

# Applications of stochastic mixed-integer second-order cone optimization

Baha Alzalg<sup>1,2</sup> · Hadjer Alioui<sup>1</sup>

Monday 22nd November, 2021 (12:56) \*

**Abstract** Second-order cone programming problems are a tractable subclass of convex optimization problems that can be solved using polynomial algorithms. In the last decade, stochastic second-order cone programming problems have been studied, and efficient algorithms for solving them have been developed. The mixed-integer version of these problems is a new class of interest to the optimization community and practitioners, in which certain variables are required to be integers. In this paper, we describe five applications that lead to stochastic mixed-integer second-order cone programming problems. Additionally, we present solution algorithms for solving stochastic mixed-integer second-order cone programming using cuts and relaxations by combining existing algorithms for stochastic second-order cone programming with extensions of mixed-integer second-order cone programming. The applications, which are the focus of this paper, include facility location, portfolio optimization, uncapacitated inventory, battery swapping stations, and berth allocation planning. Considering the fact that mixed-integer programs are usually known to be NP-hard, bringing applications to the surface can detect tractable special cases and inspire for further algorithmic improvements in the future.

**Keywords** Second-order cone programming · Mixed-integer programming · Stochastic programming · Applications · Algorithms

## 1 Introduction

Challenges, restrictions, and affects such as uncertainty, integrality, and conicity arise naturally in real-world applications. For example, the (deterministic) mixed-integer second-order cone programming (DMISOCP for short) models presented in [1] have proved to be useful in dealing with a variety of applications that involve integrality and conicity. For another example, stochastic mixed-integer linear programming (SMILP for short) [2] has been demonstrated to be effective in many applications involving integrality and uncertainty (see [2, Section 1] and the references contained therein). For a third example, the stochastic second-order cone programming (SSOCP for short) models described in [3] and the stochastic semidefinite programming models described in [4] have proved to be useful in dealing with uncertainty and conicity in many applications. In this paper, we show that uncertainty, integrality, and conicity can all naturally coexist together in the context of what can be termed as the stochastic mixed-integer second-order cone programming (SMISOCP for short).

The second-order cone is defined to contain all vectors whose first component must be at least as large as the Euclidean norm of the subvector of the remaining components. Second-order cone programming problems are a tractable subclass of convex optimization problems in which the objective function is optimized over the intersection of a linear affine space and the Cartesian product of second-order cones.

---

✉ B. Alzalg (Corresponding author)  
b.alzalg@ju.edu.jo

✉ H. Alioui  
haj9180341@ju.edu.jo

1 Department of Mathematics, The University of Jordan, Amman, Jordan 11942

2 Department of Computer Science and Engineering, The Ohio State University, Columbus, OH 43210

Interior-point algorithms are considered one of the most efficient methods developed for solving this subclass of convex optimization problems in polynomial-time (see [5–7], for example). The mixed-integer version of these problems, acronymized above as DMISOCP, requires some variables to be restricted to integers. Although it is NP-hard in general, DMISOCP can be quickly solved by applying algorithmic techniques that use scenario-based cuts and relaxations (see [1], for example). DMISOCP was introduced to make optimal decisions in applications with certainty in information. In some applications, we cannot completely determine the modal since it is dependent on some information that is not available at the moment of making optimal decisions. An extra dimension of difficulty is added if we allow uncertainty in data defining DMISOCP problems. This yields the stochastic version of these problems, acronymized above as SMISOCP, which is algorithmically much less mature compared with SSOCP, SMILP, and even DMISOCP. In the present paper, we will demonstrate how SMISOCP applies to a variety of application areas by describing five applications leading to two-stage SMISOCP models. Specifically, we present a stochastic discrete facility location problem, a portfolio optimization with CVaR and diversification constraints, a stochastic joint uncapacitated location-inventory problem, an optimal infrastructure problem for electric vehicles with battery swap technology, and an optimal random berth allocation problem with uncertain handling time.

Since this paper mainly introduces SMISOCP from a practical point of view, we do not provide each application model with a specific algorithm to solve it. Nevertheless, a solution method for the generic SMISOCP is presented in this paper to combine existing efficient algorithms for SSOCP [8–12] with extensions of DMISOCP algorithms [13–18]. Related aspects are tackled in an algorithmic companion paper [19]. It is our belief that bringing the applications proposed in this research to the surface can detect tractable special cases for extra algorithmic improvements in the future.

This paper is structured as follows. In Section 2, we introduce some notations followed by the definition of the two-stage SMISOCP problem with recourse. Section 3 contains the main results of the paper. Here, we describe different applications and their formulations as SMISOCP models. In Section 4, we first give a brief review of the solution algorithms available for solving SSOCP, then we present a solution method for solving SMISOCP. Section 5 provides some concluding remarks.

## 2 Formulation

In this part, we start by introducing certain notations, then we define the two-stage SMISOCOP problem.

### 2.1 Notations and basics

We utilize “;” and “,” for adjoining matrices and vectors in a column and a row, respectively. As an illustration, if  $x$  and  $y$  are vectors, the following expressions are equivalent:  $(x^T, y^T)^T = (x; y)$ .

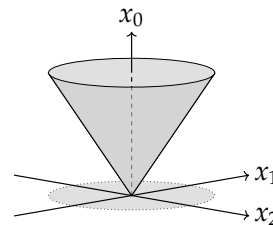
For each vector  $x \in \mathbb{R}^n$  indexed from 0, we denote  $\bar{x}$  for the sub-vector containing components 1 through  $n - 1$ ; therefore  $x = (x_0; \bar{x})$ .

The  $n$ th-dimensional second-order cone (also known as the Lorentz or quadratic cone) is defined as

$$\mathcal{Q}_n := \{x = (x_0; \bar{x}) \in \mathcal{E}_n : x_0 \geq \|\bar{x}\|\},$$

where  $\mathcal{E}_n = \mathbb{R} \times \mathbb{R}^{n-1}$  and  $\|\cdot\|$  denotes the Euclidean norm.

The graph to the right shows the 3rd-dimensional second-order cone  $\mathcal{Q}_3$ . The cone  $\mathcal{Q}_n$  is closed, pointed (i.e., it does not contain a pair of opposite nonzero vectors) and convex with nonempty interior in  $\mathbb{R}^n$ . It is also known that  $\mathcal{Q}_n$  is self-dual (i.e. it equals its dual cone) and its automorphism group acts transitively on its interior. Given these properties, it is clear that  $\mathcal{Q}_n$  is a symmetric cone (see [20]).



If  $x \in \mathbb{R}^n$  and  $n$  is known from the context, we denote  $x \geq \mathbf{0}$  to mean that  $x \in \mathcal{Q}_n$ . If  $x \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \dots \times \mathbb{R}^{k_r}$  and  $k_1, k_2, \dots, k_r$  are positive integers known from the context, we denote  $x \succeq_r \mathbf{0}$  to mean that  $x \in \mathcal{Q}_{k_1} \times \mathcal{Q}_{k_2} \times \dots \times \mathcal{Q}_{k_r}$ .

$\dots \times Q_{k_r}$ . Therefore,  $x \succeq_r \mathbf{0}$  if and only if  $x$  is partitioned conformally as  $x = (x_1; x_2; \dots; x_r)$  and  $x_i \succeq \mathbf{0}$  for each  $i = 1, 2, \dots, r$ . We also write  $x \succeq_r y$  or  $y \preceq_r x$  to mean that  $x - y \succeq_r \mathbf{0}$ .

As a tool for algorithmic development, in Section 4 we will use the Jordan multiplication  $\circ : \mathcal{E}_n \rightarrow \mathcal{E}_n$ , which is defined as

$$x \circ y = (x_0; \hat{x}) \circ (y_0; \hat{y}) := (x^T y; x_0 \hat{y} + y_0 \hat{x}). \quad (1)$$

In building the model of Subsection 3.5, we will make use of the  $n$ th-dimensional rotated quadratic cone, which is given as [5]

$$\mathcal{R}_n := \{x = (x_0; x_1; \hat{x}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} : x_0 x_1 \geq \|\hat{x}\|^2, x_0 \geq 0, x_1 \geq 0\}.$$

A hyperbolic constraint on  $x \in \mathbb{R}^n$  is one that satisfies the inequality  $x_0 x_1 \geq \|\hat{x}\|^2$ . The cone  $\mathcal{R}_n$  is derived by rotating the second-order cone  $Q_n$  in the  $x_0 x_1$ -plane by forty-five degrees. Note that

$$x_0 + x_1 \geq \|(x_0 - x_1; 2\hat{x})\| \iff (x_0 + x_1)^2 \geq \|(x_0 - x_1; 2\hat{x})\|^2 \iff 2x_0 x_1 \geq -2x_0 x_1 + 4\|\hat{x}\|^2 \iff x_0 x_1 \geq \|\hat{x}\|^2. \quad (2)$$

This means that the hyperbolic constraint  $x_0 x_1 \geq \|\hat{x}\|^2$  is equivalent to the second-order cone constraint  $(x_0 + x_1; x_0 - x_1; 2\hat{x}) \succeq \mathbf{0}$ .

## 2.2 SMISOCP problem definition

Let  $n, m, k, l, p$  and  $q$  be positive integers. Let also  $k_1, k_2, \dots, k_r$  and  $l_1, l_2, \dots, l_s$  be positive integers such that  $k = \sum_{i=1}^r k_i$  and  $l = \sum_{j=1}^s l_j$ . An SMISOCP with recourse is described using deterministic data  $A \in \mathbb{R}^{k \times n}$ ,  $b \in \mathbb{R}^k$  and  $c \in \mathbb{R}^n$ , as well as random data  $T \in \mathbb{R}^{l \times n}$ ,  $W \in \mathbb{R}^{l \times m}$ ,  $h \in \mathbb{R}^l$  and  $d \in \mathbb{R}^m$ . The realization of this data depend on an underlying outcome  $\omega$  in an event space  $\Omega$  with a known probability function  $P$ . Given this data, the two-stage SMISOCP with recourse is the problem

$$\begin{aligned} \min \quad & c^T x + \mathbb{E}[Q(x, \omega)] & \text{where } Q(x, \omega) \text{ is} & \min \quad & d(\omega)^T y \\ \text{s.t.} \quad & Ax \succeq_r b, & \text{the minimum value} & \text{s.t.} \quad & W(\omega)y \succeq_s h(\omega) - T(\omega)x, \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}, & \text{of the problem} & & y \in \mathbb{Z}^q \times \mathbb{R}^{m-q}, \end{aligned} \quad (3)$$

and

$$\mathbb{E}[Q(x, \omega)] := \int_{\Omega} Q(x, \omega) P(d\omega).$$

Here,  $x$  and  $y$  are the decision variables for the first- and second-stage, respectively.

Due to the integrality constraints on  $x$  and  $y$ , the SMISOCP problem (3) is nonconvex and NP-hard in general. Below we list three special cases of this optimization problem. See also Figure 1 which visualizes the conceptual relationships between the SMISOCP problem and its special cases.

- If  $p = q = 0$ , then no integrality constraints are imposed on  $x$  and  $y$ . In this case, the SMISOCP problem (3) is reduced to the two-stage SSOCP problem with recourse [3]. SSOCP is a convex optimization problem that can be solved in several methods in polynomial time (see [9–12]).

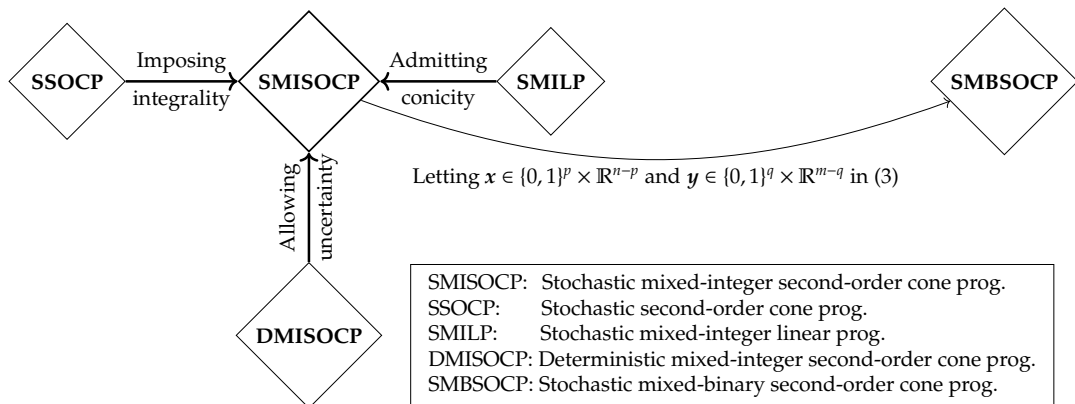


Figure 1: Conceptual relationships between SMISOCP and other relevant optimization problems.

- If  $k = r$  and  $l = s$ , then the conic constraints  $Ax \succeq_r \mathbf{b}$  and  $W(\omega)\mathbf{y} \succeq_s \mathbf{h}(\omega) - T(w)\mathbf{x}$  are reduced to linear constraints  $Ax \geq \mathbf{b}$  and  $W(\omega)\mathbf{y} \geq \mathbf{h}(\omega) - T(w)\mathbf{x}$ . In fact,  $Ax \succeq_k \mathbf{b}$  if and only if the vector  $Ax - \mathbf{b}$  is nonnegative, i.e., all its components belong to the first-dimensional second-order cone  $\mathcal{Q}_1 := \{t \in \mathbb{R} : t \geq 0\}$ . Similarly,  $W(\omega)\mathbf{y} \succeq_l \mathbf{h}(\omega) - T(w)\mathbf{x}$  if and only if  $W(\omega)\mathbf{y} + T(w)\mathbf{x} - \mathbf{h}(\omega) \geq \mathbf{0}$ . Therefore, the SMISOCP problem (3) is reduced to the two-stage SMILP problems with recourse in this case [2]. In the literature, this class of optimization problems has been studied widely. Solution methods for solving SMILP can be found in [2, 21–34].
- If all data are known with certainty, the second-stage problem disappears. In this case, the SMISOCP problem (3) reduces to the DMISOCP problem [1]. Solution methods for solving DMISOCP can be found in [1, 13–15] and the references contained therein. In particular, cutting planes are one of the most successful ways to deal with the integrality constraints in this optimization problem. Solution methods available in [1, 13, 14, 16] use scenario-based cuts for DMISOCP. The results in [15] show that branch and bound is also suited for solving DMISOCP.

Additionally, if the integrality constraints on  $\mathbf{x}$  and  $\mathbf{y}$  are restricted to be binary, we have  $\mathbf{x} \in \{0, 1\}^p \times \mathbb{R}^{n-p}$  and  $\mathbf{y} \in \{0, 1\}^q \times \mathbb{R}^{m-q}$ . In this case, the SMISOCP problem (3) may be termed as a two-stage stochastic mixed-binary second-order cone programming (SMBSOCP) problem with recourse. SMBSOCP is known to be as “hard” as the SMISOCP problem.

We refer the reader to two papers that deal with important special cases of or related to the SMISOCP problem (3): The first paper is by Luo and Mehrotra [35] which extends the work of Sen and Serali [32] for SMILP and proposes a decomposition method for (3) in which  $\mathbf{x} \in \{0, 1\}^p \times \mathbb{R}^{n-p}$  in the first-stage problem in (3) and  $\mathbf{y} \in \mathbb{Z}^q \times \mathbb{R}^{m-q}$  in the second-stage problem in (3). The second paper is by Bansal and Zhang [36] which proposes scenario-based cuts for (3) in which  $n = p$  and  $k = r$  in the first-stage problem in (3) and the  $l_2$ -norms are generalized to  $l_p$ -norms in the second-stage problem in (3).

### 3 Applications

Each of the following subsections will describe an application accompanied with SMISOCP model formulation.

#### 3.1 Stochastic discrete facility location problem

In a facility location problem (FLP for short), we are interested in choosing a location or locations to construct a new facility or new facilities so that desired distances from the new facility(ies) to existing facilities are minimized. In a Euclidean FLP, the desired distances are Euclidean distances. FLP arises in many applications including, but not limited to, locating regional campuses, educational sites, public health clinics, pharmacies and hospitals, placing servers in probabilistic networks and data centers, establishing and co-locating wireless telecommunication facilities, clustering data of high dimensionality, performing any type of location analysis for placement of new facilities of varying sizes and densities, and so on.

FLP arises also in many applications where the uncertainty is an important challenge. To realize this more, we describe a concrete example of the stochastic version of this application. We consider how to distribute new appropriate facilities among COVID-19 healthcare existing facilities across a small country, like Jordan, keeping in mind that COVID-19 is very unpredictable and easily spread. Given the pandemic circumstances, the healthcare existing facilities are either fixed or random. The fixed facilities are the constructed clinics or hospitals in the Jordanian cities. The random facilities are movable healthcare facilities equipped with the necessary medical supplies which enable patients to be treated immediately. See the picture in Figure 2 and the brief discussion provided to the left of this picture details about the random facilities. The location of a new facility can be a specific location for dropping off supplies, staff housing units, a healthcare unit with experts in treating mood and anxiety disorders, a fully functional advanced comprehensive unit, etc. In order to guarantee accuracy, the practitioners will use only the most recent information on the community’s COVID-19 epidemic outbreak. This may require increasing the initial number of random facilities. This increase, based on the number of fixed facilities and the initial number of random facilities, results in an additional cost.

The movable healthcare facilities are random existing facilities that can be moved from their sites to other sites depending on the prediction of the COVID-19 spreading rate in a model country. In the stochastic COVID-19 healthcare facility location problem, we know the specific locations of the fixed healthcare facilities, but we do not know the specific locations of the random healthcare facilities at the time of formulation. Instead, we know the realizations of the random facilities with a predefined probability  $p \in (0, 1)$ .

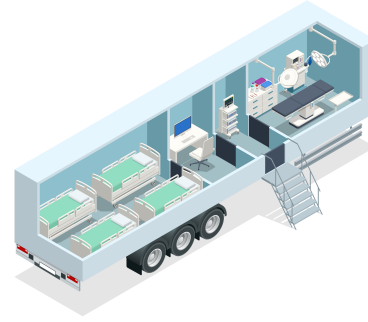


Figure 2: A random healthcare facility in the stochastic COVID-19 healthcare facility location problem.

Stochastic FLP arises in many applications where the integrality is an important restriction. Below we describe another, but discrete, concrete example of this generic application. In a discrete facility location problem with a risk of disruption, we specify a set of demand points (customers) to service and a set of potential facility points (or sites) to serve them. At the time of formulation, not all the potential facility points have constructed facilities, nor have opened facilities. Indeed, some constructed facilities are unavailable or closed due to disruptions caused by some factors such as weather, labor strikes, pandemic outbreak, natural disasters, terrorist attacks, etc. The discretion lies in the determination of whether a potential facility point has a constructed facility. The uncertainty lies in the determination of whether some of the constructed facilities are open. We are required to pick a subset of potential sites for facility construction and to choose a subset of facilities to open so that the sum of the distances between any demand point and its nearest opened facility, and the sum of the opening costs of the selected facilities are both minimized. Figure 3 shows a 49-demand-points solution to this problem in an interactive USA map.

In addition to the above two concrete examples, many other examples can also be referenced. Generally speaking, the locations of some existing facilities are not totally specified in a stochastic FLP. Such unspecified locations are dependent upon information that was not available at the time of making the decision and will not become available until a later time. In a discrete FLP, some or all new facilities must not be placed in any location, but must be located in specific locations, called nodes, in the solution space. The (deterministic) Euclidean FLP is often presented as an application of (deterministic) second-order cone programming (see [37], for example). The stochastic Euclidean FLP was presented in [3] as an application of SSOC. In this subsection, we present the stochastic discrete version of the generic Euclidean FLP as an application of SMISOCP.

★ Demand points.

○ Potential/unselected facility points.

⊙ Opened facilities with uncertainty.†

● Opened facilities with certainty.

† The constructed facilities that are open with uncertainty have randomness in terms of openness. Right now, the realizations of such facilities are unknown, but they will become known at some point in the future. The model contains these realizations with a prescribed probability  $p \in (0, 1)$ .

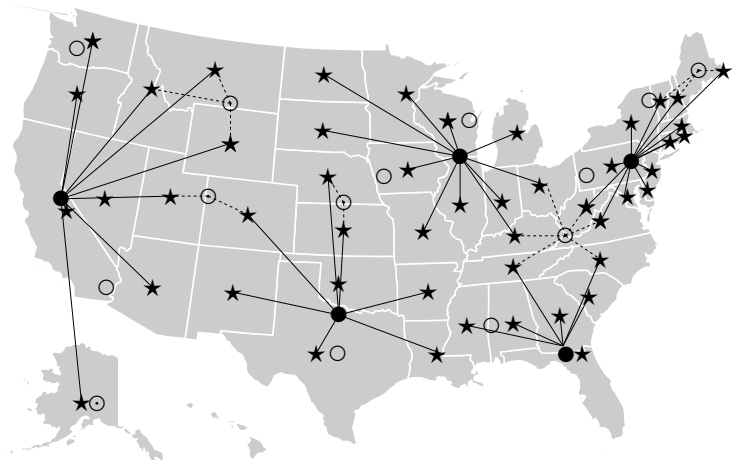


Figure 3: A facility location solution for a 49-demand-points problem shown in an interactive USA map.

In summary, let  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r$  be fixed points in  $\mathbb{R}^n$  symbolizing the coordinates of  $r$  existing fixed facilities, and  $\tilde{\mathbf{a}}_1(\omega), \tilde{\mathbf{a}}_2(\omega), \dots, \tilde{\mathbf{a}}_s(\omega)$  be random points in  $\mathbb{R}^n$  that represent the coordinates of  $s$  random facilities on which their realizations are dependent on an underlying outcome  $\omega$  in an event space  $\Omega$  with a known probability function  $P$ . Let us assume that right now the realizations of  $s$  random facilities are not known, and over the course of time their realizations will be known.

The goal is to add  $m$  new facilities,  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m \in \mathbb{R}^n$ , that minimize each of the following sums:

- The weighted sums of the Euclidean distance between each one of the new facilities,  $\mathbf{x}_j$  for  $j = 1, 2, \dots, m$ , and each one of the existing fixed facilities,  $\mathbf{a}_i$  for  $i = 1, 2, \dots, r$ . This requirement can be met by adding the inequalities  $t_{ij} \geq \|\mathbf{x}_j - \mathbf{a}_i\|$  for  $i = 1, 2, \dots, r$  and  $j = 1, 2, \dots, m$ . Equivalently, using our notations, this is satisfied by adding the constraints

$$(t_{1j}; \mathbf{x}_j - \mathbf{a}_1; \dots; t_{rj}; \mathbf{x}_j - \mathbf{a}_r) \succeq_r \mathbf{0}, \quad \text{for } j = 1, 2, \dots, m,$$

where the double summation  $\sum_{j=1}^m \sum_{i=1}^r w_{ij} t_{ij}$  is minimized. The parameter  $w_{ij}$  specifies the weight of the  $i$ th existing facility and the  $j$ th new facility, for  $i = 1, 2, \dots, r$  and  $j = 1, 2, \dots, m$ .

- The anticipated weighted sums of the Euclidean distance between each one of the new facilities and the realization of each one of the random facilities. This requirement can be met by adding the constraints

$$(\tilde{t}_{1j}; \mathbf{x}_j - \tilde{\mathbf{a}}_1(\omega); \dots; \tilde{t}_{sj}; \mathbf{x}_j - \tilde{\mathbf{a}}_s(\omega)) \succeq_s \mathbf{0}, \quad j = 1, 2, \dots, m,$$

where the double summation  $\sum_{j=1}^m \sum_{i=1}^s \tilde{w}_{ij}(\omega) \tilde{t}_{ij}$  is minimized. Here,  $\tilde{w}_{ij}$  denotes the weight associated with the  $i$ th random facility and  $j$ th new facilities, for  $i = 1, 2, \dots, s$  and  $j = 1, 2, \dots, m$ .

- The weighted sums of the Euclidean distance between any two new facilities. This requirement can be met by adding the constraints

$$(\hat{t}_{j(j+1)}; \mathbf{x}_j - \mathbf{x}_{j+1}; \dots; \hat{t}_{jm}; \mathbf{x}_j - \mathbf{x}_m) \succeq_{(m-j)} \mathbf{0}, \quad \text{for } j = 1, 2, \dots, m-1,$$

where the double summation  $\sum_{j=2}^m \sum_{k=1}^{j-1} \hat{w}_{jk} \hat{t}_{jk}$  is minimized. Here,  $\hat{w}_{jk}$  denotes the weight associated with the  $j$ th and  $k$ th new facilities, for  $k = 1, 2, \dots, j-1$  and  $j = 2, 3, \dots, m$ . We point out that if no interaction exists among the new facilities, we omit these constraints from our model because this requirement is no longer present.

Notably, our decision should be made prior to the availability of realizations of the  $s$  random facilities. Only the most recent information concerning random facilities is utilized to ensure accuracy. This may need an increasing or decreasing number of new facilities depending on the availability of the most recent data on random facilities. For simplicity, let us assume that the number of new facilities was previously determined and fixed (which is  $m$ ).

One variant of this application is to require, in addition to the above three requirements, each new facility  $\mathbf{x}_j$  to be placed at specific locations within the solution space. Such specific locations are represented by fixed points  $\mathbf{v}_1^{(j)}, \mathbf{v}_2^{(j)}, \dots, \mathbf{v}_p^{(j)} \in \mathbb{R}^n$ . That is,  $\mathbf{x}_j \in \{\mathbf{v}_1^{(j)}, \mathbf{v}_2^{(j)}, \dots, \mathbf{v}_p^{(j)}\}$  for each  $j = 1, 2, \dots, m$ . This leads to the following SMBSOCP model.

$$\begin{aligned} \min \quad & \sum_{j=1}^m \sum_{i=1}^r w_{ij} t_{ij} + \sum_{j=2}^m \sum_{k=1}^{j-1} \hat{w}_{jk} \hat{t}_{jk} + \mathbb{E}[Q(\mathbf{x}_1; \dots; \mathbf{x}_m, \omega)] \\ \text{s.t.} \quad & (t_{1j}; \mathbf{x}_j - \mathbf{a}_1; \dots; t_{rj}; \mathbf{x}_j - \mathbf{a}_r) \succeq_r \mathbf{0}, \quad j = 1, 2, \dots, m, \\ & (\hat{t}_{j(j+1)}; \mathbf{x}_j - \mathbf{x}_{j+1}; \dots; \hat{t}_{jm}; \mathbf{x}_j - \mathbf{x}_m) \succeq_{(m-j)} \mathbf{0}, \quad j = 1, 2, \dots, m-1, \\ & z_1^{(j)} \mathbf{v}_1^{(j)} + z_2^{(j)} \mathbf{v}_2^{(j)} + \dots + z_p^{(j)} \mathbf{v}_p^{(j)} = \mathbf{x}_j, \quad j = 1, 2, \dots, m, \\ & z_1^{(j)} + z_2^{(j)} + \dots + z_p^{(j)} = 1, \quad j = 1, 2, \dots, m, \\ & (\mathbf{z}^{(1)}; \mathbf{z}^{(2)}; \dots; \mathbf{z}^{(m)}; \mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_m) \in \{0, 1\}^{mp} \times \mathbb{R}^{mn}, \end{aligned} \tag{4}$$

where  $\mathbb{E}[Q(\mathbf{x}_1; \dots; \mathbf{x}_m; \mathbf{y}, \omega)] := \int_{\Omega} Q(\mathbf{x}_1; \dots; \mathbf{x}_m; \mathbf{y}, \omega)P(d\omega)$ , and  $Q(\mathbf{x}_1; \dots; \mathbf{x}_m, \omega)$  is the minimum value of the problem

$$\begin{aligned}
\min \quad & \sum_{j=1}^m \sum_{i=1}^s \tilde{w}_{ij}(\omega) \tilde{t}_{ij} + \sum_{j=2}^m \sum_{k=1}^{j-1} \hat{w}_{jk} \hat{t}_{jk} \\
\text{s.t.} \quad & (\tilde{t}_{1j}; \mathbf{x}_j - \tilde{\mathbf{a}}_1(\omega); \dots; \tilde{t}_{sj}; \mathbf{x}_j - \tilde{\mathbf{a}}_s(\omega)) \geq_s \mathbf{0}, \quad j = 1, 2, \dots, m, \\
& (\hat{t}_{j(j+1)}; \mathbf{x}_j - \mathbf{x}_{j+1}; \dots; \hat{t}_{jm}; \mathbf{x}_j - \mathbf{x}_m) \geq_{(m-j)} \mathbf{0}, \quad j = 1, 2, \dots, m-1, \\
& \mathbf{y}_1^{(j)} \mathbf{v}_1^{(j)} + \mathbf{y}_2^{(j)} \mathbf{v}_2^{(j)} + \dots + \mathbf{y}_p^{(j)} \mathbf{v}_p^{(j)} = \mathbf{x}_j, \quad j = 1, 2, \dots, m, \\
& \mathbf{y}_1^{(j)} + \mathbf{y}_2^{(j)} + \dots + \mathbf{y}_p^{(j)} = 1, \quad j = 1, 2, \dots, m, \\
& (\mathbf{y}^{(1)}; \mathbf{y}^{(2)}; \dots; \mathbf{y}^{(m)}; \mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_m) \in \{0, 1\}^{mp} \times \mathbb{R}^{mn}.
\end{aligned} \tag{5}$$

Another variant of this application is to require some of the new facilities to attain integer values, i.e.,  $x_j$  has to be integer-valued for each  $j \in \Delta \subset \{1, 2, \dots, m\}$ . Without loss of generality, let us assume that  $x_1, x_2, \dots, x_{|\Delta|} \in \mathbb{Z}^n$ . In this case, we are interested in the following SMISOCP model.

$$\begin{aligned}
\min \quad & \sum_{j=1}^m \sum_{i=1}^r w_{ij} t_{ij} + \sum_{j=2}^m \sum_{k=1}^{j-1} \hat{w}_{jk} \hat{t}_{jk} + \mathbb{E} [Q(\mathbf{x}_1; \dots; \mathbf{x}_m, \omega)] \\
\text{s.t.} \quad & (t_{1j}; \mathbf{x}_j - \mathbf{a}_1; \dots; t_{rj}; \mathbf{x}_j - \mathbf{a}_r) \geq_r \mathbf{0}, \quad j = 1, 2, \dots, m, \\
& (\hat{t}_{j(j+1)}; \mathbf{x}_j - \mathbf{x}_{j+1}; \dots; \hat{t}_{jm}; \mathbf{x}_j - \mathbf{x}_m) \geq_{(m-j)} \mathbf{0}, \quad j = 1, 2, \dots, m-1, \\
& (\mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_m) \in \mathbb{Z}^{|\Delta|n} \times \mathbb{R}^{(m-|\Delta|)n},
\end{aligned} \tag{6}$$

where  $Q(\mathbf{x}_1; \dots; \mathbf{x}_m, \omega)$  is the minimum value of the problem

$$\begin{aligned}
\min \quad & \sum_{j=1}^m \sum_{i=1}^s \tilde{w}_{ij}(\omega) \tilde{t}_{ij} + \sum_{j=2}^m \sum_{k=1}^{j-1} \hat{w}_{jk} \hat{t}_{jk} \\
\text{s.t.} \quad & (\tilde{t}_{1j}; \mathbf{x}_j - \tilde{\mathbf{a}}_1(\omega); \dots; \tilde{t}_{sj}; \mathbf{x}_j - \tilde{\mathbf{a}}_s(\omega)) \geq_s \mathbf{0}, \quad j = 1, 2, \dots, m, \\
& (\hat{t}_{j(j+1)}; \mathbf{x}_j - \mathbf{x}_{j+1}; \dots; \hat{t}_{jm}; \mathbf{x}_j - \mathbf{x}_m) \geq_{(m-j)} \mathbf{0}, \quad j = 1, 2, \dots, m-1, \\
& (\mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_m) \in \mathbb{Z}^{|\Delta|n} \times \mathbb{R}^{(m-|\Delta|)n}.
\end{aligned} \tag{7}$$

As mentioned in Section 2, a solution for a model such as that in (4) and (5) is now possible. Specifically, Model (4, 5) can be solved by the algorithm proposed in [35]. Model (6, 7) is also intractable for an extensive form formulation (see the form (35)).

### 3.2 Portfolio optimization with CVaR and diversification constraints

Portfolio optimization is one of the most important applications of second-order cone programming. One of its paradigms is maximization of the expected return over a single period subject to conditional value-at-risk (CVaR for short) constraints. These constraints are introduced to limit the downside risk of the tracking portfolio. In our application model, we also consider diversification-by-sector constraints, which are used to limit the minimum amount invested in each asset from each economic sector, and buy-in threshold constraints, which are used to prevent the investor from investing very small amounts in a single asset. The last two types of constraints mentioned above were considered in [38].

Portfolio optimization is frequently framed as an application of (deterministic) second-order cone programming (see [35], for example). The stochastic perspective of this application was presented in [3] as an application of SSOCP by maximizing expected returns over two periods or stages. The discrete version of this application was presented in [1] as an application of DMISOCP by adding diversification-by-sector constraints. The application model presented in this subsection combines both perspectives by adding diversification-by-sector constraints and buy-in threshold constraints over two stages, which results as an SMISOCP model.

We consider cash (index 0) and  $n$  risky assets from  $L$  distinct sectors in our portfolio during a two-stage period. Let  $\mathbf{x}^+, \mathbf{x}^- \in \mathbb{R}^{n+1}$  be the buy and sell transactions, respectively, throughout the first stage. We utilize

the following parameters over the first stage: we use  $\xi \in \{0, 1\}^L$  to indicate if each sector has sufficient investments, use  $\delta \in \{0, 1\}^n$  for buy-in threshold constraints, use  $\mathbf{u} \in \mathbb{R}^{n+1}$  to denote the current portfolio holdings, and let  $\mathbf{p} \in \mathbb{R}^{n+1}$  denote the expected rates of return. For simplicity, we also assume that  $\mathbf{p}$  is Gaussian with known mean  $\bar{\mathbf{p}}$  and covariance  $\Sigma$ , so the return over this stage is the Gaussian random variable  $r = \mathbf{p}^\top(\mathbf{u} + \Delta\mathbf{x})$  with mean  $\bar{r} = \bar{\mathbf{p}}^\top(\mathbf{u} + \Delta\mathbf{x})$  and variance  $\sigma = (\mathbf{u} + \Delta\mathbf{x})^\top \Sigma (\mathbf{u} + \Delta\mathbf{x})$ , where the difference  $\Delta\mathbf{x} := \mathbf{x}^+ - \mathbf{x}^-$  is the trading shares generate cash in the first stage.

Let also  $\mathbf{y}^+, \mathbf{y}^- \in \mathbb{R}^{n+1}$  be the buy and sell transactions, respectively, throughout the second stage. We use the following parameters over the second stage: We use  $\zeta \in \{0, 1\}^L$  to denote whether if each sector has sufficient investments, use  $\eta \in \{0, 1\}^n$  for buy-in threshold constraints, use  $\mathbf{v} \in \mathbb{R}^{n+1}$  to denote the current portfolio holdings, and let  $\mathbf{q}(\omega) \in \mathbb{R}^{n+1}$  denote the random expected rates of return whose realization depends on an underlying outcome  $\omega$  in an event space  $\Omega$  with known probability function  $P$ . For simplicity, we also let  $\mathbf{q}(\omega)$  be Gaussian with known mean  $\bar{\mathbf{q}}(\omega)$  and covariance  $\tilde{\Sigma}(\omega)$ , so the return over this stage is the Gaussian random variable  $\tilde{r}(\omega) = \mathbf{q}^\top(\omega)(\mathbf{v} + \Delta\mathbf{y})$  with mean  $\bar{\tilde{r}}(\omega) = \bar{\mathbf{q}}^\top(\omega)(\mathbf{v} + \Delta\mathbf{y})$  and variance  $\tilde{\sigma}(\omega) = (\mathbf{v} + \Delta\mathbf{y})^\top \tilde{\Sigma}(\omega)(\mathbf{v} + \Delta\mathbf{y})$ , where the difference  $\Delta\mathbf{y} := \mathbf{y}^+ - \mathbf{y}^-$  is the trading shares generate cash in the second stage.

We penalize the buy and sell transactions throughout the two stages by quadratic convex transaction costs of Garleanu and Pedersen [39]. The transaction costs associated with the first and second stages are given by  $\Delta\mathbf{x}^\top \Lambda \Delta\mathbf{x}$  and  $\Delta\mathbf{y}^\top \tilde{\Lambda}(\omega) \Delta\mathbf{y}$ , respectively, where  $\Lambda, \tilde{\Lambda}(\omega) \in \mathbb{R}^{(n+1) \times (n+1)}$  are symmetric positive definite matrices measuring the level of trading costs. The matrix  $\Lambda$  is obtained as a positive multiple of  $\Sigma$ , and  $\tilde{\Lambda}(\omega)$  is obtained as a positive multiple of  $\tilde{\Sigma}(\omega)$ .

Assume that we do not know the Gaussian vector's realization  $\mathbf{q}(\omega)$  at the present time, but that it will be known at a certain point in time in the future. Our objective is to select the optimal trading strategy to maximize the expected total return at the end-of-second-stage. Our choices of portfolios involve the Markowitz trade-off between their expected return and investment market risk (or volatility) [40].

From the above discussion, it can be seen that the objective function of the first-stage problem is

$$f_1(\mathbf{x}, \omega) = \Delta\mathbf{x}^\top \Lambda \Delta\mathbf{x} + \bar{\mathbf{p}}^\top(\mathbf{u} + \Delta\mathbf{x}) + \mathbb{E}[Q(\Delta\mathbf{x}, \omega)],$$

where  $Q(\Delta\mathbf{x}, \omega)$  is the maximum value of the second-stage problem, which has the objective function

$$f_2(\mathbf{y}, \omega) = \Delta\mathbf{y}^\top \tilde{\Lambda}(\omega) \Delta\mathbf{y} + \bar{\mathbf{q}}^\top(\omega)(\mathbf{v} + \Delta\mathbf{y}).$$

It is beneficial to provide continuous relaxations of the above objective functions that replace their quadratic forms by linear forms. This can be achieved via including only linear and second order cone constraints as is shown below:

$$\begin{aligned} f_1(\mathbf{x}, \omega) &= \Delta\mathbf{x}^\top \Lambda \Delta\mathbf{x} + \bar{\mathbf{p}}^\top \Delta\mathbf{x} + \bar{\mathbf{p}}^\top \mathbf{u} + \mathbb{E}[Q(\mathbf{x}, \omega)] \\ &= \Delta\mathbf{x}^\top \Lambda \Delta\mathbf{x} + \frac{1}{4} \bar{\mathbf{p}}^\top \Lambda \bar{\mathbf{p}} - \frac{1}{4} \bar{\mathbf{p}}^\top \Lambda \bar{\mathbf{p}} + \frac{1}{2} \bar{\mathbf{p}}^\top \Delta\mathbf{x} + \frac{1}{2} \Delta\mathbf{x}^\top \bar{\mathbf{p}} + \bar{\mathbf{p}}^\top \mathbf{u} + \mathbb{E}[Q(\mathbf{x}, \omega)] \\ &= \left( \Lambda^{1/2} \Delta\mathbf{x} + \frac{1}{2} \Lambda^{-1/2} \bar{\mathbf{p}} \right)^\top \left( \Lambda^{1/2} \Delta\mathbf{x} + \frac{1}{2} \Lambda^{-1/2} \bar{\mathbf{p}} \right) - \frac{1}{4} \bar{\mathbf{p}}^\top \Lambda \bar{\mathbf{p}} + \bar{\mathbf{p}}^\top \mathbf{u} + \mathbb{E}[Q(\mathbf{x}, \omega)] \\ &= \left\| \Lambda^{1/2} \Delta\mathbf{x} + \frac{1}{2} \Lambda^{-1/2} \bar{\mathbf{p}} \right\|^2 - \frac{1}{4} \bar{\mathbf{p}}^\top \Lambda \bar{\mathbf{p}} + \bar{\mathbf{p}}^\top \mathbf{u} + \mathbb{E}[Q(\mathbf{x}, \omega)]. \end{aligned}$$

Similarly, the objective function of the second-stage problem can be stated as

$$f_2(\mathbf{y}, \omega) = \left\| \tilde{\Lambda}^{1/2}(\omega) \Delta\mathbf{y} + \frac{1}{2} \tilde{\Lambda}^{-1/2}(\omega) \bar{\mathbf{q}}(\omega) \right\|^2 - \frac{1}{4} \bar{\mathbf{q}}^\top(\omega) \tilde{\Lambda}(\omega) \bar{\mathbf{q}}(\omega) + \bar{\mathbf{q}}^\top(\omega) \mathbf{v}.$$

To avoid falling below the expected wealth levels, we need to add  $M$  CVaR constraints at the end of the first stage, and  $N$  CVaR constraints at the end of the second stage. For  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, N$ , let  $\alpha_i$  and  $\tilde{\alpha}_j$  represent unwanted expected wealth levels at the end of the first and second stages, respectively, and let  $\beta_i$  and  $\tilde{\beta}_j$  denote the first and second stage's minimal probability, respectively. Assuming the above data is given, our goal is to maximize the end-of-second-stage expected total return less transaction costs. In order to limit our risk, each CVaR constraint  $i$ ,  $i = 1, 2, \dots, M$ , asks that our expected wealth at the end of the first stage is above  $\alpha_i$  with a probability of at least  $\beta_i$ , and each CVaR constraint  $j$ ,  $j = 1, 2, \dots, N$ , asks that our expected wealth at the end of the second stage is above  $\tilde{\alpha}_j$  with a probability of at least  $\tilde{\beta}_j$ .



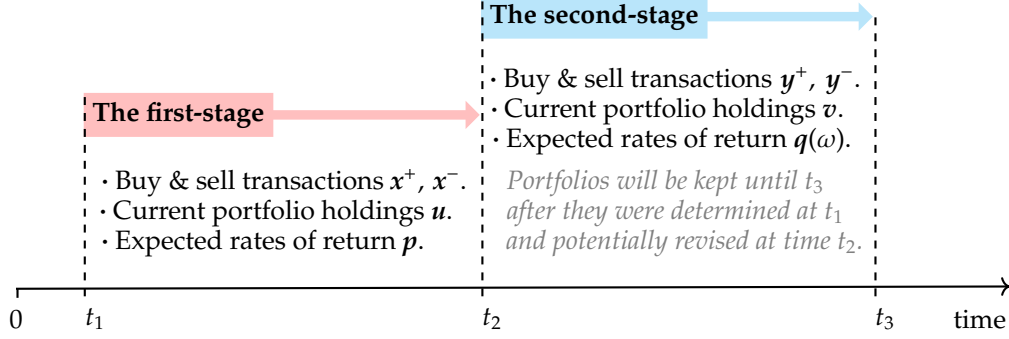


Figure 4: Two-stage portfolio optimization with CVaR constraints.

Given the above data, we need to determine the decision variables  $x^+$  and  $x^-$  so that the CVaR constraints

$$P(r \geq \alpha_i) \geq \beta_i, \quad i = 1, 2, \dots, M, \quad (8)$$

are satisfied at the end of the first stage, and determine the decision variables  $y^+$  and  $y^-$  so that the CVaR constraints

$$P(\tilde{r}(\omega) \geq \tilde{\alpha}_j) \geq \tilde{\beta}_j, \quad j = 1, 2, \dots, N, \quad (9)$$

are satisfied at the end of the second stage. These determinations need to be made before the realizations of the random expected rates of return become available. To be more precise, assume that  $0 < t_1 < t_2 < t_3$  are given times. Let  $t_1$  and  $t_2$  be the starting times of the first and second stages, respectively, and  $t_2$  and  $t_3$  be the ending times of these two stages, respectively, as shown in Figure 4. At time  $t_1$ , the investor is given the vector of returns  $p$  and is required to determine the portfolio vectors  $x^\pm$ . At time  $t_2$ , the investor can revise some, if not all, entries of the portfolio vectors  $x^\pm$  to get the portfolio vectors  $y^\pm$ . The revised portfolios  $y^\pm$  are kept until time  $t_3$ . The objective is the determination of the portfolios at time  $t_1$  in anticipation of the revision of some or all of them at time  $t_2$ .

The CVaR constraints (8) are equivalent to the second-order cone constraint

$$\left( \bar{p}^T(u + \Delta x) - \alpha_1; \Phi^{-1}(\beta_1)\Sigma^{1/2}(u + \Delta x); \dots; \bar{p}^T(u + \Delta x) - \alpha_M; \Phi^{-1}(\beta_M)\Sigma^{1/2}(u + \Delta x) \right) \succeq_M \mathbf{0}, \quad (10)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt$$

is the cumulative distribution function for a standard normal random variable.

A common proof of the above reformulation of the CVaR constraints (8) is as follows (see also [1, 41]): We can write the CVaR constraint  $P(r \geq \alpha_i) \geq \beta_i$  as

$$P\left(\frac{r - \bar{r}}{\sqrt{\sigma}} \geq \frac{\alpha_i - \bar{r}}{\sqrt{\sigma}}\right) \geq \beta_i, \quad i = 1, 2, \dots, M.$$

Since  $(r - \bar{r})/\sqrt{\sigma}$  is a zero mean unit variance Gaussian random variable, the above probability is simply  $1 - \Phi((\alpha_i - \bar{r})/\sqrt{\sigma}) \geq \beta_i, i = 1, 2, \dots, M$ . Using the symmetry of the standard distribution function and in light of the equality  $\sqrt{\sigma} = \|\Sigma^{1/2}(u + \Delta x)\|$ , the conicality constraint (10) follows from the following equivalences, for  $i = 1, 2, \dots, M$ .

$$\begin{aligned} 1 - \Phi((\alpha_i - \bar{r})/\sqrt{\sigma}) \geq \beta_i &\iff \alpha_i - \bar{r} \leq \Phi^{-1}(1 - \beta_i) \sqrt{\sigma} \\ &\iff \bar{r} - \alpha_i \geq \Phi^{-1}(\beta_i) \sqrt{\sigma} \iff \left( \bar{r} - \alpha_i; \Phi^{-1}(\beta_i) \left( \Sigma^{1/2}(u + \Delta x) \right) \right) \geq \mathbf{0}. \end{aligned}$$

Similarly, one can show that the CVaR constraints (9) are equivalent to the conicality constraint

$$\left( \bar{q}^T(\omega)(v + \Delta y) - \tilde{\alpha}_1; \Phi^{-1}(\tilde{\beta}_1)\tilde{\Sigma}^{1/2}(\omega)(v + \Delta y); \dots; \bar{q}^T(\omega)(v + \Delta y) - \tilde{\alpha}_N; \Phi^{-1}(\tilde{\beta}_N)\tilde{\Sigma}^{1/2}(\omega)(v + \Delta y) \right) \succeq_N \mathbf{0}.$$

It is now time to introduce the previously mentioned diversification constraints, which force the investor to diversify its portfolios at each stage by buying assets in at least  $L_{\min}$  different sectors. In the first stage, we link every asset  $i$  with a sector  $k$  such that the sets  $S_k, 1 \leq k \leq L$ , of assets affiliated with a sector  $k$  produce an exact partition of  $\{1, 2, \dots, n\}$ . We also associate a variable  $\xi_k \in \{0, 1\}$  with each sector  $k$ , so it is equal to 1 if and only if  $s_{\min}$  is a minimum predefined level of the amount of sector  $k$  over the first stage. That is,

$$\xi_k = 1 \iff \mathbf{1}_{S_k}^\top (\mathbf{u} + \Delta \mathbf{x}) = \sum_{i \in S_k} (u_i + \Delta x_i) \geq s_{\min},$$

where  $\mathbf{1}_{S_k} \in \{0, 1\}^{n+1}$  is the indicator vector of  $S_k$  whose  $j$ th entry has value 1 if  $j \in S_k$  and 0 otherwise. Additionally, our portfolio's total allocation of assets to an economic sector  $k$  cannot be greater than one.

As a constraint involving a binary variable  $\xi_k$ , this can be formulated as

$$s_{\min} \xi_k \leq \mathbf{1}_{S_k}^\top (\mathbf{u} + \Delta \mathbf{x}) \leq s_{\min} + (1 - s_{\min}) \xi_k.$$

In the second stage, any asset  $j$  is associated with a sector  $k$  as for those of the first stage, and a variable  $\zeta_k \in \{0, 1\}$  is associated with each sector  $k$  as follows:

$$\zeta_k = 1 \iff \mathbf{1}_{S_k}^\top (\mathbf{v} + \Delta \mathbf{y}) = \sum_{j \in S_k} (v_j + \Delta y_j) \geq \tilde{s}_{\min},$$

where  $\tilde{s}_{\min}(\omega)$  is a minimum predefined level of the amount of sector  $k$  over the second stage, whose realization depends on an underlying outcome  $\omega$  in an event space  $\Omega$  with known probability function  $P$ . Similarly, as a constraint involving a binary variable  $\zeta_k$ , this can be formulated as

$$\tilde{s}_{\min}(\omega) \zeta_k \leq \mathbf{1}_{S_k}^\top (\mathbf{v} + \Delta \mathbf{y}) \leq \tilde{s}_{\min}(\omega) + (1 - \tilde{s}_{\min}(\omega)) \zeta_k.$$

In order to satisfy the diversification requirements to detain "representative" positions in at least  $L_{\min}$  sectors, we should additionally add the cardinality constraints

$$\mathbf{1}^\top \boldsymbol{\xi} = \sum_{k=1}^L \xi_k \geq L_{\min}, \quad \text{and} \quad \mathbf{1}^\top \boldsymbol{\zeta} = \sum_{k=1}^L \zeta_k \geq L_{\min},$$

where  $\mathbf{1}$  is the vector of ones of an appropriate dimension.

We now consider the buy-in threshold constraints. As mentioned at the beginning of this subsection, these constraints are introduced to avoid the investor from investing very small amounts in a single asset at each stage. As constraints involving binary variables  $\boldsymbol{\delta}$  and  $\boldsymbol{\eta}$ , these requirements can be formulated as

$$u_{\min} \boldsymbol{\delta} \leq \mathbf{u} + \Delta \mathbf{x} \leq \boldsymbol{\delta}, \quad \text{and} \quad v_{\min} \boldsymbol{\eta} \leq \mathbf{v} + \Delta \mathbf{y} \leq \boldsymbol{\eta},$$

where  $u_{\min}$  and  $v_{\min}$  are predetermined proportion of the available capital budgets to be invested.

Our portfolio model is based on the simplest assumption that we require full portfolio allocations in the stocks at the end of each stage, i.e.,  $\mathbf{1}^\top (\mathbf{u} + \Delta \mathbf{x}) = \sum_{j=0}^n (u_j + \Delta x_j) = 1$  and  $\mathbf{1}^\top (\mathbf{v} + \Delta \mathbf{y}) = \sum_{j=0}^n (v_j + \Delta y_j) = 1$ . Additionally, for our portfolio vectors, we also assume that we can take limited short positions for each nonliquid asset to allow for short sales of them and enhance returns during volatility. This adds the constraints  $\mathbf{u} + \Delta \mathbf{x} \geq -\mathbf{s}$  and  $\mathbf{v} + \Delta \mathbf{y} \geq -\tilde{\mathbf{s}}$ , where  $\mathbf{s}$  and  $\tilde{\mathbf{s}}$  represent the short position limits at the first and second stages, respectively, for each nonliquid asset. Finally, we need the variables associated with the buy and sell transactions to be nonnegative at each stage, i.e.,  $\mathbf{x}^\pm \geq \mathbf{0}$  and  $\mathbf{y}^\pm \geq \mathbf{0}$ .

Given the above data and requirements, the stochastic portfolio optimization problem with CVaR and

diversification constraints over two stages can be cast as the following SMBSOCP model.

$$\begin{aligned}
\max \quad & h_0 - \frac{1}{4}\bar{\mathbf{p}}^\top \Lambda \bar{\mathbf{p}} + \bar{\mathbf{p}}^\top \mathbf{u} + \mathbb{E}[Q(\mathbf{x}^+; \mathbf{x}^-, \omega)] \\
\text{s.t.} \quad & \left( h_0; \Lambda^{1/2}(\mathbf{x}^+ - \mathbf{x}^-) + \frac{1}{2}\Lambda^{-1/2}\bar{\mathbf{p}} \right) \geq \mathbf{0}, \\
& \left( \bar{\mathbf{p}}^\top (\mathbf{u} + \mathbf{x}^+ - \mathbf{x}^-) - \alpha_i; \Phi^{-1}(\beta_i)\Sigma^{1/2}(\mathbf{u} + \mathbf{x}^+ - \mathbf{x}^-) \right) \geq \mathbf{0}, \quad i = 1, 2, \dots, M, \\
& s_{\min}\xi_k \leq \mathbf{1}_{S_k}^\top (\mathbf{u} + \mathbf{x}^+ - \mathbf{x}^-) \leq s_{\min} + (1 - s_{\min})\xi_k, \quad k = 1, 2, \dots, L, \\
& \mathbf{1}^\top \xi \geq L_{\min}, \\
& u_{\min}\delta \leq \mathbf{u} + \mathbf{x}^+ - \mathbf{x}^- \leq \delta, \\
& \mathbf{1}^\top (\mathbf{u} + \mathbf{x}^+ - \mathbf{x}^-) = 1, \\
& \mathbf{u} + \mathbf{x}^+ - \mathbf{x}^- \geq -\mathbf{s}, \\
& \mathbf{x}^+, \mathbf{x}^- \geq \mathbf{0}, \\
& \xi \in \{0, 1\}^L, \\
& \delta \in \{0, 1\}^n,
\end{aligned} \tag{11}$$

where  $Q(\mathbf{x}^+; \mathbf{x}^-, \omega)$  is the maximum value of the problem

$$\begin{aligned}
\max \quad & \tilde{h}_0 - \frac{1}{4}\tilde{\mathbf{q}}^\top(\omega)\tilde{\Lambda}(\omega)\tilde{\mathbf{q}}(\omega) + \tilde{\mathbf{q}}^\top(\omega)\mathbf{v} \\
\text{s.t.} \quad & \left( \tilde{h}_0; \tilde{\Lambda}^{1/2}(\omega)(\mathbf{y}^+ - \mathbf{y}^-) + \frac{1}{2}\tilde{\Lambda}^{-1/2}(\omega)\tilde{\mathbf{q}}(\omega) \right) \geq \mathbf{0}, \\
& \left( \tilde{\mathbf{q}}^\top(\omega)(\mathbf{v} + \mathbf{y}^+ - \mathbf{y}^-) - \tilde{\alpha}_j; \Phi^{-1}(\tilde{\beta}_j)\tilde{\Sigma}^{1/2}(\omega)(\mathbf{v} + \mathbf{y}^+ - \mathbf{y}^-) \right) \geq \mathbf{0}, \quad j = 1, 2, \dots, N, \\
& \tilde{s}_{\min}(\omega)\zeta_k \leq \mathbf{1}_{S_k}^\top (\mathbf{v} + \mathbf{y}^+ - \mathbf{y}^-) \leq \tilde{s}_{\min}(\omega) + (1 - \tilde{s}_{\min}(\omega))\zeta_k, \quad k = 1, 2, \dots, L, \\
& \mathbf{1}^\top \zeta \geq L_{\min}, \\
& v_{\min}\eta \leq \mathbf{v} + \mathbf{y}^+ - \mathbf{y}^- \leq \eta, \\
& \mathbf{1}^\top (\mathbf{v} + \mathbf{y}^+ - \mathbf{y}^-) = 1, \\
& \mathbf{v} + \mathbf{y}^+ - \mathbf{y}^- \geq -\tilde{\mathbf{s}}, \\
& \mathbf{y}^+, \mathbf{y}^- \geq \mathbf{0}, \\
& \zeta \in \{0, 1\}^L, \\
& \eta \in \{0, 1\}^n,
\end{aligned} \tag{12}$$

and

$$\mathbb{E}[Q(\mathbf{x}^+; \mathbf{x}^-, \omega)] = \int_{\Omega} Q(\mathbf{x}^+; \mathbf{x}^-, \omega) P(d\omega).$$

As indicated earlier, since Model (11, 12) has binary variables, a solution for such a model is now possible by applying the decomposition algorithm developed in [35].

### 3.3 Stochastic joint uncapacitated location-inventory problem

Inventory management is important in numerous industries such as E-commerce, consumer goods, car renting, food distribution, or any industry or trade that deals with inventories. We consider the basic joint uncapacitated location-inventory problem over two periods (or stages) with some uncertainty in data relating to retailers. More specifically, we consider a supply chain in which assets are shipped by suppliers from distribution centers to fixed retail stores over Stage I, and are also shipped to fixed and random retail stores over Stage II (see Figure 5). A given period or stage can be a few months or a year for example. Shen et al. [42] and Daskin et al. [43] studied the same problem over one period with nonlinearity arising from safety stock costs. Atamturk et. al [44] were the first to formulate the one-stage joint location-inventory model proposed in [42, 43] as a DMISOCP model.

Our problem is the following. We are given a number of candidate locations for opening distribution centers. We are also given a number of existing retailers over Stage I and a number of existing and/or random retailers over Stage II, each with uncertain product demand. At each stage, it is necessary to decide the number of distribution centers that need to be opened, the kind of retailers to be allocated to each center, in addition to the required level of safety stock to be maintained in order to minimize total inventory costs, opening location, and shipment while providing a certain level of service. In each stage in the model, we assume that the shipments are made directly from distribution centers to retailers, and that the demand is independent and Gaussian at each retailer.

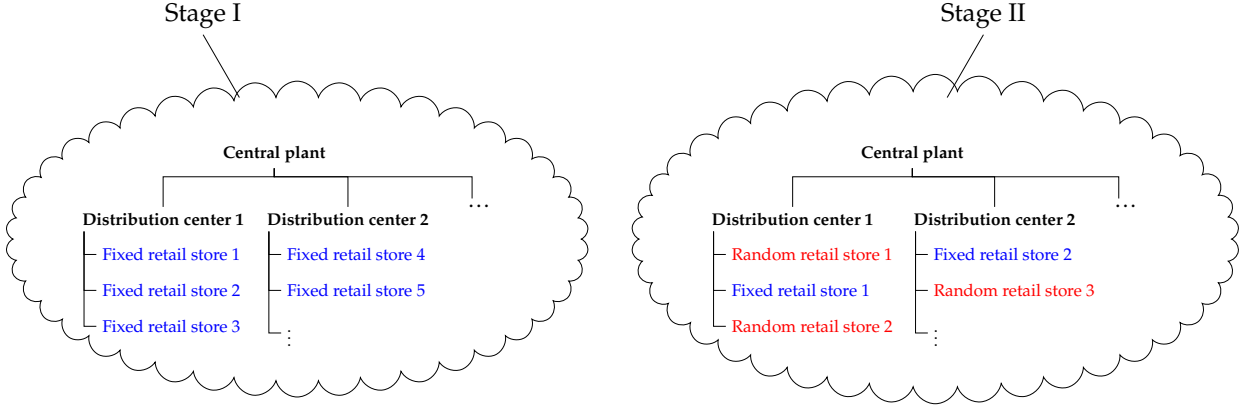


Figure 5: A two-stage stochastic inventory distribution process.

In building the model, the distributional constraint that we consider in each stage is that each retail store is supplied from exactly one open distribution center. To model this constraint, let  $I$  be the set of fixed existing retailers in the first stage, and  $\tilde{I} \subseteq I$  be the set of fixed existing retailers in the second stage. Let also  $J$  be the set of candidate locations for opening distribution centers, and  $K$  be the set of random retailers in the second stage whose realization depends on an underlying outcome  $\omega$  in an event space  $\Omega$  with known probability function  $P$ . We introduce the decision variables  $x, y^{(I)}, \dots, y^{(I|I)}, z^{(1)}, \dots, z^{(K)} \in \{0, 1\}^{|J|}$ , which are defined as follow:

$$\begin{aligned}
 x_j &= \begin{cases} 1, & \text{if location } j \text{ is opened for a new distribution center,} \\ 0, & \text{otherwise;} \end{cases} \\
 y_j^{(i)} &= \begin{cases} 1, & \text{if a fixed retailer } i \text{ is assigned to a new distribution center located at location } j, \\ 0, & \text{otherwise;} \end{cases} \\
 z_j^{(k)} &= \begin{cases} 1, & \text{if a random retailer } k \text{ is assigned to a new distribution center located at location } j, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

Given the above decision variables, the requirement that each retailer is supplied from exactly one distribution center and the requirement that retailers are only assigned to open distribution centers are modeled in Stage I as

$$\sum_{j \in J} y_j^{(i)} = 1, \text{ and } y_j^{(i)} \leq x_j, \quad i \in I, j \in J,$$

respectively, and are modeled in Stage II as

$$\sum_{j \in J} y_j^{(i)} = \sum_{j \in J} z_j^{(k)} = 1, \text{ and } y_j^{(i)}, z_j^{(k)} \leq x_j, \quad i \in \tilde{I}, k \in K, j \in J,$$

respectively.

In Stage I, we minimize the following costs:

- The fixed cost of locating the  $j$ th distribution center for  $j \in J$ . This objective term can be written as  $f_j x_j$ , where  $f_j$  is the monthly fixed cost of locating a new distribution center at location  $j$ .
- The cost of shipping from the central plant to the  $j$ th distribution center for  $j \in J$ .  $a_j$  represents the unit cost of shipment from the central plant to the  $j$ th distribution center,  $\mu_i$  is the mean of daily demand at the  $i$ th retailer, and  $\beta$  is the weight associated with the inventory costs. After that, this objective term is written as

$$\beta \sum_{i \in I} a_j \mu_i y_j^{(i)}.$$

- The cost of shipping from the  $j$ th distribution center to retailers, for  $j \in J$ . Let  $d_j^{(i)}$  be the unit cost of shipment from the  $j$ th distribution center to the  $i$ th retailer, then this objective term is written as

$$\beta \sum_{i \in I} d_j^{(i)} \mu_i y_j^{(i)}.$$

- The expected working inventory cost at the  $j$ th distribution center for  $j \in J$ . Following Shen et al. [42], we describe the process by which this objective term is derived. The expected working inventory cost includes the total shipment cost per year, the fixed cost of placing  $n$  orders per month, and the average working inventory cost. This objective term is written as

$$F_j n + \beta v \left( \frac{D_j}{n} \right) n + \frac{\theta h D_j}{2n},$$

where  $D_j = \sum_{i \in I} \mu_i y_j^{(i)}$  is the monthly expected demand at the  $j$ th distribution center,  $F_j$  is the fixed cost of placing an order at the  $j$ th distribution center per month,  $v_j(x)$  is the shipment cost function for  $x$  units from the central plant to the  $j$ th distribution center,  $h$  is the unit inventory holding cost per month, and  $\theta$  is the weight associated with the transportation costs.

Assume that  $v_j(\cdot)$  is linear in its parameters and is specified as  $v_j(x) = a_j x + g_j$ , where  $g_j$  is the fixed cost per shipment from the central plant to the  $j$ th distribution center. Then we have the expression:

$$F_j n + \beta a_j D_j + \beta g_j n + \frac{\theta h D_j}{2n}. \quad (13)$$

Taking the derivative of the above expression with respect to  $n$ , we obtain  $F_j + \beta g_j - \theta h D_j / 2n^2$ . Equating this to zero and solving for  $n$ , we get  $n = \sqrt{\theta h D_j / 2(F_j + \beta g_j)}$ . By substituting  $n$  into (13), we get  $\sqrt{2\theta h D_j (F_j + \beta g_j)} + \beta a_j D_j$ . Note that  $\beta a_j D_j = \beta \sum_{i \in I} a_j \mu_i y_j^{(i)}$  already exists in the objective. Therefore, this objective term is reduced to

$$\sqrt{2\theta h (F_j + \beta g_j)} \sqrt{\sum_{i \in I} \mu_i y_j^{(i)}}.$$

- The expected safety stock cost at the  $j$ th distribution center for  $j \in J$ . Let  $\sigma_i$  be the standard deviation of daily demand at the  $i$ th retailer,  $z_\alpha$  be the standard normal deviation associated with service level  $\alpha$ , and  $L_j$  be the lead time in days at the  $j$ th distribution center. Then this objective term is written as

$$z_\alpha \theta h \sqrt{L_j} \sqrt{\sum_{i \in I} \sigma_i^2 y_j^{(i)}}.$$

Similarly, in Stage II, the following parameters are used for  $i \in \tilde{I}, k \in K$  and  $j \in J$ :

$\tilde{f}_j(\omega)$ : the monthly fixed cost of locating a new distribution center at location  $j$ ,

$\tilde{a}_j(\omega)$ : the unit cost of shipment from the central plant to the  $j$ th distribution center,

$\mu_i$ : the mean of daily demand at the  $i$ th fixed retailer,

$\tilde{\mu}_k(\omega)$ : the expected mean of daily demand at the  $k$ th random retailer,

$\tilde{\beta}(\omega)$ : the weight associated with the inventory costs,

$d_j^{(i)}$ : the expected unit cost of shipment from the  $j$ th distribution center to the  $i$ th fixed retailer,

$\tilde{d}_j^{(k)}(\omega)$ : the unit cost of shipment from the  $j$ th distribution center to the  $k$ th random retailer,

$\tilde{F}_j(\omega)$ : the fixed cost of placing an order at the  $j$ th distribution center per month,

$\tilde{h}(\omega)$ : the unit inventory holding cost per month,

$\tilde{\theta}(\omega)$ : the weight associated with the transportation costs,

$\tilde{g}_j(\omega)$ : the fixed cost per shipment from the central plant to the  $j$ th distribution center,

$\sigma_i$ : the standard deviation of daily demand at the  $i$ th fixed retailer,

$\tilde{\sigma}_k(\omega)$ : the expected standard deviation of daily demand at the  $k$ th random retailer,  
 $\tilde{z}_\alpha(\omega)$ : the standard normal deviation associated with service level  $\alpha$ ,  
 $\tilde{L}_j(\omega)$ : the lead time in days at the  $j$ th distribution center,  
 $d_j(\omega) := \sum_{i \in \tilde{I}} \mu_i y_j^{(i)} + \sum_{k \in K} \tilde{\mu}_k(\omega) z_j^{(k)}(\omega)$ ; the expected monthly demand at the  $j$ th distribution center,  
 $\tilde{v}_j(x, \omega) := \tilde{a}_j(\omega)x + \tilde{g}_j(\omega)$ ; a shipment cost function for  $x$  units from the central plant to distribution center  $j$ .

The realizations of the above unspecified parameters depend on an underlying outcome  $\omega$  in an event space  $\Omega$  with a known probability function  $P$ . Such undetermined parameters are dependent on information that was not available at the time of making the decision, and will become available at a later point in time. At the beginning of Stage II, the investor can revise some, if not all of these parameters. The revised parameters are kept until the end of Stage II. Likewise, in Stage II the following costs, which can be derived as those costs in Stage I, are minimized for  $i \in \tilde{I}, k \in K$  and  $j \in J$ :

- The expected fixed cost of locating the  $j$ th distribution center:  $\tilde{f}_j(\omega)x_j$ .
- The expected cost of shipping from the central plant to the  $j$ th distribution center:

$$\tilde{\beta}(\omega)\tilde{a}_j(\omega) \left( \sum_{i \in \tilde{I}} \mu_i y_j^{(i)} + \sum_{k \in K} \tilde{\mu}_k(\omega) z_j^{(k)} \right).$$

- The expected cost of shipping from the  $j$ th distribution center to (fixed and random) retailers:

$$\tilde{\beta}(\omega) \left( \sum_{i \in \tilde{I}} d_j^{(i)} \mu_i y_j^{(i)} + \sum_{k \in K} \tilde{d}_j^{(k)}(\omega) \tilde{\mu}_k(\omega) z_j^{(k)} \right).$$

- The expected working inventory cost at the  $j$ th distribution center:

$$\sqrt{2\tilde{\theta}(\omega)\tilde{h}(\omega) \left( \tilde{F}_j(\omega) + \tilde{\beta}(\omega)\tilde{g}_j(\omega) \right)} \sqrt{\sum_{i \in \tilde{I}} \mu_i y_j^{(i)} + \sum_{k \in K} \tilde{\mu}_k(\omega) z_j^{(k)}}.$$

- The expected safety stock cost at the  $j$ th distribution center:

$$\tilde{z}_\alpha(\omega) \tilde{\theta}(\omega) \tilde{h}(\omega) \sqrt{\tilde{L}_j(\omega)} \sqrt{\sum_{i \in \tilde{I}} \sigma_i^2 y_j^{(i)} + \sum_{k \in K} \tilde{\sigma}_k^2(\omega) z_j^{(k)}}.$$

The objective is to minimize these costs at the beginning of Stage I in anticipation of the revision of some or all of the unspecified parameters at the beginning of Stage II.

Let also

$$\begin{aligned}
 \hat{d}_j^{(i)} &= \beta \left( d_j^{(i)} + a_j \right) \mu_i, i \in I, j \in J, & \hat{d}_j^{(k)}(\omega) &= \tilde{\beta}(\omega) \left( \tilde{d}_j^{(k)}(\omega) + \tilde{a}_j(\omega) \right) \tilde{\mu}_k(\omega), k \in K, j \in J, \\
 p_j &= \sqrt{2\theta h(F_j + \beta g_j)}, j \in J, & \tilde{p}_j(\omega) &= \sqrt{2\tilde{\theta}(\omega)\tilde{h}(\omega) \left( \tilde{F}_j(\omega) + \tilde{\beta}(\omega)\tilde{g}_j(\omega) \right)}, j \in J, \\
 q_j &= z_\alpha \theta h \sqrt{\tilde{L}_j}, j \in J, & \tilde{q}_j(\omega) &= \tilde{z}_\alpha(\omega) \tilde{\theta}(\omega) \tilde{h}(\omega) \sqrt{\tilde{L}_j(\omega)}, j \in J.
 \end{aligned}$$

The description of the two-stage stochastic joint location-inventory problem and the mathematical modeling of its constraints and objective shown above lead to the following two-stage stochastic nonlinear program.

$$\begin{aligned}
 \min \quad & \sum_{j \in J} \left( f_j x_j + \sum_{i \in \tilde{I}} \hat{d}_j^{(i)} y_j^{(i)} + p_j \sqrt{\sum_{i \in \tilde{I}} \mu_i y_j^{(i)}} + q_j \sqrt{\sum_{i \in \tilde{I}} \sigma_i^2 y_j^{(i)}} \right) + \mathbb{E} \left[ Q(\mathbf{x}; \mathbf{y}^{(1)}; \dots; \mathbf{y}^{(I)}, \omega) \right] \\
 \text{s.t.} \quad & \sum_{j \in J} y_j^{(i)} = 1, \quad i \in I, \\
 & y_j^{(i)} \leq x_j, \quad i \in I, j \in J, \\
 & x_j, y_j^{(i)} \in \{0, 1\}, \quad i \in I, j \in J,
 \end{aligned} \tag{14}$$

where  $\mathbb{E}[Q(\mathbf{x}; \mathbf{y}^{(1)}; \dots; \mathbf{y}^{(l)}, \omega)] = \int_{\Omega} Q(\mathbf{x}; \mathbf{y}^{(1)}; \dots; \mathbf{y}^{(l)}, \omega) P(d\omega)$ , and  $Q(\mathbf{x}; \mathbf{y}^{(1)}; \dots; \mathbf{y}^{(l)}, \omega)$  is the minimum value of the problem

$$\begin{aligned}
\min \quad & \sum_{j \in J} \left( \tilde{f}_j(\omega) x_j + \sum_{i \in \tilde{I}} \hat{d}_j^{(i)} y_j^{(i)} + \sum_{k \in K} \hat{d}_j^{(k)}(\omega) z_j^{(k)} + \tilde{p}_j(\omega) \sqrt{\sum_{i \in \tilde{I}} \mu_i y_j^{(i)} + \sum_{k \in K} \tilde{\mu}_k(\omega) z_j^{(k)}} \right. \\
& \left. + \tilde{q}_j(\omega) \sqrt{\sum_{i \in \tilde{I}} \sigma_i^2 y_j^{(i)} + \sum_{k \in K} \tilde{\sigma}_k^2(\omega) z_j^{(k)}} \right) \\
\text{s.t.} \quad & \sum_{j \in J} y_j^{(i)} + \sum_{j \in J} z_j^{(k)} = 1, \quad i \in \tilde{I}, k \in K, \\
& y_j^{(i)} \leq x_j, \quad i \in \tilde{I}, j \in J, \\
& z_j^{(k)} \leq x_j, \quad k \in K, j \in J, \\
& x_j, y_j^{(i)}, z_j^{(k)} \in \{0, 1\}, \quad i \in \tilde{I}, k \in K, j \in J.
\end{aligned} \tag{15}$$

The nonlinear terms in (14) and (15) can be handled by introducing the auxiliary variables  $s_j, \tilde{s}_j, t_j, \tilde{t}_j \geq 0$  and using the fact that  $(y_j^{(i)})^2 = y_j^{(i)}$  and that  $(z_j^{(k)})^2 = z_j^{(k)}$  for  $i \in I$  and  $k \in K$ . Given this, Problem (14, 15) can be cast as the following two-stage SMBSOCP model.

$$\begin{aligned}
\min \quad & \mathbf{f}^\top \mathbf{x} + \sum_{i \in I} \hat{\mathbf{d}}^{(i)\top} \mathbf{y}^{(i)} + \mathbf{p}^\top \mathbf{s} + \mathbf{q}^\top \mathbf{t} + \mathbb{E} \left[ Q(\mathbf{x}; \mathbf{y}^{(1)}; \dots; \mathbf{y}^{(l)}, \omega) \right] \\
\text{s.t.} \quad & (s_j; \sqrt{\mu_1} y_j^{(1)}; \sqrt{\mu_2} y_j^{(2)}; \dots; \sqrt{\mu_{|I|}} y_j^{(|I|)}) \geq \mathbf{0}, \quad j \in J, \\
& (t_j; \sigma_1 y_j^{(1)}; \sigma_2 y_j^{(2)}; \dots; \sigma_{|I|} y_j^{(|I|)}) \geq \mathbf{0}, \quad j \in J, \\
& \mathbf{1}^\top \mathbf{y}^{(i)} = 1, \quad i \in I, \\
& \mathbf{y}^{(i)} - \mathbf{x} \leq \mathbf{0}, \quad i \in I, \\
& \mathbf{x}, \mathbf{y}^{(i)} \in \{0, 1\}^{|I|}, \quad i \in I, \\
& \mathbf{s}, \mathbf{t} \geq \mathbf{0},
\end{aligned} \tag{16}$$

where  $Q(\mathbf{x}; \mathbf{y}^{(1)}; \dots; \mathbf{y}^{(l)}, \omega)$  is the minimum value of the problem

$$\begin{aligned}
\min \quad & \tilde{f}^\top(\omega) \mathbf{x} + \sum_{i \in \tilde{I}} \hat{\mathbf{d}}^{(i)\top} \mathbf{y}^{(i)} + \sum_{k \in K} \hat{\mathbf{d}}^{(k)\top}(\omega) \mathbf{z}^{(k)} + \tilde{\mathbf{p}}^\top(\omega) \tilde{\mathbf{s}} + \tilde{\mathbf{q}}^\top(\omega) \tilde{\mathbf{t}} \\
\text{s.t.} \quad & (\tilde{s}_j; \sqrt{\mu_1} y_j^{(1)}; \dots; \sqrt{\mu_{|\tilde{I}|}} y_j^{(|\tilde{I}|)}; \sqrt{\mu_1(\omega)} z_j^{(1)}; \dots; \sqrt{\mu_{|K|}(\omega)} z_j^{(|K|)}) \geq \mathbf{0}, \quad j \in J, \\
& (\tilde{t}_j; \sigma_1 y_j^{(1)}; \dots; \sigma_{|\tilde{I}|} y_j^{(|\tilde{I}|)}; \tilde{\sigma}_1(\omega) z_j^{(1)}; \dots; \tilde{\sigma}_{|K|}(\omega) z_j^{(|K|)}) \geq \mathbf{0}, \quad j \in J, \\
& \mathbf{1}^\top \mathbf{y}^{(i)} + \mathbf{1}^\top \mathbf{z}^{(k)} = 1, \quad i \in \tilde{I}, k \in K, \\
& \mathbf{y}^{(i)} - \mathbf{x} \leq \mathbf{0}, \quad i \in \tilde{I}, \\
& \mathbf{z}^{(k)} - \mathbf{x} \leq \mathbf{0}, \quad k \in K, \\
& \mathbf{x}, \mathbf{y}^{(i)}, \mathbf{z}^{(k)} \in \{0, 1\}^{|I|}, \quad i \in \tilde{I}, k \in K, \\
& \tilde{\mathbf{s}}, \tilde{\mathbf{t}} \geq \mathbf{0}.
\end{aligned} \tag{17}$$

We emphasize that Model (16, 17) accepts only binary variables and is intractable for an extensive form formulation. Therefore, a solution for such a model is now possible by applying the decomposition algorithm developed in [35].

### 3.4 Optimal infrastructure problem for electric vehicles with battery swap technology

Electric vehicles are today an attractive option for many car shoppers and very affordable to them, but the operation of charging the electric vehicles is still the most significant challenge faced by drivers because it requires a significant amount of time. To overcome this obstacle, a new infrastructure strategy has been developed recently, seeking to exchange depleted batteries in electric vehicles with full ones in the middle of long trips. In this part, we describe the problem of creating infrastructure, offering service for electric vehicles with battery swap mode, and providing coverage for stations with certain and uncertain arrival rates over two stages.

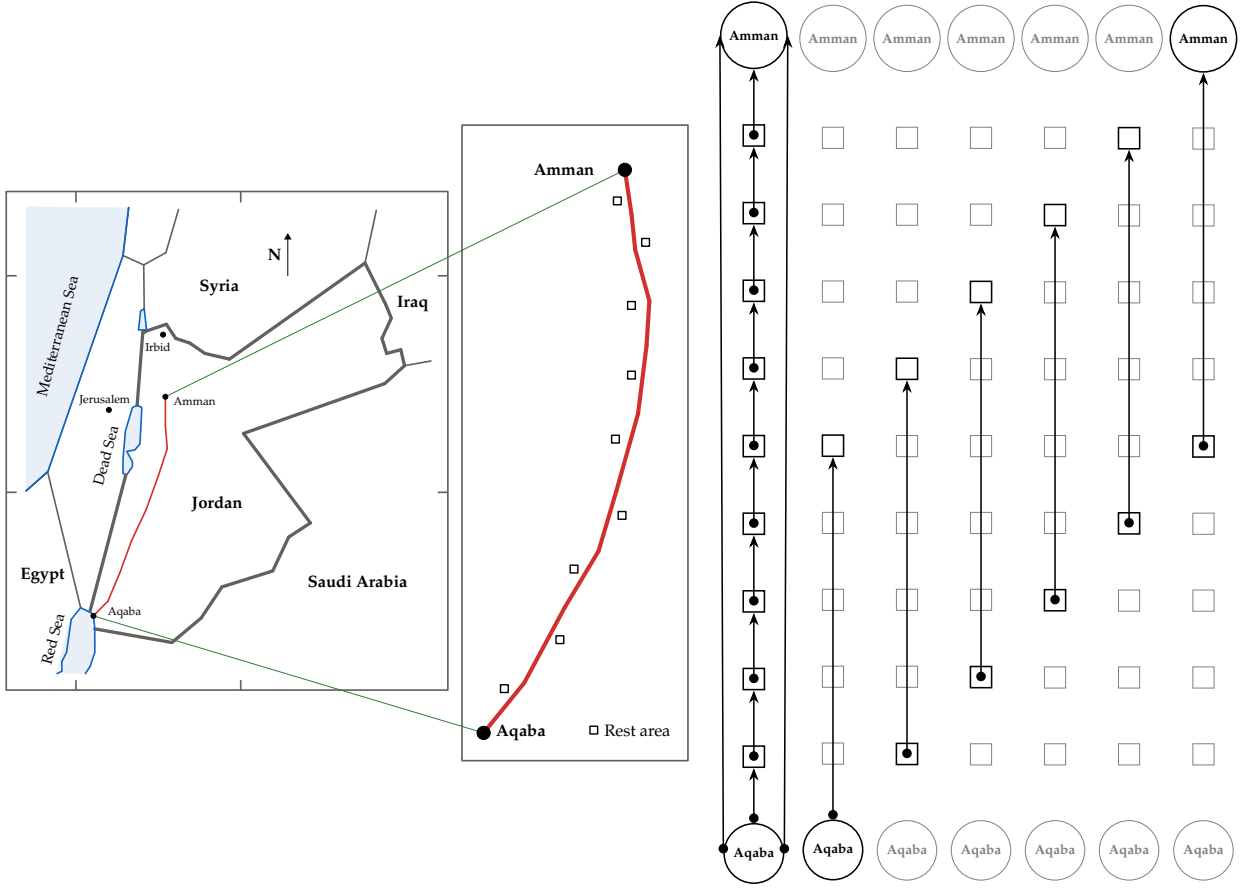


Figure 6: The map to the left shows the desert highway path from the Jordanian city of Aqaba to the capital Amman consisting of 10 linked segments which are defined between 9 adjacent exits. The graph to the right shows 6 corresponding subpaths according to the given definition.

We point out that the corresponding single-stage problem was described in [45] as an application of DMISOCP. The objective is to build a number of battery swapping stations at strategic locations along an existing freeway network.

We consider an existing freeway network connecting a number of cities. Let  $\mathcal{P}$  be the set of intercity travel paths on the network, and  $\mathcal{Q}$  be the set of subpaths of paths  $p \in \mathcal{P}$  and length longer than  $d$ , where  $d$  is the maximum travel distance allowed with one full vehicle charge. Any electric vehicle traveling a long distance needs to get access to a swap station before its battery runs out. Therefore, it must visit at least one station along each subpath of a travel path that is longer than a distance of  $d/2$  in a round trip, or longer than a distance of  $d$  in a one-way trip. For example, Figure 6 shows a highway between two Jordanian cities that consists of 10 linked segments, which are defined between 9 adjacent exits, and 6 corresponding subpaths of the path. The first subpath consisting of segments 1-5, and the next subpaths consisting of segments 2-6, 3-7, 4-8, 5-9, and 6-10. Each one of these subpaths must pass through at least one swapping station.

The number of electric vehicles traveling along the network is known along some portion of paths and is uncertain along other portions of paths. Let  $J$  be the set of candidate locations for swapping stations with known and fixed demand, and  $\tilde{J}$  be the set of candidate locations for swapping stations with uncertain demand. We consider a problem over two stages as follows. In Stage I, the values of the arrival rates at swapping stations at all sites  $j \in J$  are precisely known, but such values at swapping stations at all sites  $j \in \tilde{J}$  are not precisely known. Therefore, all swapping stations at sites  $j \in J$  are built in Stage I. At the beginning of Stage II, the realizations of the uncertain arrival rates are precisely observed, hence we can revise some information about swapping stations under uncertainty to build all stations and meet demand.



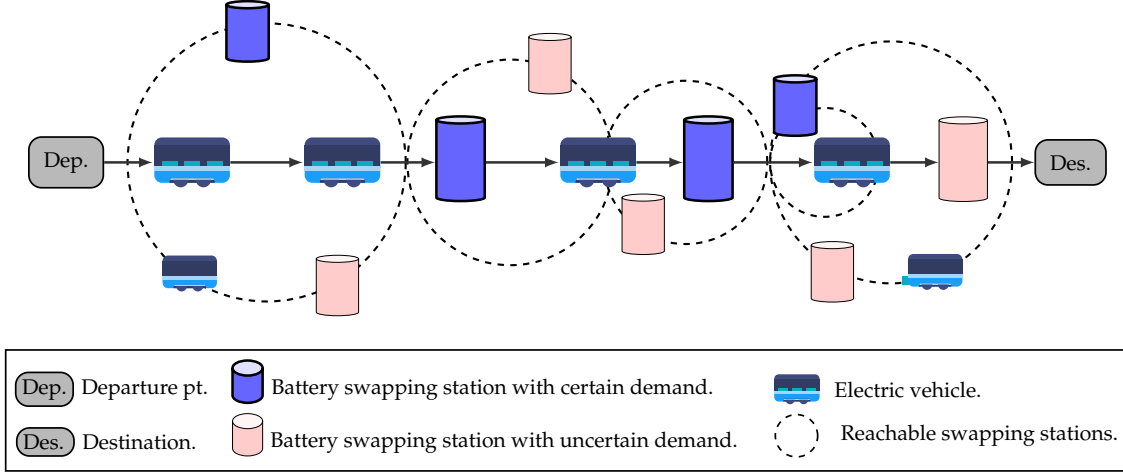


Figure 7: A concrete example of battery swapping station locations.

See Figure 7, which shows a concrete example of this application. We introduce the decision variables  $x, \mathbf{y}^{(p)}, \mathbf{z}^{(q)} \in \{0, 1\}^{|\mathcal{J}|+|\tilde{\mathcal{J}}|}$ , for  $p \in \mathcal{P}$  and  $q \in \mathcal{Q}$ , which are defined for  $j \in J \cup \tilde{J}$  as follow:

$$\begin{aligned}
 x_j &= \begin{cases} 1, & \text{if a swapping station is located at site } j, \\ 0, & \text{otherwise;} \end{cases} \\
 y_j^{(p)} &= \begin{cases} 1, & \text{if an electric vehicle traveling along a path } p \text{ will visit swapping station at site } j, \\ 0, & \text{otherwise;} \end{cases} \\
 z_j^{(q)} &= \begin{cases} 1, & \text{if an electric vehicle traveling along a subpath } q \text{ will visit swapping station at site } j, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

We assume that the swap mode of electric vehicles traveling along the path  $p \in \mathcal{P}$  enter the freeway network via a Poisson process with a rate  $\lambda^{(p)}$ . Accordingly, at a swapping station at site  $j \in \tilde{J}$ , electric vehicles arrive and request for swapping service according to a Poisson process with rate  $\lambda_j = \sum_{p \in \mathcal{P}} \lambda^{(p)} y_j^{(p)}$  which is a precisely known quantity now in Stage II. The arrival rates  $\lambda^{(p)}, p \in \mathcal{P}$ , depend on a number of mutually independent random factors,  $\xi_l(\omega), l = 1, 2, \dots, L$ , known as the primitive uncertainties. In Stage I, only the means (denoted by  $\mu_l, l = 1, \dots, L$ ), covariance matrix (denoted by  $\Sigma$  with the individual standard deviations  $\sigma_l, l = 1, \dots, L$ ), and supports of the element of the random vector  $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega), \dots, \xi_L(\omega))$  are known, but not the precise distribution. At the beginning of Stage II, the realization of  $\xi(\omega)$  is precisely observed.

We also use the following parameters for  $j \in J \cup \tilde{J}, p \in \mathcal{P}$  and  $q \in \mathcal{Q}$ :

- $f_j$ : the annualized fixed cost incurred if location  $j$  is opened for a swapping station,
- $h$ : the annualized holding cost per battery,
- $a_j^{(q)}$ : a binary parameter indicating whether a station at site  $j$  is along the  $q$ th subpath ( $= 1$ ) or not ( $= 0$ ),
- $b_{p,q}$ : a binary parameter indicating whether the  $q$ th subpath is part of the  $p$ th path ( $= 1$ ) or not ( $= 0$ ),
- $I_j(\mathbf{y})$ : the number of batteries to be stocked at a station at site  $j$  given the assignments of stations to paths,
- $g_j$ : the maximum number of batteries that can be safely accommodated by the electric grid at site  $j$ .

In stage I, we assume that the quantities  $I_j(\mathbf{y})$  are known and fixed for  $j \in J$ , but they are random for  $j \in \tilde{J}$  and their realizations depend on an underlying outcome  $\omega$  in an event space  $\Omega$  with a known probability function  $P$ . In other words, the quantities  $I_j$  depend on both the choice locations and demand realizations when  $j \in J$ , and do not depend on the chosen of locations nor on demand realizations when  $j \in \tilde{J}$ . This is represented as follows:  $I_j(\mathbf{y}) = I_j$  for  $j \in J$ , and  $I_j(\mathbf{y}) = I_j(\mathbf{y}, \omega)$  for  $j \in \tilde{J}$ . Specifically, in Stage I, the charging service provider knows the precise values of the arrival rates at swapping stations at all sites  $j \in J$ , but does not know such values precisely at swapping stations at all sites  $j \in \tilde{J}$ . After constructing all the swapping stations at sites  $j \in J$  in Stage I and precisely observing the realizations of the uncertainties at the beginning

of Stage II, the service provider can revise some, if not all of the quantities  $I_j(\mathbf{y}, \omega)$ ,  $j \in \tilde{J}$ , in order to stock batteries at swapping stations at all sites and meet demand.

In building the model, we make sure that the following constraints are satisfied:

- Electric vehicles traveling any subpath  $q \in \mathcal{Q}$  need to visit at least one swapping station. This can be modeled as  $\sum_{j \in J \cup \tilde{J}} a_j^{(q)} z_j^{(q)} \geq 1$  for each  $q \in \mathcal{Q}$ .
- If a subpath  $q \in \mathcal{Q}$ , with a swapping station at site  $j$ , is a part of multiple paths  $p \in \mathcal{P}$ , then each of those paths inherit the station at  $j$  along  $q$ . This can be modeled as  $y_j^{(p)} \geq b_{p,q} z_j^{(q)}$  for  $j \in J \cup \tilde{J}$ ,  $q \in \mathcal{Q}$ ,  $p \in \mathcal{P}$ .
- Electric vehicles are assigned to open stations only. This can be modeled as  $y_j^{(p)} \leq x_j$  for  $j \in J \cup \tilde{J}$ ,  $p \in \mathcal{P}$ .
- The number of batteries at a station  $j \in J$  is bounded by the maximum allowable number of simultaneous parallel recharges permitted by the electric grid. This can be modeled as  $I_j \leq g_j$  for  $j \in J$ .
- With a high probability of at least  $1 - \epsilon_g$ , the number of batteries at a station  $j \in \tilde{J}$  is bounded by the maximum allowable number of simultaneous parallel recharges permitted by the electric grid. This constraint acts on the worst-case (infimum) probability for all possible distributions in  $\mathbb{F}$ , where  $\mathbb{F}$  is the family of all possible distributions with the specified means and variances. This can be modeled as  $\inf_{\mathbb{P} \in \mathbb{F}} P_{\mathbb{P}}(I_j(\mathbf{y}, \omega) \leq g_j) \geq 1 - \epsilon_g$ ,  $j \in \tilde{J}$ .

We consider the family  $\mathbb{F}$  to be nonempty, i.e., the given means and variances are calculated using a univariate distribution.

The objective of the model is to minimize the following costs:

- The cost of opening a swapping station at site  $j \in J \cup \tilde{J}$  and equipping it with swapping machinery. This objective term is written as  $f_j x_j$  for  $j \in J \cup \tilde{J}$ .
- The annualized holding cost at a swapping station at site  $j \in J$ . This objective term is in the first-stage problem and is written as  $h I_j$  for  $j \in J$ .
- The expected annualized holding cost at a swapping station at site  $j \in \tilde{J}$  under the worst possible (i.e., supremum) distribution over  $\mathbb{F}$ . This objective term is in the second-stage problem and is written as  $h \sup_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}}[I_j(\mathbf{y}, \omega)]$ ,  $j \in \tilde{J}$ .

The description of the two-stage stochastic optimal infrastructure planning problem for electric vehicles with battery swap mode and the mathematical modeling of its constraints and objective shown above lead to the following two-stage stochastic nonlinear program.

$$\begin{aligned}
\min \quad & \sum_{j \in J} (f_j x_j + h I_j) + \mathbb{E}[Q(\mathbf{x}; \mathbf{y}; \mathbf{z}, \omega)] \\
\text{s.t.} \quad & y_j^{(p)} \geq b_{p,q} z_j^{(q)}, & j \in J, p \in \mathcal{P}, q \in \mathcal{Q}, \\
& y_j^{(p)} \leq x_j, & j \in J, p \in \mathcal{P}, \\
& I_j \leq g_j, & j \in J, \\
& x_j, y_j^{(p)}, z_j^{(q)} \in \{0, 1\}, & j \in J, p \in \mathcal{P}, q \in \mathcal{Q},
\end{aligned} \tag{18}$$

where  $\mathbb{E}[Q(\mathbf{x}; \mathbf{y}; \mathbf{z}, \omega)] = \int_{\Omega} Q(\mathbf{x}; \mathbf{y}; \mathbf{z}, \omega) P(d\omega)$ , and  $Q(\mathbf{x}; \mathbf{y}; \mathbf{z}, \omega)$  is the minimum value of the problem

$$\begin{aligned}
\min \quad & \sum_{j \in \tilde{J}} \left( f_j x_j + h \sup_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}}[I_j(\mathbf{y}, \omega)] \right) \\
\text{s.t.} \quad & \sum_{j \in J \cup \tilde{J}} a_j^{(q)} z_j^{(q)} \geq 1, & q \in \mathcal{Q}, \\
& y_j^{(p)} \geq b_{p,q} z_j^{(q)}, & j \in \tilde{J}, p \in \mathcal{P}, q \in \mathcal{Q}, \\
& y_j^{(p)} \leq x_j, & j \in \tilde{J}, p \in \mathcal{P}, \\
& \inf_{\mathbb{P} \in \mathbb{F}} P_{\mathbb{P}}(I_j(\mathbf{y}, \omega) \leq g_j) \geq 1 - \epsilon_g, & j \in \tilde{J}, \\
& x_j, y_j^{(p)}, z_j^{(q)} \in \{0, 1\}, & j \in \tilde{J}, p \in \mathcal{P}, q \in \mathcal{Q}.
\end{aligned} \tag{19}$$

Our aim now is to formulate both the nonlinear term in problem (19) in regards to its objective function and the chance constraint (the fourth constraint) in (19) as second-order cone constraints. We make the following assumptions, which are quite standard [45].

**Assumption 1** The path-based demand arrival rate,  $\lambda^{(p)}$ , is a linear function of a number of mutually independent random factors,  $\xi_l(\omega), l = 1, 2, \dots, L$ . That is,  $\lambda^{(p)} = \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega)$ , where  $\hat{\lambda}_l^{(p)}, l = 0, 1, \dots, L$ , are known constants.

**Assumption 2** In at least  $\alpha (> 0.5)$  proportion of the swap requests, the electric vehicle picks up a battery that has been recharged for at least  $t$  time units.

Assumption 1 is for convenience of obtaining tractable constraints. The requirement in Assumption 2 is equivalent to requiring that the number of Poisson demand arrivals in a period of  $t$  time units is smaller than the quantity  $I_j(\mathbf{y}, \omega)$  with probability  $\alpha$ . We approximate the Poisson demand with a Normal distribution that matches the mean and variance. This approximation is quite standard (see for example [45]), and is accurate especially when the demand rate is large, which is the case in practice on freeway networks. To satisfy these service requirements, the number of batteries needed is approximately

$$I_j(\mathbf{y}, \omega) = t\lambda_j(\mathbf{y}, \omega) + \Phi^{-1}(\alpha) \sqrt{t\lambda_j(\mathbf{y}, \omega)}, \quad (20)$$

where  $\Phi(\cdot)$  is the cumulative distribution function of standard Normal. Therefore, for  $j \in \tilde{J}$ , we have

$$\begin{aligned} \sup_{\mathbf{P} \in \mathbb{F}} \mathbb{E}_{\mathbf{P}}[I_j(\mathbf{y}, \omega)] &= \sup_{\mathbf{P} \in \mathbb{F}} \mathbb{E}_{\mathbf{P}} \left[ t\lambda_j + \Phi^{-1}(\alpha) \sqrt{t\lambda_j} \right] \\ &= \sup_{\mathbf{P} \in \mathbb{F}} \mathbb{E}_{\mathbf{P}} \left[ t \sum_{p \in \mathcal{P}} \lambda^{(p)} y_j^{(p)} + \Phi^{-1}(\alpha) \sqrt{t \sum_{p \in \mathcal{P}} \lambda^{(p)} y_j^{(p)}} \right] \\ &= \sup_{\mathbf{P} \in \mathbb{F}} \mathbb{E}_{\mathbf{P}} \left[ t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)} + \Phi^{-1}(\alpha) \sqrt{t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)}} \right] \\ &= t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)} + \Phi^{-1}(\alpha) \sqrt{t} \sup_{\mathbf{P} \in \mathbb{F}} \mathbb{E}_{\mathbf{P}} \left[ \sqrt{\sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)^2}} \right] \\ &\leq t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)} + \Phi^{-1}(\alpha) \Psi \sqrt{t} \sqrt{\sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)^2}}, \end{aligned}$$

where the second and third equalities follow from the definitions of  $\lambda_j$  and  $\lambda^{(p)}$ , respectively, the fourth equality follows from the fact that  $y_j^{(p)^2} = y_j^{(p)}$ , and the value of  $\Psi$  was obtained from Proposition 2 in [45] and is given by

$$\Psi = \sqrt{\max_{1 \leq l \leq L} \left\{ \frac{\xi_l(\omega)}{\mu_l} \right\}} - \frac{\max_{1 \leq l \leq L} \left\{ \frac{\xi_l(\omega)}{\mu_l} \right\} - 1}{\sqrt{\max_{1 \leq l \leq L} \left\{ \frac{\xi_l(\omega)}{\mu_l} \right\} + \sqrt{1 - \left( \min_{1 \leq l \leq L} \left\{ \frac{\sigma_l}{\mu_l} \right\} \right)^2} \left( L \left( \max_{1 \leq l \leq L} \left\{ \frac{\xi_l(\omega)}{\mu_l} \right\} - 1 \right) \right)}}.$$

It immediately follows that minimizing the quantity  $\sup_{\mathbf{P} \in \mathbb{F}} \mathbb{E}_{\mathbf{P}}[I_j(\mathbf{y}, \omega)]$  is equivalent to minimizing a variable, say  $\tilde{v}_j$ , subject to the constraint

$$\tilde{v}_j \geq t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)} + \Phi^{-1}(\alpha) \Psi \sqrt{t} \sqrt{\sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)^2}}, \quad j \in \tilde{J}. \quad (21)$$

For  $j \in \tilde{J}$ , let  $\mathbf{y}_j$  be the vector in  $\mathbb{R}^{|\mathcal{P}|}$  whose  $p$ th component is  $y_j^{(p)}$  for  $p \in \mathcal{P}$ , and  $D$  be the diagonal matrix in  $\mathbb{R}^{|\mathcal{P}| \times |\mathcal{P}|}$  whose  $(p$ th,  $p$ th) entry is  $(\sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l)^{1/2}$  for  $p \in \mathcal{P}$ . Then the constraint in (21) can be written as the

second-order cone constraint

$$\left( \tilde{v}_j - t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)}, \Phi^{-1}(\alpha) \Psi \sqrt{t} D \mathbf{y}_j \right) \geq \mathbf{0}, \quad j \in \tilde{J}.$$

Now, we formulate the chance constraint in (19) as a second-order cone constraint. Note that, for  $j \in \tilde{J}$ , we have

$$\begin{aligned} \inf_{\mathbb{P} \in \mathbb{F}} P_{\mathbb{P}} \left( I_j(\mathbf{y}, \omega) \leq g_j \right) &= \inf_{\mathbb{P} \in \mathbb{F}} P_{\mathbb{P}} \left( t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)} + \Phi^{-1}(\alpha) \sqrt{t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)}} \leq g_j \right) \\ &= \inf_{\mathbb{P} \in \mathbb{F}} P_{\mathbb{P}} \left( \left( \sqrt{t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)}} + \frac{\Phi^{-1}(\alpha)}{2} \right)^2 \leq g_j + \frac{\Phi^{-1}(\alpha)^2}{4} \right) \\ &= \inf_{\mathbb{P} \in \mathbb{F}} P_{\mathbb{P}} \left( \sqrt{t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)}} \leq \sqrt{g_j + \frac{\Phi^{-1}(\alpha)^2}{4}} - \frac{\Phi^{-1}(\alpha)}{2} \right) \\ &= \inf_{\mathbb{P} \in \mathbb{F}} P_{\mathbb{P}} \left( \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)} \leq \hat{g}_j \right), \end{aligned}$$

where the second and the fourth equalities follow from observing that the number of batteries needed, as specified in (20), is a strictly increasing function in the demand rate  $\lambda_j$ , and  $\hat{g}_j$  is a constant that is given by

$$\hat{g}_j = \frac{1}{t} \left( \sqrt{g_j + \frac{\Phi^{-1}(\alpha)^2}{4}} - \frac{\Phi^{-1}(\alpha)}{2} \right)^2.$$

It immediately follows that the chance constraint  $P(\sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)} \leq \hat{g}_j) \geq 1 - \epsilon_g, j \in \tilde{J}$ , can be written as

$$P \left( \frac{1}{\sqrt{\sigma}} \left( \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)} - \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)} \right) \leq \frac{1}{\sqrt{\sigma}} \left( \hat{g}_j - \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)} \right) \right) \geq 1 - \epsilon_g, \quad j \in \tilde{J}.$$

Because  $\frac{1}{\sqrt{\sigma}} (\sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \xi_l(\omega) y_j^{(p)} - \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)})$  is a zero mean unit variance Gaussian random variable, the above probability is simply  $\Phi((\hat{g}_j - \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)}) / \sqrt{\sigma}) \geq 1 - \epsilon_g, j \in \tilde{J}$ . Since  $\sqrt{\sigma} = \|\Sigma^{1/2} \mathbf{y}_j\|$ , the given chance constraint can be written as

$$\hat{g}_j - \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)} \geq \Phi^{-1}(1 - \epsilon_g) \sqrt{\sigma}, \quad \text{or equivalently} \quad \left( \hat{g}_j - \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)}, \Phi^{-1}(1 - \epsilon_g) \Sigma^{1/2} \mathbf{y}_j \right) \geq \mathbf{0}.$$

In the second stage problem, as we used the parameters  $\tilde{v}_j, j \in \tilde{J}$ , we also use  $v_j$  to denote the previously known quantities  $I_j, j \in J$ , in the first-stage problem.

Given the above modeling framework, Problem (18, 19) can be cast as the following two-stage SMBSOCP model.

$$\begin{aligned} \min \quad & \sum_{j \in J} (f_j x_j + h v_j) + \mathbb{E} [Q(\mathbf{x}; \mathbf{y}; \mathbf{z}, \omega)] \\ \text{s.t.} \quad & \mathbf{y}_j^{(p)} \geq b_{p,q} z_j^{(q)}, & j \in J, p \in \mathcal{P}, q \in \mathcal{Q}, \\ & \mathbf{y}_j^{(p)} \leq x_j, & j \in J, p \in \mathcal{P}, \\ & v_j \leq g_j, & j \in J, \\ & x_j, \mathbf{y}_j^{(p)}, z_j^{(q)} \in \{0, 1\}, & j \in J, p \in \mathcal{P}, q \in \mathcal{Q}, \end{aligned} \tag{22}$$

where  $Q(x; y, z, \omega)$  is the minimum value of the problem

$$\begin{aligned}
\min \quad & \sum_{j \in \tilde{J}} (f_j x_j + h \tilde{v}_j) \\
\text{s.t.} \quad & \left( \tilde{v}_j - t \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)}, \Phi^{-1}(\alpha) \Psi \sqrt{t} D y_j \right) \geq \mathbf{0}, \quad j \in \tilde{J}, \\
& \sum_{j \in \tilde{J} \cup \tilde{J}} a_j^{(q)} z_j^{(q)} \geq 1, \quad q \in \mathcal{Q}, \\
& y_j^{(p)} \geq b_{p,q} z_j^{(q)}, \quad j \in \tilde{J}, p \in \mathcal{P}, q \in \mathcal{Q}, \\
& y_j^{(p)} \leq x_j, \quad j \in \tilde{J}, p \in \mathcal{P}, \\
& \left( \hat{g}_j - \sum_{p \in \mathcal{P}} \sum_{l=1}^L \hat{\lambda}_l^{(p)} \mu_l y_j^{(p)}, \Phi^{-1}(1 - \epsilon_g) \Sigma^{1/2} y_j \right) \geq \mathbf{0}, \quad j \in \tilde{J}, \\
& x_j, y_j^{(p)}, z_j^{(q)} \in \{0, 1\}, \quad j \in \tilde{J}, p \in \mathcal{P}, q \in \mathcal{Q}.
\end{aligned} \tag{23}$$

Because the constraints in the first-stage problem (22) are linear with integer variables and without the use of second-order cone relaxations, a solution for Model (22, 23) is now possible. One way to approach this model is by applying scenario-based cuts proposed in [36].

### 3.5 Optimal random berth allocation problem with uncertain handling time

In recent years, container shipping has seen rapid growth especially during the pandemic of COVID-19. For instance, Figure 8 shows that, contrary to expectations, container shipping in Shanghai has quickly bounced back from its slowdown at the beginning of the pandemic. Vessel emissions and fuel cost are today real challenges experienced in the berth allocation, and their reduction is a common target and a key priority in the port-shipping coordination. In this part, we consider this berthing allocation problem.

Du et al. [46] studied the determination of the optimal berthing positions and order for vessels waiting at a container terminal to minimize the vessels' waiting and fuel consumption. They regard the vessels handling times as previously-known (deterministic) times when formulating this problem. Due to some uncertainties, such as mechanical problems, system failures, bad weather, and route changes, the scheduled vessels handling times may diverge from the deterministic ones, which leads to significant changes of other terminal operations and makes the baseline berthing plan no longer feasible. In this part, we consider the optimal random berth allocation problem, in which the vessels handling times and the required departure times of the vessels have known deterministic and random frames.

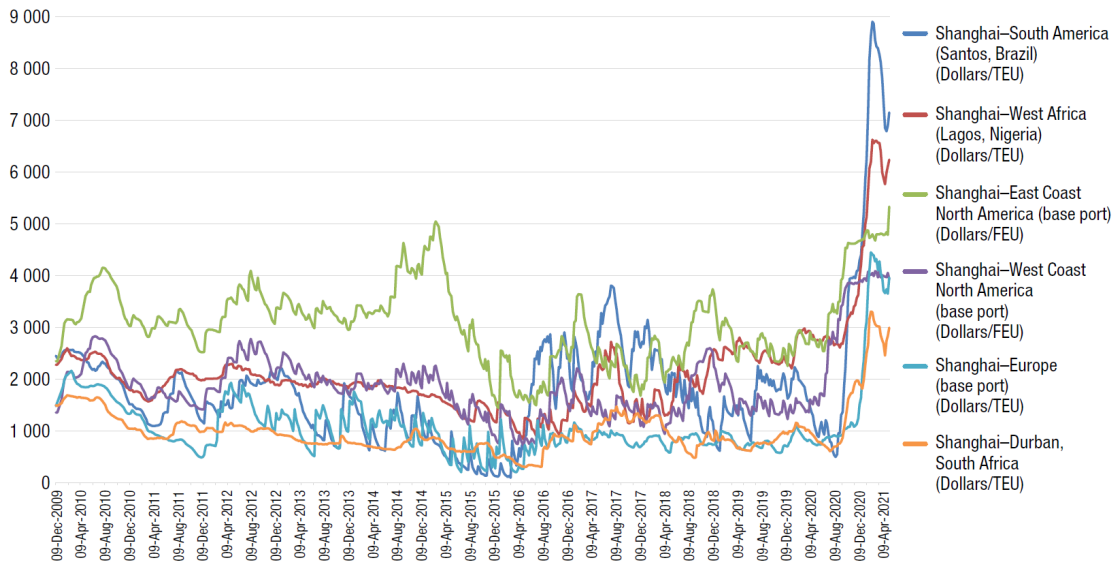


Figure 8: Shanghai containerized freight index, December 2009 - April 2021. (Abbreviations: FEU, 40-foot equivalent unit; TEU, 20-foot equivalent unit.)

In the berth plan, we assume that we are given  $f$  vessels, say  $V_1, \dots, V_f$ , whose handling times, say  $h_1, \dots, h_f$ , are precisely known at the time of formulation. We also assume that we are given  $r$  vessels, say  $V_{f+1}, \dots, V_{f+r}$ , whose handling times, say  $h_{f+1}(\omega), \dots, h_{f+r}(\omega)$ , are not precisely known at the time of formulation, and their realizations depends on an underlying outcome  $\omega$  in an event space  $\Omega$  with a known probability function  $P$ .

We presume that at present we do not know the realizations of the handling times and the realizations of the requested departure times of the vessels  $V_{f+1}, \dots, V_{f+r}$  at the present, and that at some point in time in the future the realizations of these times become known. We also assume that we need to determine the leftmost berthing position of the vessels  $V_1, \dots, V_f, V_{f+1}, \dots, V_{f+r}$ , and their start time of berthing given the handling times  $h_1, \dots, h_f$  and the realizations of the handling times  $h_{f+1}(\omega), \dots, h_{f+r}(\omega)$ . However, this decision must be made before the realizations of the handling times  $h_j(\omega), j = f + 1, \dots, f + r$ , and the realizations of the requested departure times become available. Therefore, when these realizations become available, the start times of berthing that have already been determined may or may not be optimal. At that stage, we assume that it is permitted to modify the start time of berthing of the vessels  $V_{f+1}, \dots, V_{f+r}$  (but not their leftmost berthing positions), as necessary, to ensure that all berthing start times are optimal.

Define the following decision variables for  $i, k = 1, \dots, f, f + 1, \dots, f + r$  ( $i \neq k$ ):

$x_i$ : the leftmost berthing position of the vessel  $V_i$ ,

$y_i$ : the start time of berthing of the vessel  $V_i$ ,

$a_i$ : the arrival time of the vessel  $V_i$ ,

$\alpha_{ik}$ : a binary variable indicating whether  $V_i$  is positioned left of  $V_k$  along the wharf ( $= 1$ ) or not ( $= 0$ ),

$\beta_{ik}$ : a binary variable indicating whether  $V_i$  is positioned below  $V_k$  along the timeline ( $= 1$ ) or not ( $= 0$ ).

We also use the following parameters:

$l_i$ : the length of vessel  $V_i$ , for  $i = 1, 2, \dots, f + r$ ,

$L$ : the wharf length of the container terminal,

$h_i$ : the actual handling time of vessel  $V_i$ , for  $i = 1, 2, \dots, f$ ,

$h_j(\omega)$ : the expected handling time of vessel  $V_j$ , for  $j = f + 1, f + 2, \dots, f + r$ ,

$d_i$ : the requested departure time of vessel  $V_i$ , for  $i = 1, 2, \dots, f$ ,

$d_j(\omega)$ : the expected requested departure time of vessel  $V_j$ , for  $j = f + 1, f + 2, \dots, f + r$ .

In building the model, we make sure that the following constraints are satisfied. See also Figure 9 which visually shows an example clarifying some of these constraints graphically.

- All vessels should be berthed within the wharf's boundary. This can be written as  $x_i + l_i \leq L$  for  $i = 1, 2, \dots, f + r$ .
- For  $i, k = 1, 2, \dots, f + r, i \neq k$ , if vessel  $V_i$  is positioned along the wharf to the left of vessel  $V_k$ , the rightmost berthing position of vessel  $V_i$  must be to the left of the leftmost berthing position of vessel  $V_k$ . Otherwise, the rightmost berthing position of vessel  $V_i$  must be limited by the leftmost berthing position of vessel  $V_k$  plus the wharf length of the container terminal. This can be modeled as

$$x_i + l_i \leq x_k + L(1 - \alpha_{ik}), \text{ for } i, k = 1, 2, \dots, f + r, i \neq k. \quad (24)$$

For example, in Figure 9, it can be seen that the inequality in (24) is satisfied for the vessels  $V_i, V_k$  and  $V_m$  as well as for the vessels  $V_i, V_j$  and  $V_m$ , but it is not satisfied for the vessels  $V_k$  and  $V_j$ .

- A vessel should not be berthed before its arrival time. This can be written as  $a_i \leq y_i$ , for  $i = 1, 2, \dots, f + r$ . See Figure 9. As a matter of fact, the deterministic and stochastic times in Figure 9 are existing on two different axes for more focused picture, but indeed they both occur on the same timeline.
- For  $i = 1, 2, \dots, f$  and  $k = 1, 2, \dots, f + r, i \neq k$ , if vessel  $V_i$  is positioned below vessel  $V_k$  along the timeline, the departure time of vessel  $V_i$  must precede the berthing time of vessel  $V_k$ . Otherwise, the berthing time of vessel  $V_k$  should not be limited by the departure time of vessel  $V_i$ . Let  $M$  be an arbitrary large constant, then this can be modeled as

$$y_i + h_i \leq y_k + M(1 - \beta_{ik}), \text{ for } 1 \leq i \leq f, 1 \leq k \leq f + r, i \neq k. \quad (25)$$

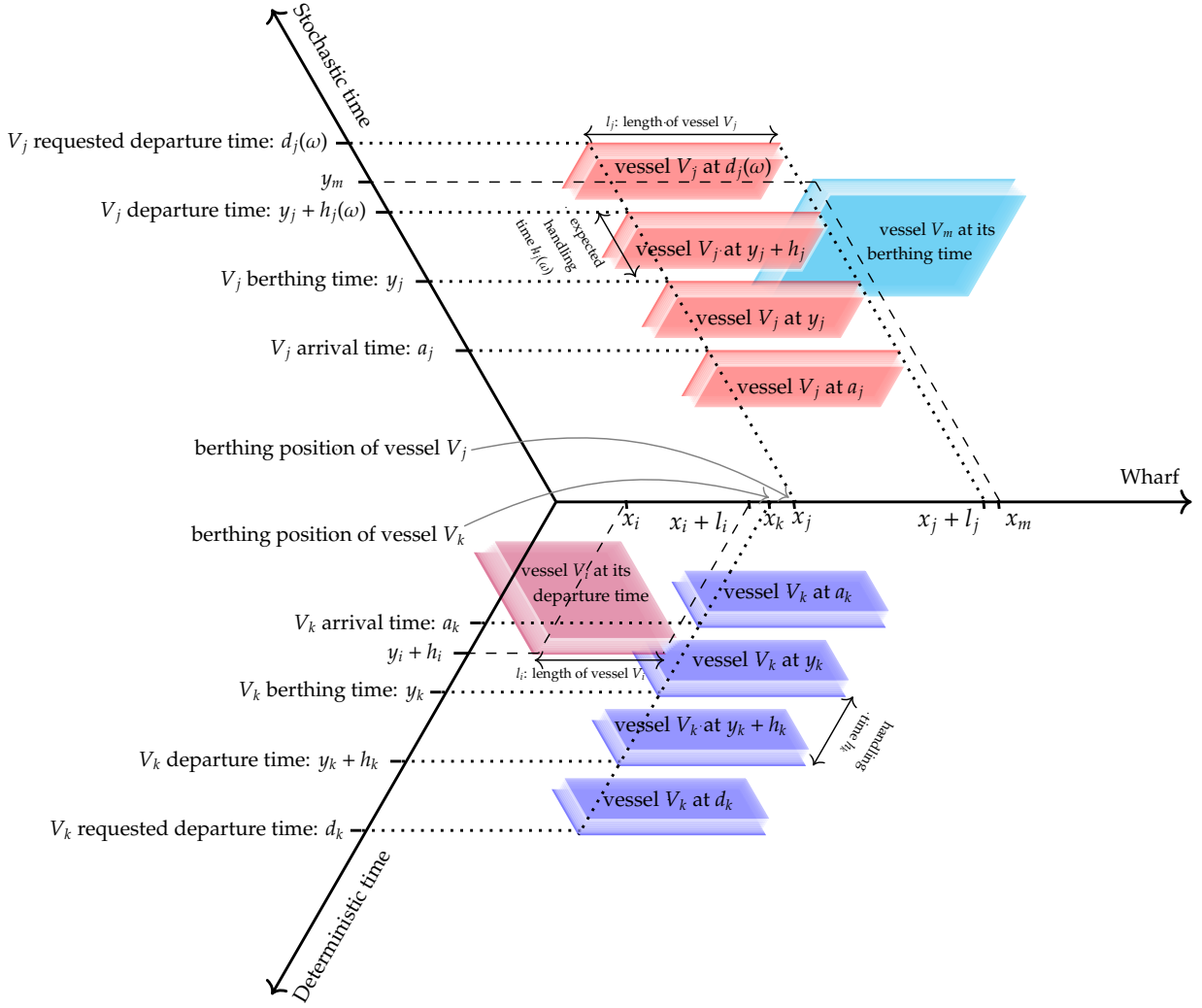


Figure 9: A graphical illustration of random berth allocation with uncertain handling time.

For example, in Figure 9, it can be seen that vessels  $V_i$  and  $V_k$  satisfy the inequality in (25).

- For  $j = f + 1, f + 2, \dots, f + r$  and  $k = 1, 2, \dots, f + r, i \neq k$ , if vessel  $V_j$  is positioned below vessel  $V_k$  along the timeline, the expected departure time of vessel  $V_j$  must precede the berthing time of vessel  $V_k$ . Otherwise, the expected departure time of vessel  $V_j$  should not be limited by the berthing time of vessel  $V_k$ . Let  $M$  be an arbitrary large constant, then this can be modeled as

$$y_j + h_j(\omega) \leq y_k + M(1 - \beta_{jk}), \text{ for } f + 1 \leq j \leq f + r, 1 \leq k \leq f + r, j \neq k. \quad (26)$$

For example, in Figure 9, it can be seen that vessels  $V_j$  and  $V_m$  satisfy the inequality in (26).

- To save fuel, the sailing speed of each vessel should be adjusted so that its arrival time at the terminal is in a given interval. This can be written as  $\underline{a}_i \leq a_i \leq \bar{a}_i$  for  $i = 1, 2, \dots, f + r$ .
- Finally, to avoid duplication and the constraints (24)–(26) among vessels, we add the constraint  $1 \leq \alpha_{ik} + \alpha_{ki} + \beta_{ik} + \beta_{ki} \leq 2$ , for  $1 \leq i, k \leq f + r, i < k$ .

We have two objectives: Minimizing the total departure delay and minimizing the fuel consumption.

The total departure delay is given by

$$\sum_{i=1}^f (y_i + h_i - d_i)^+ + \sum_{j=f+1}^{f+r} (y_j + h_j(\omega) - d_j(\omega))^+, \quad (27)$$

where  $\kappa^+ := \max\{\kappa, 0\}$  for  $\kappa \in \mathbb{R}$ . In the model, this objective is scaled with a weight parameter, say  $\lambda$ , for convenience.

The fuel consumption is given by [46]

$$\sum_{i=1}^f (c_i a_i + \hat{c}_i n_i^{\mu_i} a_i^{1-\mu_i}) + \sum_{j=f+1}^{f+r} (c_j(\omega) a_j(\omega) + \hat{c}_j(\omega) n_j^{\mu_j} a_j^{1-\mu_j}(\omega)). \quad (28)$$

This function is obtained for each vessel using regression analysis. Here,  $c_i, \hat{c}_i, c_j(\omega)$  and  $\hat{c}_j(\omega)$  are the regression coefficients,  $n_i$  (respectively,  $n_j$ ) is the distance of vessel  $V_i$  (respectively, vessel  $V_j$ ) from the terminal, and  $\mu_i, \mu_j \in \{3.5, 4, 4.5\}$  for each  $i = 1, 2, \dots, f$  and  $j = f+1, f+2, \dots, f+r$ .

By introducing the auxiliary variables  $t_i, t_j, q_i, q_j \geq 0$ , for  $i = 1, 2, \dots, f$  and  $j = f+1, f+2, \dots, f+r$ , and adding the constraints

$$\begin{aligned} y_i + h_i - d_i &\leq t_i, & a_i^{1-\mu_i} &\leq q_i, & 1 \leq i \leq f, \\ y_j + h_j(\omega) - d_j(\omega) &\leq t_j, & a_j^{1-\mu_j}(\omega) &\leq q_j, & f+1 \leq j \leq f+r, \end{aligned}$$

we can rewrite the summations in (27) and (28) as

$$\sum_{i=1}^f t_i + \sum_{j=f+1}^{f+r} t_j \quad \text{and} \quad \sum_{i=1}^f (c_i a_i + \hat{c}_i n_i^{\mu_i} q_i) + \sum_{j=f+1}^{f+r} (c_j a_j(\omega) + \hat{c}_j n_j^{\mu_j} q_j),$$

respectively.

Given the above modeling framework, the two-stage optimal random berth allocation problem is formulated as the following two-stage stochastic mixed-binary nonlinear programming model.

$$\begin{aligned} \min \quad & \sum_{i=1}^f (c_i a_i + \hat{c}_i n_i^{\mu_i} q_i) + \lambda \sum_{i=1}^f t_i + \mathbb{E}[Q(\mathbf{x}; \mathbf{y}; \boldsymbol{\alpha}; \boldsymbol{\beta}, \omega)] \\ \text{s.t.} \quad & x_i + l_i \leq L, & & 1 \leq i \leq f, \\ & x_i + l_i \leq x_k + L(1 - \alpha_{ik}), & & 1 \leq i \leq f, 1 \leq k \leq f+r, i \neq k, \\ & y_i + h_i \leq t_i + d_i, & & 1 \leq i \leq f, \\ & y_i + h_i \leq y_k + M(1 - \beta_{ik}), & & 1 \leq i \leq f, 1 \leq k \leq f+r, i \neq k, \\ & \underline{a}_i \leq a_i \leq \bar{a}_i, & & 1 \leq i \leq f, \\ & 0 \leq a_i \leq y_i, & & 1 \leq i \leq f, \\ & 1 \leq a_i^{\mu_i-1} q_i, & & 1 \leq i \leq f, \\ & 1 \leq \alpha_{ik} + \alpha_{ki} + \beta_{ik} + \beta_{ki} \leq 2, & & 1 \leq i \leq f, 1 \leq k \leq f+r, i < k, \\ & x_i, t_i, q_i \geq 0, & & 1 \leq i \leq f, \\ & \alpha_{ik}, \beta_{ik} \in \{0, 1\}, & & 1 \leq i \leq f, 1 \leq k \leq f+r, i \neq k, \end{aligned} \quad (29)$$

where  $\mathbb{E}[Q(\mathbf{x}; \mathbf{y}; \boldsymbol{\alpha}; \boldsymbol{\beta}, \omega)] = \int_{\Omega} Q(\mathbf{x}; \mathbf{y}; \boldsymbol{\alpha}; \boldsymbol{\beta}, \omega) P(d\omega)$ , and  $Q(\mathbf{x}; \mathbf{y}; \boldsymbol{\alpha}; \boldsymbol{\beta}, \omega)$  is the minimum value of the problem

$$\begin{aligned} \min \quad & \sum_{j=f+1}^{f+r} (c_j a_j(\omega) + \hat{c}_j n_j^{\mu_j} q_j) + \lambda \sum_{j=f+1}^{f+r} t_j \\ \text{s.t.} \quad & x_j + l_j \leq L, & & f+1 \leq j \leq f+r, \\ & x_j + l_j \leq x_k + L(1 - \alpha_{jk}), & & f+1 \leq j \leq f+r, 1 \leq k \leq f+r, j \neq k, \\ & y_j + h_j(\omega) \leq t_j + d_j(\omega), & & f+1 \leq j \leq f+r, \\ & y_j + h_j(\omega) \leq y_k + M(1 - \beta_{jk}), & & f+1 \leq j \leq f+r, 1 \leq k \leq f+r, j \neq k, \\ & \underline{a}_j \leq a_j \leq \bar{a}_j, & & f+1 \leq j \leq f+r, \\ & 0 \leq a_j \leq y_j, & & f+1 \leq j \leq f+r, \\ & 1 \leq a_j^{\mu_j-1} q_j, & & f+1 \leq j \leq f+r, \\ & 1 \leq \alpha_{jk} + \alpha_{kj} + \beta_{jk} + \beta_{kj} \leq 2, & & f+1 \leq j \leq f+r, 1 \leq k \leq f+r, j < k, \\ & x_j, t_j, q_j \geq 0, & & f+1 \leq j \leq f+r, \\ & \alpha_{jk}, \beta_{jk} \in \{0, 1\}, & & f+1 \leq j \leq f+r, 1 \leq k \leq f+r, j \neq k. \end{aligned} \quad (30)$$



The nonlinear terms in (29) and (30) appear in the seventh set of constraints in each, and they can be handled by transferring them into hyperbolic constraints and then using (2) to rewrite the resulting hyperbolic constraints as a group of second-order cone constraints as follows [46]:

When  $\mu_i = 3.5$ , the inequality  $1 \leq a_i^{\mu_i-1} q_i$  is  $1 \leq a_i^{5/2} q_i$  with  $a_i, q_i > 0$ . Squaring both sides, we get  $1 \leq a_i^5 q_i^2$ , which is equivalent to hyperbolic constraints:

$$u_{i1}^2 \leq a_i, \quad u_{i2}^2 \leq u_{i1} q_i, \quad 1 \leq a_i u_{i2}, \quad a_i, q_i > 0, \quad u_{i1}, u_{i2} \geq 0. \quad (31)$$

Using (2), the hyperbolic constraints in (31) are equivalent to the following group of second-order cone constraints

$$\|(2u_{i1}; a_i - 1)\| \leq a_i + 1, \quad \|(2u_{i2}; u_{i1} - q_i)\| \leq u_{i1} + q_i, \quad \|(2; a_i - u_{i2})\| \leq a_i + u_{i2}, \quad a_i, q_i > 0, \quad u_{i1}, u_{i2} \geq 0. \quad (32)$$

Similarly, when  $\mu_i = 4$ , the inequality  $1 \leq a_i^3 q_i$  is equivalent to the following group of second-order cone constraints

$$\begin{aligned} \|(2u_{i1}; a_i - 1)\| &\leq a_i + 1, & \|(2u_{i2}; u_{i1} - q_i)\| &\leq u_{i1} + q_i, & \|(2u_{i3}; a_i - u_{i2})\| &\leq a_i + u_{i2}, \\ \|(2u_{i4}; u_{i1} - 1)\| &\leq u_{i1} + 1, & \|(2; u_{i3} - u_{i4})\| &\leq u_{i3} + u_{i4}, & a_i, q_i > 0, & u_{i1}, u_{i2}, u_{i3}, u_{i4} \geq 0, \end{aligned} \quad (33)$$

and when  $\mu_i = 4.5$ , the inequality  $1 \leq a_i^{7/2} q_i$  is equivalent to the following group of second-order cone constraints

$$\|(2u_{i1}; a_i - 1)\| \leq a_i + 1, \quad \|(2u_{i2}; u_{i1} - q_i)\| \leq u_{i1} + q_i, \quad \|(2; a_i - u_{i2})\| \leq a_i + u_{i2}, \quad a_i, q_i > 0, \quad u_{i1}, u_{i2} \geq 0. \quad (34)$$

Constraints (32)–(34) were obtained for each  $i = 1, 2, \dots, f$ . Similar constraints can also be obtained for each  $j = f + 1, f + 2, \dots, f + r$ .

Similar to other models built in this paper, in light of the above conic transformation, a solution for Model (29, 30) is now possible by applying the decomposition algorithm developed in [35].

## 4 Algorithms

As mentioned in Section 2, there are available solution methods for SSOCP, SMILP, and DMISOCP. These optimization problems are algorithmically more mature than the more general case, the SMISOCP problem. Because the present paper captures a variety of its important applications, in this section we present a solution method for SMISOCP, leaving the development of more specific methodologies for a future research.

In the modeling process, the input requires that we solve the stochastic program for a finite number of scenarios. For this reason, in practice, the case of interest is when random parameters have a finite number of realizations. Therefore, we assume that the event space  $\Omega$  is discrete, and has finitely many outcomes with known probabilities.

Let  $p(\omega)$  represent the probability of occurrence for  $\omega \in \Omega$ . The equivalent deterministic formulation of the SMISOCP problem (3) is as follows:

$$\begin{aligned} \min \quad & c^\top x & + \quad & p(1)d(1)^\top y_1 & + \quad & p(2)d(2)^\top y_2 & + \quad \dots & + \quad & p(|\Omega|)d(|\Omega|)^\top y_{|\Omega|} \\ \text{s.t.} \quad & Ax & & & & & & & & \geq_r \quad b, \\ & T(1)x & + \quad & W(1)y_1 & & & & & & \geq_s \quad h(1), \\ & T(2)x & & & + \quad & W(2)y_2 & & & & \geq_s \quad h(2), \\ & \vdots & & & & & \ddots & & & \vdots \\ & T(|\Omega|x) & & & & & & + \quad & W(|\Omega|)y_{|\Omega|} & \geq_s \quad h(|\Omega|), \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}, \quad y_1 \in \mathbb{Z}^q \times \mathbb{R}^{m-q}, \quad y_2 \in \mathbb{Z}^q \times \mathbb{R}^{m-q}, \quad \dots & & & & & & & & y_{|\Omega|} \in \mathbb{Z}^q \times \mathbb{R}^{m-q}. \end{aligned} \quad (35)$$

Problem (35) is a huge-scale DMISOCP problem, and therefore, we may not even be able to solve Problem (35) using state-of-the-art optimization solvers. However, the advantage that makes the formulation in (35) different from other formulations that involve uncertainty is the nice structure that can be seen inside. We can use the solution methods available in the literature for DMISOCP, particularly cuts and relaxations, to

solve the resulting extensive formulation, and this can be successfully achieved by exploiting the special structure of Problem (35). In this connection, we emphasize that all application models proposed in Section 3 can be formulated in the form of Problem (35).

In Subsection 4.1, we give a brief review for the solution algorithms available for solving SSOCP with more elaboration on the homogeneous self-dual method. In Subsection 4.2, we describe a solution algorithm for solving SMISOCP using cuts and relaxations by combining the existing method in Subsection 4.1 for SSOCP with extensions of DMISOCP.

#### 4.1 Homogeneous interior-point methods for SSOCP

Numerous interior-point algorithms for SSOP have been developed recently. We briefly outline these algorithms ordered chronologically. In 2014, Alzalg [10] proposed a homogeneous self-dual interior-point algorithm was proposed for SSOCP. Later in the same year, Alzalg [11] proposed a solution to the problem based on a logarithmic barrier decomposition-based interior-point algorithm (see also [8]). In 2015, Alzalg [9] proposed a volumetric barrier decomposition-based interior-point algorithm for solving the problem. In 2018, Alzalg et al. [12] proposed an infeasible self-dual interior-point algorithm for solving the problem. While the methodologies in [10, 12] are based on the deterministic equivalence, which is the extensive formulation of a stochastic program that forms an equivalent large one-stage problem containing all constraints and all scenarios, the methodologies in [9, 11] are based on Bender's decomposition, which decomposes a stochastic program into stages where, at each stage, variables at preceding stages are considered as constraints so that the subproblem at the current stage is easier to solve.

In this part, we give a brief overview of the homogeneous self-dual algorithm [10] used to solve the underlying SSOCP. The continuous relaxation of (35) is obtained by the same formulation as (35), but with  $x \in \mathbb{R}^n$  and  $y_1, y_2, \dots, y_{|\Omega|} \in \mathbb{R}^m$ . If we re-define  $d(k)$  to be  $p(k)d(k)$  for each  $k = 1, 2, \dots, |\Omega|$ , the SSOCP has the form

$$\begin{aligned}
\min \quad & c^\top x + d(1)^\top y_1 + d(2)^\top y_2 + \dots + d(|\Omega|)^\top y_{|\Omega|} \\
\text{s.t.} \quad & Ax \geq_r b, \\
& T(1)x + W(1)y_1 \geq_s h(1), \\
& T(2)x + W(2)y_2 \geq_s h(2), \\
& \vdots \\
& T(|\Omega|)x + W(|\Omega|)y_{|\Omega|} \geq_s h(|\Omega|).
\end{aligned} \tag{36}$$

Note that we associate a different decision variable  $y_k$  with each realization to be slightly more general. The dual of (36) is

$$\begin{aligned}
\max \quad & b^\top u + h(1)^\top v_1 + h(2)^\top v_2 + \dots + h(|\Omega|)^\top v_{|\Omega|} \\
\text{s.t.} \quad & A^\top u + T(1)^\top v_1 + T(2)^\top v_2 + \dots + T(|\Omega|)^\top v_{|\Omega|} = c, \\
& W(1)^\top v_1 = d(1), \\
& W(2)^\top v_2 = d(2), \\
& \vdots \\
& W(|\Omega|)^\top v_{|\Omega|} = d(|\Omega|), \\
& u \geq_r \mathbf{0}, \quad v_1 \geq_s \mathbf{0}, \quad v_2 \geq_s \mathbf{0}, \quad \dots \quad v_{|\Omega|} \geq_s \mathbf{0}.
\end{aligned} \tag{37}$$

The homogeneous model for the pair (36,37) is

$$\begin{aligned}
Ax - p - \tau b &= \mathbf{0}, \\
T(k)x + W(k)y_k - q_k - \tau h(k) &= \mathbf{0}, \quad k = 1, 2, \dots, |\Omega|, \\
-A^\top u - \sum_{k=1}^{|\Omega|} T(k)^\top v_k + \tau c &= \mathbf{0}, \\
-W(k)^\top v_k + \tau d(k) &= \mathbf{0}, \quad k = 1, 2, \dots, |\Omega|, \\
b^\top u + \sum_{k=1}^{|\Omega|} h(k)^\top v_k - c^\top x - \sum_{k=1}^{|\Omega|} d(k)^\top y_k - \kappa &= 0, \\
p \geq_r \mathbf{0}, \quad u \geq_r \mathbf{0}, \quad q_k \geq_s \mathbf{0}, \quad v_k \geq_s \mathbf{0}, \quad k = 1, 2, \dots, |\Omega|, \\
\tau, \kappa &\geq 0.
\end{aligned} \tag{38}$$

The following system defines the search direction system corresponding to (38).

$$\begin{aligned}
A\Delta x - \Delta p - \Delta\tau b &= \eta r_{p0}, \\
T(k)\Delta x + W(k)\Delta y_k - \Delta q_k - \Delta\tau h(k) &= \eta r_{pk}, \quad k = 1, 2, \dots, |\Omega|, \\
-A^\top \Delta u - \sum_{k=1}^{|\Omega|} T(k)^\top \Delta v_k + \Delta\tau c &= \eta r_{d0}, \\
-W(k)^\top \Delta v_k + \Delta\tau d(k) &= \eta r_{dk}, \quad k = 1, 2, \dots, |\Omega|, \\
b^\top \Delta u + \sum_{k=1}^{|\Omega|} h(k)^\top \Delta v_k - c^\top \Delta x - \sum_{k=1}^{|\Omega|} d(k)^\top \Delta y_k - \Delta\kappa &= \eta r_g, \\
\kappa \Delta\tau + \tau \Delta\kappa &= \gamma \mu - \tau \kappa, \\
\Delta p \circ u + p \circ \Delta u &= \gamma \mu e_0 - p \circ u, \\
\Delta q_k \circ v_k + q_k \circ \Delta q_k &= \gamma \mu e_k - q_k \circ v_k, \quad k = 1, \dots, |\Omega|,
\end{aligned} \tag{39}$$

where  $\gamma$  and  $\eta$  are two parameters, the product “ $\circ$ ” is the Jordan multiplication defined in (1), and

$$\begin{aligned}
r_{p0} &= p + \tau b - Ax, \\
r_{pk} &= q_k + \tau h(k) - T(k)x - W(k)y_k, \quad k = 1, 2, \dots, |\Omega|, \\
r_{d0} &= A^\top u + \sum_{k=1}^{|\Omega|} T(k)^\top v_k - \tau c, \\
r_{dk} &= W(k)^\top v_k - \tau d(k), \quad k = 1, 2, \dots, |\Omega|, \\
r_g &= \kappa - b^\top u - \sum_{k=1}^{|\Omega|} h(k)^\top v_k + c^\top x + \sum_{k=1}^{|\Omega|} d(k)^\top y_k, \\
\mu &= \frac{1}{2(r + |\Omega|s) + 1} \left( p^\top u + \sum_{k=1}^{|\Omega|} q_k^\top v_k + \tau \kappa \right).
\end{aligned}$$

We point out that a scaling may be needed to guarantee that we iterate on the interior of the underlying second-order cone. Common examples of such a scaling are the HRVW/KSH/M direction (introduced by Kojima et al. [47] and Helmsberg et al. [48] independently and then rediscovered by Monteiro [49]), the dual HRVW/KSH/M direction (introduced by Kojima et al. [47] and rediscovered by Monteiro [49]), and the NT direction (introduced by Nesterov and Todd [50,51]).

The work in [10] describes an efficient method to find the search directions as a solution to (39). This method exploits the special structure in (36, 37) and decomposes into  $|\Omega|$  smaller computations which can be performed in parallel. The results in [10] show that this method is both theoretically and computationally efficient. In particular, the number of arithmetic operations in each iteration when computing the search direction grows cubically as a function of  $n, m, k$  and  $l$ , and linearly as a function of  $|\Omega|$  (see [10]).

## 4.2 Gomory cuts and tight relaxations for SMISOCP

In this part, we introduce two algorithmic developments for SMISOCP. The first development is based on the work of Cezik and Iyengar [16], which proposes Gomory cuts for solving (deterministic) mixed-integer conic programs by extending well-known techniques from mixed-integer linear programming to mixed-integer conic programs. In this part, we particularly extend the Chvátal-Gomory approach for generating cuts to the equivalent deterministic formulation (35). We introduce the following feasibility sets.

$$\begin{aligned}
\mathcal{F}_1 &:= \{x \in \mathbb{Z}_+^n : Ax \geq_r b\}; \\
\mathcal{F}_2^{(k)}(x) &:= \{y_k \in \mathbb{Z}_+^m : W(k)y_k \geq_s h(k) - T(k)x\} \text{ for } k = 1, 2, \dots, |\Omega|; \\
\mathcal{F} &:= \{(x, y_1, \dots, y_{|\Omega|}) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^m \times \dots \times \mathbb{Z}_+^m : Ax \geq_r b, W(k)y_k \geq_s h(k) - T(k)x, k = 1, \dots, |\Omega|\}.
\end{aligned} \tag{40}$$

The extension in this part is based on the following two equivalences (see also [16]):

$$\begin{aligned}
Ax \geq_r b &\iff (A^\top u)^\top x \geq u^\top b, & \forall u \geq_r \mathbf{0}, \\
T(k)x + W(k)y_k \geq_s h(k) &\iff (W(k)^\top v_k)^\top y_k \geq v_k^\top (h(k) - T(k)x), & \forall v_k \geq_s \mathbf{0}, \quad k = 1, 2, \dots, |\Omega|.
\end{aligned}$$

Let  $\mathbf{a}_i$  and  $\mathbf{w}_j(k)$  denote the  $i$ th column of  $A$  and  $j$ th column of  $W(k)$ , respectively, for  $k = 1, 2, \dots, |\Omega|$ . We have the following workflow which can be applied iteratively to deterministically compute the convex hull of the feasibility set  $\mathcal{F}$  of the formulation (see also [16, Section 2]).

**Workflow 1 (Chvátal-Gomory Procedure)** *We consider the following three steps for our setting:*

(i) Choose  $\mathbf{u} \geq_r \mathbf{0}$  and  $\mathbf{v}_k \geq_s \mathbf{0}$  for  $k = 1, 2, \dots, |\Omega|$ . Then

$$(A^\top \mathbf{u})^\top \mathbf{x} = \sum_{i=1}^n (\mathbf{a}_i^\top \mathbf{u}) x_i \geq \mathbf{u}^\top \mathbf{b}, \quad \text{and} \quad (W(k)^\top \mathbf{v}_k)^\top \mathbf{y}_k = \sum_{j=1}^m (\mathbf{w}_j(k)^\top \mathbf{v}_k) y_{kj} \geq \mathbf{v}_k^\top (\mathbf{h}(k) - T(k)\mathbf{x}), \quad k = 1, 2, \dots, |\Omega|.$$

(ii) Without loss of generality, assume that  $\mathbf{x} \geq \mathbf{0}$  and  $\mathbf{y}_k \geq \mathbf{0}$ ,  $k = 1, 2, \dots, |\Omega|$ . Then

$$\sum_{i=1}^n \lceil \mathbf{a}_i^\top \mathbf{u} \rceil x_i \geq \mathbf{u}^\top \mathbf{b}, \quad \text{and} \quad \sum_{j=1}^m \lceil \mathbf{w}_j(k)^\top \mathbf{v}_k \rceil y_{kj} \geq \mathbf{v}_k^\top (\mathbf{h}(k) - T(k)\mathbf{x}), \quad k = 1, 2, \dots, |\Omega|.$$

(iii) By the integrality of  $\mathbf{x}$  and  $\mathbf{y}_k$ ,  $k = 1, 2, \dots, |\Omega|$ , it follows that

$$\sum_{i=1}^n \lceil \mathbf{a}_i^\top \mathbf{u} \rceil x_i \geq \lceil \mathbf{u}^\top \mathbf{b} \rceil, \quad \text{and} \quad \sum_{j=1}^m \lceil \mathbf{w}_j(k)^\top \mathbf{v}_k \rceil y_{kj} \geq \lceil \mathbf{v}_k^\top (\mathbf{h}(k) - W(k)\mathbf{x}) \rceil, \quad k = 1, 2, \dots, |\Omega|.$$

Therefore, we add the above  $1 + |\Omega|$  valid linear inequalities to the second-order cone constraints  $A\mathbf{x} \geq_r \mathbf{b}$  and  $T(k)\mathbf{x} + W(k)\mathbf{y}_k \geq_s \mathbf{h}(k)$ ,  $k = 1, 2, \dots, |\Omega|$ .

The following proposition is due to [16, Lemma 1].

**Proposition 1** *Assume that the feasibility set  $\mathcal{F}$  is bounded. Then every valid inequality for the convex hull of  $\mathcal{F}$  can be obtained by repeating Workflow 1 a finite number of times.*

The second development investigates the hierarchies of relaxations studied by Lovász and Schrijver [18] and Balas et al. [17]. Fix subsets  $\mathcal{P} \subseteq \{1, 2, \dots, p\}$  and  $\mathcal{Q} \subseteq \{1, 2, \dots, q\}$  with sizes  $|\mathcal{P}| = \tilde{p} \leq p$  and  $|\mathcal{Q}| = \tilde{q} \leq q$ . Without loss of generality, we can consider that  $\mathcal{Q} = \{1, 2, \dots, \tilde{q}\}$  and  $\mathcal{P} = \{1, 2, \dots, \tilde{p}\}$ . Let  $\mathbf{x} \in \mathcal{F}_1$  and  $\mathbf{y}(k) := \mathbf{y}_k \in \mathcal{F}_2^{(k)}(\mathbf{x})$ ,  $k = 1, 2, \dots, |\Omega|$ , where  $\mathcal{F}_1$  and  $\mathcal{F}_2^{(k)}(\cdot)$  are defined in (40). Let also

$$U^{(0)} = [\mathbf{u}_1^{(0)}, \mathbf{u}_2^{(0)}, \dots, \mathbf{u}_{\tilde{p}}^{(0)}], \quad U^{(1)} = [\mathbf{u}_1^{(1)}, \mathbf{u}_2^{(1)}, \dots, \mathbf{u}_{\tilde{p}}^{(1)}] \in \mathbb{R}^{(m+1) \times \tilde{p}},$$

and

$$V^{(0)}(k) = [\mathbf{v}_1^{(0)}(k), \mathbf{v}_2^{(0)}(k), \dots, \mathbf{v}_{\tilde{q}}^{(0)}(k)], \quad V^{(1)}(k) = [\mathbf{v}_1^{(1)}(k), \mathbf{v}_2^{(1)}(k), \dots, \mathbf{v}_{\tilde{q}}^{(1)}(k)] \in \mathbb{R}^{(m+1) \times \tilde{q}}, \quad \text{for } k = 1, 2, \dots, |\Omega|,$$

where

$$\mathbf{u}_i^{(0)} = (1 - x_i) \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}, \quad \mathbf{u}_i^{(1)} = x_i \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}, \quad \text{for } i = 1, 2, \dots, \tilde{p},$$

and

$$\mathbf{v}_j^{(0)}(k) = (1 - y_j(k)) \begin{bmatrix} 1 \\ \mathbf{y}(k) \end{bmatrix}, \quad \mathbf{v}_j^{(1)}(k) = y_j(k) \begin{bmatrix} 1 \\ \mathbf{y}(k) \end{bmatrix}, \quad \text{for } j = 1, 2, \dots, \tilde{q}, \quad \text{and } k = 1, 2, \dots, |\Omega|.$$

Let  $U_{\mathcal{P}}^{(1)}$  and  $V_{\mathcal{Q}}^{(1)}(k)$  be the submatrices of the matrices  $U^{(1)}$  and  $V^{(1)}(k)$  corresponding to the indices in  $\mathcal{P}$  and  $\mathcal{Q}$ , respectively,  $k = 1, 2, \dots, |\Omega|$ . Then, the following problem relaxes Problem (35) with all binary variables.

$$\begin{aligned}
\min \quad & \mathbf{c}^\top \mathbf{x} + \sum_{k=1}^{|\Omega|} d(k)^\top y(k) \\
\text{s.t.} \quad & \sum_{i=1}^n u_i^{(0)} \mathbf{a}_i \geq_r u_{0_t}^{(0)} \mathbf{b}, & t = 1, 2, \dots, \tilde{p}, \\
& \sum_{i=1}^n u_i^{(1)} \mathbf{a}_i \geq_r u_{0_t}^{(1)} \mathbf{b}, & t = 1, 2, \dots, \tilde{p}, \\
& \sum_{j=1}^m v_{jj}^{(0)}(k) \mathbf{w}_j(k) \geq_s v_{0_j}^{(0)}(k) (\mathbf{h}(k) - T(k)\mathbf{x}), & j = 1, 2, \dots, \tilde{q}, k = 1, 2, \dots, |\Omega|, \\
& \sum_{j=1}^m v_{jj}^{(1)}(k) \mathbf{w}_j(k) \geq_s v_{0_j}^{(1)}(k) (\mathbf{h}(k) - T(k)\mathbf{x}), & j = 1, 2, \dots, \tilde{q}, k = 1, 2, \dots, |\Omega|, \\
& u_i^{(0)} + u_i^{(1)} = \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}, & i = 1, 2, \dots, \tilde{p}, \\
& v_j^{(0)}(k) + v_j^{(1)}(k) = \begin{bmatrix} 1 \\ \mathbf{y}(k) \end{bmatrix}, & j = 1, 2, \dots, \tilde{q}, k = 1, 2, \dots, |\Omega|, \\
& u_u^{(0)} = 0, & i = 1, 2, \dots, \tilde{p}, \\
& v_{jj}^{(0)}(k) = 0, & j = 1, 2, \dots, \tilde{q}, k = 1, 2, \dots, |\Omega|, \\
& u_u^{(1)} = u_{0_t}^{(1)}, & t = 1, 2, \dots, \tilde{p}, \\
& v_{jj}^{(1)}(k) = v_{0_j}^{(1)}(k), & j = 1, 2, \dots, \tilde{q}, k = 1, 2, \dots, |\Omega|, \\
& U_{\mathcal{P}}^{(1)} = U_{\mathcal{P}}^{(1)\top}, \\
& V_{\Omega}^{(1)}(k) = V_{\Omega}^{(1)\top}(k), & k = 1, 2, \dots, |\Omega|.
\end{aligned} \tag{41}$$

Sherali-Adams [52, 53] and later Lasserre [54] introduced hierarchies of the relaxations to present tighter relaxation of the underlying feasibility set. To extend their work to our setting, let  $\mathbf{u}$  be a vector indexed by the empty set, subsets  $\mathcal{R} \subseteq \mathcal{P}$ , and sets of the form  $\mathcal{R} \cup \{i\}, \mathcal{R} \subseteq \mathcal{P}, i \notin \mathcal{P}$ . Let also  $\mathbf{v}$  be a vector indexed by the empty set, subsets  $\mathcal{S} \subseteq \Omega$ , and sets of the form  $\mathcal{S} \cup \{j\}, \mathcal{S} \subseteq \Omega, j \notin \Omega$ . The vectors  $\mathbf{u} \in \mathbb{R}^{(n-\tilde{p}+1)2^{\tilde{p}}}$  and  $\mathbf{v}(k) \in \mathbb{R}^{(m-\tilde{q}+1)2^{\tilde{q}}}$  are defined as follow:

$$\begin{aligned}
u_{\mathcal{J}} &= \begin{cases} 1, & \mathcal{J} = \emptyset; \\ \prod_{i \in \mathcal{J}} x_i, & \mathcal{J} = \mathcal{R} \subseteq \mathcal{P}, \mathcal{J} = \mathcal{R} \cup \{i\}, \mathcal{R} \subseteq \mathcal{P}, i \notin \mathcal{P}, \end{cases} \\
v_{\mathcal{J}}(k) &= \begin{cases} 1, & \mathcal{J} = \emptyset; \\ \prod_{j \in \mathcal{J}} y_j(k), & \mathcal{J} = \mathcal{S} \subseteq \Omega, \mathcal{J} = \mathcal{S} \cup \{j\}, \mathcal{S} \subseteq \Omega, j \notin \Omega, \end{cases} \text{ for } k = 1, 2, \dots, |\Omega|.
\end{aligned}$$

For all subsets  $\mathcal{J} \subseteq \mathcal{P}$ , and subsets  $\mathcal{J} \subseteq \Omega$ , define  $\xi_0^{\mathcal{J}} \in \mathbb{R}$  and  $\zeta_0^{\mathcal{J}}(k) \in \mathbb{R}$  as

$$\begin{aligned}
\xi_0^{\mathcal{J}} &= \prod_{i \in \mathcal{J}} x_i \prod_{i \in \mathcal{P} \setminus \mathcal{J}} (1 - x_i) = \sum_{\mathcal{J} \subseteq \mathcal{R} \subseteq \mathcal{P}} (-1)^{|\mathcal{P} \setminus \mathcal{R}|} u_{\mathcal{R}} \geq 0, \\
\zeta_0^{\mathcal{J}}(k) &= \prod_{j \in \mathcal{J}} y_j(k) \prod_{j \in \Omega \setminus \mathcal{J}} (1 - y_j(k)) = \sum_{\mathcal{J} \subseteq \mathcal{S} \subseteq \Omega} (-1)^{|\Omega \setminus \mathcal{S}|} v_{\mathcal{S}}(k) \geq 0, \text{ for } k = 1, 2, \dots, |\Omega|,
\end{aligned}$$

and define  $\xi^{\mathcal{J}} \in \mathbb{R}^n$  and  $\zeta^{\mathcal{J}}(k) \in \mathbb{R}^m$  as

$$\begin{aligned}
\xi_v^{\mathcal{J}} &= \prod_{i \in \mathcal{J}} x_i \prod_{i \in \mathcal{P} \setminus \mathcal{J}} (1 - x_i) = \sum_{\mathcal{J} \subseteq \mathcal{R} \subseteq \mathcal{P}} (-1)^{|\mathcal{P} \setminus \mathcal{R}|} u_{\mathcal{R} \cup \{v\}}, \text{ for } v = 1, 2, \dots, n, \\
\zeta_v^{\mathcal{J}}(k) &= \prod_{j \in \mathcal{J}} y_j(k) \prod_{j \in \Omega \setminus \mathcal{J}} (1 - y_j(k)) = \sum_{\mathcal{J} \subseteq \mathcal{S} \subseteq \Omega} (-1)^{|\Omega \setminus \mathcal{S}|} v_{\mathcal{S} \cup \{v\}}(k), \text{ for } v = 1, 2, \dots, m, \text{ for } k = 1, 2, \dots, |\Omega|.
\end{aligned}$$

Table 1: Comparing the number of variables and constraints of Relaxations (41) and (42).

	Number of variables	Number of conic constraints
Relaxation (41)	$\mathcal{O}(n \mathcal{P}  + m \Omega  \Omega )$	$\mathcal{O}( \mathcal{P}  +  \Omega  \Omega )$
Relaxation (42)	$\mathcal{O}(n2^{ \mathcal{P} } + m \Omega 2^{ \Omega })$	$\mathcal{O}(2^{ \mathcal{P} } +  \Omega 2^{ \Omega })$

Multiplying the second-order cone constraints  $Ax \succeq_r \mathbf{b}$  and  $W(k)y(k) \succeq_s \mathbf{h}(k) - T(k)x$  by the products  $\prod_{i \in \mathcal{J}} x_j \prod_{i \in \mathcal{P} \setminus \mathcal{J}} (1 - x_j)$  and  $\prod_{j \in \mathcal{J}} y_j(k) \prod_{j \in \Omega \setminus \mathcal{J}} (1 - y_j(k))$ , respectively, we obtain

$$\begin{aligned} A\xi^{\mathcal{J}} - \mathbf{b}\xi_0^{\mathcal{J}} &\succeq_r \mathbf{0}, & \mathcal{J} \subseteq \mathcal{P}, \\ T(k)x \zeta_0^{\mathcal{J}}(k) + W(k)\zeta^{\mathcal{J}}(k) - \mathbf{h}(k)\zeta_0^{\mathcal{J}}(k) &\succeq_s \mathbf{0}, & \mathcal{J} \subseteq \Omega, k = 1, 2, \dots, |\Omega|. \end{aligned}$$

Then, the following problem relaxes the SMBSOCP version of Problem (35) (i.e., Problem (35) in which  $x \in \{0, 1\}^p \times \mathbb{R}^{n-p}$  and  $y(k) \in \{0, 1\}^q \times \mathbb{R}^{m-q}$ ).

$$\begin{aligned} \min \quad & \mathbf{c}^T x + d(1)^T y(1) + d(2)^T y(2) + \dots + d(|\Omega|)^T y(|\Omega|) \\ \text{s.t.} \quad & A\xi^{\mathcal{J}} \succeq_r \mathbf{b}\xi_0^{\mathcal{J}}, & \mathcal{J} \subseteq \mathcal{P}, \\ & T(1)x \zeta_0^{\mathcal{J}}(1) + W(1)\zeta^{\mathcal{J}}(1) \succeq_s \mathbf{h}(1)\zeta_0^{\mathcal{J}}(1), & \mathcal{J} \subseteq \Omega, \\ & T(2)x \zeta_0^{\mathcal{J}}(2) + W(2)\zeta^{\mathcal{J}}(2) \succeq_s \mathbf{h}(2)\zeta_0^{\mathcal{J}}(2), & \mathcal{J} \subseteq \Omega, \\ & \vdots & \vdots \\ & T(|\Omega|)x \zeta_0^{\mathcal{J}}(|\Omega|) + W(|\Omega|)\zeta^{\mathcal{J}}(|\Omega|) \succeq_s \mathbf{h}(|\Omega|)\zeta_0^{\mathcal{J}}(|\Omega|), & \mathcal{J} \subseteq \Omega, \\ & x_v = u_{[v]}, \quad y_v(1) = v_{[v]}(1), \quad y_v(2) = v_{[v]}(2), \quad \dots \quad y_v(|\Omega|) = v_{[v]}(|\Omega|), & 1 \leq v \leq n, \quad 1 \leq v \leq m. \end{aligned} \tag{42}$$

Finally, we apply the solution technique described in Subsection 4.1 to solve Problem (42).

From Table 1, we can see that the number of variables and constraints grow, as a function of  $\mathcal{P}$  and  $\Omega$ , linearly in Relaxation (41) and exponentially in Relaxation (42).

## 5 Conclusion

Stochastic mixed-integer second-order cone programming is an important class of optimization problems that includes stochastic second-order cone programming and stochastic mixed-integer linear programming as special cases. In this paper, we have described five different applications that lead to two-stage stochastic mixed-integer second-order cone programming, and this has been achieved by relaxing assumptions that include deterministic information to incorporate random data. Application areas include facility location, portfolio optimization, uncapacitated inventory, battery swapping stations, and berth allocation planning. It is important to indicate that all of the optimization models that have been developed in this study are intractable for an extensive form formulation. Therefore, solutions for such application models are now possible by applying either the decomposition algorithm developed recently in [35] or the scenario-based cuts proposed recently in [36].

Developing other and more specialized algorithms to solve the generic two-stage stochastic mixed-integer second-order cone programming problem is an important subject of study. In this paper, we have started this study by describing solution methods for solving the generic stochastic mixed-integer second-order cone programming problem using cuts and relaxations by combining the existing homogeneous interior-point algorithm for stochastic second-order cone programming with extensions of mixed-integer second-order cone programming. We emphasize that the work presented in this paper is far from being comprehensive, but we believe that it would rather shape a basis for further research studies in second-order cone programs over integers under uncertainty. In particular, it is our firm belief that this research is a rich source for stochastic mixed-integer second-order cone programming models, and it can be greatly utilized to implement numerous future algorithms for solving this class of optimization problems.

**Acknowledgment** The authors thank Blake Whithman from the Ohio State University for reading the manuscript and pointing out misprints.

**Credits** Figure 2 is from <https://colab.research.google.com>. The source of Figure 8 is UNCTAD calculations, based on data from Clarksons Research, Shipping Intelligence Network Time Series.

## References

1. Benson HY, Seğlam Ü (2013) Mixed-integer second-order cone programming: A survey. *INFORMS TutORials Oper. Res.* 13–36.
2. Küçükyavuz S, Sen S (2017) An introduction to two-stage stochastic mixed-integer programming. *Oper. Res. Rev.* 1–27.
3. Alzalg BM (2012) Stochastic second-order cone programming: Applications models. *Appl. Math. Model.* 36(10):5122–5134.
4. Zhu Y, Ariyawansa KA (2011) A preliminary set of applications leading to stochastic semidefinite programs and chance-constrained semidefinite programs. *Appl. Math. Model.* 35(5):2425–2442.
5. Alizadeh F, Goldfarb D (2003) Second-order cone programming. *Math. Program.* 95(1):3–51.
6. Alzalg BM (2019) A primal-dual interior-point method based on various selections of displacement step for symmetric optimization. *Comput. Optim. Appl.* 72:363–390.
7. Alzalg BM (2020) A logarithmic barrier interior-point method based on majorant functions for second-order cone programming. *Optim. Lett.* 14:729–746.
8. Alzalg BM, Ariyawansa KA (2014) Logarithmic barrier decomposition-based interior-point methods for stochastic symmetric programming. *J. Math. Anal. Appl.* 409(2):973–995.
9. Alzalg BM (2015) Volumetric barrier decomposition algorithms for stochastic quadratic second-order cone programming. *Appl. Math. Comput.* 256:494–508.
10. Alzalg BM (2014) Homogeneous self-dual algorithms for stochastic second-order cone programming. *J. Optim. Theory Appl.* 163(1):148–164.
11. Alzalg BM (2014) Decomposition-based interior point methods for stochastic quadratic second-order cone programming. *App. Math. Comput.* 249:1–18.
12. Alzalg BM, Badarneh K, Ababneh A (2019) An infeasible interior-point algorithm for stochastic second-order cone optimization. *J. Optim. Theory Appl.* 181(1):324–346.
13. Drewes S, Ulbrich S (2012) Subgradient based outer approximation for mixed integer second order cone programming. In: *Mixed Integer Nonlinear Programming*. (Springer. New York, NY). 41–59.
14. Duran MA, Grossmann IE (1986) An outer-approximation algorithm for a class of mixed-integer nonlinear programs. *Math. program.* 36(3):307–339.
15. Vielma JP, Ahmed S, Nemhauser GL (2008) A lifted linear programming branch-and-bound algorithm for mixed-integer conic quadratic programs. *INFORMS. J. Comput.* 20(3):438–450.
16. Cezik MT, Iyengar G (2005) Cuts for mixed 0-1 conic programming. *Math. Program.* 104(1):179–202.
17. Balas E, Ceria S, Cornuéjols G (1993) A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Math. Program.* 58(1):295–324.
18. Lovász L, Schrijver A (1991) Cones of matrices and set-functions and 0-1 optimization. *SIAM J. Optim.* 1(2):166–190.
19. Alioui H, Alzalg BM, Mixed-integer stochastic second-order cone optimization using interior-point methods, Working paper.
20. Faraut J, Korányi A (1994) *Analysis on Symmetric Cones*. (Oxford Mathematical Monographs).
21. Haneveld K, Klein WK, Van der Vlerk, Maarteen H (1999) Stochastic integer programming: General models and algorithms. *Annals Oper. Res.* 85:39–57.

22. Beraldi P, Ruszczyński A (2002) A branch and bound method for stochastic integer programs under probabilistic constraints. *Optim. Methods Soft.* 17(3):359–382.
23. Carøe CC, Tind J (1997) A cutting-plane approach to mixed 0-1 stochastic integer programs. *Eur. J. Oper. Res.* 101(2):306–316.
24. Carøe CC, Tind J (1998) L-shaped decomposition of two-stage stochastic programs with integer recourse. *Math. Program.* 83(1):451–464.
25. Gade D, Hackbeil G, Ryan SM, Watson JP, Wets RJB, Woodruff DL (2016) Obtaining lower bounds from the progressive hedging algorithm for stochastic mixed-integer programs. *Math. Program.* 157(1):47–67.
26. Gade D, Küçükyavuz S, Sen S (2014) Decomposition algorithms with parametric Gomory cuts for two-stage stochastic integer programs. *Math. Program.* 144(1):39–64.
27. Laporte G, Louveaux FV (1993) The integer L-shaped method for stochastic integer programs with complete recourse. *Oper. Res. Lett.* 13(3):133–142.
28. Qi Y, Sen S (2017) The ancestral Benders’ cutting plane algorithm with multi-term disjunctions for mixed-integer recourse decisions in stochastic programming. *Math. Program.* 161:193–235.
29. Sen S (2010) Stochastic mixed-integer programming algorithms: Beyond Benders’ decomposition. In: *Wiley Encyclopedia of Operations Research and Management Science*.
30. Sen S, Hight JL (2005) The C3 theorem and a D2 algorithm for large scale stochastic mixed integer programming: Set convexification. *Math. Program.* 104(1):1–20.
31. Sherali HD, Fraticelli BMP (2002) A modification of Benders’ decomposition algorithm for discrete subproblems: An approach for stochastic programs with integer recourse. *J. Glob. Optim.* 22(1):319–342.
32. Sen S, Sherali HD (2006) Decomposition with branch-and-cut approaches for two-stage stochastic mixed-integer programming. *Math. Program.* 106(2):203–223.
33. Zhang M, Küçükyavuz S (2004) Finitely convergent decomposition algorithms for two-stage stochastic pure integer programs. *SIAM J. Optim.* 24(4):1933–1951.
34. Bansal M, Huang KL, Mehrotra S (2018) Tight second stage formulations in two-stage stochastic mixed integer programs. *SIAM J. Optim.* 28(1):788–819.
35. Luo F, Mehrotra S (2021) A decomposition method for distributionally-robust two-stage stochastic mixed-integer conic programs. *Math. Program.* 1–45.
36. Bansal M, Zhang Y (2021) Scenario-based cuts for structured two-stage stochastic and distributionally robust p-order conic mixed integer programs. *J. Glob. Optim.* 1–43.
37. Vanderbei RV, Yurttan H (1998) Using LOQO to solve second order cone programming problems. Report SOR (Princeton University, Princeton, USA) 98–09.
38. Bonami P, Lejeune M (2009) An exact solution approach for portfolio optimization problems under stochastic and integer constraints. *Oper. Res.* 57 (3): 650–670.
39. Gârleanu NB, Pedersen LH (2009) Dynamic trading with predictable returns and transaction costs. *J. Finance* 68(6):2309–2340.
40. Markowitz HM (1959) *Portfolio Selection: Efficient Diversification of Investments*. (John Wiley & Sons, New York, NY).
41. Lobo MS, Vandenberghe L, Boyd S, Lebret H (1998) Application of second-order cone programming. *Linear Algebra Appl.* 284:193–228.
42. Shen Z-JM, Coullard C, Daskin MS (2003) A joint location-inventory model. *Trans. Sci.* 37(1):40–55.



43. Daskin MS, Coullard CR, Shen Z-JM (2002) An inventory-location model: Formulation, solution algorithm and computational results. *Annals Res. Oper.* 110:83–106.
44. Atamturk A, Berenguer G, Shen Z-JM (2012) A conic integer programming approach to stochastic joint location-inventory problems. *Oper. Res.* 60(2):366–381.
45. Mak HY, Rong Y, Shen Z-JM (2013) Infrastructure planning for electric vehicles with battery swapping. *Manag. Sci.* 59(7):1557–1575.
46. Du Y, Chen Q, Quan X, Long L, Fung RY (2011) Berth allocation considering fuel consumption and vessel emissions. *Trans. Res. E-Log.* 47(6):1021–1037.
47. Kojima M, Shindoh S, Hara S (1997) Interior-point methods for the monotone semidefinite linear complementary problem in symmetric matrices. *SIAM J. Optim.* 7(1):86–125.
48. Helmberg C, Rendl F, Vanderbei RJ, Wolkowicz H (1996) An interior-point method for semidefinite programming. *SIAM J. Optim.* 6(2):342–361.
49. Monteiro RDC (1997) Primal-dual path-following algorithms for semidefinite programming. *SIAM J. Optim.* 7(3):663–678.
50. Nesterov YE, Todd MJ (1997) Self-scaled barriers and interior-point methods for convex programming. *Math. Oper. Res.* 22(1):1–42.
51. Nesterov YE, Todd MJ (1998) Primal-dual interior-point methods for self-scaled cones. *SIAM J. Optim.* 8(2):324–364.
52. Sherali HD, Adams WP (1990) A hierarchy of relaxations between the continuous and convex hull representations of zero-one programming problems. *SIAM J. Disc. Math.* 3(3):411–430.
53. Sherali HD, Adams WP (1994) A hierarchy of relaxations for mixed-integer zero-one programming problems. *Discrete Appl. Math.* 52(1):83–106.
54. Lasserre JB (2001) Global optimization with polynomials and the methods of moments. *SIAM J. Optim.* 11(3):796–817.