

The equilateral small octagon of maximal width

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Abstract

A small polygon is a polygon of unit diameter. The maximal width of an equilateral small polygon with $n = 2^s$ vertices is not known when $s \geq 3$. This paper solves the first open case and finds the optimal equilateral small octagon. Its width is approximately 3.24% larger than the width of the regular octagon: $\cos(\pi/8)$. In addition, the paper proposes a family of equilateral small n -gons, for $n = 2^s$ with $s \geq 4$, whose widths are within $O(1/n^4)$ of the maximal width.

Keywords Convex geometry, equilateral polygons, isodiametric problem, maximal width

1 Introduction

The *diameter* of a polygon is the largest Euclidean distance between pairs of its vertices. A polygon is said to be *small* if its diameter equals one. The *width* of a polygon for a given direction is the distance between two parallel lines perpendicular to this direction and supporting the polygon from below and above. The width of a polygon is the minimum width over all directions. The diameter graph of a small polygon is defined as the graph with the vertices of the polygon, and an edge between two vertices exists only if the distance between these vertices equals one. Figures 1, 2 and 3 represent diameter graphs of small quadrilaterals, hexagons and octagons, respectively. The solid lines are the edges of the graphs, and the dashed lines simply delimit the polygons.

For an integer $n \geq 3$, the maximal width of an equilateral small n -gon is unknown when $n = 2^s$ and $s \geq 3$ [1]. The present paper solves the first open case and gives the optimal equilateral small 8-gon. Thus, the main result is the following:

Theorem 1. *If w_8^* denote the maximal width among all equilateral small 8-gons then*

$$w_8^* = \frac{t_0^2 + t_0}{2\sqrt{t_0^3 + t_0^2 - 2t_0 + 1}} + \varepsilon = 0.9537763006 \dots + \varepsilon,$$

where $t_0 = 0.7682191676 \dots$ is the unique positive root of the polynomial equation $t^5 - 6t^3 + 3t^2 + 10t - 7 = 0$ and $0 \leq \varepsilon \leq 10^{-6}$. The value w_8^* is only attained by the 8-gon F_8 illustrated in Figure 3c.

In addition, for $n = 2^s$ with integer $s \geq 4$, a tight lower bound on the maximal width is proposed by constructing a family of equilateral small n -gons whose widths are greater than that of the regular small n -gons.

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Theorem 2. Suppose $n = 2^s$ with integer $s \geq 4$. Let $\overline{W}_n := \cos \frac{\pi}{2n}$ denote an upper bound on the width $W(\mathbb{P}_n)$ of a small n -gon \mathbb{P}_n [1]. Let \mathbb{R}_n denote the regular small n -gon. Then there exists an equilateral small n -gon \mathbb{G}_n such that

$$\overline{W}_n - W(\mathbb{G}_n) = \frac{2\pi^4}{3n^4} + O\left(\frac{1}{n^6}\right)$$

and

$$W(\mathbb{G}_n) - W(\mathbb{R}_n) = \frac{3\pi^2}{8n^2} + O\left(\frac{1}{n^4}\right).$$

The remainder of this paper is organized as follows. Section 2 recalls principal results on the maximal width of small polygons. Section 3 studies the equilateral small octagon and gives the proof of Theorem 1. Theorem 2 is proved in Section 4 and Section 5 concludes the paper.

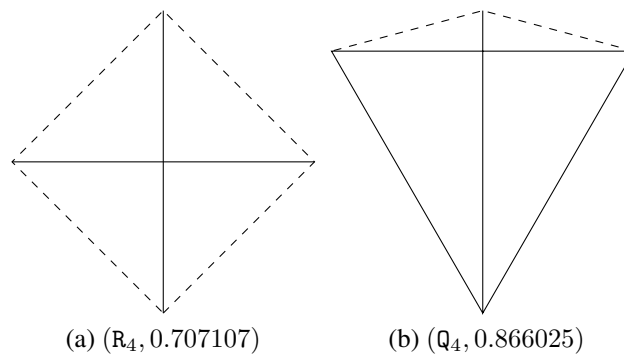


Figure 1: Two small 4-gons $(\mathbb{P}_4, W(\mathbb{P}_4))$: (a) Regular 4-gon; (b) An optimal non-equilateral 4-gon [2]

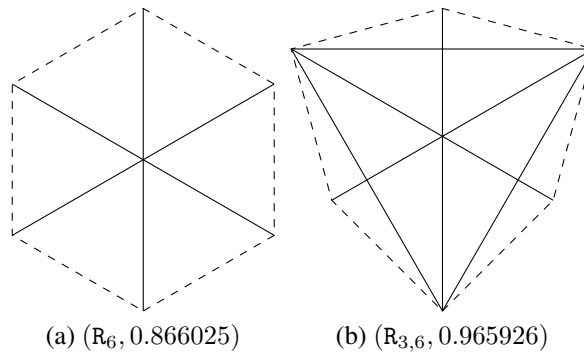


Figure 2: Two equilateral small 6-gons $(\mathbb{P}_6, W(\mathbb{P}_6))$: (a) Regular 6-gon; (b) Reinhardt 6-gon [3]

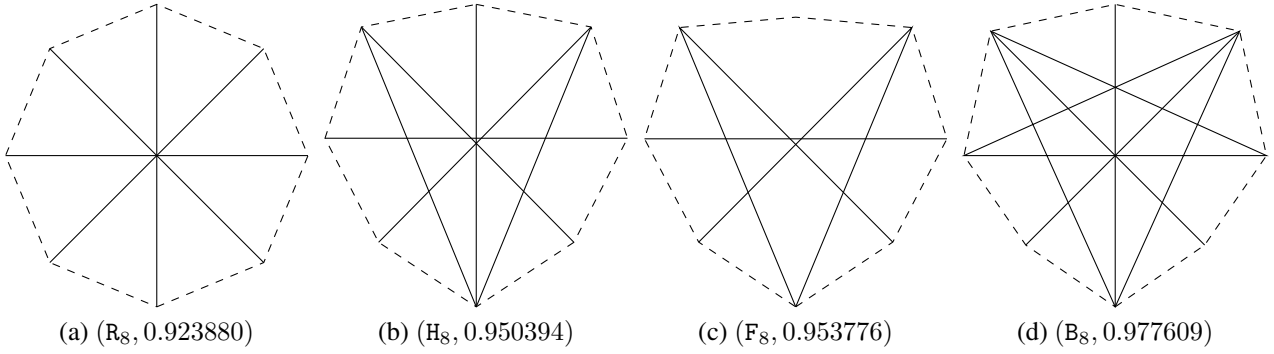


Figure 3: Four small 8-gons ($P_8, W(P_8)$): (a) Regular 8-gon; (b) Equilateral 8-gon of maximal perimeter [4]; (c) Equilateral 8-gon of maximal width; (d) A non-equilateral 8-gon of maximal width [5]

2 Widths of small polygons

Let $W(P)$ denote the width of a polygon P . For a given integer $n \geq 3$, the regular small n -gon R_n satisfies

$$W(R_n) = \begin{cases} \cos \frac{\pi}{2n} & \text{if } n \text{ is odd,} \\ \cos \frac{\pi}{n} & \text{if } n \text{ is even.} \end{cases}$$

When n has an odd factor m , consider the family of equilateral small n -gons constructed as follows:

1. Transform the regular small m -gon R_m into a Reuleaux m -gon by replacing each edge by a circle's arc passing through its end vertices and centered at the opposite vertex;
2. Add at regular intervals $n/m - 1$ vertices within each arc;
3. Take the convex hull of all vertices.

These n -gons are denoted $R_{m,n}$ and their widths satisfy $W(R_{m,n}) = \cos \frac{\pi}{2n}$. The 6-gon $R_{3,6}$ is illustrated in Figure 2b.

Theorem 3 (Bezdek and Fodor [1]). *For all $n \geq 3$, let W_n^* denote the maximal width among all small n -gons, w_n^* the maximal width among all equilateral ones, and $\overline{W}_n := \cos \frac{\pi}{2n}$.*

- *When n has an odd factor, $w_n^* = W_n^* = \overline{W}_n$ is achieved by finitely many equilateral n -gons [6–8], including $R_{m,n}$. The optimal n -gon $R_{m,n}$ is unique if m is prime and $n/m \leq 2$.*
- *When $n = 2^s$ with integer $s \geq 2$, $W(R_n) < W_n^* < \overline{W}_n$.*

When $n = 2^s$, W_n^* is known for $s \leq 3$ and for the equilateral cases, w_n^* is known only for $s = 2$. Bezdek and Fodor [1] proved that $W_4^* = \sqrt{3}/2$, and this value is achieved by infinitely many non-equilateral small 4-gons, including Q_4 represented in Figure 1b. Audet, Hansen, Messine, and Ninin [5] found that $W_8^* = \frac{1}{4}\sqrt{10 + 2\sqrt{7}}$, which is also achieved by infinitely many non-equilateral small 8-gons, including B_8 represented in Figure 3d. For the equilateral 4-gons, it is trivial that $w_4^* = W(R_4) = \sqrt{2}/2 < W_4^*$.

For $n = 2^s$ with $s \geq 4$, tight lower bounds on W_n^* may be obtained analytically. Bingane [9, 10] proved that, for $n = 2^s$ with $s \geq 4$,

$$W_n^* \geq \cos \left(\frac{\pi}{2n} + \frac{1}{2} \arctan \left(\tan \frac{2\pi}{n} \tan \frac{\pi}{n} \right) - \frac{1}{2} \arcsin \left(\frac{\sin(2\pi/n) \sin(\pi/n)}{\sqrt{4 \sin^2(\pi/n) + \cos(4\pi/n)}} \right) \right),$$

which implies

$$\overline{W}_n - W_n^* \leq \frac{\pi^5}{4n^5} + O\left(\frac{1}{n^7}\right).$$

3 The equilateral small octagon of maximal width

3.1 Definitions

Cartesian coordinates are used to describe an n -gon P_n . The vertex \mathbf{v}_k , $k \in \{0, 1, \dots, n-1\}$, is positioned at abscissa x_k and ordinate y_k . Sums or differences of coordinate indices are taken modulo n . Without any loss of generality, the vertex \mathbf{v}_0 is placed at the origin: $x_0 = y_0 = 0$, the n -gon P_n belongs to the half-plane $y \geq 0$ and the vertices \mathbf{v}_k , $k \in \{1, 2, \dots, n-1\}$, are arranged in a counterclockwise order as illustrated in Figure 4, i.e., $x_i y_{i+1} \geq y_i x_{i+1}$ for all $i \in \{1, 2, \dots, n-2\}$. The n -gon P_n is small if $\max_{i,j} \|\mathbf{v}_i - \mathbf{v}_j\| = 1$. It is equilateral if $\|\mathbf{v}_i - \mathbf{v}_{i-1}\| = c$ for all $i \in \{1, 2, \dots, n\}$.

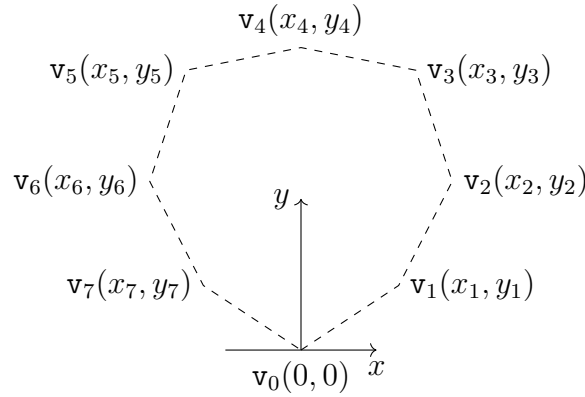


Figure 4: Definition of variables: Case of $n = 8$ vertices

Let $i \in \{1, 2, \dots, n\}$. The distance h_{ik} between a vertex \mathbf{v}_k , $k \in \{0, 1, \dots, n-1\}$, and the line containing the side $\mathbf{v}_{i-1}\mathbf{v}_i$ is

$$h_{ik} := \frac{(x_{i-1} - x_k)(y_i - y_k) - (y_{i-1} - y_k)(x_i - x_k)}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}}.$$

The *height* h_i associated to the side $\mathbf{v}_{i-1}\mathbf{v}_i$ is defined as the maximal value among h_{ik} and we denote k_i the index of a vertex \mathbf{v}_{k_i} such that $h_{ik_i} = h_i$. Then the width of the n -gon P_n is given by

$$W(P_n) = \min_{i=1,2,\dots,n} h_i.$$

We define the *height graph* of the n -gon P_n as the graph with the vertices $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ of P_n , and the edges $\mathbf{v}_{i-1} - \mathbf{v}_{k_i}$ and $\mathbf{v}_i - \mathbf{v}_{k_i}$ for all $i \in \{1, 2, \dots, n\}$. By construction, the length of an edge of the height graph is greater than or equal to $W(P_n)$. Remark that the diameter graph of P_n is a subgraph of its height graph. Figure 5 shows height graphs of some small polygons, the edges are represented by solid and dotted segments. The solid lines illustrate pairs of vertices which are unit distance apart. The dotted lines connect pairs of vertices that are a distance greater or equal to the width of the polygon but less than one. The dashed lines are not part of the graph, as they simply represent the sides of the polygons.

3.2 Bounds on the maximal width

A trivial lower bound on the maximal width w_8^* is provided by the regular 8-gon R_8 . However, the convex equilateral small 8-gon of maximal perimeter H_8 [4], illustrated in Figure 3b, provides a tighter

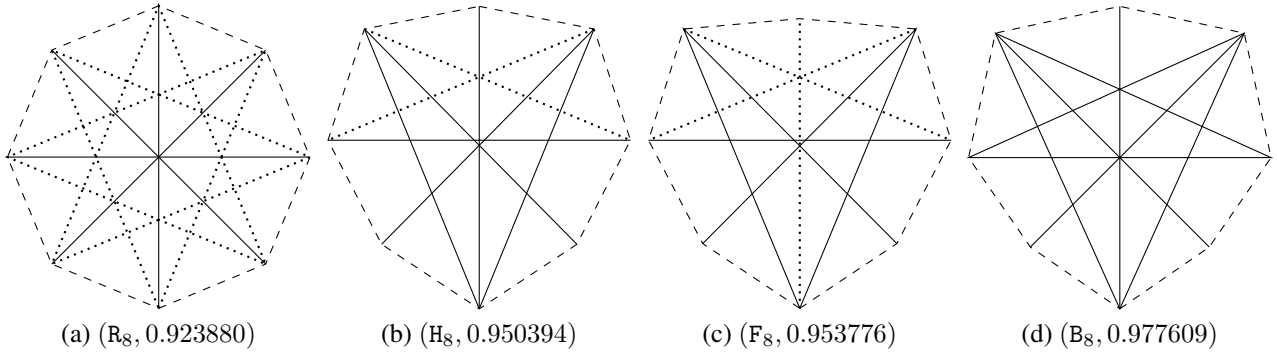


Figure 5: Height graphs of some small 8-gons ($P_8, W(P_8)$): (a) Regular 8-gon; (b) Equilateral 8-gon of maximal perimeter [4]; (c) Equilateral 8-gon of maximal width; (d) A non-equilateral 8-gon of maximal width [5]

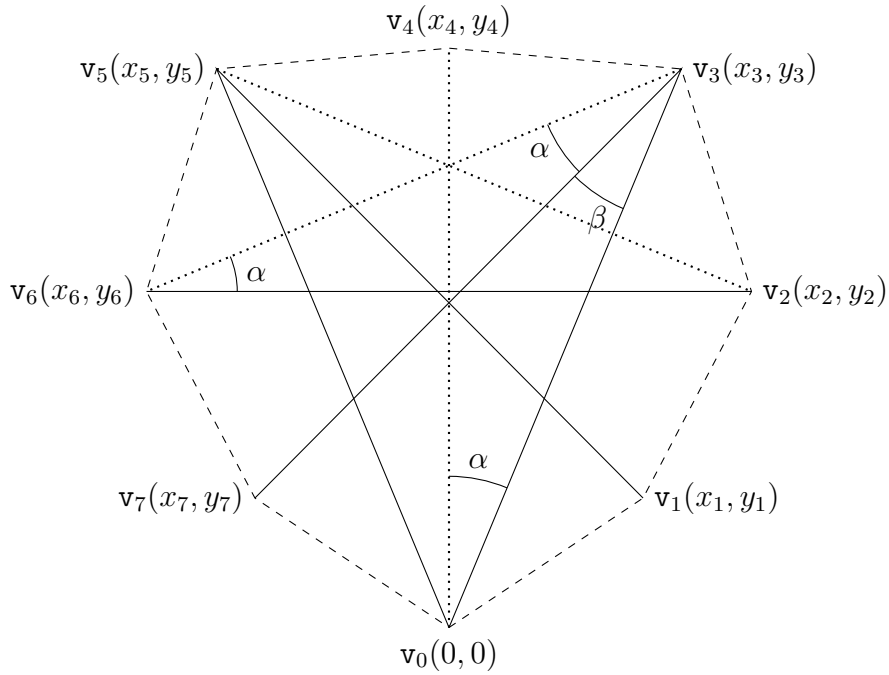


Figure 6: Configuration of the 8-gon providing a lower bound on the maximal width w_8^*

bound: $W(H_8) = 0.9503943246\dots > W(R_8) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$. This lower bound is improved in Proposition 1.

Proposition 1. *There exists an equilateral small 8-gon F_8 such that*

$$W(F_8) = 0.9537763006\dots > W(H_8),$$

where H_8 is the convex equilateral small octagon from [4] that maximizes the perimeter.

Proof. Consider an equilateral small 8-gon P_8 having the following height graph: a cycle $v_0 - v_3 - v_6 - v_2 - v_5 - v_0$ of length five, plus three pendant edges $v_0 - v_4$, $v_3 - v_7$, and $v_5 - v_1$, as illustrated in Figure 6. In addition, P_8 is axially symmetrical with respect to the edge $v_0 - v_4$ and the three edges represented by dotted lines possess the same length d , i.e., $\|v_0 - v_4\| = \|v_2 - v_5\| = \|v_3 - v_6\| = d \leq 1$.

Place the vertex v_4 at $(0, d)$ in the plane. Let $\alpha := \angle v_4 v_0 v_3 \in (0, \pi/6)$ and $\beta := \angle v_0 v_3 v_7 \in (0, \pi/8)$. Since P_8 is equilateral, we have $\|v_4 - v_3\| = \|v_0 - v_7\|$, which implies

$$1 + d^2 - 2d \cos \alpha = 2 - 2 \cos \beta. \quad (1)$$

Since P_8 is symmetric, we have from the cycle $v_0 - v_3 - v_6 - v_2 - v_5 - v_0$,

$$3\alpha + \beta = \pi/2. \quad (2)$$

Since the edge $v_6 - v_2$ is horizontal and $\|v_6 - v_2\| = 1$, we also have $x_6 = -1/2 = -x_2$, which yields

$$d = \frac{2 \sin \alpha + 1}{2 \sin(2\alpha + \beta)}. \quad (3)$$

The system of equations (1), (2), and (3) has a solution

$$(\alpha_0, \beta_0, d_0) = (0.3942432313\dots, 0.3880666326\dots, 0.9575669263\dots).$$

We can show that $t_0 = 2 \sin \alpha_0 = 0.7682191676\dots$ is the unique positive root of the polynomial equation

$$t^5 - 6t^3 + 3t^2 + 10t - 7 = 0.$$

Let F_8 denote the 8-gon obtained by setting $(\alpha, \beta, d) = (\alpha_0, \beta_0, d_0)$. The width of F_8 is

$$W(F_8) = \frac{d_0 \sin \alpha_0}{2 \sin(\beta_0/2)} = 0.9537763006\dots > W(H_8).$$

By construction, F_8 is small and its vertices are given by

$$\begin{aligned} x_0 &= 0, & y_0 &= 0, \\ x_1 &= 0.3208100713\dots = -x_7, & y_1 &= 0.2140003477\dots = y_7, \\ x_2 &= 1/2 & &= -x_6, & y_2 &= 0.5554768772\dots = y_6, \\ x_3 &= 0.3841095838\dots = -x_5, & y_3 &= 0.9232875108\dots = y_5, \\ x_4 &= 0, & y_4 &= 0.9575669263\dots \end{aligned}$$

□

From Theorem 3, $w_8^* < W_8^* < \overline{W}_8$. A tighter upper bound is obtained by using Theorem 4.

Theorem 4 (Audet, Hansen, and Messine [11]). *For all $n \geq 3$, let $L(P_n)$ denote the perimeter of an n -gon P_n and $W(P_n)$ its width. Then $\frac{W(P_n)}{L(P_n)} \leq \frac{1}{2n} \cot \frac{\pi}{2n}$.*

Since the perimeter of any convex equilateral small octagon, including those of maximal width w_8^* , is bounded above by $L(H_8)$, it follows that

$$w_8^* < \frac{L(H_8)}{16} \cot \frac{\pi}{16} < 0.9727 < W_8^*,$$

with $L(H_8) = 3.0956093174\dots > L(R_8) = 4\sqrt{2 - \sqrt{2}}$.

3.3 Height graph

Let P_8^* be an equilateral small 8-gon of maximal width, i.e., $W(P_8^*) = w_8^*$, and let $\underline{w} := W(F_8)$. If c denotes the sides length of P_8^* then

$$c < \bar{c} := \frac{L(H_8)}{8} < 0.3870.$$

A lower bound on c is deduced from Theorem 4:

$$c \geq 2w_8^* \tan \frac{\pi}{16} \geq 2\underline{w} \tan \frac{\pi}{16} > 0.3794.$$

Next, we study the height graph of the optimal octagon P_8^* . Combining the upper bound on c with the lower bound on w_8^* , one gets that the furthest vertex v_{k_i} to the side $v_{i-1}v_i$ must be the third neighbor of the side. This leads to the following lemma:

Lemma 1. For all $i \in \{1, 2, \dots, 8\}$, $k_i \in \{i + 3, i + 4\}$.

Proof. Since $c < \bar{c} < 0.3870$ and $w_8^* \geq \underline{w} > 0.9537 > 2\bar{c}$, we have $k_i \notin \{i + 1, i + 2, i + 5, i + 6\}$. \square

The next lemma shows that two edges of the optimal height graph cannot be disjoint.

Lemma 2. For all $i \in \{1, 2, \dots, 8\}$, $\min\{\|v_i - v_{i+3}\|, \|v_{i+4} - v_{i-1}\|\} < w_8^*$.

Proof. Without loss of generality, assume that for $i = 1$, $\min\{\|v_1 - v_4\|, \|v_5 - v_0\|\} \geq w_8^* > 0.9537$. It follows that

$$1 \geq \max\{\|v_1 - v_4\|, \|v_5 - v_0\|\} \geq \sqrt{w_8^{*2} + c^2} > \sqrt{0.9537^2 + 0.3794^2} > 1.0263.$$

This contradiction implies that at most one of the two diagonals may exceed w_8^* . \square

Combining Lemma 1 and Lemma 2, one obtains additional conditions that the furthest vertices must satisfy. This leads to the following proposition:

Proposition 2. For all $i \in \{1, 2, \dots, 8\}$, $k_i = i + 3 \Leftrightarrow k_{i+4} = i$ and $k_i = i + 4 \Leftrightarrow k_{i+4} = i - 1$.

Proof. Without loss of generality, assume $i = 1$. From Lemma 1, $k_1 \in \{4, 5\}$.

1. If $k_1 = 4$ then $\|v_1 - v_4\| \geq w_8^*$. It follows, from Lemma 2, $\|v_5 - v_0\| < w_8^*$, which implies $k_5 \neq 0$. Finally, from Lemma 1, $k_5 = 1$.
2. If $k_5 = 1$ then $\|v_1 - v_4\| \geq w_8^*$. It follows, from Lemma 2, $\|v_5 - v_0\| < w_8^*$, which implies $k_1 \neq 5$. Finally, from Lemma 1, $k_1 = 4$.

A similar reasoning yields $k_1 = 5 \Leftrightarrow k_5 = 0$. \square

The next corollary details the structure of the optimal height graph.

Corollary 1. The height graph of P_8^* has either a cycle of length 7 with one pendant edge, or a cycle of length 5 with three pendant edges from three non-consecutive vertices of the cycle.

Proof. The result follows from Proposition 2. \square

From Corollary 1, there are only two possible height graphs of P_8^* , both represented in Figure 7 and given by the vector of furthest vertices $\mathbf{k} = (k_1, k_2, \dots, k_8)$: k_i is the index of the furthest vertex to the side $v_{i-1}v_i$. The first configuration $\mathbf{k}^a = (5, 6, 7, 0, 0, 1, 2, 3)$ in Figure 7a has a cycle of length 7 with one pendant edge and the second $\mathbf{k}^b = (5, 5, 6, 0, 0, 2, 3, 3)$ in Figure 7b has a cycle of length 5 with three pendant edges.

In both configurations \mathbf{k}^a and \mathbf{k}^b , there exists an index $i \in \{1, 2, \dots, 8\}$ such that $k_i = i + 3$ and $k_{i+3} = i - 1$. This property allows to further tighten bounds of some attributes of the optimal octagon in Proposition 3.

Proposition 3. For all $i \in \{1, 2, \dots, 8\}$, if $k_i = i + 3$ and $k_{i+3} = i - 1$ then $c > 0.3844$, $w_8^* < 0.9564$, and $\max\{\|v_{i-1} - v_{i+2}\|, \|v_i - v_{i+3}\|\} < 0.9644$.

Proof. Without any loss of generality, assume that, for $i = 1$, $k_1 = 4$ and $k_4 = 0$. Let $a := \|v_0 - v_3\|$, $b := \|v_1 - v_4\|$, and $d := \|v_0 - v_4\|$. From the triangles $v_0v_3v_4$ and $v_4v_0v_1$, we have

$$\begin{aligned} a^2 &= d^2 + c^2 - 2c\sqrt{d^2 - h_4^2}, \\ b^2 &= d^2 + c^2 - 2c\sqrt{d^2 - h_1^2}, \end{aligned}$$

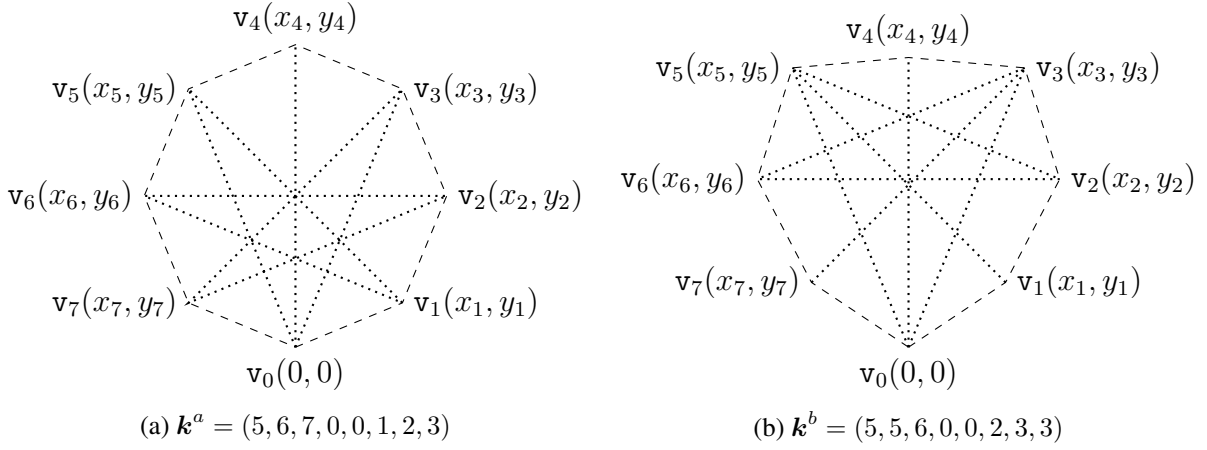


Figure 7: Possible optimal height graphs $\mathbf{k} = (k_1, k_2, \dots, k_8)$: (a) Configuration \mathbf{k}^a ; (b) Configuration \mathbf{k}^b

respectively. Therefore

$$\min\{a, b\} \geq \sqrt{d^2 + c^2 - 2c\sqrt{d^2 - w_8^{*2}}}.$$

Applying Ptolemy's inequality to the quadrilateral $v_0v_1v_3v_4$ yields

$$ab \leq c^2 + \|v_1 - v_3\|d.$$

Since $\|v_1 - v_3\| \leq 2c$, it follows that

- $d^2 + c^2 - 2c\sqrt{d^2 - w_8^{*2}} \leq c^2 + 2cd \Rightarrow c \geq \frac{d^2}{2d+2\sqrt{d^2-w_8^{*2}}} \geq \frac{1}{2+2\sqrt{1-w^2}} > 0.3844$,
- $d^2 + c^2 - 2c\sqrt{d^2 - w_8^{*2}} \leq c^2 + 2cd \Rightarrow w_8^* \leq \frac{d\sqrt{4cd-d^2}}{2c} \leq \frac{\sqrt{4\bar{c}-1}}{2\bar{c}} < 0.9564$,
- $\max\{a, b\} \leq \frac{c^2+2cd}{\sqrt{d^2+c^2-2c\sqrt{d^2-w_8^{*2}}}} \leq \frac{\bar{c}^2+2\bar{c}}{\sqrt{1+\bar{c}^2-2\bar{c}\sqrt{1-w^2}}} < 0.9644$.

□

3.4 Numerical results

Now, mathematical optimization tools are used to show that the octagon F_8 is optimal. For $n = 8$, the equilateral case of the maximal width problem can be formulated as follows:

$$\max_{\mathbf{x}, \mathbf{y}, c, \mathbf{h}, w} w \tag{4a}$$

$$\text{s. t. } (x_j - x_i)^2 + (y_j - y_i)^2 \leq 1 \quad \forall 1 \leq i < j \leq 7, \tag{4b}$$

$$x_i^2 + y_i^2 \leq 1 \quad \forall 1 \leq i \leq 7, \tag{4c}$$

$$y_i \geq 0 \quad \forall 1 \leq i \leq 7, \tag{4d}$$

$$x_i y_{i+1} - y_i x_{i+1} \geq 0 \quad \forall 1 \leq i \leq 6, \tag{4e}$$

$$c^2 = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 \quad \forall 1 \leq i \leq 8, \tag{4f}$$

$$w \leq h_i \quad \forall 1 \leq i \leq 8, \tag{4g}$$

$$ch_i = (x_{i-1} - x_{k_i})(y_i - y_{k_i}) - (y_{i-1} - y_{k_i})(x_i - x_{k_i}) \quad \forall 1 \leq i \leq 8, \tag{4h}$$

$$x_4 = 0, \tag{4i}$$

$$0.384462 \leq c \leq 0.386952, \tag{4j}$$

$$w \geq 0.953776. \tag{4k}$$

Problem (4) was solved on the NEOS Server 6.0 using AMPL with Couenne 0.5.8. AMPL codes have been made available at <https://github.com/cbingane/optigon>. The solver Couenne [12] is a branch-and-bound algorithm that aims at finding global optima of nonconvex mixed-integer nonlinear optimization problems. The results support the following keypoints:

- Under configuration $\mathbf{k}^a = (5, 6, 7, 0, 0, 1, 2, 3)$, Couenne certified in 0.32 s that Problem (4) is infeasible, i.e., there is no equilateral small octagon under configuration \mathbf{k}^a having a width greater than \underline{w} .
- Under configuration $\mathbf{k}^b = (5, 5, 6, 0, 0, 2, 3, 3)$, Couenne solved Problem (4) in 3.18 s. The optimal value obtained is

$$w^* = 0.95377630014\dots$$

and the vertices of the optimal octagon are

$$\begin{aligned} x_0^* &= 0, & y_0^* &= 0, \\ x_1^* &= 0.320810 &= -x_7^*, & y_1^* = 0.214000 &= y_7^*, \\ x_2^* &= 0.500000 &= -x_6^*, & y_2^* = 0.555477 &= y_6^*, \\ x_3^* &= 0.384110 &= -x_5^*, & y_3^* = 0.923288 &= y_5^*, \\ x_4^* &= 0, & y_4^* &= 0.957567. \end{aligned}$$

Within the limit of the numerical computations, this corresponds to the octagon F_8 .

Proposition 4. *The width of an equilateral small octagon does not exceed the value $\underline{w} + 10^{-6}$.*

Proof. By replacing the lower bound $w \geq 0.953776$ on the variable w by $w \geq 0.953777$ in Problem (4), Couenne certified in 2.05 s that there is no equilateral small octagon whose width exceeds the value 0.953777. \square

Let \mathbf{z}^* be an optimal solution of a maximization problem

$$\max_{\mathbf{z} \in \Omega} g(\mathbf{z}),$$

where $\Omega \subset \mathbb{R}^n$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}$. For $\varepsilon^* \geq 0$, $\mathbf{z}^* \in \Omega$ is said ε^* -unique if

$$g(\mathbf{z}^*) > \max_{\mathbf{z} \in \Omega, \|\mathbf{z} - \mathbf{z}^*\| \geq \varepsilon} g(\mathbf{z})$$

for any $\varepsilon \geq \varepsilon^*$. This definition was introduced in [4] to show that H_8 is the near-unique convex equilateral small octagon of maximal perimeter.

Proposition 5. *The optimal octagon F_8 is ε^* -unique, with $\varepsilon^* = 1.5 \times 10^{-4}$, using the Euclidian norm.*

Proof. Let (x_i^*, y_i^*) , $i = 0, 1, \dots, 7$, be the vertices coordinates of F_8 . Solving with Couenne showed in 2.80 s that adding the constraint

$$\sum_{i=1}^7 (x_i - x_i^*)^2 + (y_i - y_i^*)^2 \geq 2.25 \times 10^{-8}$$

to Problem (4) makes it infeasible due to the lower bound on the variable w . \square

4 Bounds on the maximal width of equilateral small 2^s -gons with integer $s \geq 4$

This section provides a family of equilateral small n -gons, with $n = 2^s$ and $s \geq 4$, whose widths are within $O(1/n^4)$ of the optimal value.

For any $n = 2^s$ with $s \geq 4$, consider the n -gon G_n , illustrated in Figure 8 for $n = 16$ and $n = 32$. Its height graph has the edge $v_0 - v_{\frac{n}{2}}$ as axis of symmetry and can be described by the $(3n/4 - 1)$ -length cycle $v_0 - v_{\frac{n}{2}-1} - \dots - v_{\frac{3n}{4}+1} - v_{\frac{n}{4}} - v_{\frac{3n}{4}} - v_{\frac{n}{4}-1} - \dots - v_{\frac{n}{2}+1} - v_0$ and the pendant edges $v_0 - v_{\frac{n}{2}}$, $v_{4k-1} - v_{4k-1+\frac{n}{2}}$, $k = 1, \dots, n/8$. The edges in dotted lines have all the same length d .

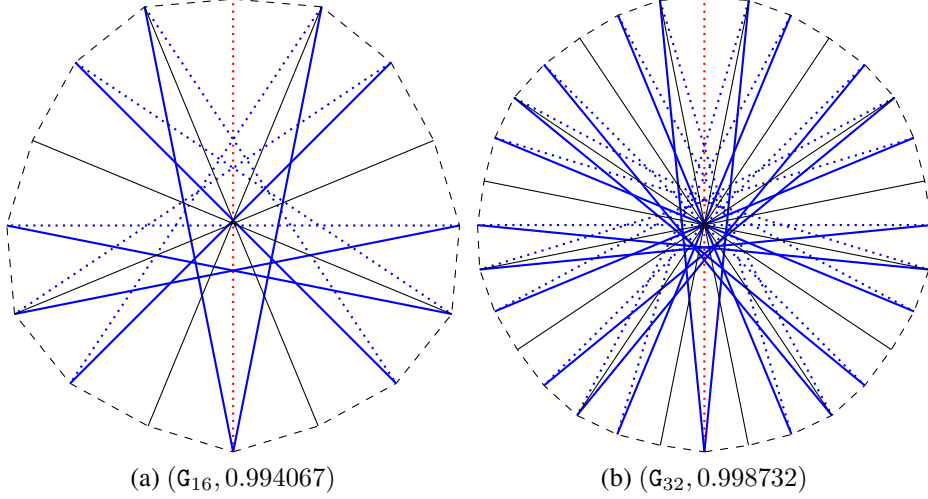


Figure 8: Polygons $(G_n, W(G_n))$ defined in Theorem 2: (a) Hexadecagon G_{16} ; (b) Triacontadigon G_{32}

Place the vertex $v_{\frac{n}{2}}$ at $(0, d)$ in the plane. Let $\alpha := \angle v_{\frac{n}{2}} v_0 v_{\frac{n}{2}-1}$ and $\beta := \angle v_0 v_{\frac{n}{2}-1} v_{n-1}$. Since G_n is equilateral, we have $\|v_{\frac{n}{2}} - v_{\frac{n}{2}-1}\| = \|v_0 - v_{n-1}\|$, which implies

$$1 + d^2 - 2d \cos \alpha = 2 - 2 \cos \beta. \quad (5)$$

Since G_n is symmetric, we have from the cycle $v_0 - \dots - v_{\frac{n}{4}} - v_{\frac{3n}{4}} - \dots - v_0$,

$$\frac{n}{8}(3\alpha + \beta) = \frac{\pi}{2}. \quad (6)$$

Since the edge $v_{\frac{n}{4}} - v_{\frac{3n}{4}}$ is horizontal and $\|v_{\frac{n}{4}} - v_{\frac{3n}{4}}\| = d$, we also have $x_{\frac{n}{4}} = \frac{d}{2} = -x_{\frac{3n}{4}}$, which yields after simplifications,

$$d = \frac{2 \cos(2\alpha + \beta) + 1}{2 \cos \alpha + \cos(3\alpha + \beta)}. \quad (7)$$

An asymptotic analysis produces that, for large n , the system of equations (5), (6), and (7) has a solution $(\alpha_0(n), \beta_0(n), d_0(n))$ satisfying

$$\begin{aligned} \alpha_0(n) &= \frac{\pi}{n} + \frac{\pi^5}{6n^5} + \frac{\pi^7}{12n^7} + O\left(\frac{1}{n^9}\right), \\ \beta_0(n) &= \frac{\pi}{n} - \frac{\pi^5}{2n^5} - \frac{\pi^7}{4n^7} + O\left(\frac{1}{n^9}\right), \\ d_0(n) &= 1 - \frac{4\pi^4}{3n^4} - \frac{7\pi^6}{3n^6} + O\left(\frac{1}{n^8}\right). \end{aligned}$$

Table 1: Widths of G_n

n	$W(R_n)$	$W(F_n)$	$W(G_n)$	\overline{W}_n	$\frac{W(G_n) - W(R_n)}{\overline{W}_n - W(R_n)}$
16	0.9807852804	0.9915310059	0.9940673080	0.9951847267	0.9224
32	0.9951847267	0.9983271244	0.9987316811	0.9987954562	0.9823
64	0.9987954562	0.9996398418	0.9996949197	0.9996988187	0.9957
128	0.9996988187	0.9999173147	0.9999244595	0.9999247018	0.9989
256	0.9999247018	0.9999802514	0.9999811602	0.9999811753	0.9997

By setting $(\alpha, \beta, d) = (\alpha_0(n), \beta_0(n), d_0(n))$, the width of G_n is

$$W(G_n) = \frac{d_0(n) \sin \alpha_0(n)}{2 \sin(\beta_0(n)/2)} = 1 - \frac{\pi^2}{8n^2} - \frac{85\pi^4}{128n^4} - \frac{92801\pi^6}{46080n^6} + O\left(\frac{1}{n^8}\right)$$

and

$$\begin{aligned} \overline{W}_n - W(G_n) &= \frac{2\pi^4}{3n^4} + \frac{145\pi^6}{72n^6} + O\left(\frac{1}{n^8}\right), \\ W(G_n) - W(R_n) &= \frac{3\pi^2}{8n^2} - \frac{271\pi^4}{384n^4} + O\left(\frac{1}{n^6}\right). \end{aligned}$$

By construction, G_n is small. This completes the proof of Theorem 2. \square

All polygons presented in this work and in [9, 10, 13, 14] were implemented as a MATLAB package: OPTIGON, which is freely available on GitHub [15]. One can also find an algorithm developed in [16] to find an estimate of the maximal area of a small n -gon when $n \geq 6$ is even.

Table 1 gives the widths of G_n , along with the upper bounds \overline{W}_n and the widths of R_n . We also report the widths of F_n , illustrated in Figure 9. Polygons F_n can be seen as a generalization of the optimal 8-gon F_8 and we can show that

$$\overline{W}_n - W(F_n) = \frac{\pi^3}{2n^3} - \frac{5\pi^5}{8n^5} + O\left(\frac{1}{n^6}\right)$$

for all $n = 2^s$ and $s \geq 3$. The height graph of F_n in Figure 9 has the vertical edge as axis of symmetry and can be described by a cycle of length $n/2 + 1$, plus $n/2 - 1$ additional pendant edges, arranged so that all but two particular vertices of the cycle have a pendant edge. In Table 1, G_n provides a tighter lower bound on the maximal width w_n^* compared to R_n or F_n . The fraction $\frac{W(G_n) - W(R_n)}{\overline{W}_n - W(R_n)}$ of the length of the interval $[W(R_n), \overline{W}_n]$ containing $W(G_n)$ shows that $W(G_n)$ approaches \overline{W}_n much faster than $W(R_n)$ does as n increases. Indeed, $W(G_n) - W(R_n) \sim 3\pi^2/(8n^2)$ for large n .

5 Conclusion

We found with precision 10^{-6} the equilateral small octagon of maximal width and showed that there is no other optimal octagon outside a ball of radius 1.5×10^{-4} . In addition, for each $n = 2^s$ where $s \geq 4$ is an integer, we constructed an equilateral small n -gon whose width is within $2\pi^4/(3n^4) + O(1/n^6)$ of the maximal width and exceeds that of the regular polygon.

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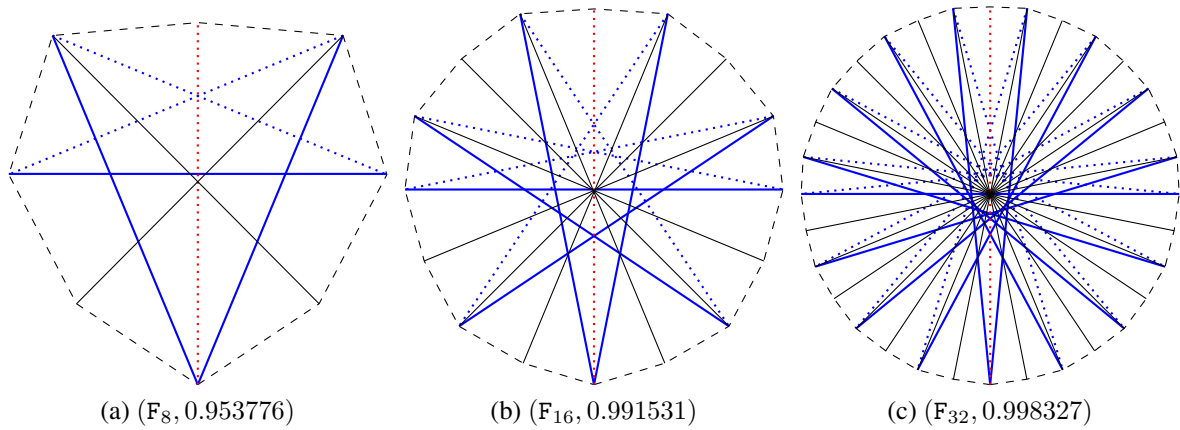


Figure 9: A generalization of the optimal octagon $(F_n, W(F_n))$: (a) Optimal octagon F_8 ; (b) Hexadecagon F_{16} ; (c) Triacontadigon F_{32}

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