Strong duality of a conic optimization problem with a single hyperplane and two cone constraints

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Abstract

Strong (Lagrangian) duality of general conic optimization problems (COPs) has long been studied and its profound and complicated results appear in different forms in a wide range of literatures. As a result, characterizing the known and unknown results can sometimes be difficult. The aim of this article is to provide a unified and geometric view of strong duality of COPs for the known results. For our framework, we employ a COP minimizing a linear function in a vector variable $\boldsymbol{x}$ subject to a single hyperplane constraint $\boldsymbol{x} \in H$ and two cone constraints $\boldsymbol{x} \in \mathbb{K}_1$, $\boldsymbol{x} \in \mathbb{K}_2$. It can be identically reformulated as a simpler COP with the single hyperplane constraint $\boldsymbol{x} \in H$ and the single cone constraint $\boldsymbol{x} \in \mathbb{K}_1 \cap \mathbb{K}_2$. This simple COP and its dual as well as their duality relation can be represented geometrically, and they have no duality gap without any constraint qualification. The dual of the original target COP is equivalent to the dual of the reformulated COP if the Minkowski sum of the duals of the two cones $\mathbb{K}_1$ and $\mathbb{K}_2$ is closed or if the dual of the reformulated COP satisfies a certain Slater condition. Thus, these two conditions make it possible to transfer all duality results, including the existence and/or boundedness of optimal solutions, on the reformulated COP to the ones on the original target COP, and further to the ones on a standard primal-dual pair of COPs with symmetry.

Key words. Duality, conic optimization problems, simple conic optimization problems, generalizing the Slater condition, closedness of the Minkowski sum of two cones

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1 Introduction

It is well-known that strong duality of conic optimization problems (COPs), including semidefinite programs (SDPs) [3, 11, 17], second order programs [2, 7], doubly nonnegative programs [4, 18] and copositive programs [5, 6] depends on their representation. In this paper, we consider a primal COP of the following form:

$$\eta_p \;=\; \inf\;\{\langle \boldsymbol{q},\, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathbb{K}_1,\; \boldsymbol{x} \in \mathbb{K}_2,\;\; \langle \boldsymbol{h},\, \boldsymbol{x}\rangle = 1\}\,. \tag{1}$$

Here,

$$\left.\begin{array}{l} \mathbb{V} = \text{a finite dimensional vector space endowed with an inner product} \\ \qquad \langle \boldsymbol{x},\, \boldsymbol{y}\rangle \text{ and a norm } \|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x},\, \boldsymbol{x}\rangle} \text{ for every } \boldsymbol{x},\; \boldsymbol{y} \in \mathbb{V}, \\ \mathbb{K}_1,\; \mathbb{K}_2 = \text{a nonempty closed convex cone in } \mathbb{V},\; \boldsymbol{q} \in \mathbb{V},\; \boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{K}_1^*, \\ \mathbb{K}^* = \text{the dual } \{\boldsymbol{y} \in \mathbb{V} : \langle \boldsymbol{x},\, \boldsymbol{y}\rangle \geq 0 \text{ for every } \boldsymbol{x} \in \mathbb{K}\} \text{ of a cone } \mathbb{K} \subset \mathbb{V}. \end{array}\right\} \tag{2}$$

A general form of COP with inequality constraints can be converted into COP (1) where the requirement $\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{K}_1^*$ above is naturally fulfilled. The conversion is described in Section 4.

The (Lagrangian) dual of COP (1) is written as

$$\begin{aligned} \eta_d &= \sup\,\{t_0 : \boldsymbol{q} - \boldsymbol{h}t_0 \in \mathbb{K}_1^* + \mathbb{K}_2^*\} \\ &= \sup\,\{t_0 : \boldsymbol{q} - \boldsymbol{h}t_0 - \boldsymbol{y}_2 \in \mathbb{K}_1^*,\; \boldsymbol{y}_2 \in \mathbb{K}_2^*\}\,, \end{aligned} \tag{3}$$

where $\mathbb{K}_1^* + \mathbb{K}_2^*$ denotes the Minkowski sum $\{\boldsymbol{y}_1 + \boldsymbol{y}_2 : \boldsymbol{y}_1 \in \mathbb{K}_1^*, \boldsymbol{y}_2 \in \mathbb{K}_2^*\}$ of $\mathbb{K}_1^*$ and $\mathbb{K}_2^*$. If primal COP (1) (or dual COP (3)) is feasible, η_p (or η_d) takes either a finite value or $-\infty$ (or ∞). If they are infeasible, we set $\eta_p = \infty$ and $\eta_d = -\infty$. Thus, the well-known weak duality $\eta_d \leq \eta_p$ holds even if one of them is infeasible. We simply write $-\infty < \eta_p < \infty$ (or $-\infty < \eta_d < \infty$) for the case where primal COP (1) (or dual COP (3)) is feasible and has a finite optimal value η_p (or η_d). We say that *strong duality* holds if $-\infty < \eta_d = \eta_p < \infty$, and primal COP (1) (or dual COP (3)) is *solvable* if η_p (or η_d) is attained by a feasible solution.

Given a closed convex cone $\mathbb{K} \subset \mathbb{V}$, primal COP (1) remains identical regardless of how $\mathbb{K}$ is decomposed into the intersection $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$ of two closed convex cones $\mathbb{K}_1$ and $\mathbb{K}_2$. Dual COPs (3) with different decompositions, however, may yield different duality results. For instance, when we take $\mathbb{K}_1 = \mathbb{K}$ and $\mathbb{K}_2 = \mathbb{V}$, there is no duality gap as we will see in Section 2. In the case where $\mathbb{K}_1 \neq \mathbb{V}$ and $\mathbb{K}_2 \neq \mathbb{V}$, $\eta_d < \eta_p$ can occur.

We introduce three conditions to characterize duality relations between COP (1) and COP (3)

- Cl: $\mathbb{K}_1^* + \mathbb{K}_2^*$ is closed.
- Ri: There exists a $\widetilde{t} \in \mathbb{R}$ such that $\widetilde{\boldsymbol{y}} \equiv \boldsymbol{q} - \boldsymbol{h}\widetilde{t}$ lies in relint$(\mathbb{K}_1^* + \mathbb{K}_2^*)$, the relative interior of $(\mathbb{K}_1^* + \mathbb{K}_2^*)$ with respect to the subspace spanned by $(\mathbb{K}_1^* + \mathbb{K}_2^*)$.
- Po: $\mathbb{K}_1 \cap \mathbb{K}_2$ is pointed.

It is known that the closure of $\mathbb{K}_1^* + \mathbb{K}_2^*$ coincides with $(\mathbb{K}_1 \cap \mathbb{K}_2)^*$. Hence Condition Cl can be restated as $\mathbb{K}_1^* + \mathbb{K}_2^* = (\mathbb{K}_1 \cap \mathbb{K}_2)^*$. We establish the following result.

Theorem 1.1. *Assume that $-\infty < \eta_p < \infty$ or $-\infty < \eta_d < \infty$ holds.*

(i) If Condition Cl *or Condition* Ri *is satisfied, then $-\infty < \eta_p = \eta_d < \infty$ holds.*

(ii) If Condition Cl *is satisfied, then dual* COP (3) *is solvable.*

(iii) The set of optimal solutions of primal COP (1) *is nonempty and bounded if and only if Conditions* Ri *and* Po *are satisfied.*

(iv) The set of optimal solutions of primal COP (1) *is nonempty and unbounded if* Po *is not satisfied (, i.e., $\mathbb{K}_1 \cap \mathbb{K}_2$ is not pointed) and Condition* Ri *is satisfied.*

Assertion (i) indicates that both Conditions Cl and Ri are equally important to guarantee strong duality between primal COP (1) and dual COP (3). We note that "Conditions Ri and Po" in assertion (iii) can be replaced with a single condition that "there exists a $\widetilde{t} \in \mathbb{R}$ such that $\boldsymbol{q} - \boldsymbol{h}\widetilde{t} \in \mathrm{int}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ (the interior of $\mathbb{K}_1^* + \mathbb{K}_2^*$)". (See Lemma 3.6). The most significant (and straightforward) consequence of assertions (iii) and (i) of the theorem is:

Corollary 1.2. *If the set of optimal solutions (or the feasible region) of primal* COP (1) *is nonempty and bounded, then $-\infty < \eta_p = \eta_d < \infty$.*

Existing work and contribution of the paper

Strong duality of COPs including SDPs has been widely studied [1, 10, 11, 14, 15, 13, 16], and various results on their duality have been shown in different forms. In particular, Fenchel's duality theorem presented in Rockafellar [13] is quite general and (implicitly) covers a standard strong duality theorem of COPs under Slater's condition ([13, Theorem 31.4]). Nesterov and Nemiroskii also gave a comprehensive discussion on duality for general COPs in their book [11], and presented a strong duality result under a Slater condition (see Theorem 4.2.1 of [11]). Luo, Sturm and Zhang [10] discussed the boundedness of the optimal solution set of COPs in addition to their strong duality under Slater's condition. Shapiro proposed the closedness of a certain cone, instead of Slater condition, under which strong duality was established for a fairly general COPs [16, Propositions 2.6 and 2.8]. His closed condition has been playing a crucial role in the recent development of duality theory of COPs (see, [1] and [12]).

The main purpose of this paper is to provide a unified and geometric overview of known strong duality results. A unique feature of the proposed approach in this paper is that strong duality of general COPs is obtained with a very simple form of COPs described geometrically by a cone, a hyperplane and a line. More precisely, we introduce the following two simple COPs:

$$\begin{aligned} \mathrm{P}(\mathbb{K}): \quad & \zeta_p(\mathbb{K}) = \inf\{\langle \boldsymbol{q},\, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathbb{K},\ \langle \boldsymbol{h},\, \boldsymbol{x}\rangle = 1\}, \\ \mathrm{D}(\mathbb{J}): \quad & \zeta_d(\mathbb{J}) = \sup\{t : \boldsymbol{q} - \boldsymbol{h}t \in \mathbb{J}\}. \end{aligned}$$

Here $\mathbb{K}$ and $\mathbb{J}$ are convex cones in a finite dimensional vector space $\mathbb{V}$, $\boldsymbol{q} \in \mathbb{V}$ and $\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{J}$. If we take $\mathbb{J} = \mathbb{K}^*$, then COPs $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$ serves as a primal-dual pair. The feasible region of primal COP $\mathrm{P}(\mathbb{K})$ can be geometrically represented as the intersection of the hyperplane $H \equiv \{\boldsymbol{x} \in \mathbb{V} : \langle \boldsymbol{h},\, \boldsymbol{x}\rangle = 1\}$ and the cone $\mathbb{K}$. Also the feasible region of dual COP $\mathrm{D}(\mathbb{K}^*)$ can be viewed as the intersection of the 1-dimension line $\{\boldsymbol{q} - \boldsymbol{h}t : t \in \mathbb{R}\}$ with the dual cone $\mathbb{K}^*$ of $\mathbb{K}$, where $\boldsymbol{h}$ is a normal vector of the hyperplane H. This geometrical representation is an essential feature of the primal-dual pair of COPs $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$, which makes it possible to geometrically interpret not only strong duality relation, but also duality gap on the pair as we see in seven graphs in Figure 1. These graphic interpretations are important contributions of this paper as they show basic essentials of strong duality geometrically. In particular, strong duality $-\infty < \zeta_p(\mathbb{K}) = \zeta_d(\mathbb{K}^*) < \infty$ holds whenever either primal COP $\mathrm{P}(\mathbb{K})$ or dual COP $\mathrm{D}(\mathbb{K}^*)$ has a finite optimal value *without any condition* (Theorem 2.1, (i)). If we choose $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$ and $\mathbb{J} = \mathbb{K}_1^* + \mathbb{K}_2^*$, then the pair of COPs $\mathrm{P}(\mathbb{K}_1 \cap \mathbb{K}_2)$ and $\mathrm{D}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ represent the primal-dual pair of COPs (1) and (3). All assertions (i), (ii), (iii) and (iv) of Theorem 1.1 on COPs (1) and (3) follow from duality relations between COPs $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$. Although they could be derived from some existing results referred above, the detailed derivation is not included here due to its complexity. Instead, we prefer to present a unified and geometric treatment of the strong duality through COPs $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$.

The simple primal-dual pair of COPs $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$ was originally introduced in [8] as a Lagrangian-doubly nonnegative (DNN) relaxation of a class of quadratic optimization problems (QOPs), and their strong duality was studied in [4, Lemma 2.5]. Some relation of their results and Theorem 1.1 will be discussed in Section 5.

Outline of the paper

In Section 2, we discuss strong duality of the primal-dual pair of simple COPs $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$, and establish Theorem 2.1. This theorem may be regarded as a special case of Theorem 1.1 for $\mathbb{K}_1 = \mathbb{K}$ and $\mathbb{K}_2 = \mathbb{V}$, and makes it easier to directly handle Theorem 1.1. Assertions of Theorem 2.1 are illustrated in Figure 1. Based on Theorem 2.1, the main theorem of the paper, Theorem 1.1, is proved in Section 3. A geometric implication of Conditions Cl and Ri is illustrated in Figure 2. In addition, some lemmas are presented to relate Conditions Cl and Ri to Slater type conditions. In Section 4, we apply Theorem 1.1 to a standard primal-dual pair of COPs with symmetry, and present a slight extension of the strong duality results given in [16, Propositions 2.6 and 2.8]. We conclude in Section 5 with remarks on the strong duality result in [8, 4] for the pair of COPs of $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$ which was originally proposed as a Lagrangian-DNN relaxation of a class of QOPs.

Notation and symbols

For every convex subset C of a finite dimensional space $\mathbb{V}$, $\mathrm{int}C$, $\mathrm{linspan}C$, $\mathrm{relint}C$, $\mathrm{cl}C$ and $C^{\perp}$ denote the interior of C, the linear subspace spanned by C (*i.e.*, the smallest linear subspace containing C), the relative interior of C with respect to $\mathrm{linspan}C$, the closure of C and the orthogonal complement of $\mathrm{linspan}C$, respectively.

2 Strong duality between COPs $\mathbf{P}(\mathbb{K})$ and $\mathbf{D}(\mathbb{K}^*)$

In this section, we establish Theorem 2.1 which leads to assertions (i), (ii), (iii) and (iv) of our main theorem, Theorem 1.1. Throughout this section we assume that $\mathbb{K}$ is a closed convex cone in a finite dimensional vector space $\mathbb{V}$, $\boldsymbol{q} \in \mathbb{V}$ and $\mathbf{0} \neq \boldsymbol{h} \in \mathbb{K}^*$. To facilitate connecting assertions of Theorem 2.1 with those of Theorem 1.1, we modify Conditions Cl, Ri and Po with an arbitrary convex cone $\mathbb{J} \subset \mathbb{V}$.

Cl'($\mathbb{J}$): $\mathbb{J}$ is closed.

Ri'($\mathbb{J}$): There exists a $\widetilde{t} \in \mathbb{R}$ such that $\widetilde{\boldsymbol{y}} \equiv \boldsymbol{q} - \boldsymbol{h}\widetilde{t} \in \text{relint}\mathbb{J}$.

Po'($\mathbb{J}$): $\mathbb{J}$ is pointed.

Note that original Conditions Cl, Ri and Po can be rewritten as Cl'($\mathbb{K}_1^* + \mathbb{K}_2^*$), Ri'($\mathbb{K}_1^* + \mathbb{K}_2^*$) and Po'($\mathbb{K}_1 \cap \mathbb{K}_2$), respectively.

Theorem 2.1. *Assume that* $-\infty < \zeta_p(\mathbb{K}) < \infty$ *or* $-\infty < \zeta_d(\mathbb{K}^*) < \infty$ *holds (as in Figure 1 (a), (b), (c) or (e)).*

- *(i)* $-\infty < \zeta_p(\mathbb{K}) = \zeta_d(\mathbb{K}^*) < \infty$ *holds.*
- *(ii) Dual* COP D($\mathbb{K}^*$) *is solvable.*
- *(iii) The set of optimal solutions of primal* COP P($\mathbb{K}$) *is nonempty and bounded if and only if Conditions* Ri'($\mathbb{K}^*$) *and* Po'($\mathbb{K}$) *are satisfied (as in Figure 1 (a) and (e)).*
- *(iv) The set of optimal solutions of primal* COP P($\mathbb{K}$) *is nonempty and unbounded if* Po'($\mathbb{K}$) *is not satisfied (i.e.,* $\mathbb{K}$ *is not pointed) and Condition* Ri'($\mathbb{K}^*$) *is satisfied (as in Figure 1 (c)).*

The geometric interpretation of assertions (i), (ii), (iii) and (iv) is illustrated with seven two-dimensional examples in Figure 1 and Table 1 to help the reader understand them. A proof of Theorem 2.1 is presented after a basic lemma below.

Lemma 2.2. *Let* $\mathbb{J} \subset \mathbb{V}$ *be a nonempty closed convex cone. Then Conditions* Ri'($\mathbb{J}^*$) *and* Po'($\mathbb{J}$) *hold if and only if*

$$\text{there exists a } \widetilde{t} \in \mathbb{R} \text{ such that } \widetilde{\boldsymbol{y}} \equiv \boldsymbol{q} - \boldsymbol{h}\widetilde{t} \in \text{int}(\mathbb{J}^*). \tag{4}$$

Proof. We know that $\mathbb{L} \subset \mathbb{J}$ holds for a linear subspace $\mathbb{L}$ if and only if $\mathbb{J}^* \subset \mathbb{L}^L$. Hence $\mathbb{J}$ is pointed if and only if $\text{int}(\mathbb{J}^*) \neq \emptyset$. This implies the desired result. □

Proof of Theorem 2.1 (i) and (ii): Let $\boldsymbol{d} \in \text{relint}\mathbb{K}$. It follows from $\boldsymbol{h} \in \mathbb{K}^*$ that $\langle \boldsymbol{h}, \boldsymbol{d} \rangle \geq 0$. Assume that $\langle \boldsymbol{h}, \boldsymbol{d} \rangle = 0$. Since $\boldsymbol{d} \in \text{relint}\mathbb{K}$, for every $\boldsymbol{x} \in \mathbb{K}$, there is a $\mu > 1$ such that $(1-\mu)\boldsymbol{x} + \mu\boldsymbol{d} \in \mathbb{K}$ (see [13, Theorem 6.4]). Hence $\mathbf{0} \leq \langle \boldsymbol{h}, (1-\mu)\boldsymbol{x} + \mu\boldsymbol{d} \rangle = (1-\mu)\langle \boldsymbol{h}, \boldsymbol{x} \rangle$ for every $\boldsymbol{x} \in \mathbb{K}$, which together with $\boldsymbol{h} \in \mathbb{K}^*$ implies that $\boldsymbol{h} \in \mathbb{K}^\perp$. Therefore P($\mathbb{K}$) is infeasible or $\zeta_p(\mathbb{K}) = +\infty$, and either $\zeta_d(\mathbb{K}^*) = +\infty$ (if $\boldsymbol{q} \in \mathbb{K}^*$) or $\zeta_d(\mathbb{K}^*) = -\infty$ (if $\boldsymbol{q} \notin \mathbb{K}^*$). This contradicts the assumption, and we have shown that $\langle \boldsymbol{h}, \boldsymbol{d} \rangle > 0$. Now, let $\tilde{\boldsymbol{x}} = \boldsymbol{d}/\langle \boldsymbol{h}, \boldsymbol{d} \rangle$. Then $\tilde{\boldsymbol{x}}$ is a feasible solution of P($\mathbb{K}$) satisfying Slater's condition, *i.e.* $\tilde{\boldsymbol{x}} \in \text{relint}\mathbb{K}$ and $\langle \boldsymbol{h}, \tilde{\boldsymbol{x}} \rangle = 1$. By the standard strong duality theorem (for example, [10, Theorem 7], [13, Theorem 31.4]), assertions (i) and (ii) hold. □

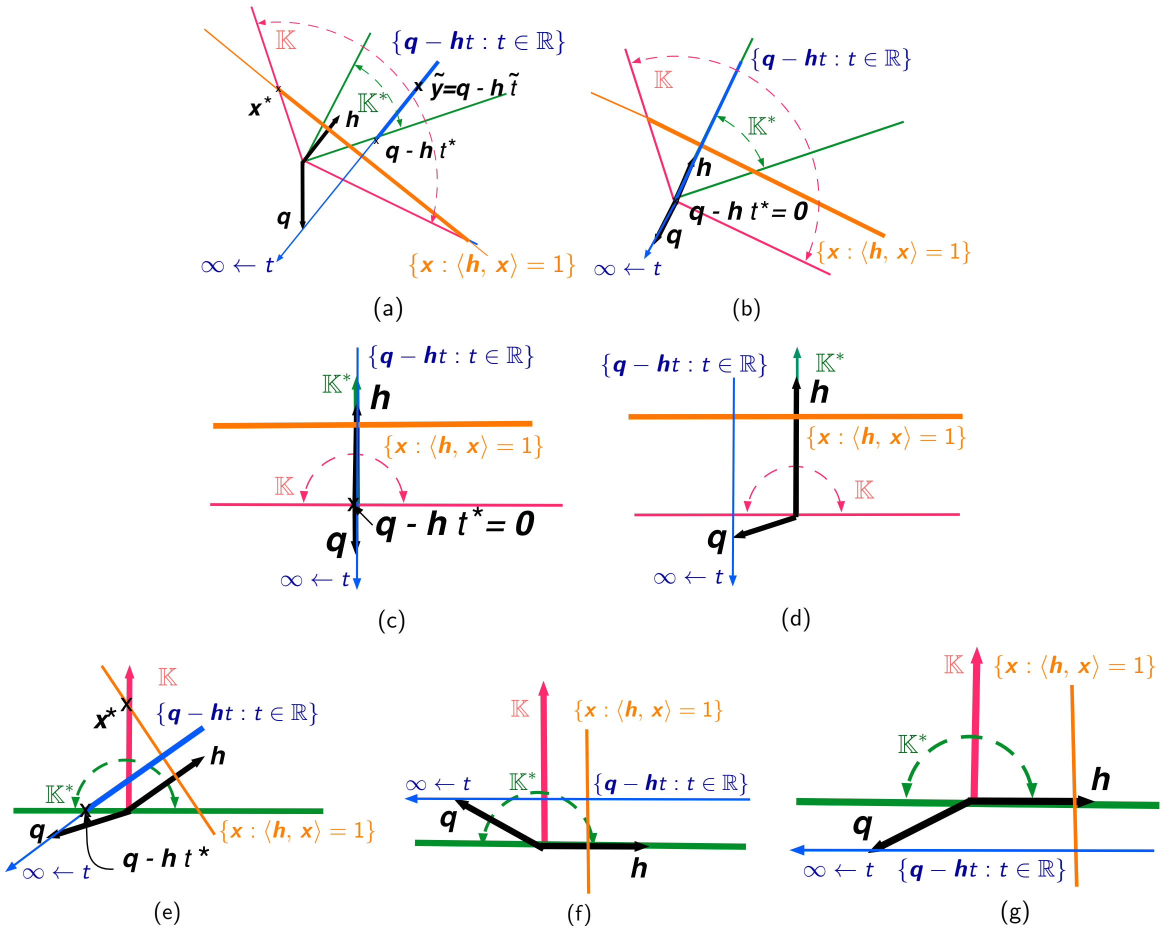

Figure 1: Cases (a), (b), (c) and (e) satisfy the assumption of Theorem 2.1 (hence $\zeta_p(\mathbb{K}) = \zeta_d(\mathbb{K}^*)$ holds by Theorem 2.1 (i) and D($\mathbb{K}^*$) is solvable), while cases (d), (f) and (g) do not. Cases (a) and (e) satisfy Conditions Ri'($\mathbb{K}^*$) and Po'($\mathbb{K}$); hence the set of optimal solutions of P($\mathbb{K}$) is nonempty and bounded by Theorem 2.1 (ii). In case (b), Po'($\mathbb{K}$) is satisfied and both P($\mathbb{K}$) and D($\mathbb{K}^*$) are solvable, but Condition Ri'($\mathbb{K}^*$) is violated. Hence we know that Condition Ri'($\mathbb{K}^*$) is merely a sufficient condition for the solvability of P($\mathbb{K}$) but not necessary. Case (b) is sensitive to the change in $\boldsymbol{q}$: if the direction of $\boldsymbol{q}$ is perturbed slightly, then either Ri'($\mathbb{K}^*$) together with Po'($\mathbb{K}$) is satisfied or D($\mathbb{K}^*$) gets infeasible (or, either P($\mathbb{K}$) has a unique optimal solution at $\boldsymbol{x}^*$ or $\zeta_p(\mathbb{K}) = -\infty$). Case (c) satisfies Condition Ri'($\mathbb{K}^*$) but does not satisfy Condition Po'($\mathbb{K}$); hence the set of solutions of P($\mathbb{K}$) is nonempty and unbounded. Case (d) is an example for $-\infty = \zeta_d(\mathbb{K}^*) = \zeta_p(\mathbb{K})$, and case (f) for $\zeta_d(\mathbb{K}^*) = \zeta_p(\mathbb{K}) = \infty$. In case (g), both P($\mathbb{K}$) and D($\mathbb{K}^*$) are infeasible; hence $-\infty = \zeta_d(\mathbb{K}^*) < \zeta_p(\mathbb{K}) = \infty$. Table 1 summarizes all cases, where cases designated by $\times$ cannot occur by weak duality or (i) of Theorem 2.1.

Proof of Theorem 2.1 (iii): We already by (ii) know that $-\infty < \zeta_p(\mathbb{K}) = \zeta_d)\mathbb{K}^*) < \infty$. Also, by Lemma 2.2, we may replace Conditions Ri'($\mathbb{K}^*$) and Po'($\mathbb{K}$) with (4).

"if part": We assume that (4) holds and on the contrary that the set of optimal solutions of P($\mathbb{K}$) is either empty or unbounded, and we show a contradiction. Since

		P(𝕂)		
		Feasible		Infeasible
		$-\infty < \zeta_p(\mathbb{K})$	$-\infty = \zeta_p(\mathbb{K})$	$\zeta_p(\mathbb{K}) = \infty$
D($\mathbb{K}^*$)	Feasible $\zeta_d(\mathbb{K}^*) < \infty$	(a),(b),(c),(e)	$\times$(by Th.2.1 (i))	$\times$(by Th.2.1 (i))
	Feasible $\zeta_d(\mathbb{K}^*) = \infty$	$\times$(by Th.2.1 (i))	$\times$(by w.duality)	(f)
	Infeasible $\zeta_d(\mathbb{K}^*) = -\infty$	$\times$(by Th.2.1 (i))	(d)	(g)

Table 1: Cases (a) through (g) correspond to those of Figure 1, respectively. Cases designated by $\times$ cannot occur by weak duality or (i) of Theorem 2.1.

$-\infty < \zeta_p(\mathbb{K}) < \infty$ by assumption, there exist an $\epsilon \geq 0$ and a sequence of feasible solutions $\{\boldsymbol{x}^k\}$ of P($\mathbb{K}$) such that

$$\boldsymbol{x}^k \in \mathbb{K},\ \langle \boldsymbol{q},\, \boldsymbol{x}^k \rangle \leq \zeta_p(\mathbb{K}) + \epsilon,\ \langle \boldsymbol{h},\, \boldsymbol{x}^k \rangle = 1 \tag{5}$$

for $k = 1, 2, \ldots$, and $\|\boldsymbol{x}^k\| \to \infty$ as $k \to \infty$. (If the set of optimal solutions is nonempty, we can take $\epsilon = 0$ and $\langle \boldsymbol{q},\, \boldsymbol{x}^k \rangle = \zeta_p(\mathbb{K})$). We may assume without loss of generality that $\boldsymbol{x}^k/\|\boldsymbol{x}^k\| \in \mathbb{K}$ converges to some $\Delta\boldsymbol{x} \in \mathbb{K}$ with $\|\Delta\boldsymbol{x}\| = 1$. Dividing the relation (5) by $\|\boldsymbol{x}^k\|$ and taking the limit as $k \to \infty$, we see that

$$\Delta\boldsymbol{x} \in \mathbb{K},\ \|\Delta\boldsymbol{x}\| = 1,\ \langle \boldsymbol{q},\, \Delta\boldsymbol{x} \rangle \leq 0,\ \langle \boldsymbol{h},\, \Delta\boldsymbol{x} \rangle = 0.$$

By (4), if we take a sufficiently small $\delta > 0$ such that $\widetilde{\boldsymbol{y}} - \delta\Delta\boldsymbol{x} = \boldsymbol{q} - \boldsymbol{h}\widetilde{t} - \delta\Delta\boldsymbol{x} \in \mathbb{K}^*$, then $0 \leq \langle \boldsymbol{q} - \boldsymbol{h}\widetilde{t} - \delta\Delta\boldsymbol{x},\, \Delta\boldsymbol{x} \rangle \leq -\delta < 0$, which is a contradiction.

"only if part": Assume on the contrary that the line $S = \{\boldsymbol{z} = \boldsymbol{q} - \boldsymbol{h}t : t \in \mathbb{R}\}$ does not intersect with int($\mathbb{K}^*$). By the separation theorem on convex sets (see, for example, [13, Theorem 11.3]), there exist an $\alpha \in \mathbb{R}$ and a nonzero $\Delta\boldsymbol{x} \in \mathbb{V}$ such that

$$\langle \boldsymbol{y},\, \Delta\boldsymbol{x} \rangle \geq \alpha \geq \langle \boldsymbol{z},\, \Delta\boldsymbol{x} \rangle \text{ if } \boldsymbol{y} \in \mathbb{K}^* \text{ and } \boldsymbol{z} \in S.$$

Since $\mathbb{K}^*$ is a closed cone containing $\boldsymbol{0} \in \mathbb{V}$, we see that

$$\begin{aligned} &\langle \boldsymbol{y},\, \Delta\boldsymbol{x} \rangle \geq 0 \text{ for every } \boldsymbol{y} \in \mathbb{K}^*;\ \text{hence } \boldsymbol{0} \neq \Delta\boldsymbol{x} \in \mathbb{K}, \\ &0 \geq \alpha \geq \langle \boldsymbol{q} - \boldsymbol{h}t,\, \Delta\boldsymbol{x} \rangle \text{ for every } t \in \mathbb{R};\ \text{hence } \langle \boldsymbol{h},\, \Delta\boldsymbol{x} \rangle = 0 \text{ and } 0 \geq \langle \boldsymbol{q},\, \Delta\boldsymbol{x} \rangle. \end{aligned}$$

Let $\boldsymbol{x}^*$ be an optimal solution of P($\mathbb{K}$) whose existence is guaranteed by the assumption of the "only if" part. Then,

$$\begin{aligned} &\boldsymbol{x}^* + \mu\Delta\boldsymbol{x} \in \mathbb{K},\ \langle \boldsymbol{h},\, \boldsymbol{x}^* + \mu\Delta\boldsymbol{x} \rangle = 1; \\ &(\text{hence } \boldsymbol{x}^* + \mu\Delta\boldsymbol{x} \text{ is a feasible solution of P}(\mathbb{K})), \\ &\zeta_p(\mathbb{K}) = \langle \boldsymbol{q},\, \boldsymbol{x}^* \rangle \geq \langle \boldsymbol{q},\, \boldsymbol{x}^* + \mu\Delta\boldsymbol{x} \rangle \geq \langle \boldsymbol{q},\, \boldsymbol{x}^* \rangle = \zeta_p(\mathbb{K}) \end{aligned}$$

hold for all $\mu \geq 0$. This implies that $\{\boldsymbol{x}^* + \mu\Delta\boldsymbol{x} : \mu \geq 0\}$ forms an unbounded ray in the set of optimal solutions of P($\mathbb{K}$). This contradicts to the assumption of "only if" part. □

Proof of Theorem 2.1 (iv): Assume that $\mathbb{K}$ is not pointed. We know by (iii) that the set of optimal solution of P($\mathbb{K}$) is either empty or unbounded. Thus it suffices to show that the set of optimal solution is nonempty if Condition Ri'($\mathbb{K}^*$) is satisfied. To show this, we embed the primal COP P($\mathbb{K}$) to an equivalent COP P($\mathbb{J}$) in a subspace $\mathbb{L}$ of $\mathbb{V}$, to which we will apply assertion (iii). We take the subspace of $\mathbb{V}$ spanned by $\mathbb{K}^*$ for the subspace $\mathbb{L}$. Let Π denote the orthogonal projection from $\mathbb{V}$ onto $\mathbb{L}$, and define the convex cone $\mathbb{J} = \{\Pi(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{K}\}$. Then we observe that

$$\begin{aligned}&\mathbb{J} \subset \mathbb{L},\ \mathbb{K}^* \subset \mathbb{L},\ \boldsymbol{h} \in \mathbb{K}^* \subset \mathbb{L},\ \mathbb{L}^\perp \subset \mathbb{K} \text{ and} \\ &\boldsymbol{q} \in \mathbb{L} \text{ (since otherwise } \zeta_p(\mathbb{K}) = -\infty).\end{aligned}$$

Hence we may regard $\mathbb{L}$ as the entire space on which the primal-dual pair P($\mathbb{J}$) and D($\mathbb{J}^*$) are defined, where $\mathbb{J}^* = \{\boldsymbol{y} \in \mathbb{L} : \langle \boldsymbol{x}, \boldsymbol{y}\rangle \geq 0 \text{ for every } \boldsymbol{x} \in \mathbb{J}\} = \mathbb{K}^*$. Furthermore, for every feasible solution $\boldsymbol{x}$ of P($\mathbb{K}$),

$$1 = \langle \boldsymbol{h}, \boldsymbol{x}\rangle = \langle \boldsymbol{h}, \Pi(\boldsymbol{x})\rangle,\ \Pi(\boldsymbol{x}) \in \mathbb{J} \text{ and } \langle \boldsymbol{q}, \boldsymbol{x}\rangle = \langle \boldsymbol{q}, \Pi(\boldsymbol{x})\rangle.$$

This implies the equivalence of P($\mathbb{K}$) to P($\mathbb{J}$) in the sense that $\boldsymbol{x}$ is a feasible solution of P($\mathbb{K}$) if and only if its projection $\Pi(\boldsymbol{x})$ onto $\mathbb{L}$ is a feasible solution of P($\mathbb{J}$) with the same objective value $\langle \boldsymbol{q}, \Pi(\boldsymbol{x})\rangle = \langle \boldsymbol{q}, \boldsymbol{x}\rangle$. By construction, $\mathbb{J}$ is pointed with respect the space $\mathbb{L}$. By assertion (iii), the set of optimal solutions of P($\mathbb{J}$) is nonempty and bounded if and only if Condition Ri'($\mathbb{J}^*$) (= Condition Ri'($\mathbb{K}^*$)) holds. Therefore, the set of optimal solutions of P($\mathbb{K}$) is nonempty if Condition Ri'($\mathbb{K}^*$) holds. □

Remark 2.3. The dual version of (iii) was given in [10, Theorem 5].

3 Proofs of Theorem 1.1 and related results

Throughout this section, we use the notation and symbols given in (2). We need the following lemmas to prove Theorem 1.1.

Lemma 3.1. *Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be nonempty closed convex cones in $\mathbb{V}$. Then $(\mathbb{K}_1 \cap \mathbb{K}_2)^* =$ cl$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ and* $\mathrm{relint}((\mathbb{K}_1 \cap \mathbb{K}_2)^*) = \mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*)$.

Proof. Assertion $(\mathbb{K}_1 \cap \mathbb{K}_2)^* = \mathrm{cl}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ is well-known (see, for example, [12]). Assertion $\mathrm{relint}((\mathbb{K}_1 \cap \mathbb{K}_2)^*) = \mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ follows from $\mathrm{cl}(\mathbb{K}_1^* + \mathbb{K}_2^*) = (\mathbb{K}_1 \cap \mathbb{K}_2)^*$ and the convexity of $\mathbb{K}_1^* + \mathbb{K}_2^*$. □

Lemma 3.2. *Let $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$. Assume that $-\infty < \eta_d < \infty$ and Condition* Ri'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ *is satisfied. Then $\eta_d = \zeta_d(\mathbb{K}_1^* + \mathbb{K}_2^*) = \zeta_d(\mathbb{K}^*)$.*

Proof. The identity $\eta_d = \zeta_d(\mathbb{K}_1^* + \mathbb{K}_2^*)$ holds by definition. Since $\zeta_d(\mathbb{K}_1^* + \mathbb{K}_2^*) \leq \zeta_d(\mathbb{K}^*)$ holds by $\mathbb{K}_1^* + \mathbb{K}_2^* \subset \mathbb{K}^*$, it suffices to show that $\zeta_d(\mathbb{K}^*) \leq \zeta_d(\mathbb{K}_1^* + \mathbb{K}_2^*)$. Let t be a feasible solution of dual COP D($\mathbb{K}^*$); $\boldsymbol{q} - \boldsymbol{h}t \in \mathbb{K}^* = \mathrm{cl}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ by Lemma 3.1. By Condition Ri'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$, we know that $\boldsymbol{q} - \boldsymbol{h}\widetilde{t} \in \mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*)$. Hence the convex combination $(1-\epsilon)(\boldsymbol{q} - \boldsymbol{h}t) + \epsilon(\boldsymbol{q} - \boldsymbol{h}\widetilde{t})$ with any small $\epsilon \in (0, 1]$ lies in $\mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*)$. It follows that

$$\mathbb{K}_1^* + \mathbb{K}_2^* \supset \mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*) \ni (1-\epsilon)(\boldsymbol{q} - \boldsymbol{h}t) + \epsilon((\boldsymbol{q} - \boldsymbol{h}\widetilde{t}) = \boldsymbol{q} - \boldsymbol{h}(t + \epsilon(\widetilde{t} - t))$$

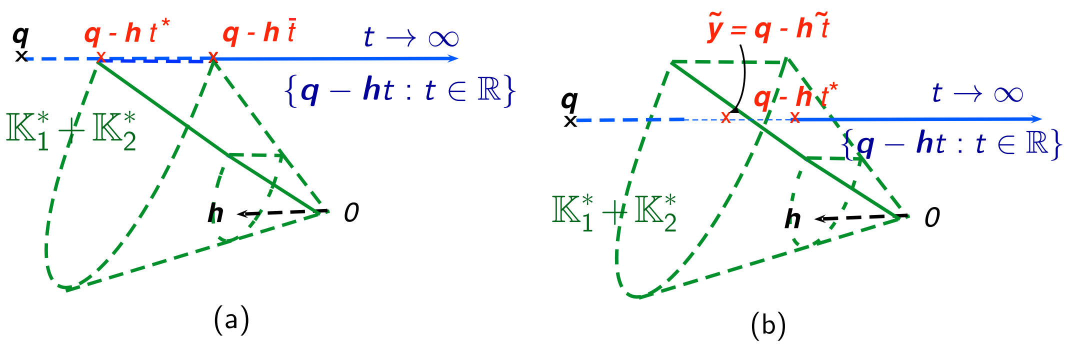

Figure 2: Illustration for assertion (i) of Theorem 1.1

holds for any small $\epsilon \in (0,1]$. This implies that $(t + \epsilon(\widetilde{t} - t))$ is a feasible solution of dual COP $\mathrm{D}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ for any small $\epsilon \in (0,1]$. Therefore we conclude that $\zeta_d(\mathbb{K}^*) \leq \zeta_d(\mathbb{K}_1^* + \mathbb{K}_2^*)$. □

Proof of Theorem 1.1 (i): Let $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$. Then primal COP (1) and $\mathrm{P}(\mathbb{K})$ are identical (hence $\eta_p = \zeta_p(\mathbb{K})$). If Condition Cl is satisfied, then dual COP (3) and $\mathrm{D}(\mathbb{K}^*)$ are identical (hence $\eta_d = \zeta_d(\mathbb{K}^*)$) by Lemma 3.1, and if Condition Ri is satisfied, then $\eta_d = \zeta_d(\mathbb{K}^*)$ holds by Lemma 3.2. Therefore, assertion (i) follows from assertion (i) of Theorem 2.1.

Proof of Theorem 1.1 (ii): Let $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$. Then, by Lemma 3.2, primal COP (1) and dual COP (3) are identical to $\mathrm{P}(\mathbb{K})$ and $\mathrm{D}(\mathbb{K}^*)$, respectively. Hence assertion (ii) follows from assertion (ii) of Theorem 2.1. □

Proof of Theorem 1.1 (iii) and (iv): Let $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$. Then we can replace primal COP (1) with $\mathrm{P}(\mathbb{K})$ in assertions (iii) and (iv) since they are identical. Also Po and Po'($\mathbb{K}$) are identical. By Lemma 3.1, we also see $\mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*) = \mathrm{relint}(\mathbb{K}^*)$. Hence Condition Ri'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ (= Condition Ri) holds if and only if Condition Ri'$(\mathbb{K}^*)$ holds.

"if part of (iii)": Assume that Conditions Ri'$(\mathbb{K}^*)$ and Po'$(\mathbb{K})$ are satisfied. Then $\zeta_d(\mathbb{K}_1^* + \mathbb{K}_2^*) = \zeta_d(\mathbb{K}^*)$ by Lemma 3.2. Thus, $-\infty < \zeta_p(\mathbb{K}) < \infty$ or $-\infty < \zeta_d(\mathbb{K}^*) < \infty$ by the assumption of the theorem. Therefore, the set of optimal solutions of $\mathrm{P}(\mathbb{K})$, which coincides with the set of optimal solutions of primal COP $\mathrm{P}(\mathbb{K}_1 \cap \mathbb{K}_2)$, is nonempty and bounded by (iii) of Theorem 2.1.

"only if part of (iii)": Conversely, assume that the set of optimal solutions of $\mathrm{P}(\mathbb{K}_1 \cap \mathbb{K}_2) = \mathrm{P}(\mathbb{K})$ is nonempty and bounded, which implies $-\infty < \zeta_p(\mathbb{K}) < \infty$, then Conditions Ri'$(\mathbb{K}^*)$, which is equivalent to Condition Ri, and Po'$(\mathbb{K})$ hold by (iii) of Theorem 2.1.

Assertion (iv) also follows from assertion (iv) of Theorem 2.1 since primal COP (1), η_d, Po and Ri are identical or equivalent to $\mathrm{P}(\mathbb{K})$, $\zeta_d(\mathbb{K}^*)$, Po'$(\mathbb{K})$ and Ri'$(\mathbb{K}^*)$, respectively. □

Figure 2 illustrates assertion (i) of Theorem 1.1. In case (a), neither Cl'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ nor Ri'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ is satisfied. The cone $\mathbb{K}_1^* + \mathbb{K}_2^*$ is almost open except the solid line that forms an extremal ray of the cone. The line $\{\boldsymbol{q} - \boldsymbol{h}t : t \in \mathbb{R}\}$ touches the cone at the single point $\boldsymbol{q} - \boldsymbol{h}t^*$, but its line segment between $\boldsymbol{q} - \boldsymbol{h}t^*$ and $\boldsymbol{q} - \boldsymbol{h}\bar{t}$ is included in a face of its closure $\mathrm{cl}(\mathbb{K}_1^* + \mathbb{K}_2^*) = (\mathbb{K}_1 \cap \mathbb{K})^*$. Hence $\zeta_d(\mathbb{K}_1^* + \mathbb{K}_2^*) = t^* < \bar{t} = \zeta_d((\mathbb{K}_1 \cap \mathbb{K}_2)^*) = \zeta_p(\mathbb{K}_1 \cap \mathbb{K}_2)$, which implies that a duality gap $\bar{t} - t^* > 0$ exists between primal COP $\mathrm{P}(\mathbb{K}_1 \cap \mathbb{K}_2)$ and its dual $\mathrm{D}(\mathbb{K}_1^* + \mathbb{K}_2^*)$.

The cone $\mathbb{K}_1^* + \mathbb{K}_2^*$ in case (b) is the same as in case (a), but the line $\{\boldsymbol{q} - \boldsymbol{h}t : t \in \mathbb{R}\}$ penetrates the interior of the cone, so that Ri'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ is satisfied, where $\widetilde{\boldsymbol{y}} = \boldsymbol{q} - \boldsymbol{h}\widetilde{t} \in \mathrm{int}(\mathbb{K}_1^* + \mathbb{K}_2^*)$. Both $\mathrm{D}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ and $\mathrm{D}((\mathbb{K}_1 \cap \mathbb{K}_2)^*)$ share the common optimal value t^* although $\boldsymbol{q} - \boldsymbol{h}t^* \not\in \mathbb{K}_1^* + \mathbb{K}_2^*$ and $\mathrm{D}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ has no feasible solution that attains the optimal value t^*. In this case, there is no duality gap between primal COP $\mathrm{P}(\mathbb{K}_1 \cap \mathbb{K}_2)$ and its dual $\mathrm{D}(\mathbb{K}_1^* + \mathbb{K}_2^*)$.

Remark 3.3. Theorem 1.1 could be proved by using [16, Propositions 2.6 and 2.8] and [10, Theorems 5 and 7]. But we need to convert COPs $\mathrm{P}(\mathbb{K}_1 \cap \mathbb{K}_2)$ and $\mathrm{D}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ into primal-dual pairs of COPs in different forms. See also Section 4.

Necessary and/or sufficient conditions for the closedness of $\mathbb{K}_1^* + \mathbb{K}_2^*$ were throughly studied by Pataki [12]. We refer to some of them to derive three important cases where Condition Cl holds. A closed convex cone $\mathbb{J} \subset \mathbb{V}$ is called nice if $\mathbb{J}^* + F^{\perp}$ is closed for every face F of $\mathbb{J}$. Polyhedral cones, positive semidefinite cones and doubly nonnegative cones are known to be nice. For every closed convex cone $\mathbb{J}$ and $\boldsymbol{x} \in \mathbb{J}$, let $\mathrm{dir}(\boldsymbol{x}, \mathbb{J})$ denote the convex cone $\{\boldsymbol{w} : \boldsymbol{x} + t\boldsymbol{w} \in \mathbb{J} \text{ for some } t > 0\}$ and $\mathrm{face}(\boldsymbol{x}, \mathbb{J})$ the smallest face of $\mathbb{J}$ containing $\boldsymbol{x}$.

Lemma 3.4. *Let $\bar{\boldsymbol{x}} \in relint(\mathbb{K}_1 \cap \mathbb{K}_2)$, $F_1 = face(\bar{\boldsymbol{x}}, \mathbb{K}_1)$ and $F_2 = face(\bar{\boldsymbol{x}}, \mathbb{K}_2)$. Then*

$$dir(\bar{\boldsymbol{x}}, \mathbb{K}_1) \cap dir(\bar{\boldsymbol{x}}, \mathbb{K}_2) = (cl\ dir(\bar{\boldsymbol{x}}, \mathbb{K}_1)) \cap (cl\ dir(\bar{\boldsymbol{x}}, \mathbb{K}_2)) \tag{6}$$

is a necessary condition for $\mathbb{K}_1^* + \mathbb{K}_2^*$ to be closed. If in addition $\mathbb{K}_1^* + F_1^{\perp}$ and $\mathbb{K}_2^* + F_2^{\perp}$ are closed — in particular, if $\mathbb{K}_1$ and $\mathbb{K}_2$ are both nice — then (6) is a sufficient condition.

Proof. The assertion of the lemma is included in Theorem 5.1 of [12]. In fact, (6) is given there as one of four equivalent necessary conditions for the closedness of $\mathbb{K}_1^* + \mathbb{K}_2^*$, which are sufficient under the additional assumption mentioned above. □

Lemma 3.5. *Condition* Cl *holds (i.e., $\mathbb{K}_1^* + \mathbb{K}_2^*$ is closed) if one of the following conditions (i), (ii) and (iii) is satisfied.*

(i) $\mathbb{K}_1$ and $\mathbb{K}_2$ are polyhedral, i.e., $\mathbb{K}_i = \{\boldsymbol{x} \in \mathbb{V} : \langle \boldsymbol{a}_i^j, \boldsymbol{x} \rangle \geq 0\ (j = 1, \ldots, \ell_i\}$ for some $\boldsymbol{a}_i^j \in \mathbb{V}$ $(j = 1, 2, \ldots, \ell_i, i = 1, 2)$.

(ii) There exists an $\widetilde{\boldsymbol{x}} \in (\mathrm{int}\mathbb{K}_1) \cap \mathbb{K}_2$.

(iii) The cone $\mathbb{K}_1$ is nice, the cone $\mathbb{K}_2$ is polyhedral, and there exists an $\widetilde{\boldsymbol{x}} \in (\mathrm{relint}\mathbb{K}_1) \cap \mathbb{K}_2$.

Proof. (i) In this case, we see that

$$(\mathbb{K}_1 \cap \mathbb{K}_2)^* \ = \ \left\{ \boldsymbol{y} = \sum_{j=1}^{\ell_1} \boldsymbol{a}_1^j u_j + \sum_{k=1}^{\ell_2} \boldsymbol{a}_2^k v_k : \begin{array}{l} u_j \geq 0\ (j = 1, \ldots, \ell_1), \\ v_k \geq 0\ (k = 1, \ldots, \ell_2) \end{array} \right\} \ = \ \mathbb{K}_1^* + \mathbb{K}_2^*.$$

(ii) Let $\widehat{\boldsymbol{x}} \in \mathrm{relint}\mathbb{K}_2$. Since $\widetilde{\boldsymbol{x}} \in (\mathrm{int}\mathbb{K}_1) \cap \mathbb{K}_2$, we can take a sufficiently small $\epsilon > 0$ such that $\bar{\boldsymbol{x}} = (1 - \epsilon)\widetilde{\boldsymbol{x}} + \epsilon\widehat{\boldsymbol{x}} \in (\mathrm{int}\mathbb{K}_1) \cap \mathrm{relint}\mathbb{K}_2 \subset \mathrm{relint}(\mathbb{K}_1 \cap \mathbb{K}_2)$. We see from

$\bar{\boldsymbol{x}} \in \text{int}\mathbb{K}_1$ and $\bar{\boldsymbol{x}} \in \text{relint}\mathbb{K}_2$ that $F_i = \text{face}(\bar{\boldsymbol{x}}, \mathbb{K}_i) = \mathbb{K}_i$, which implies $\mathbb{K}_i^* + F_i^{\perp} = \mathbb{K}_i^*$, and $\text{dir}(\bar{\boldsymbol{x}}, \mathbb{K}_i) = \text{linspan}\mathbb{K}_i$ $(i = 1, 2)$. Since $\mathbb{K}_i^*$ and $\text{linspan}\mathbb{K}_i$ $(i = 1, 2)$ are closed, $\mathbb{K}_1^* + \mathbb{K}_2^*$ is closed by Lemma 3.4.

(iii) Let $\hat{\boldsymbol{x}} \in \text{relint}(\mathbb{K}_1 \cap \mathbb{K}_2)$. Then $\bar{\boldsymbol{x}} = (\widetilde{\boldsymbol{x}} + \hat{\boldsymbol{x}})/2 \in (\text{relint}\mathbb{K}_1) \cap \text{relint}(\mathbb{K}_1 \cap \mathbb{K}_2)$ since $\widetilde{\boldsymbol{x}} \in \text{relint}\mathbb{K}_1$, $\hat{\boldsymbol{x}} \in \mathbb{K}_1$, $\hat{\boldsymbol{x}} \in \text{relint}(\mathbb{K}_1 \cap \mathbb{K}_2)$ and $\widetilde{\boldsymbol{x}} \in \mathbb{K}_1 \cap \mathbb{K}_2$ hold. Since both cones $\mathbb{K}_1$ and $\mathbb{K}_2$ are nice, it suffices to show (6). We see from $\widetilde{\boldsymbol{x}} \in \text{relint}\mathbb{K}_1$ that $\text{dir}(\bar{\boldsymbol{x}}, \mathbb{K}_1) = \text{linspan}\mathbb{K}_1$. Hence $\text{dir}(\bar{\boldsymbol{x}}, \mathbb{K}_1) = \text{cl dir}(\bar{\boldsymbol{x}}, \mathbb{K}_1)$. Since $\mathbb{K}_2$ is a polyhedral cone by the assumption, $\text{dir}(\bar{\boldsymbol{x}}, \mathbb{K}_2)$ becomes polyhedral, hence $\text{dir}(\bar{\boldsymbol{x}}, \mathbb{K}_2) = \text{cl dir}(\bar{\boldsymbol{x}}, \mathbb{K}_2)$. Therefore (6) holds. □

The lemma below provides Slater type sufficient conditions for Conditions $\text{Ri}(\mathbb{K}_1^* + \mathbb{K}_2^*)$ and $\text{Po}(\mathbb{K}_1 \cap \mathbb{K}_2)$.

Lemma 3.6.

(i) Conditions Ri *and* Po *holds simultaneously if and only if there exists a* $\widetilde{t} \in \mathbb{R}$ *such that* $\boldsymbol{q} - \boldsymbol{h}\widetilde{t} \in \textit{int}(\mathbb{K}_1^* + \mathbb{K}_2^*)$.

(ii) Assume that $\widetilde{\boldsymbol{y}}_1 \equiv \boldsymbol{q} - \boldsymbol{h}\widetilde{t} - \widetilde{\boldsymbol{y}}_2 \in \textit{int}(\mathbb{K}_1^*)$ *for some* $(\widetilde{t}, \widetilde{\boldsymbol{y}}_2) \in \mathbb{R} \times \mathbb{K}_2^*$. *Then Conditions* Ri *and* Po *hold.*

Proof. Assertion (i) follows from Lemmas 2.2 and 3.1. We only prove assertion (ii). It suffices to show that $\widetilde{\boldsymbol{y}} \equiv \widetilde{\boldsymbol{y}}_1 + \widetilde{\boldsymbol{y}}_2 = \boldsymbol{q} - \boldsymbol{h}\widetilde{t} \in \text{int}(\mathbb{K}_1^* + \mathbb{K}_2^*)$. Since $\widetilde{\boldsymbol{y}}_1 \in \text{int}\mathbb{K}_1^*$, we can take a positive number ϵ such that $\boldsymbol{y}_1 \in \mathbb{K}_1^*$ if $\|\boldsymbol{y}_1 - \widetilde{\boldsymbol{y}}_1\| \leq \epsilon$. Assume that $\boldsymbol{y} \in \mathbb{V}$ and $\|\boldsymbol{y} - \widetilde{\boldsymbol{y}}\| \leq \epsilon$. Let $\boldsymbol{y}_1 = \widetilde{\boldsymbol{y}}_1 + (\boldsymbol{y} - \widetilde{\boldsymbol{y}})$. Then $\boldsymbol{y} = \boldsymbol{y}_1 + \widetilde{\boldsymbol{y}}_2$, $\|\boldsymbol{y}_1 - \widetilde{\boldsymbol{y}}_1\| = \|\boldsymbol{y} - \widetilde{\boldsymbol{y}}\| \leq \epsilon$ (hence $\boldsymbol{y}_1 \in \mathbb{K}_1^*$), and $\widetilde{\boldsymbol{y}}_2 \in \mathbb{K}_2^*$ by assumption. Therefore, $\boldsymbol{y} \in \mathbb{K}_1^* + \mathbb{K}_2^*$ and we have shown that $\widetilde{\boldsymbol{y}} \in \text{int}(\mathbb{K}_1^* + \mathbb{K}_2^*)$. □

4 Strong duality in a symmetric primal-dual pair of COPs

Let $\mathbb{E}_p$ and $\mathbb{E}_d$ be finite dimensional vector spaces with inner products, which are both denoted by $\langle \cdot, \cdot \rangle$, and $\mathbb{J}_p$ and $\mathbb{J}_d$ closed convex cones of $\mathbb{E}_p$ and $\mathbb{E}_d$, respectively. Let $\boldsymbol{b} \in \mathbb{E}_d$, $\boldsymbol{c} \in \mathbb{E}_p$, $\mathcal{A}$ a linear map from $\mathbb{E}_p$ into $\mathbb{E}_d$. We denote the adjoint of $\mathcal{A}$ by $\mathcal{A}^*$. We consider the following primal-dual pair of COPs:

$$\theta_p = \inf \left\{ \langle \boldsymbol{c}, \boldsymbol{u} \rangle : \boldsymbol{u} \in \mathbb{J}_p, \ \mathcal{A}\boldsymbol{u} - \boldsymbol{b} \in \mathbb{J}_d \right\}. \tag{7}$$

$$\theta_d = \sup \left\{ \langle \boldsymbol{b}, \boldsymbol{v} \rangle : \boldsymbol{v} \in \mathbb{J}_d^*, \ \boldsymbol{c} - \mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^* \right\}. \tag{8}$$

The following three convex cones play essential roles in characterizing strong duality of the primal-dual pair above in the subsequent discussion.

$$\begin{aligned}
\mathbb{M}_p &= \left\{ (\alpha, \boldsymbol{v} - \mathcal{A}\boldsymbol{u}) : \alpha \geq \langle \boldsymbol{c}, \boldsymbol{u} \rangle, \ \boldsymbol{u} \in \mathbb{J}_p, \ \boldsymbol{v} \in \mathbb{J}_d \right\}, \\
\mathbb{N}_p &= \left\{ \boldsymbol{v} - \mathcal{A}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{J}_p, \ \boldsymbol{v} \in \mathbb{J}_d \right\}, \\
\mathbb{N}_p^* &= \{ \boldsymbol{v} \in \mathbb{J}_d^* : -\mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^* \}.
\end{aligned}$$

We derive the following result from Theorem 1.1.

Theorem 4.1. *Assume that $-\infty < \theta_p < \infty$ or $-\infty < \theta_d < \infty$. Then the assertions below hold.*

(i) If $\mathbb{M}_p$ is closed or $-\boldsymbol{b} \in \mathrm{relint}\mathbb{N}_p$, then $-\infty < \theta_p = \theta_d < \infty$ holds.

(ii) If $\mathbb{M}_p$ is closed, then primal COP (7) is solvable.

(iii) The set of optimal solutions of dual COP (8) is nonempty and bounded if and only if $-\boldsymbol{b} \in \mathrm{int}\mathbb{N}_p$, which implies that $\mathbb{N}_p^*$ is pointed, holds.

(iv) The set of optimal solutions of dual COP (8) is nonempty and unbounded if $\mathbb{N}_p^*$ is not pointed and $-\boldsymbol{b} \in \mathrm{relint}\mathbb{N}_p$ holds.

Remark 4.2. $\mathbb{M}_p$ and $\mathbb{N}_p$ were originally introduced by Shapiro [16] to establish strong duality between COPs (7) and (8). Theorem 4.1 is a slight modification (and extension) of [16, Propositions 2.6 and 2.8]. We note that Shapiro in [16] dealt with the case where the spaces $\mathbb{E}_p$ and $\mathbb{E}_d$ can be a general vector (not necessarily finite dimensional) space. Theorem 4.1 could be proved by [16, Propositions 2.6 and 2.8] and [10, Theorem 5].

As a corollary, we obtain:

Corollary 4.3. *If the set of optimal solutions (or the feasible region) of primal* COP (7) *or dual* COP (8) *is nonempty and bounded, then $-\infty < \theta_p = \theta_d < \infty$.*

Proof. Assume that the set of optimal solutions of dual COP (8) is nonempty and bounded. Then, by the assumption and (iii) of Theorem 4.1, we see that $-\infty < \theta_d < \infty$ and $-\boldsymbol{b} \in \mathrm{int}\mathbb{N}_p$ hold. By (i) of Theorem 4.1, we obtain that $-\infty < \theta_p = \theta_d < \infty$. Since the primal-dual pair of COPs (7) and (8) is symmetric, we can derive the same conclusion even if we regard COP (8) as primal and COP (7) as dual. See the discussions in Sections 4.1 and 4.2. This corollary could also be obtained from Corollary 1.2 □

We convert primal COP (7) to COP (1) in Sections 4.1, and dual COP (8) to COP (1) in Section 4.2. The proof of Theorem 4.1 is stated in Section 4.3 where the latter conversion given in Section 4.2 is utilized. Although Sections 4.1 is not relevant to the proof of Theorem 4.1, it may clarify the discussion in Section 4.2 and the dual counter part of Theorem 4.1. The discussion in Section 4.1 will help the reader to derive such counter part. Table 2 summarizes the duality results shown in Theorems 4.1, Corollary 4.3 and the results on their duals (see Section 4.1).

4.1 Conversion of the primal-dual pair of COP (7) and COP (8) to the primal-dual pair of COP (1) and COP (3)

Let

$$\left.\begin{array}{rcl} \mathbb{V} & = & \mathbb{R} \times \mathbb{E}_p = \{(x_0, \boldsymbol{u}) : x_0 \in \mathbb{R},\ \boldsymbol{u} \in \mathbb{E}_p\}, \\ \mathbb{K}_1 & = & \mathbb{R}_+ \times \mathbb{J}_p,\ \mathbb{K}_2 = \{(x_0, \boldsymbol{u}) \in \mathbb{V} : x_0 \in \mathbb{R},\ \mathcal{A}\boldsymbol{u} - \boldsymbol{b}x_0 \in \mathbb{J}_d\}, \\ \boldsymbol{q} & = & (0, \boldsymbol{c}) \in \mathbb{V},\ \boldsymbol{h} = (1, \boldsymbol{0}) \in \mathbb{K}_1^*. \end{array}\right\} \quad (9)$$

It should be noted that the symbols $\mathbb{V}$, $\mathbb{K}_1$, $\mathbb{K}_2$, $\boldsymbol{q}$ and $\boldsymbol{h}$ above are defined locally in this section. The same symbols will be also defined locally in Section 4.2 with different

Sufficient conditions for strong duality $\theta_p = \theta_d$	The sets of primal and dual optimal solutions pSol	dSol
$\mathbb{M}_p$: closed	nonempty	?
$\mathbb{N}_p^*$: pointed & $-\boldsymbol{b} \in \mathrm{relint}\mathbb{N}_p$	?	nonempty & bounded
$\mathbb{N}_p^*$: not pointed & $-\boldsymbol{b} \in \mathrm{relint}\mathbb{N}_p$	?	nonempty & unbounded
dSol : nonempty & bounded	?	nonempty & bounded
$\mathbb{M}_d$: closed	?	nonempty
$\mathbb{N}_d^*$: pointed & $\boldsymbol{c} \in \mathrm{relint}\mathbb{N}_d$	nonempty & bounded	?
$\mathbb{N}_d^*$: not pointed & $\boldsymbol{c} \in \mathrm{relint}\mathbb{N}_d$	nonempty & unbounded	?
pSol : nonempty & bounded	nonempty & bounded	?

Table 2: A summary of strong duality $-\infty < \theta_p = \theta_d < \infty$, solvability of primal COP (7) and solvability of dual COP (8). For all cases, we assume that $-\infty < \theta_p < \infty$ or $-\infty < \theta_d < \infty$ holds. "pSol" and "dSol" denote the sets of optimal solutions of COP (7) and COP (8), respectively. If $\mathbb{N}_p^*$ ($\mathbb{N}_d^*$) is pointed, then $\mathrm{relint}\mathbb{N}_p = \mathrm{int}\mathbb{N}_p$ ($\mathrm{relint}\mathbb{N}_d = \mathrm{int}\mathbb{N}_d$). $\mathbb{M}_p = \left\{(\alpha, \boldsymbol{v} - \mathcal{A}\boldsymbol{u}) : \alpha \geq \langle \boldsymbol{c}, \boldsymbol{u} \rangle, \ \boldsymbol{u} \in \mathbb{J}_p, \ \boldsymbol{v} \in \mathbb{J}_d\right\}$, $\mathbb{N}_p = \left\{\boldsymbol{v} - \mathcal{A}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{J}_p, \ \boldsymbol{v} \in \mathbb{J}_d\right\}$, $\mathbb{N}_p^* = \{\boldsymbol{v} \in \mathbb{J}_d^* : -\mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^*\}$, $\mathbb{M}_d = \left\{(\beta, \boldsymbol{u} + \mathcal{A}^*\boldsymbol{v}) : \beta \leq \langle \boldsymbol{b}, \boldsymbol{v} \rangle, \ \boldsymbol{u} \in \mathbb{J}_p^*, \boldsymbol{v} \in \mathbb{J}_d^*\right\}$, $\mathbb{N}_d = \left\{\boldsymbol{u} + \mathcal{A}^*\boldsymbol{v} : \boldsymbol{u} \in \mathbb{J}_p^*, \boldsymbol{v} \in \mathbb{J}_d^*\right\}$, $\mathbb{N}_d^* = \{\boldsymbol{u} \in \mathbb{J}_p : \mathcal{A}\boldsymbol{u} \in \mathbb{J}_d\}$. "?" means that the existence of an optimal solution is not guaranteed.

meanings. As the inner product of $\boldsymbol{x} = (x_0, \boldsymbol{u}) \in \mathbb{V}$, $\boldsymbol{y} = (t_0, \boldsymbol{w}) \in \mathbb{V}$, we define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_0 t_0 + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$. Then,

$$\begin{aligned} &\mathbb{K}_1^* = \mathbb{R}_+ \times \mathbb{J}_p^*, \ \mathbb{K}_2^* = \{(\langle -\boldsymbol{b}, \boldsymbol{v} \rangle, \mathcal{A}^*\boldsymbol{v}) : \boldsymbol{v} \in \mathbb{J}_d^*\}, \\ &\boldsymbol{u} \in \mathbb{J}_p, \ \mathcal{A}\boldsymbol{u} - \boldsymbol{b} \in \mathbb{J}_d \text{ if and only if } \boldsymbol{x} = (x_0, \boldsymbol{u}) \in \mathbb{K}_1, \ \boldsymbol{x} \in \mathbb{K}_2, \ \langle \boldsymbol{h}, \boldsymbol{x} \rangle = 1. \end{aligned}$$

Thus, we can rewrite COP (7) as COPs (1). We also see that

$$\begin{aligned} \boldsymbol{q} - \boldsymbol{h}t - \boldsymbol{y}_2 &= (0, \boldsymbol{c}) - (t, 0) - (\langle -\boldsymbol{b}, \boldsymbol{v} \rangle, \mathcal{A}^*\boldsymbol{v}) \\ &= (\langle \boldsymbol{b}, \boldsymbol{v} \rangle - t, \boldsymbol{c} - \mathcal{A}^*\boldsymbol{v}) \\ &\quad \text{if } \boldsymbol{y}_2 = (\langle -\boldsymbol{b}, \boldsymbol{v} \rangle, \mathcal{A}^*\boldsymbol{v}) \in \mathbb{K}_2^* \ \text{ for some } \boldsymbol{v} \in \mathbb{J}_d^*; \text{ hence} \\ \langle \boldsymbol{b}, \boldsymbol{v} \rangle \geq t &\text{ and } \boldsymbol{c} - \mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^* \text{ for some } \boldsymbol{v} \in \mathbb{J}_d^* \text{ if and only if } \boldsymbol{q} - \boldsymbol{h}t \in \mathbb{K}_1^* + \mathbb{K}_2^*. \end{aligned}$$

Therefore, COP (8) can be rewritten as COP (3). In this case, $\mathbb{K}_1^* + \mathbb{K}_2^*$ is represented as $\left\{(-\langle \boldsymbol{b}, \boldsymbol{v} \rangle + s, \mathcal{A}^*\boldsymbol{v} + \boldsymbol{u}) : s \in \mathbb{R}_+, \ \boldsymbol{u} \in \mathbb{J}_p^*, \boldsymbol{v} \in \mathbb{J}_d^*\right\}$. Obviously, $\mathbb{K}_1^* + \mathbb{K}_2^*$ is closed if and only if

$$\mathbb{M}_d \ = \ \left\{(\beta, \boldsymbol{u} + \mathcal{A}^*\boldsymbol{v}) : \beta \leq \langle \boldsymbol{b}, \boldsymbol{v} \rangle, \ \boldsymbol{u} \in \mathbb{J}_p^*, \boldsymbol{v} \in \mathbb{J}_d^*\right\}$$

is closed. The lemma below converts Conditions Ri (= Ri'($\mathbb{K}_1^* + \mathbb{K}_2^*$)) and Po (=Po'($\mathbb{K}_1 \cap \mathbb{K}_2$)) on COP (3) to the corresponding conditions on COP (8), respecctively.

Lemma 4.4. *Assume that $\mathbb{V}$, $\mathbb{K}_1$, $\mathbb{K}_2$, $\boldsymbol{q}$ and $\boldsymbol{h}$ are given in (9).*

(i) Condition Ri'($\mathbb{K}_1^* + \mathbb{K}_2^*$) holds if and only if $\boldsymbol{c} \in \mathrm{relint}\mathbb{N}_d$, where

$$\mathbb{N}_d = \left\{\boldsymbol{u} + \mathcal{A}^*\boldsymbol{v} : \boldsymbol{u} \in \mathbb{J}_p^*, \boldsymbol{v} \in \mathbb{J}_d^*\right\}.$$

(ii) $\mathbb{K}_1 \cap \mathbb{K}_2$ *is pointed if and only if* $\mathbb{N}_d^*$ *is pointed.*

Proof. "only if part" of (i): Assume that $\boldsymbol{q} - \boldsymbol{h}\widetilde{t} = (-\widetilde{t}, \boldsymbol{c})$ lies in $\mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*)$. Then there exists $(\widetilde{s}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}}) \in \mathbb{R}_+ \times \mathbb{J}_p^* \times \mathbb{J}_d^*$ such that

$$\begin{aligned} (-\widetilde{t}, \boldsymbol{c}) &= (\langle -\boldsymbol{b}, \widetilde{\boldsymbol{v}} \rangle + \widetilde{s}, \widetilde{\boldsymbol{u}} + \mathcal{A}^* \widetilde{\boldsymbol{v}}) \\ &\in \mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*) \\ &= \mathrm{relint} \left\{ (\langle -\boldsymbol{b}, \boldsymbol{v} \rangle + s, \boldsymbol{u} + \mathcal{A}^* \boldsymbol{v}) : s \in \mathbb{R}_+, \ \boldsymbol{u} \in \mathbb{J}_p^*, \ \boldsymbol{v} \in \mathbb{J}_d^* \right\}, \end{aligned}$$

which implies $\boldsymbol{c} = \widetilde{\boldsymbol{u}} + \mathcal{A}^* \widetilde{\boldsymbol{v}} \in \mathrm{relint}\mathbb{N}_d$.

"if part" of (i): Assume that $\boldsymbol{c} = \widetilde{\boldsymbol{u}} + \mathcal{A}^* \widetilde{\boldsymbol{v}} \in \mathrm{relint}\mathbb{N}_d$. If we take $\widetilde{s} > 0$ and let $\widetilde{t} = -(\langle -\boldsymbol{b}, \widetilde{\boldsymbol{v}} \rangle + \widetilde{s})$, then $\boldsymbol{q} - \boldsymbol{h}\widetilde{t} = (-\widetilde{t}, \boldsymbol{c}) = (\langle -\boldsymbol{b}, \widetilde{\boldsymbol{v}} \rangle + \widetilde{s}, \widetilde{\boldsymbol{u}} + \mathcal{A}^* \widetilde{\boldsymbol{v}}) \in \mathrm{relint}(\mathbb{K}_1^* + \mathbb{K}_2^*)$.

(ii): We prove contraposition of the assertion. To prove "if part', assume that $\mathbb{K}_1 \cap \mathbb{K}_2$ is not pointed. Then there exists a nonzero $(x_0, \boldsymbol{u}) \in \mathbb{V}$ such that

$$(x_0, \boldsymbol{u}) \in \mathbb{K}_1 \cap \mathbb{K}_2 \ \text{ and } \ -(x_0, \boldsymbol{u}) \in \mathbb{K}_1 \cap \mathbb{K}_2. \tag{10}$$

It follows that $x_0 = 0$, $\boldsymbol{u} \neq \boldsymbol{0}$,

$$\boldsymbol{u} \in \{\boldsymbol{u} \in \mathbb{J}_p : \mathcal{A}\boldsymbol{u} \in \mathbb{J}_d\} \ \text{ and } \ -\boldsymbol{u} \in \{\boldsymbol{u} \in \mathbb{J}_p : \mathcal{A}\boldsymbol{u} \in \mathbb{J}_d\}. \tag{11}$$

This implies $\{\boldsymbol{u} \in \mathbb{J}_p : \mathcal{A}\boldsymbol{u} \in \mathbb{J}_d\}$ is not pointed. Now, assume that $\{\boldsymbol{u} \in \mathbb{J}_p : \mathcal{A}\boldsymbol{u} \in \mathbb{J}_d\}$ is not pointed to prove "only if part". Then there exists a nonzero $\boldsymbol{u}$ satisfying (11). Let $x_0 = 0$. Then $\boldsymbol{0} \neq (x_0, \boldsymbol{u}) \in \mathbb{V}$ satisfies (10), which implies that $\mathbb{K}_1 \cap \mathbb{K}_2$ is pointed. We can easily verify the identity $\{\boldsymbol{u} \in \mathbb{J}_p : \mathcal{A}\boldsymbol{u} \in \mathbb{J}_d\} = \mathbb{N}_d^*$ as the the identity $\{\boldsymbol{v} \in \mathbb{J}_d^* : -\mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^*\} = \mathbb{N}_p^*$. ◻

4.2 Conversion of the primal-dual pair of COP (7) and COP (8) to the dual-primal pair of COP (3) and COP (1)

Since the primal-dual pair of COPs (7) and (8) is symmetric, we can interchange their role as follows:

$$\begin{aligned} \eta_p &= -\theta_d = \inf \left\{ \langle -\boldsymbol{b}, \boldsymbol{v} \rangle : \boldsymbol{v} \in \mathbb{J}_d^*, \ \boldsymbol{c} - \mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^* \right\} \\ &= \inf \left\{ \langle \widetilde{\boldsymbol{b}}, \boldsymbol{v} \rangle : \boldsymbol{v} \in \mathbb{J}_d^*, \ -\widetilde{\boldsymbol{c}} - \mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^* \right\}. \end{aligned} \tag{12}$$

$$\begin{aligned} \eta_d &= -\theta_p = \sup \left\{ \langle -\boldsymbol{c}, \boldsymbol{u} \rangle : \boldsymbol{u} \in \mathbb{J}_p, \ \mathcal{A}\boldsymbol{u} - \boldsymbol{b} \in \mathbb{J}_d \right\} \\ &= \sup \left\{ \langle \widetilde{\boldsymbol{c}}, \boldsymbol{u} \rangle : \boldsymbol{u} \in \mathbb{J}_p, \ \mathcal{A}\boldsymbol{u} + \widetilde{\boldsymbol{b}} \in \mathbb{J}_d \right\}. \end{aligned} \tag{13}$$

Here $\widetilde{\boldsymbol{b}} = -\boldsymbol{b}$ and $\widetilde{\boldsymbol{c}} = -\boldsymbol{c}$. Now we regard COP (12) induced from dual COP (8) as primal, and COP (13) induced from primal COP (7) as dual. Let

$$\left.\begin{aligned} \mathbb{V} &= \mathbb{R} \times \mathbb{E}_d = \left\{ (x_0, \boldsymbol{v}) : x_0 \in \mathbb{R}, \ \boldsymbol{v} \in \mathbb{E}_d \right\}, \\ \mathbb{K}_1 &= \mathbb{R}_+ \times \mathbb{J}_d^*, \ \mathbb{K}_2 = \left\{ (x_0, \boldsymbol{v}) \in \mathbb{V} : -\widetilde{\boldsymbol{c}}x_0 - \mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^* \right\}, \\ \boldsymbol{q} &= (0, \widetilde{\boldsymbol{b}}) \in \mathbb{V}, \ \boldsymbol{h} = (1, \boldsymbol{0}) \in \mathbb{K}_1^*. \end{aligned}\right\} \tag{14}$$

Then we can similarly show as in Section 4.1 that COPs (12) and (13) are equivalent to COPs (1) and (3), respectively. As a result, the pair of dual COP (8) and primal COPs (7) is equivalently reformulated as the primal-dual pair of COPs (1) and COPs (3) with $\mathbb{V}$, $\mathbb{K}_1$, $\mathbb{K}_2$, $\boldsymbol{q}$ and $\boldsymbol{h}$ given in (14). We also see that

$$\mathbb{K}_1^* + \mathbb{K}_2^* \;\; = \;\; \left\{(\alpha, \boldsymbol{v} - \mathcal{A}\boldsymbol{u}) : \alpha \geq \langle \boldsymbol{c}, \, \boldsymbol{u}\rangle, \; \boldsymbol{u} \in \mathbb{J}_p, \; \boldsymbol{v} \in \mathbb{J}_d\right\} = \mathbb{M}_p.$$

The following lemma can be proved similarly to Lemma 4.4.

Lemma 4.5. *Assume that $\mathbb{V}$, $\mathbb{K}_1$, $\mathbb{K}_2$, $\boldsymbol{q}$ and $\boldsymbol{h}$ are given in (14).*

(i) Condition Ri'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ *holds if and only if* $-\boldsymbol{b} \in \mathrm{relint}\mathbb{N}_p$.

(ii) $\mathbb{K}_1 \cap \mathbb{K}_2$ *is pointed if and only if* $\mathbb{N}_p^*$ *is pointed.*

4.3 Proof of Theorem 4.1

We reformulate the pair of dual COP (8) and primal COP (7) as the pair of primal COP (1) and dual COP (3) with $\mathbb{V}$, $\mathbb{K}_1$, $\mathbb{K}_2$, $\boldsymbol{q}$ and $\boldsymbol{h}$ given in (14) as discussed in the previous section, and apply Theorem 1.1 to the reformulated pair. We know that $\eta_p = -\theta_d$, $\eta_d = -\theta_p$, Condition Cl'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ is equivalent to the closedness of $\mathbb{M}_p$, Condition Ri'$(\mathbb{K}_1^* + \mathbb{K}_2^*)$ is equivalent to $-\boldsymbol{b} \in \mathrm{relint}\mathbb{N}_p$, and $\mathbb{K}_1 \cap \mathbb{K}_2$ is pointed if and only if $\{\boldsymbol{v} \in \mathbb{J}_d^* : -\mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^*\} = \mathbb{N}_p^*$ is pointed. We also know by assertion (i) of Lemma 3.6 that $-\boldsymbol{b} \in \mathrm{int}\mathbb{N}_p$ if and only if Conditions Ri and Po are satisfied. Therefore, all assertions of Theorem 4.1 follow from Theorem 1.1. □

5 Concluding remarks

In [8], Kim, Kojima and Toh formulated a DNN relaxation of a class of linearly constrained QOPs in nonnegative and binary variables as a COP of the form

$$\varphi_p \;\; = \;\; \inf\left\{\langle \boldsymbol{q}, \, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathbb{K}_1, \; \langle \boldsymbol{h}, \, \boldsymbol{x}\rangle = 1, \; \langle \boldsymbol{h}_1, \, \boldsymbol{x}\rangle = 0\right\}, \tag{15}$$

where $\mathbb{K}_1$ is a nice cone (the intersection of the doubly nonnegative cone and a linear subspace) in the space of symmetric matrices, $\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{K}_1^*$ and $\boldsymbol{h}^1 \in \mathbb{K}_1^*$. The dual of COP (15) can be written as

$$\varphi_d \;\; = \;\; \sup\left\{t : \boldsymbol{q} - \boldsymbol{h}t - \boldsymbol{h}s \in \mathbb{K}_1^*\right\}. \tag{16}$$

Under the assumption that ensures the feasible region of COP (15) is nonempty and bounded, they proved the strong duality that $-\infty < \varphi_p = \varphi_d < \infty$ [8, Lemma 3] (see also [4, Theorem 2.6] for the same assertion under a weaker assumption). By applying a Lagrangian relaxation to COPs (15) and (16), they induced the primal-dual pair of COPs of the form (1) and the form (3), which has no duality gap, with a Lagrangian multiplier parameter λ associated with the constraint $\langle \boldsymbol{h}_1, \, \boldsymbol{x}\rangle = 1$. The strong duality equality $\varphi_p = \varphi_d$ was derived by taking the limit of their optimal values with no gap as $\lambda \to \infty$. This result was used in the development of the Newton-bracketing method whose convergence is quadratic for solving the Lagrangian-DNN relaxation of the aforementioned class of QOPs in [9]. See [4, 9] for more details.

The primal-dual pair of COPs (15) and (16) can be reformulated as the primal-dual pair of COPs (1) and (3) with $\mathbb{K}_2 = \{\boldsymbol{x} \in \mathbb{V} : \langle \boldsymbol{h}_1, \boldsymbol{x} \rangle = 0\}$. Under their assumption, the set of optimal solutions of primal COP (1) is nonempty and bounded. Hence, their strong duality can be derived directly from Corollary 1.2 (or Theorem 1.1) without relying on the Lagrangian relaxation of COP (15).

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