

On the Complexity of Separation From the Knapsack Polytope^{*}

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Abstract. We close three open problems in the separation complexity of valid inequalities for the knapsack polytope. Specifically, we establish that the separation problems for extended cover inequalities, $(1, k)$ -configuration inequalities, and weight inequalities are all \mathcal{NP} -complete. We also give a number of special cases where the separation problem can be solved in polynomial time.

Key words: Knapsack polytope; Separation problem; Complexity theory

1 Introduction

The *multiple knapsack problem* is the integer programming (IP) problem

$$\max\{c^\top x \mid Ax \leq d, x \in \{0, 1\}^n\}, \quad (1)$$

where $A \in \mathbb{Z}_+^{m \times n}$, $c \in \mathbb{Z}_+^n$, and $d \in \mathbb{Z}_+^m$. When the constraint matrix A only has one row a and the right-hand side vector is a positive integer b , problem (1) is referred to as *knapsack problem*, and the convex hull of the associated feasible region, $\text{conv}(\{x \in \{0, 1\}^n \mid a^\top x \leq b\})$, is referred to as the *knapsack polytope*.

The multiple knapsack problem is a fundamental problem in discrete optimization and valid inequalities for the feasible region have been widely studied. The paper [10] provides a modern survey. In this paper, we study the complexity of the separation problem for well-known families of valid inequalities for (1).

A standard and computationally useful way for generating cuts for (1) is to generate cuts for the knapsack polytope defined by its individual constraints. Suppose a is a row of the constraint matrix A and let b be the corresponding coordinate of the right-hand side d , we denote the associated knapsack polytope by $K := \text{conv}(\{x \in \{0, 1\}^n \mid a^\top x \leq b\})$.

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Many families of valid inequalities for K are based on the notion of a *cover*, which is a subset C of $\{1, 2, \dots, n\}$ such that $\sum_{i \in C} a_i > b$. Given a cover C , the inequality

$$\sum_{i \in C} x_i \leq |C| - 1$$

is valid for K , and it is called a *cover inequality* (CI). Often, cover inequalities can be strengthened through a process called lifting, and the resulting inequalities are called *lifted cover inequalities* (LCIs) [3,8,14].

Balas [2] gave one family of LCI known as *extended cover inequalities* (ECIs), which have the form

$$\sum_{j \notin C: a_j \geq \max_{i \in C} a_i} x_j + \sum_{i \in C} x_i \leq |C| - 1.$$

A *minimal cover* is a cover C such that $\sum_{i \in C \setminus \{j\}} a_i \leq b$ for any $j \in C$. A set $N \cup \{t\}$ with $N \subsetneq \{1, \dots, n\}$ and $t \notin N$ is called a $(1, k)$ -*configuration* for $k \in \{2, \dots, |N|\}$ if $\sum_{i \in N} a_i \leq b$ and $Q \cup \{t\}$ is a minimal cover for every $Q \subseteq N$ with $|Q| = k$. Padberg [15] showed that for any $(1, k)$ -configuration $N \cup \{t\}$, the inequality

$$(|S| - k + 1)x_t + \sum_{i \in S} x_i \leq |S|$$

is valid for K for every $|S| \subseteq N$ with $|S| \geq k$. Such inequality is called a $(1, k)$ -*configuration inequality*.

Other valid inequalities for the knapsack polytope K arise from the concept of a *pack*. For the knapsack polytope K , a set $P \subseteq \{1, \dots, n\}$ is a *pack* if $\sum_{i \in P} a_i \leq b$. Given a pack P , the corresponding *pack inequality* $\sum_{i \in P} a_i x_i \leq \sum_{i \in P} a_i$ is trivially valid for K , as it is implied by the upper bound constraints $x_i \leq 1$. However, pack inequalities can be lifted in several different ways to obtain more interesting *lifted pack inequalities* (LPIs) [1]. Weismantel [17] derived the weight-inequalities, which are LPIs. To define the weight inequalities, let $r(P) := b - \sum_{i \in P} a_i$ be the *residual capacity* of the pack P . The indices $j \notin P$ with $a_j > r(P)$ are lifted to obtain the *weight inequality* (WI):

$$\sum_{i \in P} a_i x_i + \sum_{j \notin P} \max\{a_j - r(P), 0\} x_j \leq \sum_{i \in P} a_i.$$

Consider the *linear programming* (LP) *relaxation* of (1):

$$\max\{c^\top x \mid Ax \leq d, x \in [0, 1]^n\}. \quad (2)$$

For a given family \mathcal{F} of valid inequalities for (1), the associated *separation problem* is defined as follows: “Let x^* be a feasible solution to (2), does there exist an inequality in \mathcal{F} that is violated by x^* ? If so, return one such inequality from \mathcal{F} .” In this paper, we are mainly interested in the weaker decision version of the separation problem where we do not have to return a separating inequality even if it does exist, and we assume that x^* is an optimal solution to (2). In fact,

the separation of optimal solution is no harder than the separation of general feasible solution, and from the computational point of view, x^* almost always comes from the optimal solution to some linear relaxation.

The separation problem for several families of valid inequalities for the knapsack polytope has been shown to be \mathcal{NP} -complete, including CIs [13,4], and LCIs [7]. On the other hand the complexity of the separation problem for ECIs, for $(1, k)$ -configuration inequalities and for WIs are, to the best of our knowledge, unknown. It was conjectured explicitly in [5] that the separation problem for $(1, k)$ -configuration inequalities is \mathcal{NP} -hard (see also [10]). Kaparis and Letchford also stated that the separation problem seems likely to be \mathcal{NP} -hard for ECIs in [11]. Moreover, the complexity of the separation problem for WIs is also open, as mentioned in [10]. In this paper we provide positive answers to all these conjectures. Namely, we show that the separation problems for ECIs, for $(1, k)$ -configuration inequalities and for WIs are all \mathcal{NP} -complete. The first two results are proven via a reduction from the separation problem for CIs, and the separation complexity for WIs is given via the reduction from the Subset Sum Problem (SSP).

We remark that several heuristics and exact separation algorithms are present in the literature for these families of cuts. Both Gabrel and Minoux [6], Kaparis and Letchford [11] provide an exact separation algorithm for ECIs that runs in pseudo-polynomial time. Ferreira *et al.* [5] presented simple heuristics for the separation problem of $(1, k)$ -configuration inequalities. For the separation problem of WIs, Weismantel [17] proposed an exact algorithm in pseudo-polynomial time. Helmberg and Weismantel [9] further presented a fast separation heuristic for WIs, that simply inserts items into the pack P in non-increasing order of x^* value. Kaparis and Letchford [11] gave two exact algorithms and a heuristic for separating WIs, and then show how to convert these methods into heuristics for separating LPs.

Next, we formally define the separation problems considered in this paper.

Problem CI-SP

Input: $(A, d, c) \in (\mathbb{Z}_+^{m \times n}, \mathbb{Z}_+^m, \mathbb{Z}_+^n)$ and an optimal solution x^* to the LP relaxation (2).

Question: Is there a cover C with respect to some row constraint $a^T x \leq b$ of (2), such that $\sum_{i \in C} x_i^* > |C| - 1$?

Problem ECI-SP

Input: $(A, d, c) \in (\mathbb{Z}_+^{m \times n}, \mathbb{Z}_+^m, \mathbb{Z}_+^n)$ and an optimal solution x^* to the LP relaxation (2).

Question: Is there a cover C with respect to some row constraint $a^T x \leq b$ of (2), such that $\sum_{j \notin C: a_j \geq \max_{i \in C} a_i} x_j + \sum_{i \in C} x_i > |C| - 1$?

Problem CONFIG-SP

Input: $(A, d, c) \in (\mathbb{Z}_+^{m \times n}, \mathbb{Z}_+^m, \mathbb{Z}_+^n)$ and an optimal solution x^* to the LP relaxation (2).

Question: Is there a $(1, k)$ -configuration $N \cup \{t\}$ and a subset $S \subseteq N$ with

$|S| \geq k$ with respect to some row constraint $a^\top x \leq b$ of (2), such that $(|S| - k + 1)x_i^* + \sum_{i \in S} x_i^* > |S|$?

Problem WI-SP

Input: $(A, d, c) \in (\mathbb{Z}_+^{m \times n}, \mathbb{Z}_+^m, \mathbb{Z}_+^n)$ and an optimal solution x^* to the LP relaxation (2).

Question: Is there a pack P with respect to some row constraint $a^\top x \leq b$ of (2), such that $\sum_{i \in P} a_i x_i^* + \sum_{j \notin P} \max\{a_j - r(P), 0\} x_j^* > \sum_{i \in P} a_i$?

For CI-SP, we have the following classic results.

Theorem 1 ([13,4]).

- CI-SP is \mathcal{NP} -complete, even if $m = 1$.
- CI-SP is \mathcal{NP} -complete, even if x^* is an extreme point.

We will show the other three problems, ECI-SP, CONFIG-SP, and WI-SP are all \mathcal{NP} -Complete.

Clearly, the \mathcal{NP} -hardness of the above problems imply the \mathcal{NP} -hardness of the more general separation problem where x^* is a feasible, and not necessarily optimal, solution to (2). We should also remark that, since verifying if a given point violates a given inequality can be obviously done in polynomial time with respect to the input size of such point and inequality, the separation problems for these families of cuts are clearly in class \mathcal{NP} . Therefore, when we talk about the separation complexity for those cuts, we do not distinguish between \mathcal{NP} -hard and \mathcal{NP} -complete throughout this paper.

Notation. For an integer n we set $[n] := \{1, 2, \dots, n\}$. We define e_n as n -dimensional vector of ones, where we often repress the n if the vector dimension may be implied by the context. For a vector $x \in \mathbb{R}^n$ and $S \subseteq [n]$, we set $x(S) := \sum_{i \in S} x_i$.

2 Extended Cover Inequality Separation

In this section, we establish the complexity of extended cover inequality separation with a simple reduction from the cover inequality separation problem. In the case where the point to be separated is an extreme point of a polytope with a “small” number of inequalities, then extended cover inequality separation can be accomplished in polynomial time.

Theorem 2. *Problem ECI-SP is \mathcal{NP} -complete, even if $m = 1$. Furthermore, Problem ECI-SP is \mathcal{NP} -complete, even if x^* is an extreme point solution to the LP relaxation (2).*

Proof. We transform CI-SP to ECI-SP. Let $(A, d, c, x^*) \in (\mathbb{Z}_+^{m \times n}, \mathbb{Z}_+^m, \mathbb{Z}_+^n, [0, 1]^n)$ be the input to CI-SP. We proceed the proof by constructing input $(A', d', c', y^*) \in$

$(\mathbb{Z}_+^{m \times (n+1)}, \mathbb{Z}_+^m, \mathbb{Z}_+^{n+1}, [0, 1]^{n+1})$ to ECI-SP with the property that there is a yes-certificate to CI-SP with input (A, d, c, x^*) if and only if there is a yes-certificate to ECI-SP with input (A', d', c', y^*) .

In the construction, the first n columns of A' are the columns of A , while the last column of A' is the sum of all other columns; the right-hand side vector in the construction is $d' = d + Ae$; and the first n coefficients of c' remain the same as c , while the last is a large positive constant:

$$\begin{aligned} A'_{ij} &= A_{ij} \quad \forall i \in [m], \forall j \in [n] & A'_{i,n+1} &= \sum_{j=1}^n A_{ij} \quad \forall i \in [m] \\ c'_j &= c_j \quad \forall j \in [n] & c'_{n+1} &= M \\ d'_i &= d_i + \sum_{j=1}^n A_{ij} \quad \forall i \in [m]. \end{aligned}$$

The constant M is chosen to be large enough so that if x^* is an optimal solution to the linear program (2), then $y^* = (x^*, 1)$ is an optimal solution to the linear program

$$\max\{(c')^\top y \mid A'y \leq d', y \in [0, 1]^{n+1}\}. \quad (3)$$

It is a consequence of linear programming duality that selecting $M \geq (\pi^*)^\top Ae$, where π^* are optimal dual multipliers for the inequality constraints in (2), will ensure the optimality of y^* . As is known that there is an optimal solution π^* whose encoding length is of polynomial size [16], we see that M exists and its encoding size is a polynomial function of the input size of CI-SP.

Let $C \subseteq [n]$ be a cover with respect to row constraint $a^\top x \leq b$ of $Ax \leq d$ such that the associated CI does not hold at x^* , so $x^*(C) > |C| - 1$. Then $C' := C \cup \{n+1\}$ is a cover with respect to the constraint $(a, a^\top e)^\top y \leq b + a^\top e$ within $A'y \leq d'$, and the associated ECI cuts off $y^* : y^*(C') = 1 + x^*(C) > |C| = |C'| - 1$.

On the other hand, assume that C' is a cover with respect to some row constraint $a^\top y = (a, a^\top e)^\top y \leq b + a^\top e = b'$ within $A'y \leq d'$ such that the associated ECI cuts off y^* . Note that if $n+1 \notin C'$, then $\sum_{j \in C'} a'_j \leq a'([n]) = a^\top e < b + a^\top e$, and C' cannot be a cover with respect to that row constraint. Thus, $n+1 \in C'$, and the ECI of C' is just its cover inequality $y(C') \leq |C'| - 1$. By construction, the set $C := C' \setminus \{n+1\}$ is a cover with respect to the constraint $a^\top x \leq b$ within $Ax \leq d$. The ECI of C' cuts off y^* , $y^*(C') = 1 + x^*(C) > |C'| - 1 = |C|$, so $x^*(C) > |C| - 1$, and the CI from C cuts off x^* .

We have thereby shown that there is a yes-certificate to CI-SP with input (A, d, c, x^*) if and only if there is a yes-certificate to ECI-SP with input (A', d', c', y^*) . Together with Theorem 1, we establish that ECI-SP is \mathcal{NP} -complete, even if $m = 1$, and that ECI-SP is \mathcal{NP} -complete, even if x^* is an extreme point to the LP relaxation (2). For the second statement, it suffices to realize that the input $y^* = (x^*, 1)$ for ECI-SP will be an extreme point of $\{y \in [0, 1]^{n+1} \mid A'y \leq d'\}$ if x^* is an extreme point of $\{x \in [0, 1]^n \mid Ax \leq d\}$. \square

In the case that the number of rows of A is $\mathcal{O}(\log(n))$, we can solve ECI-SP in polynomial time.

Theorem 3. *If the number of constraints in the LP relaxation (2) is $\mathcal{O}(\log(n))$, and x^* is an extreme point to (2), then the separating ECI for x^* can be obtained in polynomial time if one exists.*

Proof. For a given x^* , let $I_0 = \{i \in [n] \mid x_i^* = 0\}$, $I_1 = \{i \in [n] \mid x_i^* = 1\}$ and $I_f = \{i \in [n] \mid x_i^* \in (0, 1)\}$. It is obvious to observe that, for point x^* and row constraint $a^\top x \leq b$, there exists a separating ECI from such constraint if and only if, for some $t \in [n]$, there exists a cover C with $\max_{i \in C} a_i = a_t$, such that:

$$\sum_{i: a_i \geq a_t} x_i^* + \sum_{i \in C: a_i < a_t} x_i^* > |C| - 1. \quad (4)$$

Here C can be partitioned as: $C = T_1 \cup T_f \cup T_0 \cup T$, where $T_1 = \{i \in C \mid a_i < a_t, x_i^* = 1\}$, $T_f = \{i \in C \mid a_i < a_t, x_i^* \in (0, 1)\}$, $T_0 = \{i \in C \mid a_i < a_t, x_i^* = 0\}$, and $T = \{i \in C \mid a_i = a_t\}$. Therefore, (4) can be reformed as:

$$\sum_{i: a_i \geq a_t} x_i^* > \sum_{i \in T_f} (1 - x_i^*) + |T_0 \cup T| - 1. \quad (5)$$

For a fixed $t \in [n]$, the right hand side of (5) is a constant. Since C is a cover, in order for (5) to hold, here w.l.o.g. we can let T_1 be the largest possible, which is $\{i \in [n] \mid a_i < a_t, x_i^* = 1\}$. Therefore, there exists a separating ECI from row constraint $a^\top x \leq b$ if and only if, for some $t \in [n]$ and $T_f \subseteq \{i \in [n] \mid a_i < a_t, x_i^* \in (0, 1)\}$, there is:

$$\min \left\{ \sum_{i \in S_t} z_i \mid \sum_{i \in S_t} a_i z_i \geq b_t \right\} < \sum_{i: a_i \geq a_t} x_i^* - \sum_{i \in T_f} (1 - x_i^*) + 1, \quad (6)$$

where $S_t = \{i \in [n] \mid a_i = a_t \text{ or } a_i < a_t, x_i^* = 0\}$, and $b_t = b + 1 - \sum_{i: a_i < a_t, x_i^* = 1} a_i - \sum_{i \in T_f} a_i$.

Note that for any fixed $t \in [n]$ and $T_f \subseteq \{i \in [n] \mid a_i < a_t, x_i^* \in (0, 1)\}$, the right hand side of (6) is a constant and the left hand side of (6) is simply a knapsack problem with objective coefficients being 1s. Next we present a subroutine to solve the left hand side problem of (6) in polynomial time. Here we assume $a(S_t) \geq b_t$ to guarantee that such problem is feasible, otherwise such problem is infeasible and (6) does not hold. First, sort the a_i for $i \in S_t$ in nonincreasing order: $a_{i_1} \geq \dots \geq a_{i_\ell}$, where $\ell = |S_t|$. For each $j \in [\ell]$, compute $\sigma_j = \sum_{k=1}^j a_{i_k}$. Then, let $\gamma \in [\ell]$ be such that $\sigma_{\gamma-1} < b_t$ and $\sigma_\gamma \geq b_t$, here we default $\sigma_0 = 0$. Ultimately the optimal value to the left hand side problem of (6) is γ , and optimal solution can be given by $z_{i_k} = 1$ for $k \in [\gamma]$ and 0 elsewhere. In other words, for any fixed $t \in [n]$ and $T_f \subseteq \{i \in [n] \mid a_i < a_t, x_i^* \in (0, 1)\}$, (6) can be verified in polynomial time, and if it holds, set $\{i \in [n] \mid a_i < a_t, x_i^* = 1\} \cup T_f \cup \{i_1, \dots, i_\gamma\}$ is a cover to $a^\top x \leq b$ whose associated ECI cuts off x^* .

Since x^* is an extreme point to LP relaxation (2), we know that at most m components of x^* are fractional, where m is the number of constraints in (2). Hence the number of different set $T_f \subseteq \{i \in [n] \mid a_i < a_t, x_i^* \in (0, 1)\}$ is at most 2^m , which is polynomially-many by our assumption on m . Therefore, the number of different combinations of $T_f \subseteq \{i \in [n] \mid a_i < a_t, x_i^* \in (0, 1)\}$ and $t \in [n]$ is also polynomial with respect to the input size. In the end, we have concluded that, for point x^* and row constraint $a^\top x \leq b$, if there exists a separating ECI from row constraint $a^\top x \leq b$, then it can be obtained in polynomial time. Such argument can be applied to each row constraint of (2), hence we complete the proof. \square

3 (1, k)-Configuration Inequality Separation

In this section we establish that the separation problem for (1, k)-configuration inequalities is \mathcal{NP} -Complete using a similar reduction as in the proof of Theorem 2.

Theorem 4. *Problem CONFIG-SP is \mathcal{NP} -complete, even if $m = 1$. Furthermore, Problem CONFIG-SP is \mathcal{NP} -complete, even if x^* is an extreme point solution to the LP relaxation (2).*

Proof. We transform CI-SP to CONFIG-SP. Given an input (A, d, c, x^*) to CI-SP which is in $(\mathbb{Z}_+^{m \times n}, \mathbb{Z}_+^m, \mathbb{Z}_+^n, [0, 1]^n)$, we will construct a corresponding input $(A', d', c', y^*) \in (\mathbb{Z}_+^{m \times (n+2)}, \mathbb{Z}_+^m, \mathbb{Z}_+^{n+2}, [0, 1]^{n+2})$ to CONFIG-SP such that there is a yes-certificate to CI-SP with input (A, d, c, x^*) if and only if there is a yes-certificate to CONFIG-SP with input (A', d', c', y^*) .

In the construction, the first n columns of A' are the columns of A , while the last two columns of A' are the sum of all other columns; the right-hand side vector in the construction is $d' = d + 2Ae$; and the first n coefficients of c' remain the same as c , while the last two coefficients are a large positive constant:

$$\begin{aligned} A'_{ij} &= A_{ij} \quad \forall i \in [m], \forall j \in [n] & A'_{ik} &= \sum_{j=1}^n A_{ij} \quad \forall i \in [m], \forall k \in \{n+1, n+2\} \\ c'_j &= c_j \quad \forall j \in [n] & c'_{n+1} &= c'_{n+2} = M \\ d'_i &= d_i + 2 \sum_{j=1}^n A_{ij} \quad \forall i \in [m]. \end{aligned}$$

The constant M is chosen to be large enough so that if x^* is an optimal solution to the linear program (2), then $y^* = (x^*, 1, 1)$ is an optimal solution to the linear program

$$\max\{(c')^\top y \mid A'y \leq d', y \in [0, 1]^{n+2}\}. \quad (7)$$

It is a consequence of linear programming duality that selecting $M \geq (\pi^*)^\top Ae$, where π^* are optimal dual multipliers for the inequality constraints in (2), will ensure the optimality of y^* . As is known that there is an optimal solution π^*

whose encoding length is of polynomial size [16], we see that M exists and its encoding size is a polynomial function of the input size of CI-SP.

Let $C \subseteq [n]$ be a cover with respect to the row constraint $a^\top x \leq b$ of $Ax \leq d$ such that the associated CI cuts off $x^* : x^*(C) > |C| - 1$. Let $C' := C \cup \{n+1, n+2\}$. By definition of the input (A', d') , C' is a cover with respect to the row constraint $a'^\top y = (a, a^\top e, a^\top e)^\top y \leq b + 2a^\top e = b'$ within $A'y \leq d'$, and the associated CI cuts off $y^* : y^*(C') = 2 + x^*(C) > |C| + 1 = |C'| - 1$. As every cover inequality is dominated by a minimal cover inequality, there is a minimal cover contained in C' whose associated minimal CI cuts off y^* . Every minimal CI is a special case of $(1, k)$ -configuration inequality, so we have obtained a $(1, k)$ -configuration inequality with respect to a row constraint within $A'y \leq d'$ that cuts off y^* .

To complete the proof we must show that if $N \cup \{\ell\}$ is a $(1, k)$ -configuration with respect to some row constraint $a'^\top y \leq b'$ within $A'y \leq d'$, and $S \subseteq N$ with $|S| \geq k$, such that $(|S| - k + 1)y_\ell^* + \sum_{i \in S} y_i^* > |S|$, then we can construct a cover C with respect to the some row constraint in $Ax \leq b$ such that the associated CI cuts off x^* .

By construction of (A', d') , we know that row constraint $a'^\top y \leq b'$ takes the form of $(a, a^\top e, a^\top e)^\top y \leq b + 2a^\top e$, where $a^\top x \leq b$ is a row constraint in $Ax \leq b$. First observe that in the constraint $(a, a^\top e, a^\top e)^\top y \leq b + 2a^\top e$, any cover must contain both $n+1$ and $n+2$. By definition of a $(1, k)$ -configuration, for any subset $Q \subseteq N$ with $|Q| = k$, $Q \cup \{\ell\}$ is a minimal cover. Specifically, $N \cup \{\ell\}$ is a cover. This implies that $\{n+1, n+2\} \subseteq N \cup \{\ell\}$, which means N must contain $n+1$ or $n+2$. If $k \leq |N| - 1$, then for any $i' \in N$, the set $N \cup \{\ell\} \setminus \{i'\}$ will also be a cover. However, since N contains $n+1$ or $n+2$, then when $i' = n+1$ (or $n+2$), $N \cup \{\ell\} \setminus \{i'\}$ will not be a cover. Therefore, $|N| = k$, and the associated $(1, k)$ -configuration inequality reduces to a minimal CI, so

$$y^*(N \cup \{\ell\}) > |N|. \quad (8)$$

Let $C = N \cup \{\ell\} \setminus \{n+1, n+2\}$, so $|C| = |N| - 1$. Since $N \cup \{\ell\}$ is a cover with respect to the constraint $(a, a^\top e, a^\top e)^\top y \leq b + 2a^\top e$, and $n+1, n+2 \in N \cup \{\ell\}$, we can infer that $a(C) + 2a^\top e > b + 2a^\top e$, so C is a cover with respect to the row constraint $a^\top x \leq b$ of $Ax \leq d$. Furthermore, from (8) and the definition of $y^* = (1, 1, x^*)$, we have $x^*(C) > |N| - 2 = |C| - 1$. Therefore, we end up with a cover C whose associated CI cuts off x^* .

We have thereby shown that there is a yes-certificate to CI-SP with input (A, d, c, x^*) if and only if there is a yes-certificate to CONFIG-SP with input (A', d', c', y^*) . Together with Theorem 1 our proof establishes that both that CONFIG-SP is \mathcal{NP} -complete even if $m = 1$, and that CONFIG-SP is \mathcal{NP} -complete, even if x^* is an extreme point to the LP relaxation (2). For the second statement, it suffices to realize that the input $y^* = (x^*, 1, 1)$ for CONFIG-SP will be an extreme point of $\{y \in [0, 1]^{n+2} \mid A'y \leq d'\}$ if x^* is an extreme point of $\{x \in [0, 1]^n \mid Ax \leq d\}$. \square

For any given k and $t \in [n]$, by definition of $(1, k)$ -configuration, it is unclear how to find set N such that $N \cup \{t\}$ is a $(1, k)$ -configuration efficiently. Therefore,

even if the input solution x^* only has constant number of fractional components, we do not know whether the separating $(1, k)$ -configuration inequality can be obtained in polynomial time. In fact, we conjecture it to be \mathcal{NP} -hard to produce such separating $(1, k)$ -configuration inequality.

Conjecture 1. There exists $\alpha \in \mathbb{N}$, such that CONFIG-SP is \mathcal{NP} -complete, even if the input solution x^* satisfies $|\{i \in [n] \mid x_i^* \in (0, 1)\}| \leq \alpha$.

4 Weight Inequality Separation

Our goal, in this section, is to prove that WI-SP is \mathcal{NP} -hard in its full generality, and to present special cases where it can be solved in polynomial time. For a pack P of a given knapsack constraint $a^\top x \leq b$, we denote by $C(P) := \{i \in [n] \setminus P \mid a_i > r(P)\}$. With this notation, the WI associated with P takes the form

$$\sum_{i \in P} a_i x_i + \sum_{j \in C(P)} (a_j - r(P)) x_j \leq a(P),$$

where we remind the reader that $r(P) := b - a(P)$. First, we will need the following auxiliary result.

Lemma 1. *Let $(a, b) \in \mathbb{Z}_+^{n+1}$ with $a([n])/b \notin \mathbb{Z}$, and let $x_1^* = \dots = x_n^* = b/a([n])$. Then there exists a pack P of $a^\top x \leq b$ whose associated WI separates x^* if and only if there exists a pack P' of $a^\top x \leq b$ such that $r(P') > 0$, $P' \cup C(P') = [n]$, and $|C(P')| = \lfloor a([n])/b \rfloor$.*

Proof. First, assume that there exists a pack P' of $a^\top x \leq b$ such that $r(P') > 0$, $P' \cup C(P') = [n]$, and $|C(P')| = \lfloor a([n])/b \rfloor$. We have

$$\begin{aligned} \sum_{i \in P'} a_i x_i^* + \sum_{j \in C(P')} (a_j - r(P')) x_j^* &= \sum_{i \in P' \cup C(P')} a_i x_i^* - r(P') \sum_{j \in C(P')} x_j^* \\ &= \sum_{i \in [n]} a_i x_i^* - r(P') \sum_{j \in C(P')} x_j^* \\ &= b - r(P') \cdot |C(P')| \cdot \frac{b}{a([n])} \\ &> b - r(P') \\ &= a(P'), \end{aligned}$$

where the inequality holds because $r(P') > 0$, $a([n])/b \notin \mathbb{Z}$, and $|C(P')| = \lfloor a([n])/b \rfloor$. Therefore, we know that the WI associated with pack P' separates x^* .

Next, assume that there exists a pack P of $a^\top x \leq b$ whose associated WI separates x^* . Namely:

$$f(P) := \sum_{i \in P} a_i x_i^* + \sum_{j \in C(P)} (a_j - r(P)) x_j^* - a(P) > 0.$$

Without loss of generality, we assume that P has the largest value $f(P)$ among all packs of the knapsack constraint $a^\top x \leq b$. By re-arranging the terms and using some basic algebra, we have that $f(P) > 0$ implies $r(P) > 0$ and

$$\sum_{j \in C(P)} x_j^* < \frac{\sum_{i \in P \cup C(P)} a_i x_i^* - a(P)}{r(P)}.$$

Since $\sum_{i \in P \cup C(P)} a_i x_i^* \leq b$, we obtain

$$\sum_{j \in C(P)} x_j^* < \frac{b - a(P)}{r(P)} = 1.$$

Replacing $\sum_{j \in C(P)} x_j^* = |C(P)| \cdot b/a([n])$, we have shown

$$|C(P)| < \frac{a([n])}{b}. \quad (9)$$

Now let $i' \in P$. Note that we have $r(P \setminus \{i'\}) = r(P) + a_{i'}$, so we obtain

$$\begin{aligned} f(P) - f(P \setminus \{i'\}) &= a_{i'} \left(\sum_{j \in C(P \setminus \{i'\})} x_j^* + x_{i'}^* - 1 \right) + \sum_{j \in C(P) \setminus C(P \setminus \{i'\})} (a_j - r(P)) x_j^* \\ &\in \left[a_{i'} \left(\sum_{j \in C(P \setminus \{i'\})} x_j^* + x_{i'}^* - 1 \right), a_{i'} \left(\sum_{j \in C(P)} x_j^* + x_{i'}^* - 1 \right) \right], \end{aligned}$$

where the last relation holds because $a_j > r(P)$ for every $j \in C(P)$ and $a_j - r(P) = a_j - r(P \setminus \{i'\}) + a_{i'} \leq a_{i'}$ for every $j \notin C(P \setminus \{i'\})$. Since $P \setminus \{i'\}$ is clearly also a pack, our maximality assumption on $f(P)$ implies that $f(P) \geq f(P \setminus \{i'\})$. Hence we have $a_{i'} \left(\sum_{j \in C(P)} x_j^* + x_{i'}^* - 1 \right) \geq 0$. This implies that $\sum_{j \in C(P)} x_j^* + x_{i'}^* \geq 1$ for any $i' \in P$. Since $x_1^* = \dots = x_n^* = b/a([n])$, we obtain

$$|C(P)| \geq \frac{a([n])}{b} - 1.$$

Combined with (9) and the assumption that $a([n])/b \notin \mathbb{Z}$, we have:

$$|C(P)| = \left\lfloor \frac{a([n])}{b} \right\rfloor.$$

To complete the proof, we only need to show that $P \cup C(P) = [n]$. If not, there exists $i' \in [n] \setminus (P \cup C(P))$, such that $P \cup \{i'\}$ remains a pack. Hence

$$\begin{aligned} f(P \cup \{i'\}) - f(P) &= a_{i'} \left(\sum_{j \in C(P)} x_j^* + x_{i'}^* - 1 \right) + \sum_{j \in C(P \cup \{i'\}) \setminus C(P)} (a_j - r(P \cup \{i'\})) x_j^* \\ &\in \left[a_{i'} \left(\sum_{j \in C(P)} x_j^* + x_{i'}^* - 1 \right), a_{i'} \left(\sum_{j \in C(P \cup \{i'\})} x_j^* + x_{i'}^* - 1 \right) \right]. \end{aligned}$$

Since $|C(P)| = \lfloor a([n])/b \rfloor$ and $a([n])/b \notin \mathbb{Z}$, we know that $\sum_{j \in C(P)} x_j^* + x_{i'}^* > 1$, which implies that $f(P \cup \{i'\}) > f(P)$. This contradicts the maximality assumption on $f(P)$ and the fact that $P \cup \{i'\}$ is a pack. We have thereby shown $P \cup C(P) = [n]$. \square

To prove that the separation problem WI-SP is \mathcal{NP} -hard, we establish a reduction from the *Subset Sum Problem (SSP)* to WI-SP.

Problem SSP

Input: $\alpha \in \mathbb{Z}_+^n$ and $w \in \mathbb{Z}_+$.

Question: Is there a subset $S \subseteq [n]$ such that $\alpha(S) = w$?

The SSP is among Karp's 21 \mathcal{NP} -complete problems [12]. It is simple to check that SSP is \mathcal{NP} -complete even if $w > \max(\alpha)$. We are now ready to prove that WI-SP is \mathcal{NP} -hard.

Theorem 5. *Problem WI-SP is \mathcal{NP} -complete, even if $m = 1$. Furthermore, Problem WI-SP is \mathcal{NP} -complete, even if x^* is an extreme point solution to the LP relaxation (2).*

Proof. First, we prove the first part of the statement. We show that WI-SP is \mathcal{NP} -hard even in case of a single knapsack constraint. Given an instance $(\alpha, w) \in \mathbb{Z}_+^{n+1}$ of SSP with $w > \max(\alpha)$, we construct a knapsack problem $\max\{c^\top x \mid a^\top x \leq b, x \in \{0, 1\}^{2n+2}\}$ and give an optimal solution x^* to the associated LP relaxation. The data a, b, c of the constructed knapsack problem is defined as follows:

$$\begin{aligned} a_i &:= \alpha_i + 2, & \forall i = 1, \dots, n, \\ a_{n+1} &:= w \cdot (n+1) + 2(n+1)^2 - 3n - \alpha([n]), \\ a_{n+1+j} &:= 2, & \forall j = 1, \dots, n+1, \\ b &:= w + 2n + 3, \\ c &:= a, \\ x_1^* &:= \dots := x_{2n+2}^* := \frac{w + 2n + 3}{w \cdot (n+1) + 2n^2 + 5n + 4}. \end{aligned} \tag{10}$$

It is simple to check that a, b, c are all integral, that (a, b, c, x^*) has polynomial encoding size with respect to that of (α, w) , and that $a^\top x^* = b$. Furthermore, x^* is an optimal solution to the original knapsack problem, since $c^\top x^* = a^\top x^* = b$. Hence (a, b, c, x^*) is a feasible input to WI-SP where $m = 1$. Note that $(w \cdot (n+1) + 2n^2 + 5n + 4)/(w + 2n + 3) = n + 1 + 1/(w + 2n + 3) \notin \mathbb{Z}$. Hence, we can apply Lemma 1 and obtain that there exists a separating WI for x^* if and only if there exists a pack P such that:

$$\begin{aligned} r(P) &> 0, \\ P \cup C(P) &= [2n+2], \\ |C(P)| &= \left\lfloor \frac{w \cdot (n+1) + 2n^2 + 5n + 4}{w + 2n + 3} \right\rfloor = n + 1. \end{aligned} \tag{11}$$

Claim 1. There exists a WI from constraint $a^\top x \leq b$ that separates x^* if and only if there exists a subset $S \subseteq [n]$ such that $\alpha(S) = w$.

Proof of claim. It suffices to show that there exists pack P such that (11) holds if and only if there exists a subset $S \subseteq [n]$ such that $\alpha(S) = w$.

First, we assume that P is a pack such that (11) holds. The two equations in (11) imply $|P| = 2n+2 - |C(P)| = n+1$. If $\{n+2, n+3, \dots, 2n+2\} \cap C(P) = \emptyset$, then $P \cup C(P) = [2n+2]$ implies that $\{n+2, n+3, \dots, 2n+2\} \subseteq P$, which means $P = \{n+2, n+3, \dots, 2n+2\}$ since $|P| = n+1$. However, since $w > \max(\alpha)$, we know that $2 + \max(\alpha) + 2(n+1) \leq b = w + 2n + 3$, which implies that $P \cup \{i'\}$ is a pack for any $i' \in [n]$, and this contradicts the assumption $C(P) = [2n+2] \setminus P$ of (11). Therefore, there must exist some $i' \in \{n+2, n+3, \dots, 2n+2\} \cap C(P)$. Hence $r(P) = b - a(P) < a_{i'} = 2$. Moreover, because $r(P) > 0$, we have $r(P) = 1$, which implies $a(P) = b - 1 = w + 2n + 2$. Since $a_{n+1} = w \cdot (n+1) + 2(n+1)^2 - 3n - \alpha([n]) \geq w + 2(n+1)^2 - 3n + (w \cdot n - \alpha([n])) > w + 2n + 2$, we know $n+1 \notin P$. Let $S := P \cap [n]$. We then obtain $a(S) = 2|S| + \alpha(S)$ and $a(P \setminus S) = 2(|P| - |S|) = 2(n+1 - |S|)$. Therefore, $w + 2n + 2 = a(P) = a(S) + a(P \setminus S) = \alpha(S) + 2n + 2$, which gives us $\alpha(S) = w$.

Next, we assume that S is a subset of $[n]$ with $\alpha(S) = w$. Clearly, $n+1 \notin S$. Then we define the set \tilde{S} containing $n+1 - |S|$ arbitrary indices from $\{n+2, \dots, 2n+2\}$. Then $P := S \cup \tilde{S}$ is a pack such that (11) holds. In fact, we have

$$\begin{aligned} r(P) &= b - a(P) \\ &= w + 2n + 3 - a(S) - a(\tilde{S}) \\ &= w + 2n + 3 - (2|S| + \alpha(S)) - 2(n+1 - |S|) \\ &= 1. \end{aligned}$$

This further implies $C(P) = [2n+2] \setminus P$ and $|C(P)| = 2n+2 - |P| = n+1$, since $a_i > 1$ for all $i \in [2n+2]$. Hence (11) is satisfied by pack P . \diamond

Claim 1 completes the proof of the first part of the statement, since SSP itself is \mathcal{NP} -hard.

Next, we prove the second part of the statement. We show that WI-SP is \mathcal{NP} -hard, even if x^* is an extreme point solution to the LP relaxation (2). Given an instance $(\alpha, w) \in \mathbb{Z}_+^{n+1}$ of SSP with $w > \max(\alpha)$, we construct an instance of the multiple knapsack problem $\max\{c^\top x \mid Ax \leq d, x \in \{0, 1\}^{2N}\}$ and give an optimal solution x^* to the associated LP relaxation, where $N = 2n+2$. Let G be a node-node adjacency matrix of a cycle on N nodes. The constraints of the constructed multiple knapsack problem are then defined as follows:

$$\begin{aligned} a^\top y &\leq b, \\ y_i + 2z_1 + 2z_2 + 2z_3 &\leq 3 + \epsilon, \quad \forall i \in [N], \\ Gz &\leq e_N. \end{aligned} \tag{12}$$

Here $(a, b) \in \mathbb{Z}_+^{N+1}$ is defined as in (10), $\epsilon := (w + 2n + 3) / (w \cdot (n+1) + 2n^2 + 5n + 4)$ and e_N is the N -dimensional vector with all components equal to one. Now we

define the objective vector $c := (a, e_N)$, and we let $x^* = (y^*, z^*) := (\epsilon e_N, e_N/2)$. Note that we can multiply all the rows of (12) by $w \cdot (n + 1) + 2n^2 + 5n + 4$ to get an instance of WI-SP with integral data. The instance defined here clearly has polynomial encoding size with respect to that of (α, w) .

First, we verify that this is a valid input for WI-SP. Clearly x^* is feasible. Furthermore, by summing all inequalities in $Gz \leq e_N$, it follows that x^* is an optimal solution to the LP relaxation.

Next we show that x^* is an extreme point of the polyhedron given by (12). Since $N = 2n + 2$ is even, then G is a square matrix with rank $N - 1$. We can further verify that the first $2N$ constraints in (12) give a system of $2N$ linearly independent constraints in $2N$ variables, and the only vector that satisfies all of them at equality is x^* .

Claim 2. There exists a WI from (12) that separates x^* if and only if there exists a WI from the constraint $a^\top y \leq b$ that separates y^* .

Proof of claim. First, we assume that P is a pack with respect to some constraint $a'^\top x \leq b'$ of (12) such that its corresponding WI separates x^* . If such constraint $a'^\top x \leq b'$ comes from the subsystem $Gz \leq e_N$, say $z_1 + z_2 \leq 1$, then the only WI is $z_1 + z_2 \leq 1$, which cannot be violated by x^* since x^* is a feasible point. If $a'^\top x \leq b'$ is $y_i + 2z_1 + 2z_2 + 2z_3 \leq 3 + \epsilon$ for some $i \in [N]$, then all the nonempty packs are $\{i\}, \{i, N + 1\}, \{i, N + 2\}, \{i, N + 3\}, \{N + 1\}, \{N + 2\}, \{N + 3\}$. The corresponding WIs are $y_i \leq 1$ and:

$$\begin{aligned} y_i + 2z_1 + (2 - \epsilon)(z_2 + z_3) &\leq 3, & 2z_1 + (1 - \epsilon)(z_2 + z_3) &\leq 2, \\ y_i + 2z_2 + (2 - \epsilon)(z_1 + z_3) &\leq 3, & 2z_2 + (1 - \epsilon)(z_1 + z_3) &\leq 2, \\ y_i + 2z_3 + (2 - \epsilon)(z_1 + z_2) &\leq 3, & 2z_3 + (1 - \epsilon)(z_1 + z_2) &\leq 2. \end{aligned}$$

It is simple to check that none of the above inequalities is violated by $x^* = (\epsilon e_N, e_N/2)$. Hence the constraint $a'^\top x \leq b'$ is just $a^\top y \leq b$. In other words, we have shown that if (12) admits a separating WI that separates x^* , then the constraint $a^\top y \leq b$ admits a separating WI that separates y^* .

On the other hand, any WI from the constraint $a^\top y \leq b$ is also an a WI from the entire linear system (12). We have thereby proven this claim. \diamond

Note that $y^* = \epsilon e_N$ in this proof coincides with the x^* in Claim 1. From Claim 2 and Claim 1, we have completed the proof for the second part of the statement of this theorem, since SSP is \mathcal{NP} -hard. \square

Even though the problem WI-SP is \mathcal{NP} -hard in general, in the next theorem we provide a special case where it can be solved in polynomial time, and such separating WI can be obtained in polynomial time if one exists.

Theorem 6. *Let x^* be the input solution to WI-SP. If $\max\{|S| : x_i^* \in (0, 1) \forall i \in S, x^*(S) < 1\}$ is a constant, then the separating WI can be obtained in polynomial time if one exists.*

Proof. We assume without loss of generality that $Ax \leq d$ has only a single constraint $a^\top x \leq b$, since we can always solve WI-SP with input (A, d, c, x^*) by solving the corresponding WI-SP problems for each single constraint individually. For any $P \subseteq [n]$, let $f(P) := \sum_{i \in P} a_i x_i^* + \sum_{j \in C(P)} (a_j - r(P)) x_j^* - a(P)$. Then $f(P) > 0$ implies that

$$\sum_{j \in C(P)} x_j^* < \frac{\sum_{i \in P \cup C(P)} a_i x_i^* - a(P)}{r(P)} \leq \frac{b - a(P)}{r(P)} = 1.$$

Among all the packs with the largest $f(P)$ value, let P' be one that is inclusion-wise maximal. In other words, $f(P') \geq f(P)$ for any pack P , and $f(P) = f(P')$ implies that P' is not contained in P . Let $C := C(P')$. Note that for any $i' \in P'$ we have

$$\begin{aligned} f(P') - f(P' \setminus \{i'\}) &= a_{i'} \left(\sum_{j \in C(P' \setminus \{i'\})} x_j^* + x_{i'}^* - 1 \right) + \sum_{j \in C \setminus C(P' \setminus \{i'\})} (a_j - r(P')) x_j^* \\ &\in \left[a_{i'} \left(\sum_{j \in C(P' \setminus \{i'\})} x_j^* + x_{i'}^* - 1 \right), a_{i'} \left(\sum_{j \in C} x_j^* + x_{i'}^* - 1 \right) \right]. \end{aligned}$$

Since $P' \setminus \{i'\}$ is also a pack and $f(P') \geq f(P' \setminus \{i'\})$, we have

$$\sum_{j \in C} x_j^* + x_{i'}^* \geq 1, \quad \forall i' \in P'. \quad (13)$$

On the other hand, for any $i' \in [n] \setminus (C \cup P')$:

$$\begin{aligned} f(P' \cup \{i'\}) - f(P') &= a_{i'} \left(\sum_{j \in C} x_j^* + x_{i'}^* - 1 \right) + \sum_{j \in C(P' \cup \{i'\}) \setminus C} (a_j - r(P' \cup \{i'\})) x_j^* \\ &\in \left[a_{i'} \left(\sum_{j \in C} x_j^* + x_{i'}^* - 1 \right), a_{i'} \left(\sum_{j \in C(P' \cup \{i'\})} x_j^* + x_{i'}^* - 1 \right) \right]. \end{aligned}$$

Since $i' \in [n] \setminus (C \cup P')$, the set $P' \cup \{i'\}$ is still a pack, hence $f(P' \cup \{i'\}) - f(P') \leq 0$. Furthermore, since P' is an inclusion-wise maximal pack with the largest $f(P')$ value, we have $f(P' \cup \{i'\}) - f(P') < 0$. Therefore,

$$\sum_{j \in C} x_j^* + x_{i'}^* < 1, \quad \forall i' \in [n] \setminus (C \cup P'). \quad (14)$$

From (13) and (14), we obtain

$$P' = \{i \in [n] \setminus C \mid x^*(C) + x_i^* \geq 1\}. \quad (15)$$

We have thereby shown that there exists a WI from knapsack constraint $a^\top x \leq b$ which separates x^* , if and only if there exists $C \subseteq [n]$, such that the

corresponding P' , as defined in (15), is a pack satisfying $f(P') > 0$. Therefore, WI-SP can be solved by checking whether the set $P' = \{i \in [n] \setminus C \mid x^*(C) + x_i^* \geq 1\}$ is a pack with $f(P') > 0$, for any possible $C \subseteq [n]$ with $x^*(C) < 1$.

Let $I_0 := \{i \in [n] \mid x_i^* = 0\}$ and $I_f := \{i \in [n] \mid x_i^* \in (0, 1)\}$. From the assumptions of this theorem, we know that $\alpha := \max\{|S| : x^*(S) < 1, S \subseteq I_f\}$ is a constant. For any $T \subseteq I_0$ and $S \subseteq I_f$ with $x^*(S) < 1$, it is easy to see that

$$\{i \in [n] \setminus S \mid x^*(S) + x_i^* \geq 1\} = \{i \in [n] \setminus (S \cup T) \mid x^*(S \cup T) + x_i^* \geq 1\}.$$

Hence, $\{i \in [n] \setminus C \mid x^*(C) + x_i^* \geq 1\}$ is a pack with positive f value for some $C \subseteq [n]$ with $x^*(C) < 1$, if and only if $\{i \in [n] \setminus (C \setminus I_0) \mid x^*(C \setminus I_0) + x_i^* \geq 1\}$ is a pack with positive value, where $C \setminus I_0 \subseteq I_f$ and $x^*(C \setminus I_0) = x^*(C) < 1$. Therefore, WI-SP can be solved by the following procedure:

1. For any $S \subseteq I_f$ with $x^*(S) < 1$, construct the corresponding $P' = \{i \in [n] \setminus S \mid x^*(S) + x_i^* \geq 1\}$.
2. Check if P' is a pack with $f(P') > 0$.
3. If the answer to the previous check is yes for some $S \subseteq I_f$ with $x^*(S) < 1$, then the corresponding P' works as a yes-certificate to WI-SP, and its corresponding WI separates x^* ; If the answer is no for all $S \subseteq I_f$ with $x^*(S) < 1$, then x^* cannot be separated by any WI from the knapsack constraint $a^\top x \leq b$.

Since $\alpha = \max\{|S| : x^*(S) < 1, S \subseteq I_f\}$, we have

$$|\{S \mid x^*(S) < 1, S \subseteq I_f\}| \leq \sum_{k=0}^{\alpha} \binom{n}{k} = O(n^\alpha).$$

So this above procedure can be implemented in polynomial time, and we complete the proof. \square

In particular, Theorem 6 implies that, if x^* has a constant number of fractional components, then WI-SP can be solved in polynomial time. We directly obtain the following corollary.

Corollary 1. *If the number of constraints in the LP relaxation (2) is a constant and x^* is an extreme point solution to (2), then the separating WI can be obtained in polynomial time if one exists.*

Proof. Let the number of constraints be a constant α . Since x^* is an extreme point, we know that at most α components of x^* are fractional. Hence $\max\{|S| : x_i^* \in (0, 1) \forall i \in S, x^*(S) < 1\} \leq \alpha$. The result then follows from Theorem 6. \square

References

1. Alper Atamtürk. Cover and pack inequalities for (mixed) integer programming. *Annals of Operations Research*, 139(1):21–38, 2005.

2. Egon Balas. Facets of the knapsack polytope. *Mathematical programming*, 8(1):146–164, 1975.
3. Egon Balas and Eitan Zemel. Facets of the knapsack polytope from minimal covers. *SIAM Journal on Applied Mathematics*, 34(1):119–148, 1978.
4. Carlos E Ferreira. On combinatorial optimization problems arising in computer system design. 1994.
5. Carlos E Ferreira, Alexander Martin, and Robert Weismantel. Solving multiple knapsack problems by cutting planes. *SIAM Journal on Optimization*, 6(3):858–877, 1996.
6. Virginie Gabrel and Michel Minoux. A scheme for exact separation of extended cover inequalities and application to multidimensional knapsack problems. *Operations research letters*, 30(4):252–264, 2002.
7. Z. Gu, G. L. Nemhauser, and M.W.P. Savelsbergh. Lifted cover inequalities for 0-1 integer programs: Complexity. *INFORMS Journal on Computing*, 11:117–123, 1999.
8. Zonghao Gu, George L Nemhauser, and Martin WP Savelsbergh. Lifted cover inequalities for 0-1 integer programs: Computation. *INFORMS Journal on Computing*, 10(4):427–437, 1998.
9. Christoph Helmberg and Robert Weismantel. Cutting plane algorithms for semidefinite relaxations. In *Topics in semidefinite and interior-point methods (Toronto, ON, 1996)*, volume 18 of *Fields Inst. Commun.*, pages 197–213. Amer. Math. Soc., Providence, RI, 1998.
10. Christopher Hojny, Tristan Gally, Oliver Habeck, Hendrik Lüthen, Frederic Matter, Marc E Pfetsch, and Andreas Schmitt. Knapsack polytopes: a survey. *Annals of Operations Research*, pages 1–49, 2019.
11. Konstantinos Kaparis and Adam N Letchford. Separation algorithms for 0-1 knapsack polytopes. *Mathematical programming*, 124(1-2):69–91, 2010.
12. Richard M Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer, 1972.
13. Diego Klabjan, George L Nemhauser, and Craig Tovey. The complexity of cover inequality separation. *Operations Research Letters*, 23(1-2):35–40, 1998.
14. Adam N Letchford and Georgia Souli. On lifted cover inequalities: A new lifting procedure with unusual properties. *Operations Research Letters*, 47(2):83–87, 2019.
15. Manfred W Padberg. (1, k)-configurations and facets for packing problems. *Mathematical Programming*, 18(1):94–99, 1980.
16. Alexander Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, 1986.
17. Robert Weismantel. On the 0/1 knapsack polytope. *Mathematical Programming*, 77(3):49–68, 1997.