On Componental Operators in Hilbert Space

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Abstract

We consider a Hilbert space that is a product of a finite number of Hilbert spaces and operators that are represented by "componental operators" acting on the Hilbert spaces that form the product space. We attribute operatorial properties to the componental operators rather than to the full operators. The operatorial properties that we discuss include nonexpansivity, firm nonexpansivity, relaxed firm nonexpansivity, averagedness, being a cutter, quasi-nonexpansivity, strong quasi-nonexpansivity, strict quasi-nonexpansivity and contraction.

Some relationships between operators whose componental operators have such properties and operators that have these properties on the product space are studied. This enables also to define componental fixed point sets and to study their properties. For componental contractions we offer a variant of the Banach fixed point theorem.

Our motivation comes from the desire to extend a fully-simultaneous method that takes into account sparsity of the linear system in order to accelerate convergence [Censor et al., On diagonally relaxed orthogonal projection methods, *SIAM J. Sci. Comput.* **30** (2008), 473–504]. This was originally applicable to the linear case only and gives rise to an iterative process that uses different componental operators during iterations.

1 Introduction

Let $\mathcal{H}_j$, $j \in J := \{1, 2, ..., n\}$ be real Hilbert spaces and $\mathcal{H} : = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n$ for $n \geqslant 2$. Consider an operator $T : \mathcal{H} \to \mathcal{H}$ that maps the Hilbert space $\mathcal{H}$ into itself and is represented by

$$T(x) = ((T(x))_1, (T(x))_2, ..., (T(x))_n), \tag{1}$$

where, for all $j \in J$, $(T(\cdot))_j : \mathcal{H} \to \mathcal{H}_j$ denotes "the j-th component operator of $T(\cdot)$".

In this paper we attribute operatorial properties to the component operators of an operator T. To explain what we mean, let us take, by way of example, the well-known nonexpansivness property. We define that the operator T is "*j-nonexpansive* (j-NE)" if for all $x, y \in \mathcal{H}$

$$\|(T(x))_j - (T(y))_j\|_j \leq \|x_j - y_j\|_j, \tag{2}$$

for some $j \in J$, where $\|\cdot\|_j$ is the norm in $\mathcal{H}_j$. If T is j-NE for all $j \in J$ then we say that T is "component-wise NE (CW-NE)".

Obviously, such definitions imply that if T is CW-NE then it is NE but not vice versa, making the set of CW-NE operators a proper subset of the NE operators. In this sense, a CW-NE operator is "stronger" than an NE operator which is not necessarily CW-NE.

We look at various operatorial properties of the components operators of an operator T and investigate their relationships with such properties of the operator T itself. To do so, we define the notions of *j-nonexpansive* (j-NE), *j-firmly nonexpansive* (j-FNE), *j-relaxed firmly nonexpansive* (j-RFNE), *j-averaged* (j-AV), *j-cutter, j-quasi-nonexpansive* (j-QNE), (ρ_j, j)*-strongly quasi-nonexpansive* ((ρ_j, j)-SQNE), *j-strictly quasi-nonexpansive* (j-sQNE) and (α_j, j)*-contraction* ((α_j, j)-CONT) and their associated "CW-X" properties, where "X" stands for any of the, above mentioned, properties: NE, FNE, RFNE, AV, cutter, QNE, SQNE, sQNE or CONT. To say that T is CW-X means that it is "j-X" for all $j \in J$.

For $j \in J$ we also define the "j-th fixed point set of T" by

$$\mathrm{Fix}^j T := \{z \in \mathcal{H} \mid (T(z))_j = z_j\} \tag{3}$$

and study its properties.

Our motivation comes from a desire to study an iterative process that uses, as the iterations proceed, different components operators of the operator T. We devote below a special section to describe this motivating topic.

Working with "blocks" of variables is a common and wide-ranging subject in the study of iterative processes. An early approach to analyzing iterative processes over Cartesian product sets and "component solution methods" can be found in the book of Bertsekas and Tsitsiklis [6, Subsection 3.1.2]. A product space formulation, which became classical by now, for feasibility-seeking and optimization problems, is the work of Pierra [14]. A kind of stochastic component solution methods were studied in [12].

Here we set forth a general framework that fits a large variety of operatorial properties. It remains to be discovered whether the subsets of operators of the form "CW-X", for any of the above mentioned properties "X", can generate stronger results in fixed point theory due to the fact that a CW-X operator is "stronger" than its associated operator with property X which is not CW-X.

The paper is structured as follows. In Subsections 2.1, 2.2 we develop our framework of componental properties of operators. In Subsection 2.3 we define α_j*-contractions* and formulate the Banach fixed point theorem for them. In Section 3 we define and study the componental regularity property, and in Section 4 we present and analyze our motivating case of extending the fully-simultaneous Diagonally-Relaxed Orthogonal Projections (DROP) method of [10], originally applicable to the linear case only.

2 Componental operators

Let $\mathcal{H}_j$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_j$ and induced norm $\| \cdot \|_j$, $j \in J := \{1, 2, ..., n\}$. Define the product Hilbert space $\mathcal{H} := \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ with inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{j=1}^{n} \langle x_j, y_j \rangle_j$ and the induced norm $\| \cdot \| : \mathcal{H} \to \mathbb{R}$ defined by $\|x\| := \sqrt{\sum_{j=1}^{n} \|x_j\|_j^2}$, where $x = (x_1, x_2, ..., x_n) \in \mathcal{H}$, $y = (y_1, y_2, ..., y_n) \in \mathcal{H}$ with $x_j, y_j \in \mathcal{H}_j$, $j \in J$. Let $T : \mathcal{H} \to \mathcal{H}$, i.e.,

$$T(x) = ((T(x))_1, (T(x))_2, ..., (T(x))_n), \tag{4}$$

where $(T(x))_j \in \mathcal{H}_j$ denotes "the j-th component operator of $T(x)$", $j \in J$. Denote by $T_\lambda := \mathrm{Id} + \lambda(T - \mathrm{Id})$ the λ-relaxation of T, where $\lambda \geq 0$ and Id denotes the identity.

2.1 Componental nonexpansive, firmly nonexpansive, relaxed firmly nonexpansive and averaged operators

Bearing in mind the well-known definitions of operators that are *nonexpansive* (NE), *firmly nonexpansive* (FNE), *relaxed firmly nonexpansive* (RFNE), or *averaged* (AV), see, e.g., [8] or [3], we introduce the following new definitions. We refer to any of these types of operators by the general name of "*componental operators*".

Definition 1 Let $j \in J$. We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is:

(i) *j-nonexpansive* (j-NE) if for all $x, y \in \mathcal{H}$

$$\|(T(x))_j - (T(y))_j\|_j \leq \|x_j - y_j\|_j; \tag{5}$$

(ii) *j-firmly nonexpansive* (j-FNE) if for all $x, y \in \mathcal{H}$

$$\langle (T(x))_j - (T(y))_j, x_j - y_j \rangle_j \geq \|(T(x))_j - (T(y))_j\|_j^2; \tag{6}$$

(iii) *j-relaxed firmly nonexpansive* (j-RFNE), if there exist a constant $\lambda \in [0, 2]$ and a j-FNE operator $U : \mathcal{H} \to \mathcal{H}$ such that $(T(\cdot))_j$ is a λ-relaxation of $(U(\cdot))_j$;

(iv) *j-averaged* (j-AV), if there exist a constant $\alpha \in (0, 1)$ and a j-NE operator U such that

$$(T(x))_j = (1 - \alpha)x_j + \alpha(U(x))_j; \tag{7}$$

(v) *component-wise NE* (CW-FNE, CW-RFNE, CW-AV) if T is j-NE (j-FNE, j-RFNE, j-AV) for all $j \in J$.

Alternatively, if we want to emphasize the constants λ in (iii) and α in (iv) explicitly, then we say that T is (λ, j)-relaxed firmly nonexpansive ((λ, j)-RFNE) in (iii) and (α, j)-averaged ((α, j)-AV) in (iv).

Clearly, a CW-NE (CW-FNE) operator is NE (FNE). Note, however, that the converse is not true.

Example 2 The operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ is NE but not CW-NE.

Fact 3 *Let $j \in J$ and let T be j-NE. Let $x, y \in \mathcal{H}$ be such that $x_j = y_j$. Then*

$$(T(x))_j = (T(y))_j. \tag{8}$$

Thus, one can define an operator $T^j : \mathcal{H}_j \to \mathcal{H}_j$ by

$$T^j(x_j) := (T(x))_j. \tag{9}$$

Proof. Equality (8) follows directly from Definition 1. ■

Fact 3 has a consequence that the component operators of a CW-NE operator T are NE operators.

Fact 4 *If an operator $T : \mathcal{H} \to \mathcal{H}$ is CW-NE, then for any $x \in \mathcal{H}$ it holds*

$$T(x) = (T^1(x_1), T^2(x_2), ..., T^n(x_n)) \tag{10}$$

and $T^j, j \in J$, are NE. Conversely, if the operators $T^j : \mathcal{H}_j \to \mathcal{H}_j$, $j \in J$, are NE then the operator T defined by (10) is CW-NE.

Proof. In view of (4), equality (10) follows from Fact 3. ■

The following corollary shows that we can set a common constant $\alpha \in (0, 1)$ and a common CW-NE operator U in the definition of a CW-AV operator T.

Corollary 5 *An operator $T : \mathcal{H} \to \mathcal{H}$ is CW-AV if and only if there is a constant $\alpha \in (0, 1)$ and a CW-NE operator U such that $T = V_\alpha := (1 - \alpha) \operatorname{Id} + \alpha U$.*

Proof. The "if" part is obvious. Suppose that T is CW-AV. Then for any $j \in J$ there are $\alpha_j \in (0, 1)$ and CW-NE operators $U_j : \mathcal{H} \to \mathcal{H}$ such that for all $x \in \mathcal{H}$ it holds

$$(T(x))_j = (1 - \alpha_j)x_j + \alpha_j(U_j(x))_j. \tag{11}$$

By Fact 4, $U_j(x) = (U_j^1(x_1), U_j^2(x_2), ..., U_j^n(x_n))$, $j \in J$, $x \in \mathcal{H}$, and all operators $U_j^i : \mathcal{H}_i \to \mathcal{H}_i$, $i, j \in J$, defined by $U_j^i(x_i) := (U_j(x))_i$ are NE. Let $\alpha := \max_{j\in J} \alpha_j$ and $\beta_j := \alpha_j/\alpha \in (0, 1]$. Define $V_j := (U_j^j)_{\beta_j} = (1 - \beta_j) \operatorname{Id} + \beta_j U_j^j$, $j \in J$. Clearly, $U_j^j = (V_j)_{\beta_j^{-1}}$, $j \in J$. Define $V : \mathcal{H} \to \mathcal{H}$ by $V(x) = (V_1(x_1), V_2(x_2), ..., V_n(x_n))$, $x \in \mathcal{H}$. The operators V_j, $j \in J$, are NE as convex combinations of NE operators U_j^j and Id. Again, by Fact 4, V is CW-NE. We have

$$(T(x))_j = (U_j^j)_{\alpha_j}(x_j) = ((V_j)_{\beta_j^{-1}})_{\alpha_j}(x_j) = (V_j)_\alpha(x_j) = (V_\alpha(x))_j. \tag{12}$$

Thus, $T = V_\alpha$, where V is CW-NE and $\alpha \in (0, 1)$.

In a similar way one can prove that we can set a common constant $\lambda \in [0, 2]$ and a common CW-FNE operator U in the definition of a CW-RFNE operator T. This yields that a CW-RFNE operator is RFNE. ■

Fact 6 *Let $j \in J$ and let $T : \mathcal{H} \to \mathcal{H}$. The following conditions are equivalent:*

(i) *T is j-FNE;*

(ii) *T_λ is j-NE for all $\lambda \in [0, 2]$;*

(iii) *T has the form $T = \frac{1}{2}(\operatorname{Id} + S)$ for some j-NE operator S, i.e., T is $(\frac{1}{2}, j)$-AV;*

(iv) *$\operatorname{Id} - T$ is j-FNE;*

(v) *The following inequality holds for all* $x, y \in \mathcal{H}$

$$\|(T(x))_j - (T(y))_j\|_j^2 \le \|x_j - y_j\|_j^2 - \|(x_j - (T(x))_j) - (y_j - (T(y))_j)\|_j^2. \quad (13)$$

Proof. Similar to [8, Theorem 2.2.10]. ■

Denote $I := \{1, 2, ..., m\}$, where $m \in \mathbb{N}$.

Corollary 7 *Let* $j \in J$*,* $T_i : \mathcal{H} \to \mathcal{H}$ *be* j*-FNE, for all* $i \in I$ *and* $w = (w_1, w_2, ..., w_m) \in \Delta_m := \{w \in \mathbb{R}^m \mid w \ge 0, \sum_{i=1}^m w_i = 1\}$*. Then the operator* $T : \mathcal{H} \to \mathcal{H}$*, defined by* $T(x) := \sum_{i=1}^m w_i T_i(x)$*, is* j*-FNE.*

Proof. Similar to [8, Corollary 2.2.20]. ■

For a j-NE operator $T_i : \mathcal{H} \to \mathcal{H}$, the operators $T_i^j : \mathcal{H}_j \to \mathcal{H}_j$ are defined in a similar way as in (9), i.e.,

$$T_i^j(x_j) := (T_i(x))_j, \; i \in I, \; j \in J. \quad (14)$$

Fact 8 *Let* $j \in J$*,* $w_{ij} \ge 0$*,* $i \in I$*,* $w_{\cdot j} := \sum_{i=1}^m w_{ij} > 0$*. If* $T_i : \mathcal{H} \to \mathcal{H}$*,* $i \in I$*, are* j*-FNE, then the operator* $T : \mathcal{H} \to \mathcal{H}$ *defined by its components* $T^j : \mathcal{H}_j \to \mathcal{H}_j$*,*

$$T^j(x_j) := x_j + \lambda \sum_{i=1}^m w_{ij}(T_i^j(x_j) - x_j), \quad (15)$$

where $\lambda \in [0, 2/w_{\cdot j}]$*, is* $(\lambda w_{\cdot j}, j)$*-RFNE.*

Proof. Clearly, T_i^j are FNEs, $i \in I$. Define an operator $S_j : \mathcal{H}_j \to \mathcal{H}_j$ by

$$S_j(x_j) := \sum_{i=1}^m v_i T_i^j(x_j), \quad (16)$$

where $v_i := \frac{w_{ij}}{w_{\cdot j}}$. Since $\sum_{i=1}^m v_i = 1$, the operator S_j is FNE, see [8, Corollary 2.2.20], and we have that $T^j = (S_j)_{\lambda w_{\cdot j}}$, the $(\lambda w_{\cdot j})$-relaxation of S_j, i.e., T^j is $(\lambda w_{\cdot j}, j)$-RFNE. ■

Corollary 9 *Let* $w_{ij} \ge 0$*,* $w_{\cdot j} := \sum_{i=1}^m w_{ij} > 0$*,* $i \in I$*,* $j \in J$*, and* $w := \max_{j \in J} w_{\cdot j}$*. If* $T_i : \mathcal{H} \to \mathcal{H}$*,* $i \in I$*, are CW-FNE, then the operator* T *defined by*

$$(T(x))_j := x_j + \lambda \sum_{i=1}^m w_{ij}((T_i(x_j))_j - x_j), \quad (17)$$

$j \in J$*, where* $\lambda \in [0, 2/w]$*, is* (λw)*-CW-RFNE.*

Proof. By Facts 4 and 8, T^j is $\lambda w_{\cdot j}$-RFNE, $j \in J$. Since $0 \leq \lambda w_{\cdot j} \leq \lambda w \leq 2$, T^j is λw-RFNE, $j \in J$. Noting that $T(x) = (T^1(x_1), T^2(x_2), ..., T^n(x_n))$, tells us that T is λw-RFNE. ■

Example 10 Consider a consistent system of linear equations $Ax = b$, where A is an $m \times n$ matrix with rows $a_i \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Consider the hyperplanes $H_i := \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle = b_i\}$, $i \in I$. A special case of an operator T defined by (17), occurs when $T_i := P_{H_i}$, the metric projection onto H_i, $w_{ij} := w_i/s_j$, $i \in I$, $i \in J$, with $w = (w_1, w_2, ..., w_m) \in \Delta_m$ and s_j being the number of nonzero elements in the j-th column of A. This special case, introduced and investigated in [10, Equation (1.11)], is our motivating example, discussed in Section 4 below.

2.2 Componental cutter, quasi-nonexpansive, strongly quasi-nonexpansive and strictly quasi-nonexpansive operators

Let $T : \mathcal{H} \to \mathcal{H}$ be defined by (4). For $j \in J$ we define the "j-th fixed point set of T" by

$$\operatorname{Fix}^j T := \{z \in \mathcal{H} \mid (T(z))_j = z_j\}. \tag{18}$$

Clearly, $\operatorname{Fix} T = \bigcap_{j \in J} \operatorname{Fix}^j T$.

Bearing in mind the well-known definitions of operators that are *cutter, quasi-nonexpansive* (QNE), *strongly quasi-nonexpansive* (SQNE), or *strictly quasi-nonexpansive* (sQNE), see, e.g., [8], we introduce the following additional new definitions of componental operators.

Definition 11 Let $j \in J$ and let $T : \mathcal{H} \to \mathcal{H}$ be an operator with $\operatorname{Fix}^j T \neq \emptyset$. We say that T is:

(i) a *j-cutter*, if for all $x \in H$ and $z \in \operatorname{Fix}^j T$ it holds that

$$\langle x_j - (T(x))_j, z_j - (T(x))_j\rangle_j \leq 0; \tag{19}$$

(ii) *j-quasi-nonexpansive* (j-QNE), if for all $x \in H$ and $z \in \operatorname{Fix}^j T$ it holds that

$$\|(T(x))_j - z_j\|_j \leq \|x_j - z_j\|_j; \tag{20}$$

(iii) *j-strongly quasi-nonexpansive* (j-SQNE), if there is a constant $\rho_j > 0$ such that for all $x \in H$ and $z \in \operatorname{Fix}^j T$ it holds that

$$\|(T(x))_j - z_j\|_j^2 \leq \|x_j - z_j\|_j^2 - \rho_j\|(T(x))_j - x_j\|_j^2; \tag{21}$$

(iv) *j-strictly quasi-nonexpansive* (j-sQNE), if for all $x \notin \operatorname{Fix}^j T$ and $z \in \operatorname{Fix}^j T$ it holds that

$$\|(T(x))_j - z_j\|_j < \|x_j - z_j\|_j; \tag{22}$$

(v) a *component-wise cutter* (CW-QNE, CW-SQNE, CW-sQNE) if T is j-cutter (j-QNE, j-SQNE, j-sQNE) for all $j \in J$.

Alternatively, if we want to emphasize the constant ρ_j in (iii) or the vector $r := (\rho_1, \rho_2, ..., \rho_n)$ in (v) explicitly, then we say that T is (ρ_j, j)-SQNE or T is r-CW-SQNE.

Note that if T is (ρ_j, j)-SQNE and $\rho_j \geq \rho > 0$ then T is (ρ, j)-SQNE. Thus, one can set the constant ρ in (v), which does not depend on j, e.g., $\rho := \min_{j \in J} \rho_j$.

Example 12 Let $U^j : \mathcal{H}_j \to \mathcal{H}_j$, $j \in J$, and define $U : \mathcal{H} \to \mathcal{H}$ by

$$U(x) := (U^1(x_1), U^2(x_2), ..., U^n(x_n)). \tag{23}$$

Clearly, $\operatorname{Fix} U = \prod_{j=1}^{n} \operatorname{Fix} U^j$. By definition, $(U(x))_j = U^j(x_j)$. Thus, it follows from Definitions 1 and 11 that U^j is NE (a cutter, QNE, SQNE), if and only if U is j-NE (j-cutter, j-QNE, j-SQNE), $j \in J$. Consequently, U^j are NE (a cutter, QNE, SQNE) for all $j \in J$, if and only if U is CW-NE (CW-cutter, CW-QNE, CW-SQNE).

Example 13 Let $S_j : \mathcal{H} \to \mathcal{H}$, $j \in J$, and define $S : \mathcal{H} \to \mathcal{H}$ by

$$S(x) := ((S_1(x))_1, (S_2(x))_2, ..., (S_n(x))_n). \tag{24}$$

Then $(S(x))_j = (S_j(x))_j$, $j \in J$. Consequently, S is a CW-cutter (r-CW-SQNE, where $r := (\rho_1, \rho_2, ..., \rho_n)$ with $\rho_j > 0$, $j \in J$) if and only if S_j is a j-cutter ((ρ_j, j)-SQNE) for all $j \in J$.

For $j \in J$ and $T : \mathcal{H} \to \mathcal{H}$ define

$$F^j(T) := \{z \in \mathcal{H} \mid \|(T(x))_j - z_j\|_j \leq \|x_j - z_j\|_j \text{ for all } x \in \mathcal{H}\} \tag{25}$$

and

$$F_j(T) := \{z_j \in \mathcal{H}_j \mid \|(T(x))_j - z_j\|_j \leq \|x_j - z_j\|_j \text{ for all } x \in \mathcal{H}\}. \tag{26}$$

Clearly,

$$F_j(T) = \bigcap_{x \in \mathcal{H}} \{z_j \in \mathcal{H}_j \mid \|(T(x))_j - z_j\|_j \leq \|x_j - z_j\|_j\}, \tag{27}$$

thus, F_j is a closed convex subset as intersection of closed half-spaces.

For a j-QNE operator T, where $j \in J$, the property expressed in Fact 3 is not true in general. Thus, contrary to an NE operator, a CW-QNE operator cannot be decomposed. Nevertheless, the fixed point set of a CW-QNE operator is a Cartesian product of some sets.

Fact 14 *Let $j \in J$.*

(i) *The following inclusion holds*

$$F^j(T) \subseteq \operatorname{Fix}^j T. \tag{28}$$

(ii) *If T is j-QNE then the converse inclusion is also true. Consequently, for a CW-QNE operator T we have*

$$\operatorname{Fix} T = \bigcap_{j=1}^{n} \operatorname{Fix}^j T = \bigcap_{j=1}^{n} F^j(T) = \prod_{j=1}^{n} F_j(T). \tag{29}$$

(iii) *If $T : \mathcal{H} \to \mathcal{H}$ is a CW-cutter (CW-QNE, CW-SQNE, CW-sQNE), then T is a cutter (QNE, SQNE, sQNE).*

Proof. (i) If $F^j(T) = \emptyset$ then the inclusion in (28) is clear. Let now $F^j(T) \neq \emptyset$ and $z \in F^j(T)$. If we take $x = z$ in (25) then we obtain

$$\|(T(z))_j - z_j\|_j \leq \|z_j - z_j\|_j = 0, \tag{30}$$

i.e., $z \in \operatorname{Fix}^j T$.

(ii) Suppose that T is j-QNE and let $z \in \operatorname{Fix}^j T$. Then, by definition, $z \in F^j(T)$. This together with (i) implies the second equality in (29) if T is CW-QNE. The first and the last equalities in (29) are obvious.

(iii) follows from the definition of the inner product in $\mathcal{H}$ and from the definition of a CW-cutter (CW-QNE, CW-SQNE operator). ■

If we consider a relaxation of an operator $T : \mathcal{H} \to \mathcal{H}$ we can associate different relaxation parameters λ_j with various components $(T(\cdot))_j$. Let $y := (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n_+$. We say that the operator T_y, defined by,

$$(T_y(x))_j := x_j + \lambda_j((T(x))_j - x_j),\ j \in J, \tag{31}$$

is the *y-CW-relaxation* of T or, in short, a *CW-relaxation* of T. Clearly, if $\lambda_j = \lambda$ for all $j \in J$ then the notion of the y-CW-relaxation of T coincides with the notion of the λ-relaxation of T.

Fact 15 *Let $T : \mathcal{H} \to \mathcal{H}$ with $\operatorname{Fix} T \neq \emptyset$, let $y = (\lambda_1, \lambda_2, ..., \lambda_n) \in (0, 2]^n$. If T is a CW-cutter then its y-relaxation T_y, is r-CW-SQNE, where $r := (\rho_1, \rho_2, ..., \rho_n)$ with $\rho_j = (2 - \lambda_j)/\lambda_j$, $j \in J$.*

Proof. Suppose that T is a CW-cutter and let $z \in \operatorname{Fix} T$. For any $j \in J$ we have

$$\begin{aligned}
\|(T_y(x))_j - z_j\|_j^2 &= \|x_j + \lambda_j((T(x))_j - x_j) - z_j\|_j^2 \\
&= \|x_j - z_j\|_j^2 + \lambda_j^2 \|(T(x))_j - x_j\|_j^2 \\
&+ 2\lambda_j \langle x_j - (T(x))_j, z_j - (T(x))_j \rangle_j - 2\lambda_j \|(T(x))_j - x_j\|_j^2 \\
&\leq \|x_j - z_j\|_j^2 - \lambda_j(2 - \lambda_j)\|(T(x))_j - x_j\|_j^2 \\
&= \|x_j - z_j\|_j^2 - \frac{2 - \lambda_j}{\lambda_j} \|(T_\lambda(x))_j - x_j\|_j^2, \qquad (32)
\end{aligned}$$

which means that T_y is r-CW-SQNE. ■

Fact 16 *Let $j \in J$. A j-NE operator having a fixed point is j-QNE. Consequently, a CW-NE operator having a fixed point is CW-QNE.*

Proof. Let T be j-NE and let $z \in \operatorname{Fix}^j T$. Then $(T(z))_j = z_j$ and, for all $x \in \mathcal{H}$, we have

$$\|(T(x))_j - z_j\|_j = \|(T(x))_j - (T(z))_j\|_j \leq \|x_j - z_j\|_j. \qquad (33)$$

This means that T is j-QNE. ■

Fact 17 *Let $j \in J$, let $T_i : \mathcal{H} \to \mathcal{H}$ be j-sQNE, $i \in I$, with $\bigcap_{i \in I} F^j(T_i) \neq \emptyset$, and $T := \sum_{i=1}^m w_i T_i$, where $w \in \operatorname{ri} \Delta_m$ (ri is the relative interior). Then*

$$F^j(T) = \bigcap_{i \in I} F^j(T_i) \qquad (34)$$

and T is j-sQNE.

Proof. To prove the inclusion $\supseteq$ in (34) , let $z \in \bigcap_{i \in I} F^j(T_i)$ and $x \in \mathcal{H}$ be arbitrary. The convexity of the norm and the assumption that T_i are j-QNE, $i \in I$, yield

$$\|(T(x))_j - z_j\|_j \leq \sum_{i=1}^m w_i \|(T_i(x))_j - z_j\|_j \leq \sum_{i=1}^m w_i \|x_j - z_j\|_j = \|x_j - z_j\|_j, \qquad (35)$$

which shows that $z \in F^j(T)$.

To prove the inclusion $\subseteq$ in (34) we observe that the inclusion is clear if $\bigcap_{i \in I} F^j(T_i) = \mathcal{H}$. Suppose the opposite and let $x \notin \bigcap_{i \in I} F^j(T_i)$ and $z \in \bigcap_{i \in I} F^j(T_i)$. The convexity of the norm and the assumption that T_i are j-sQNE, $i \in I$, yield

$$\|(T(x))_j - z_j\|_j \leq \sum_{i=1}^m w_i \|(T_i(x))_j - z_j\|_j < \sum_{i=1}^m w_i \|x_j - z_j\|_j = \|x_j - z_j\|_j, \qquad (36)$$

because $\|(T_{i_0}(x))_j - z_j\|_j < \|x_j - z_j\|_j$ for some i_0, and $w_{i_0} > 0$. Now it is clear that $x \notin \operatorname{Fix}^j T = F^j(T)$, because otherwise,

$$\|x_j - z_j\|_j = \|(T(x))_j - z_j\|_j < \|x_j - z_j\|_j \qquad (37)$$

which would lead to a contradiction. ■

Fact 18 *Let $j \in J$, $T_i : \mathcal{H} \to \mathcal{H}$ be j-cutters having a common fixed point, $i \in I$ and $w \in \operatorname{ri} \Delta_m$. Then the operator $T : \mathcal{H} \to \mathcal{H}$ defined by $T(x) := \sum_{i=1}^m w_i T_i(x)$ is a j-cutter.*

Proof. Similar to [8, Corollary 2.1.49]. ∎

Corollary 19 *Let $T_i : \mathcal{H} \to \mathcal{H}$ be CW-cutters having a common fixed point, $i \in I$, and $w \in \Delta_m$. Then the operator $T : \mathcal{H} \to \mathcal{H}$ defined by $T(x) = \sum_{i=1}^m w_i T_i(x)$ is a CW-cutter. If, moreover, $w_i > 0$, $i \in I$, then $\operatorname{Fix} T = \bigcap_{i \in I} \operatorname{Fix} T_i$.*

Fact 20 *Let $w_{ij} \geq 0$, $i \in I$, $w_{\cdot j} := \sum_{i=1}^m w_{ij} > 0$, $j \in J$. If $T_i : \mathcal{H} \to \mathcal{H}$, $i \in I$, are CW-cutters having a common fixed point then the operator $T : \mathcal{H} \to \mathcal{H}$, defined by*

$$(T(x))_j := x_j + \lambda_j \sum_{i=1}^m w_{ij}((T_i(x))_j - x_j), \tag{38}$$

$j \in J$, where $\lambda_j \in (0, 2/w_{\cdot j})$, is r-CW-SQNE, where $r := (\rho_1, \rho_2, ..., \rho_n)$ with $\rho_j = (2 - \lambda_j)/\lambda_j, j \in J$.

Proof. Define an operator $S_j : \mathcal{H} \to \mathcal{H}$ by

$$S_j(x) := \sum_{i=1}^m \frac{w_{ij}}{w_{\cdot j}} T_i(x),\ j \in J. \tag{39}$$

By Fact 18, S_j is a j-cutter, $j \in J$. Define, for $x \in \mathcal{H}$,

$$S(x) = ((S_1(x))_1, (S_2(x))_2, ..., (S_n(x))_n). \tag{40}$$

The operator $S : \mathcal{H} \to \mathcal{H}$ is a CW-cutter (see Example 13). We have

$$(S(x))_j = (S_j(x))_j = \sum_{i=1}^m \frac{w_{ij}}{w_{\cdot j}} (T_i(x))_j, \tag{41}$$

and

$$(T(x))_j = x_j + \lambda w_{\cdot j}((S(x))_j - x_j). \tag{42}$$

Denote $y = (\lambda w_{\cdot 1}, \lambda w_{\cdot 2}, ..., \lambda w_{\cdot n})$. Equality (42) means that $T = S_y$, i.e., T is a y-CW-relaxation of a CW-cutter S. Thus, Fact 15 yields that the operator T is r-CW-SQNE, where $r := (\rho_1, \rho_2, ..., \rho_n)$ with $\rho_j = (2 - \lambda_j)/\lambda_j, j \in J$. ∎

2.3 Componental contractions

Definition 21 An operator $T : \mathcal{H} \to \mathcal{H}$ is a *j-contraction* (j-CONT) if, for some $j \in J$,

$$\|(T(x))_j - (T(y))_j\|_j \leq \alpha_j \|x_j - y_j\|_j, \text{ for all } x, y \in \mathcal{H}, \tag{43}$$

with an $\alpha_j \in [0,1)$. Alternatively, if we want to emphasize the constant α_j explicitly, then we say that T is (α_j, j)-contraction.

If T is an (α_j, j)-contraction for all $j \in J$ then it is an α-contraction with $\alpha := \max_{1 \leq j \leq n} \alpha_j$. That the opposite is not true follows from the counter example $T(x_1, x_2) := \left(\frac{1}{2}x_2, \frac{1}{2}x_1\right)$ which is an α-contraction, with, e.g., $\alpha = 0.5$, since

$$\|T(x) - T(y)\| = \frac{1}{2}\|x - y\|. \tag{44}$$

However, for the points $(0,0)$ and $(1,5)$ there does not exist a real number $\alpha_1 \in [0,1)$ with which T is $(\alpha_1, 1)$-contractive.

Following the Banach fixed point theorem, as reformulated in Berinde's book [5, Theorem 2.1], we formulate a componental contraction mapping principle as follows.

Theorem 22 *Let $j \in J$ and let $T : \mathcal{H} \to \mathcal{H}$ be an (α_j, j)-contraction. Then*

(i) *The j-th componental fixed point set of T is nonempty, i.e.,* $\mathrm{Fix}^j T := \{x \in \mathcal{H} \mid (T(x))_j = x_j\} \neq \emptyset$ *and the variable x_j is unique for all $x \in \mathrm{Fix}^j T$, henceforth denoted as $x_j = x_{j*}$.*

(ii) *The sequence $\{x_j^k\}_{k=0}^\infty$ of the j-th components of any sequence $\{x^k\}_{k=0}^\infty$, generated by the Picard iteration $x^{k+1} = T(x^k)$ associated with T, converges for any initial point $x^0 \in \mathcal{H}$, and*

$$\lim_{k \to \infty} x_j^k = x_{j*}. \tag{45}$$

(iii) *The following a priori and a posteriori error estimates hold:*

$$\left\|x_j^k - x_{j*}\right\|_j \leq \frac{(\alpha_j)^k}{1 - \alpha_j}\left\|x_j^0 - x_j^1\right\|_j, \text{ for all } k = 1, 2, \ldots, \tag{46}$$

$$\left\|x_j^k - x_{j*}\right\|_j \leq \frac{\alpha_j}{1 - \alpha_j}\left\|x_j^{k-1} - x_j^k\right\|_j, \text{ for all } k = 1, 2, \ldots. \tag{47}$$

(iv) *The rate of convergence of the sequence $\{x_j^k\}_{k=0}^\infty$, in* (ii) *above, is given by*

$$\left\|x_j^k - x_{j*}\right\|_j \leq \alpha_j \left\|x_j^{k-1} - x_{j*}\right\|_j \leq (\alpha_j)^k \left\|x_j^0 - x_{j*}\right\|_j, \text{ for all } k = 1, 2, \ldots. \tag{48}$$

Proof. This can be proved exactly along the lines of the proof in [5, Theorem 2.1]. Alternatively, one can introduce an operator $U : \mathcal{H} \to \mathcal{H}$ by $U(x) := (0, ..., 0, T_j(x), 0, ..., 0)$ and apply [5, Theorem 2.1] to it. ■

To justify Theorem 22 we build an example of an operator which is not an α-contraction but is an α_j-contractions for some indices $j \in J$, but not all. The original Banach fixed point theorem would not apply to them but our theorem would.

Example 23 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x_1, x_2) = \begin{pmatrix} T_1(x_1, x_2) \\ T_2(x_1, x_2) \end{pmatrix} := \begin{pmatrix} \frac{x_1}{2} + 3 \\ 8x_2 \end{pmatrix}. \tag{49}$$

This T is an α_1-contraction with, e.g., $\alpha_1 = 1/2$ because

$$|T_1(x_1, x_2) - T_1(y_1, y_2)| = \frac{1}{2} |x_1 - y_1| . \tag{50}$$

But T is not an α-contraction because

$$\|T(x) - T(y)\|^2 = \frac{1}{4}(x_1 - y_1)^2 + 64(x_2 - y_2)^2 \tag{51}$$

and there is no $\alpha \in [0, 1)$ for which

$$\frac{1}{4}(x_1 - y_1)^2 + 64(x_2 - y_2)^2 \leq \alpha^2 \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right) \tag{52}$$

for all $x, y \in \mathbb{R}^2$.

3 Regularity of component-wise quasi-nonexpansive operators

The notions of asymptotic regularity of sequences and operators play a central role in fixed point theory, see, e.g., [3] or [8]. We define next a notion of componental regularity.

Definition 24 Let $T : \mathcal{H} \to \mathcal{H}$ be a QNE operator and let $j \in J$.

(i) We say that T is *j-weakly regular* (j-WR) if, for any sequence $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$ and some $y \in \mathcal{H}$,

$$x_j^k \rightharpoonup y_j \text{ and } \lim_{k\to\infty} \|(T(x^k))_j - x_j^k\|_j = 0 \Longrightarrow y \in \operatorname{Fix}^j T. \tag{53}$$

(ii) If T is j-weakly regular for all $j \in J$, then we say that T is *CW-weakly regular* (CW-WR).

(iii) If (53) holds after all j indices are removed from it then we say that T is weakly regular (WR) (cf. [9, Definition 3.1]).

The weak regularity of an operator T means that $T - \mathrm{Id}$ is demi-closed at 0 (cf. [13]).

Fact 25 *A QNE operator $T : \mathcal{H} \to \mathcal{H}$ is CW-WR if and only if T is WR.*

Proof. Suppose that T is CW-WR. Let $x^k \rightharpoonup y$ and $\lim_{k\to\infty} \|T(x^k) - x^k\| = 0$. Then $x_j^k \rightharpoonup y_j$ and $\lim_{k\to\infty} \|T(x^k)_j - x_j^k\|_j = 0$ for all $j \in J$. Since T is j-WR, $y \in \operatorname{Fix}^j T$, for all $j \in J$, thus, $y \in \bigcap_{j\in J} \operatorname{Fix}^j T = \operatorname{Fix} T$ which means that T is WR. The converse can be proved similarly. ■

Theorem 26 *Let $T : \mathcal{H} \to \mathcal{H}$ be an r-CW-SQNE and CW-WR operator, where $r = (\rho_1, \rho_2, ..., \rho_n)$ with $\rho_j > 0, j \in J$, and let the sequence $\{x^k\}_{k=0}^{\infty}$ be generated by the iteration*

$$x_j^{k+1} = (T(x^k))_j \ k \geq 0, \tag{54}$$

$j \in J$, where $x^0 \in \mathcal{H}$ is arbitrary. Then $\{x^k\}_{k=0}^{\infty}$ converges weakly to some $x^* \in \operatorname{Fix} T$.

Proof. By assumption, T is ρ_j-SQNE, $j \in J$, i.e., for any $z = (z_1, z_2, ..., z_n) \in \operatorname{Fix} T = \bigcap_{j\in J} \operatorname{Fix}^j T$ it holds

$$\|x_j^{k+1} - z_j\|_j^2 = \|(T(x^k))_j - z_j\|_j^2 \leq \|x_j^k - z_j\|_j^2 - \rho_j \|(T(x^k))_j - x_j^k\|_j^2. \tag{55}$$

That Fix$T$ is nonempty follows from Definition 11. Using standard arguments this leads to the boundedness of x_j^k, $j \in J$ and to $\|(T(x^k))_j - x_j^k\|_j \to 0$ as $k \to \infty$. Thus, there exists a subsequence $\{x_j^{n_k}\}_{k=0}^{\infty}$ which converges weakly to some x_j^*, $j \in J$. Let $x^* = (x_1^*, x_2^*, ..., x_n^*)$. Because T is j-weakly regular, we have $x^* \in \operatorname{Fix}^j T$, $j \in J$. This and Fact 14(ii) yields $x^* \in \operatorname{Fix} T$. ■

4 Nonlinear DROP as a simultaneous projection method with component-wise weights

Many problems in mathematics, in physical sciences and in real-world applications can be modeled as a *convex feasibility problem* (CFP); i.e., a problem of finding a point $x^* \in Q := \cap_{i=1}^m Q_i$ in the intersection of finitely many closed convex sets

$Q_i \subseteq \mathbb{R}^n$ in the finite-dimensional Euclidean space. The literature on this subject is enormously large, see, e.g., [2] or [3] and [8] and references therein.

Fully-simultaneous (parallel) algorithmic schemes for the CFP employ iterative steps of the form

$$x^{k+1} = x^k + \lambda_k \left(\sum_{i=1}^{m} w_i \left(P_{Q_i}(x^k) - x^k \right) \right), \tag{56}$$

where $P_\Omega(x)$ stands for the *orthogonal (nearest Euclidean distance) projection* of a point x onto the closed convex set Ω, the parameters $\{w_i\}_{i=1}^m$ are a *system of weights* such that $w_i > 0$ for all $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m w_i = 1$, and the *relaxation parameters* $\{\lambda_k\}_{k=0}^\infty$ are user-chosen and, in most convergence analyses, must remain in a certain fixed interval, in order to guarantee convergence.

For linear equations, represented by hyperplanes, i.e.,

$$Q_i = H_i := \left\{ x \in \mathbb{R}^n \mid \langle a^i, x \rangle = b_i \right\}, \tag{57}$$

for $i = 1, 2, \ldots, m$, the orthogonal projection $P_i(z)$ of a point $z \in \mathbb{R}^n$ onto H_i is

$$P_i(z) \ = \ z + \frac{b_i - \langle a^i, z \rangle}{\|a^i\|_2^2} a^i, \tag{58}$$

where $a^i = (a_j^i)_{j=1}^n \in \mathbb{R}^n$, $a^i \neq 0$, and $b_i \in \mathbb{R}$ are the given data of the linear equations and $\|\cdot\|_2$ is the Euclidean norm. The iterative steps of (56) then take the form

$$x^{k+1} \ = \ x^k + \lambda_k \sum_{i=1}^{m} w_i \frac{b_i - \langle a^i, x^k \rangle}{\|a^i\|_2^2} a^i. \tag{59}$$

This algorithm was first proposed by Cimmino [11] (read about the profound impact of this paper on applied scientific computing in Benzi's paper [4]) and generalized to convex sets by Auslender [1].

For the case of equal weights $w_i = 1/m$ (59) becomes

$$x^{k+1} \ = \ x^k + \frac{\lambda_k}{m} \sum_{i=1}^{m} \frac{b_i - \langle a^i, x^k \rangle}{\|a^i\|_2^2} a^i. \tag{60}$$

When the $m \times n$ system matrix $A = (a_j^i)$ is sparse, as often happens in some important real-world applications, only a relatively small number of the elements $\{a_j^1, a_j^2, \ldots, a_j^m\}$ in the j-th column of A are nonzero, but in (60) the sum of their contributions is divided by the relatively large m – slowing down the progress of the algorithm. This observation led us, in [10], to consider replacement of the factor $1/m$ in (60) by a factor that depends only on the number of *nonzero* elements in the set $\{a_j^1, a_j^2, \ldots, a_j^m\}$. Specifically, for each $j = 1, 2, \ldots, n$, we denoted by s_j the number of nonzero elements in column j of the matrix A, assuming that all columns of A are nonzero, thus $s_j \neq 0$, for all j. Then we replaced (60) by

$$x_j^{k+1} = x_j^k + \frac{\lambda_k}{s_j} \sum_{i=1}^{m} \frac{b_i - \langle a^i, x^k \rangle}{\|a^i\|_2^2} a_j^i \,, \text{ for } j = 1, 2, \ldots, n. \tag{61}$$

This iterative formula is the backbone of the proposed fully-simultaneous Diagonally-Relaxed Orthogonal Projections (DROP) method for linear equations in [10]. Certainly, if the matrix A is sparse, then the s_j values will be much smaller than m and using them instead of the large m will enlarge the additive updates in the iterative process and lead to faster initial convergence.

This leads naturally to the question whether the weights w_i in (56) be allowed to depend on the component index j as iterations proceed, without loosing the guaranteed convergence of the algorithm? Or, phrased mathematically, may the iterations proceed according to

$$x_j^{k+1} = x_j^k + \lambda_k \left(\sum_{i=1}^{m} w_{ij} \left(\left(P_{Q_i}(x^k) \right)_j - x_j^k \right) \right), \text{ for } j = 1, 2, \ldots, n, \tag{62}$$

where the parameters $\{w_{ij}\}_{i=1}^m$ form n *systems of weights* such that, for $j = 1, 2, \ldots, n$, $w_{ij} \geq 0$ for all $i = 1, 2, \ldots, m$, and $\sum_{i=1}^m w_{ij} = 1$? If such component-wise relaxation is possible then we could use it to exploit sparsity of the underlying problem and to control asynchronous (block) iterations.

The convergence of such a scheme, like (61), for the linear case is studied in [10] but the general case of (62) remained unsettled.

With $T_i := P_{Q_i}$, for all $i = 1, 2, \ldots, m$, and with a fixed $\lambda \in (0, 2]$, such that $\lambda_k = \lambda$, for all $k \geq 0$, Eq. (62) describes the iterative process in (54) with the operator in (38). The following theorem yields the weak convergence of sequences generated by the iterative process (62).

Theorem 27 *Let $w_{ij} \geq 0$ with $\sum_{i=1}^m w_{ij} = 1$, $i \in I$, $j \in J$, and $\lambda \in (0, 2)$. If $T_i : \mathcal{H} \to \mathcal{H}$ are CW-cutters and CW-WR operators having a common fixed point then any sequence generated by the iterative process*

$$x_j^{k+1} = x_j^k + \lambda \left(\sum_{i=1}^{m} w_{ij} \left(\left(T_i(x^k) \right)_j - x_j^k \right) \right), \textit{ for } j = 1, 2, \ldots, n, \tag{63}$$

where $x^0 \in \mathcal{H}$ is arbitrary, converges weakly to some point $x^* \in \operatorname{Fix} T = \bigcap_{j \in J} \operatorname{Fix}^j T$.

Proof. Iteration (63) can be written as $x_j^{k+1} = (T(x^k))_j$, where T is defined by (38) with $\lambda_j = \lambda$. By Fact 20, T is r-CW-SQNE, where $r = \dfrac{2 - \lambda}{\lambda} e$ with $e = (1, 1, ..., 1) \in$

$\mathbb{R}^n$. By Fact 25 and [9, Corollary 5.3(i)], T is CW-WR. Thus, the theorem follows from Theorem 26. ■

It is, of course, possible to replace the constant relaxation parameter λ in (63) with $\lambda_k \in [\varepsilon, 2-\varepsilon]$ for some positive ε and the proof will remain true. Therefore, since the operator T in the above proof is a CW-WR operator, and so are also orthogonal projections, Theorem 27 is a generalization the convergence theorem of the DROP method, [10, Theorem 2.3].

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