j C^j + p^i x^{(S,i)}$. Note that this time the carbon emission constraint (2d) should be tight. As a result, more units can be produced either i is used along with a new resource $j \notin S$, or along with a resource $j \in S$ that was used fully utilized (in the reverse direction). Specifically, the following scenarios should be considered for each $(S, i) \in \bar{S}^1$:

(a) A new resource $j \notin S$ might be included in the combination, and this new combination can be used until either i or j is fully utilized. Note that i and j should be from different resource groups (green vs regular) since the emission constraint is tight.

(b) Production level of a resource $j \in S$ that was fully used in the combination might be decreased while the production level of i is increased until either i is fully used or j's production level reaches to zero.

(c) A new resource $j \notin S$ might be included in the combination, and the production level of j might be increased while the production level of i is decreased until either i's production level reaches to zero or j is fully utilized.

In Tables 1 and 2, we present the data necessary to define the line segment (production cost function) for each scenario given above. In these tables, the starting point represents the coordinate - quantity and cost - of the starting point in the two-dimensional plane, slope is the unit production cost for the new combination, and additional quantity is the length of the line segment in the first axis, i.e. additional quantity that can be produced with the new combination (assume that division by zero results ∞). We also give the necessary conditions for the resources under the column "Condition". We do not report the costs of the other end points of the line segments since they can be determined easily by using the data given in the tables.

While determining the possible scenarios for S or (S, i), we only consider the cases where producing more units has a positive cost with that combination since the reverse (the negative cost case) corresponds to another combination, and it will be also considered as another element in the set $\bar{S}$. Hence, the conditions given in the tables are derived due to this assumption and also due to the carbon emission constraints. Moreover, in the cases where the production level of a resource is reduced, it is possible to observe an endpoint where that resource is eliminated,

Table 2: Possible scenarios for $(S,i) \in \bar{S}^1$

Condition on j	**Starting point**	**Slope**	**Additional quantity**
$j \notin S$, $\bar{e}^i\bar{e}^j < 0$	$\left(Q\left(S,i\right), F\left(S,i\right) + f^j\right)$	$\frac{p^i\lvert\bar{e}^j\rvert + p^j\lvert\bar{e}^i\rvert}{\lvert\bar{e}^i\rvert + \lvert\bar{e}^j\rvert}$	$\min\left\{\frac{C^i - x^{(S,i)}}{\lvert\bar{e}^j\rvert}, \frac{C^j}{\lvert\bar{e}^i\rvert}\right\}\left(\lvert\bar{e}^i\rvert + \lvert\bar{e}^j\rvert\right)$
$j \in S$, $\lvert\bar{e}^j\rvert > \lvert\bar{e}^i\rvert$, $p^i\lvert\bar{e}^j\rvert > p^j\lvert\bar{e}^i\rvert$	$\left(Q\left(S,i\right), F\left(S,i\right)\right)$	$\frac{p^i\lvert\bar{e}^j\rvert - p^j\lvert\bar{e}^i\rvert}{\lvert\bar{e}^j\rvert - \lvert\bar{e}^i\rvert}$	$\min\left\{\frac{C^i - x^{(S,i)}}{\lvert\bar{e}^j\rvert}, \frac{C^j}{\lvert\bar{e}^i\rvert}\right\}\left(\lvert\bar{e}^j\rvert - \lvert\bar{e}^i\rvert\right)$
$j \notin S$, $\lvert\bar{e}^i\rvert > \lvert\bar{e}^j\rvert$, $p^j\lvert\bar{e}^i\rvert > p^i\lvert\bar{e}^j\rvert$	$\left(Q\left(S,i\right), F\left(S,i\right) + f^j\right)$	$\frac{-p^i\lvert\bar{e}^j\rvert + p^j\lvert\bar{e}^i\rvert}{\lvert\bar{e}^i\rvert - \lvert\bar{e}^j\rvert}$	$\min\left\{\frac{x^{(S,i)}}{\lvert\bar{e}^j\rvert}, \frac{C^j}{\lvert\bar{e}^i\rvert}\right\}\left(\lvert\bar{e}^i\rvert - \lvert\bar{e}^j\rvert\right)$

i.e. its production level drops to zero. In these cases, to correctly determine the cost of the end point one has to reduce the setup cost of that resource from the total cost. But we do not have to do that, since we take the lower envelope after all these production cost functions are determined.

Note that for each element in $\bar{S}$, $O(M)$ line segments might be considered. Hence, after all the production cost functions are determined for each potential breakpoint in the set $\bar{S}$, the total production cost function can be found by taking the lower envelope of these line segments by using the algorithm of Shamos and Hoey (1976) in $O\left(M|\bar{S}|logM|\bar{S}|\right)$ time, i.e. in constant time for fixed M.

Once the total production cost function is determined together with its breakpoints and slopes, one can solve the problem CLS-PC in $O\left(T^{2M'+2}\right)$ time by using the dynamic programming algorithm of Koca et al. (2014) where M' represents the number of important breakpoints of the total production cost function, i.e. the ones where the function is not concave between them.

7 CLS-PC with at most two setups at each period (CLS2-PC)

In this section, we consider an extension of CLS-PC where at most two of the resources can be used at any period, i.e. the constraints $\sum_{m=1}^{M} y_t^m \leq 2$ for $t = 1, \ldots, T$ are added to the formulation (2). This problem might arise in a production system where some of the resources should be selected at each period to allow the ones that are not selected to be used in the other production processes, or in a system where a contract with a limited number of suppliers can be signed, or a distribution system where a limited number of transportation modes should be selected. Here, the main idea is to keep the number of resources used at any period at a limited level to make the system easier to manage. Indeed, a more general problem where at most $k \leq M$ of the resources should be selected could be considered as an extension of CLS-PC, but as it is expected, the complexity of the problem increases with k.

Due to the new restriction on the number of resources that can be used at any period, in a preprocessing step, one can consider all machine pairs (m_1, m_2) where m_1 is a green resource and m_2 is a green or regular resource, and construct the production cost function for each machine pair as it is done in Section 4. Note that constructing the production cost function for any machine pair can be done in a fixed time, and since the number of machine pairs that should be considered is $\bar{M} = M_g \times M_r + M_g \times M_g$, the problem can be reduced to an instance of multi-mode lot sizing problem with piecewise linear production cost functions (MMLS-PL) where at most one resource can be used at any period in $O(M_g \max\{M_g, M_r\})$ time. Moreover, as it can be observed from Section 4, the production cost functions include at most two (important) breakpoints for each resource pair.

To the best of our knowledge, there is no algorithm in the literature for solving MMLS-PL. Indeed, the problem can be transformed into an instance of single resource lot sizing problem with piecewise linear production cost functions, and then it can be solved by the algorithms of Koca et al. (2014) or Ou (2017). But, to do that, as it is done in Section 6 for CLS-PC, one has to take the lower envelope of the piecewise linear production cost functions of resources. In this section, instead of constructing a single piecewise linear cost function, we modify the algorithm of Koca et al. (2014) to solve MMLS-PL directly in polynomial time when $\bar{M}$ is fixed.

We make use of the following concepts which are widely used in the lot sizing literature.

Definiton 1. *An interval $[k, l]$ where $1 \leq k \leq l \leq T$, $s_{k-1} = s_l = 0$ and $s_t > 0$ for $k \leq t < l$ is called a* ***regeneration interval****.*

Definiton 2. *A production period t is called a* ***fractional period*** *if the production amount x_t^m is not equal to any of the breakpoints of the production cost function of resource m.*

From now on, we call the resource pairs as resources, and we assume that the number of resources $\bar{M}$ is fixed. The number of breakpoints of the production cost function of resource r will be denoted by B_r where $B_r \leq 2$ for all $r = 1, \ldots, \bar{M}$. Let b be the vector of breakpoints such that b_{ri} is the ith breakpoint of the production cost function of resource r for $i = 1, \ldots, B_r$, $r = 1, \ldots, \bar{M}$. We assume that we have $0 < b_1 < \ldots < b_{B_r}$ for each resource $r = 1, \ldots, \bar{M}$. To represent the no production case, we define a dummy resource 0 with one breakpoint, $B_0 = 1$ and $b_{01} = 0$. We denote the total number of breakpoints as $\bar{B} = \sum_{r=0}^{\bar{M}} B_r$ and assume that it is also fixed. The unit production cost for resource r is given by c_{ri} if the production quantity x lies in the ith segment of the cost function, i.e. $b_{r,i-1} < x \leq b_{ri}$ for $i = 1, \ldots, B_r$ and $r = 1, \ldots, \bar{M}$. The segment of the production cost function of resource r where x lies is denoted by $i(r, x)$ for $0 \leq x \leq b_{r,B_r}$ and $r = 0, \ldots, \bar{M}$.

Due to the monotonicity of the production cost function of each resource, there exists an optimal solution to MMLS-PL with zero ending inventory, i.e. $s_T = 0$. Thus, there exists an optimal solution to the problem that is composed of a series of regeneration intervals that cover the planning horizon $[1, T]$. Moreover, for each regeneration interval of an optimal solution we have the following property.

Theorem 3. *There exists an optimal solution for MMLS-PL such that there exists at most one fractional period at each regeneration interval.*

Proof. Let $[k, l]$ be a regeneration interval in an optimal solution (x, s), and assume that there are more than one fractional periods in $[k, l]$. Let u and v be two consecutive fractional periods in this regeneration interval with $u < v$. Assume that resources r_1 and r_2 are used in periods u and v, respectively, and the quantities x_{ur_1} and x_{vr_2} lie in the i_1th and i_2th segments of their production cost functions, respectively, i.e. $b_{r_1,i_1-1} < x_{ur_1} < b_{r_1,i_1}$ and $b_{r_2,i_2-1} < x_{vr_2} < b_{r_2,i_2}$. Define $\Delta^1 = \min\{\min_{t=u}^{v-1} s_t, x_{ur_1} - b_{r_1,i_1-1}, b_{r_2,i_2} - x_v\} > 0$ and $\Delta^2 = \min\{b_{r_1,i_1} - x_{ur_1}, x_{vr_2} - b_{r_2,i_2-1}\} > 0$. Now consider the following two solutions (x^1, s^1) and (x^2, s^2) same with (x, s) except

$$x^1_{ur_1} = x_{ur_1} - \Delta^1,\ s^1_t = s_t - \Delta^1 \ \text{ for } t = u, \ldots, v-1,\ x^1_{vr_2} = x_{vr_2} + \Delta^1$$

$$x^2_{ur_1} = x_{ur_1} + \Delta^2,\ s^2_t = s_t + \Delta^2 \ \text{ for } t = u, \ldots, v-1,\ x^2_{vr_2} = x_{vr_2} - \Delta^2$$

Note that both solutions are feasible. Moreover, since (x, s) is optimal we should have

$$\Delta^1 \left(-c_{r_1 i_1} - \sum_{t=u}^{v-1} h_t + c_{r_2 i_2} \right) \geq 0 \text{ and } \Delta^2 \left(c_{r_1 i_1} + \sum_{t=u}^{v-1} h_t - c_{r_2 i_2} \right) \geq 0.$$

Since $\Delta^1, \Delta^2 > 0$, $c_{r_1 i_1} + \sum_{t=u}^{v-1} h_t - c_{r_2 i_2} = 0$ should hold. This means that both (x^1, s^1) and (x^2, s^2) are also optimal. Note that (x^2, s^2) is an optimal solution where the number of fractional periods in the regeneration interval $[k, l]$ is reduced by one since u or v is not a fractional period: i)if $\Delta^2 = b_{r_1,i_1} - x_{ur_1}$, then u is not a fractional period, ii) if $\Delta^2 = x_v - b_{r_2,i_2-1}$, then v is not a fractional period. This procedure can be applied until an optimal solution with the desired property is obtained, so the result follows. □

Note that these results are analogous to Theorems 1 and 2 of Koca et al. (2014) derived for the (single resource) lot sizing problem with piecewise concave production cost functions. We also use the notation of Koca et al. (2014) in our algorithm given below. The algorithm is composed of two main steps: first, the minimum cost for each regeneration interval is determined, and then the problem is solved as a shortest path problem. The details of these steps are given in the next two subsections.

7.1 Minimum cost for a regeneration interval $[k, l]$

In this part, we find the minimum cost for a regeneration interval $[k, l]$ where $s_{k-1} = s_l = 0$ and $s_t > 0$ for $t = k, \ldots, l-1$.

We define $\boldsymbol{\tau}_{ri} \in \mathbb{Z}_+$ and $\boldsymbol{\pi}_{ri} \in \mathbb{Z}_+$ to represent the number of times production occurs at the level of breakpoint b_{ri} before and after period t, respectively, for $r = 1, \ldots, \bar{M}$, $i = 1, \ldots, B_r$. Note that if τ_{ri} times b_{ri} units are produced between periods k and t for $r = 1, \ldots, \bar{M}$, $i = 1, \ldots, B_r$, and π_{ri} times b_{ri} units will be produced between periods $t+1$ and l for $r = 1, \ldots, \bar{M} - 1$, $i = 1, \ldots, B_r$, and $i = 1, \ldots, B_{\bar{M}} - 1$ when $r = \bar{M}$, then the number of times $b_{\bar{M}B_{\bar{M}}}$ units will be produced between periods $t+1$ and l can be determined as

$$\pi_{\bar{M}B_{\bar{M}}} := \left\lfloor \frac{d_{kl} - \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} - \sum_{r=1}^{\bar{M}-1} \sum_{i=1}^{B_r} \pi_{ri} b_{ri} - \sum_{i=1}^{B_{\bar{M}}-1} \pi_{\bar{M}i} b_{\bar{M}i}}{b_B} \right\rfloor$$

where $d_{kl} = \sum_{t=k}^{l} d_t$. Hence, we define $\boldsymbol{\pi}_{\bar{M}}$ as a $B_{\bar{M}}$-vector, but assume that its last component is determined by the expression given in the previous statement. Moreover, the remaining amount, after considering π and τ, will be the amount that should be produced in a fractional period, say $\rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi})$:

$$\rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi}) = d_{kl} - \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} - \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \pi_{ri} b_{ri}.$$

Note that in a single resource problem one has to determine the fractional period and the fractional production amount. As we have multiple resources, in addition to these decisions, we should also decide the resource that will produce the fractional quantity.

Since we search for an optimal solution where at each regeneration interval there exists at most one fractional period, we define two types of value functions $F_{kl}(t, \boldsymbol{\tau})$ and $G_{kl}(t, \boldsymbol{\tau}, \boldsymbol{\pi})$ to represent the cases where the interval $[k, t]$ does not contain any fractional period, and contains a single fractional period, respectively.

Let $F_{kl}(t, \boldsymbol{\tau})$ be the minimum cost for a regeneration interval $[k, l]$ during which τ_{ri} times b_{ri} units are produced at the corresponding resources and no fractional production is done between periods k and t given that $s_t > 0$ for $t = k, \ldots, l-1$. Then, for the first period k of the regeneration interval, for $r = 0, \ldots, \bar{M}$, $i = 1, \ldots, B_r$, we compute

$$F_{kl}(k, \boldsymbol{e}_{ri}) = \begin{cases} c_{ri} b_{ri} + h_k (b_{ri} - d_k), & \text{if } b_{ri} > d_k \text{ and } k < l \\ c_{ri} b_{ri}, & \text{if } b_{ri} = d_k \text{ and } k = l \\ \infty, & \text{otherwise.} \end{cases}$$

where $\boldsymbol{e}_{ri}$ is a vector of the appropriate size (which has a component for each breakpoint of each resource) where only the ith component of the rth vector is one with the convention that $\boldsymbol{e}_{01}$ represents the zero vector (no production case). For the other values of τ, i.e. $\tau \neq e_{ri}$ for all $r \in \{0, \ldots, \bar{M}\}$, $i \in \{1, \ldots, B_r\}$, we let F_{kl} to be ∞. Then, for $t \in \{k+1, \ldots, l\}$, the following recursive equations will be used

$$F_{kl}(t, \boldsymbol{\tau}) = \min_{\substack{r=0,\ldots,\bar{M}, \\ i=1,\ldots,B_r: \\ \boldsymbol{\tau} \geq \boldsymbol{e}_{ri}}} \{F_{kl}(t-1, \boldsymbol{\tau} - \boldsymbol{e}_{ri}) + c_{ri} b_{ri}\} + h_t \left(\sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} - d_{kt} \right)$$

for $\boldsymbol{\tau}_r \in \mathbb{Z}_+^{B_r}$ vectors for $r = 0, \ldots, \bar{M}$ satisfying $\sum_{r=0}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} \leq t-k+1$, $\sum_{r=0}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} > d_{kt}$ when $t < l$, and $\sum_{r=0}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} = d_{kt}$ when $t = l$. Otherwise, we set $F_{kl}(t, \boldsymbol{\tau}) = \infty$.

While determining the value of $F_{kl}(t, \boldsymbol{\tau})$, all possible production quantities for period t are considered with the vectors $\boldsymbol{e}_{ri}$ for $r = 0, \ldots, \bar{M}$, $i = 1, \ldots, B_r$. We evaluate the above recursion for increasing values of t and all possible values of $\boldsymbol{\tau}$. For a given t and $\boldsymbol{\tau}$, F_{kl} can be computed in fixed time since $\bar{M}$ and $\bar{B}$ are assumed to be fixed. As the number of possible $\boldsymbol{\tau}$ vectors is $O\left(T^{\bar{B}}\right)$, F_{kl} can be evaluated in $O\left(T^{\bar{B}+1}\right)$ time for a regeneration interval $[k, l]$.

Let $G_{kl}(t, \boldsymbol{\tau}, \boldsymbol{\pi})$ be the minimum cost for a regeneration interval $[k, l]$ during which τ_{ri} times b_{ri} units are produced and a fractional production is done in the level of $\rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi})$ between periods k and t, given that π_{ri} times b_{ri} units are produced between periods $t+1$ and l, and $s_t > 0$ for $t = k, \ldots, l-1$. Then, for the first period k of the regeneration interval, we compute

$$G_{kl}(k, \boldsymbol{e}_{01}, \boldsymbol{\pi}) = \begin{cases} \min_{r \in \bar{M}_\rho} \{c_{ri(r,\rho_{kl})}\} \rho_{kl}(\boldsymbol{e}_{01}, \boldsymbol{\pi}) + h_k(\rho_{kl}(\boldsymbol{e}_0, \boldsymbol{\pi}) - d_k), & \text{if } \rho_{kl}(\boldsymbol{e}_{01}, \boldsymbol{\pi}) > d_k \text{ and } k < l \\ \min_{r \in \bar{M}_\rho} \{c_{ri(r,\rho_{kl})}\} \rho_{kl}(\boldsymbol{e}_{01}, \boldsymbol{\pi}), & \text{if } \rho_{kl}(\boldsymbol{e}_{01}, \boldsymbol{\pi}) = d_k \text{ and } k = l \\ \infty, & \text{otherwise.} \end{cases}$$

where $\bar{M}_\rho$ is the subset of resources such that $\rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi})$ is not equal to any of the breakpoints, and $\boldsymbol{\pi}$ satisfies $d_{kl} > \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \pi_{ri} b_{ri}$ and $\sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \pi_{ri} \leq l - k$, and $\rho_{kl}(\boldsymbol{e}_{01}, \boldsymbol{\pi})$ such that $\bar{M}_\rho \neq \emptyset$. Otherwise, we let G_{kl} to be ∞. Then, for $t \in \{k+1, \ldots, l\}$, the following recursive equations will be used

$$G_{kl}(t, \boldsymbol{\tau}, \boldsymbol{\pi}) = \min \left\{ F_{kl}(t-1, \boldsymbol{\tau}) + \min_{r \in \bar{M}_\rho} \{c_{ri(r,\rho)}\} \rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi}), \min_{\substack{r=0,\ldots,\bar{M}, \\ i=1,\ldots,B_r: \\ \boldsymbol{\tau} \geq \boldsymbol{e}_{ri}}} \{G_{kl}(t-1, \boldsymbol{\tau} - \boldsymbol{e}_{ri}, \boldsymbol{\pi} + \boldsymbol{e}_{ri}) + c_{ri} b_{ri}\} \right\}$$
$$+ h_t \left(\sum_{i=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} + \rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi}) - d_{kt} \right)$$

for $\boldsymbol{\tau}_r \in \mathbb{Z}_+^{B_r}$ and $\boldsymbol{\pi}_r \in \mathbb{Z}_+^{B_r}$ satisfying: $\sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} \leq t - k$, $\sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \pi_{ri} \leq l - t$, $\sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} + \rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi}) > d_{kt}$ when $t < l$, and $\sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} + \rho_{kl}(\boldsymbol{\tau}, \boldsymbol{\pi}) = d_{kt}$ when $t = l$, $\sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} + \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \pi_{ri} b_{ri} \leq d_{kl}$ and $\bar{M}_\rho \neq \emptyset$. Otherwise, we set $G_{kl}(t, \boldsymbol{\tau}, \boldsymbol{\pi}) = \infty$.

While determining the value of $G_{kl}(t, \boldsymbol{\tau}, \boldsymbol{\pi})$, all possibilities for period t are considered: i) the fractional production quantity is produced in period t and the previous productions occur in breakpoint levels ii) fractional production quantity is produced at a previous period, and production at period t occurs at a breakpoint level of a resource. For given t, τ, and π, G_{kl} can be evaluated in a fixed time when the number of resources $\bar{M}$ and the total number of breakpoints $\bar{B}$ are fixed. We evaluate the above recursion for increasing values of t and all possible values of τ and π. Since the number of possible τ and π vectors are $O\left(T^{\bar{B}}\right)$ and $O\left(T^{\bar{B}-1}\right)$, respectively, G_{kl} can be evaluated in $O\left(T^{2\bar{B}}\right)$ time for a regeneration interval $[k, l]$.

7.2 Minimum cost for the problem MMLS-PL

Once F_{kl} and G_{kl} are evaluated for all possible t, $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$, the minimum cost for the regeneration interval $[k, l]$ can be found by

$$\mu_{kl} = \min_{\substack{r=0,\ldots,\bar{M}: \\ \boldsymbol{\tau}_r \in \{0,\ldots,T\}^{B_r}}} \{F_{kl}(l, \boldsymbol{\tau}), G_{kl}(l, \boldsymbol{\tau}, \boldsymbol{e}_{01})\}.$$

Hence, the minimum cost for a given regeneration interval can be determined in $O\left(T^{2\bar{B}}\right)$ time. Since there are $O\left(T^2\right)$ regeneration intervals, the minimum cost for all regeneration intervals can be found in $O\left(T^{2\bar{B}+2}\right)$ time.

After determining μ_{kl} for all $1 \le k \le l \le T$, we construct a directed graph $G = (N, A)$ where each node represents a period, $N = \{1, \ldots, T{+}1\}$, and each arc $(k, l{+}1)$ represents a regeneration interval $[k, l]$, $A = \{(k, l+1) : 1 \le k \le l \le T\}$. We assume that cost of arc $(k, l+1)$ is given by μ_{kl}. Then, the shortest path problem from node 1 to node $T + 1$ solves MMLS-PL. Since the shortest path problem can be solved in $O\left(T^2\right)$ time, the dominant part in the algorithm is to construct the graph G which is done in $O\left(T^{2\bar{B}+2}\right)$ time.

As a result, CLS2-PC can be solved in $O\left(T^{2\bar{B}+2}\right)$ time which is polynomial when the number of resources is fixed, and the cost and emission parameters are time-invariant.

8 Conclusion

We studied the single item capacitated multi-mode lot sizing problem with periodic carbon emission constraints. We presented several structural properties of the problem and showed that the resource capacities make the problem NP-Hard. We proposed algorithms to construct the piecewise linear total production cost functions to be able to solve the problem using the algorithms developed for the lot sizing problem with piecewise concave production cost functions. We also considered an extension of the problem where at most two resources can be used at any period, and developed a polynomial time dynamic programming algorithm to solve it when the number of resources, the cost and emission parameters and the capacities of the resources are time-invariant.

The complexity of the algorithms presented in Sections 5 and 6 are fixed when the number of resources is fixed. Alternatively, one can try to construct the total production cost functions by considering the possible amounts that can be produced in any period. But note that, this leads to another exponential algorithm when the number of resources is not fixed. Alternative and more efficient algorithms can be designed in a future research. Besides, heuristics, or algorithms to approximate the total production cost function could be proposed.

In this paper, we focused on the complexities of the problems. In a future research, computational analysis can be conducted to see the effects of the carbon emission constraints and the resource capacities. Besides, different carbon emission constraints (nonlinear, for example) might be considered instead of the ones considered in this paper.

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