Approximation algorithm for the two-stage stochastic set multicover problem with simple resource

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Abstract

We study a two-stage, finite-scenarios stochastic version of the set multicover problem, where there is uncertainty about a demand for each element to be covered and the penalty cost is imposed linearly on the shortfall in each demand. This problem is NP-hard and has an application in shift scheduling in crowdsourced delivery services. For this problem, we present an LP-based $O(\ln n)$-approximation algorithm, where n is the number of elements to be covered.

Keywords: set multicover problem, stochastic programming, approximation algorithm, crowdsourced delivery

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1. Introduction

The set cover problem (SC) is one of the most fundamental combinatorial optimization problems, where given a ground set U, a collection of subsets of U, $\mathcal{S} \subseteq 2^U$ and a cost function $c : \mathcal{S} \to \mathbb{Q}_+$, the objective is to find a minimum cost sub-collection of $\mathcal{S}$ that covers all elements of U. The set multicover problem (SMC) is a natural generalization of SC. In addition to the input of the set cover problem, given a covering requirement or demand $r_e \in \mathbb{Z}_+$ for $e \in U$, SMC is formulated as follows.

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c(S) x_S & \\ \text{subject to} & \sum_{S \in \mathcal{S}: e \in S} x_S \geq r_e & \forall e \in U, \\ & x_S \in \mathbb{Z}_+ & \forall S \in \mathcal{S}. \end{array} \tag{1}$$

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In this paper, we study a two-stage finite-scenarios stochastic program of SMC, where there is uncertainty about covering requirements. We are given a set of scenarios Ω. In scenario $\omega \in \Omega$, the covering requirement of $e \in U$ takes $r_{e,\omega} \in \mathbb{Z}_+$ and scenario ω occurs with probability $p_\omega \in \mathbb{Q}_+$. We are also given penalty $\pi_e \in \mathbb{Q}_+$ for $e \in U$. In the first stage, we have to fix decision variable x_S for $S \in \mathcal{S}$. In the second stage, for each $e \in U$, we have to pay a penalty cost π_e per unit of the shortage of the realized covering requirement of $e \in U$. The objective of the problem is to decide x_S minimizing the sum of the covering cost $\sum_{S \in \mathcal{S}} C(S)x_S$ and expected penalty cost for shortage over all the scenarios. This problem is formulated as follows.

$$\begin{aligned} \text{minimize} \quad & \sum_{S \in \mathcal{S}} c(S)x_S + \sum_{e \in U} \pi_e \sum_{\omega \in \Omega} p_\omega \max\left(r_{e,\omega} - \sum_{S \in \mathcal{S}} x_S, 0 \right) \\ \text{subject to} \quad & x_S \in \mathbb{Z}_+ \quad \forall S \in \mathcal{S}. \end{aligned} \tag{2}$$

This kind of model is called a two-stage stochastic program with simple resource [1] and thus we call this problem the stochastic set multicover problem with simple resource (Stochastic SMC).

Stochastic SMC has an application in shift scheduling in crowdsourced delivery. Crowdsourced delivery is an emerging method in delivery services to provide faster delivery and to accommodate fluctuations in demand, where individual non-employed couriers deliver packages to customers. Crowdsourced delivery is used by popular services such as Uber EATS or Amazon Flex. Recently, from viewpoints of service stability and cost reduction, a combination of deliveries by employed couriers and non-employed couriers can be effective [2, 3], where employed couriers work on an hourly basis for relatively long hours and non-employed couriers work at a relatively high salary for about one hour. In this method, a company needs to schedule shifts of employed taking into account the distribution of demand for the number of couriers and costs of couriers for each time period and this optimization problem is modeled as Stochastic SMC as follows. Let U be the set of considering time periods and $\mathcal{S}$ be the set of available shifts with cost function $c : \mathcal{S} \to \mathbb{Z}_+$, which means a total salary of an employed courier for each shift. Given some scenario $\omega \in \Omega$, $r_{e,\omega}$ is the number of necessary couriers in time period $e \in U$. For each time period $e \in U$, π_e is the salary for one non-employed courier at time period $e \in U$. Note that this situation can occur with other services that combine crowdsourcing and employees.

From the complexity viewpoint, we can show that Stochastic SMC is NP-hard by a reduction of SMC, which is known to be NP-hard, to Stochastic SMC as follows. Given an instance $(U, \mathcal{S}, r, c)$ of SMC (1), let $\Omega = \{\omega\}$ and $p_\omega = 1$. Also, for any $e \in U$, let $r'_{e,\omega} = r_e$ and π_e be some big enough positive integer. Then the instance of SMC is reduced to the instance of Stochastic SMC, $(U, \mathcal{S}, r', c, \Omega, p, \pi)$.

For SC and its stochastic variants, approximation algorithms are well studied. An algorithm is called an α-approximation algorithm for a minimization problem if it runs in polynomial time and produces a solution whose objective value is less than or equal to α times the optimal value. For SC, Chvatal [4] developed a simple greedy $H(\Delta)$-approximation algorithm, where $\Delta = \max_{S \in \mathcal{S}} |S|$ and $H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Note that $H(\Delta) = O(\ln |U|)$. It is also known that no $o(\ln |U|)$-approximation is possible in polynomial time unless P = NP[5]. Dobson [6] showed that a greedy algorithm for SMC is also $H(\Delta)$-approximation. For stochastic variants of SC, $O(\ln |U|)$-approximation algorithms are also developed. Könemann et al. [7] gave an $H(\Delta)$-approximation algorithm for the prize-collecting set cover problem, which is can be seen as a special case when $|\Omega| = 1$ and $r_{e,\omega} = 1$ in Stochastic SMC. Ravic and Sinhac [8] showed the two-stage finite-scenarios stochastic set cover problem (Stochastic SC) can be reduced to SC and there is an $H(|\Delta||\Omega|)$-approximation algorithm for this problem. Note that, in Stochastic SC, there is uncertainty about a ground set U and, in the second stage, we can choose subsets again to cover all the realized ground set $U' \subseteq U$ with a cost function different from one of the first stage. Li and Liu [9] developed an $O(\ln |U|)$-approximation algorithm for a special case of Stochastic SC. Recently, DeValve et al. [10] designed a constant-factor approximation algorithm for an assemble-to-order (ATO) problem, which has an objective function and constraints similar to Stochastic SMC.

In this paper, we present an approximation algorithm for Stochastic SMC. As a trivial approximation algorithm for Stochastic SMC, we can easily get an $H(|U||\Omega|)$-approximation algorithm from a reduction of Stochastic SMC to SMC, where a ground set is $\{(e, \omega) \mid e \in U, \omega \in \Omega\}$ and the covering requirement of (e, ω) is $r_{e,\omega}$ in SMC. This reduction is almost the same way in [8]. However, there is a gap between its approximation ratio and the best possible approximation ratio $H(\Delta)$ of SMC by Dobson [6]. In this paper, we develop an LP-based $H(\Delta)$-approximation algorithm for Stochastic SMC. Therefore it achieves the best possible approximation ratio unless P=NP. Moreover our algorithm can handle upper bound constraints of decision variables. Our algorithm first solves a natural LP relaxation problem of Stochastic SMC (3) and solves an IP problem again, where integer parts of an optimal solution of the LP are fixed, by the greedy algorithm for a variant of SMC.

2. Algorithm

In this section, we present our $H(\Delta)$-approximation algorithm for Stochastic SMC. Our algorithm is based on the fact that a subproblem, where integer parts of an optimal solution of its LP relaxation are fixed, can be reduced to a simple set multicover problem. In the following discussion, we consider the problem (2) with upper bound constraints for decision variables.

Stochastic SMC (2) is formulated as an integer linear program as follows. Ra

$$\begin{array}{lll} \text{minimize} & \sum_{S\in\mathcal{S}} c(S)x_S + \sum_{e\in U}\pi_e \sum_{\omega\in\Omega} p_{e,\omega} z_{e,\omega} & \\ \text{subject to} & \sum_{S\in\mathcal{S}:e\in S} x_S + z_{e,\omega} \geq r_{e,\omega} & \forall\omega\in\Omega\ \forall e\in U, \\ & x_S \in \mathbb{Z}_+ & \forall S\in\mathcal{S}, \\ & x_S \leq d(S) & \forall S\in\mathcal{S}, \\ & z_{e,\omega}\in\mathbb{Z}_+ & \forall\omega\in\Omega\ \forall e\in U, \end{array} \tag{3}$$

where $d : U \to \mathbb{Z}_+$ is a given upper bound function. A natural LP relaxation problem of (3) is written as follows.

$$\begin{array}{lll} \text{minimize} & \sum_{S\in\mathcal{S}} c(S)x_S + \sum_{e\in U}\pi_e \sum_{\omega\in\Omega} p_{e,\omega} z_{e,\omega} & \\ \text{subject to} & \sum_{S\in\mathcal{S}:e\in S} x_S + z_{e,\omega} \geq r_{e,\omega} & \forall\omega\in\Omega\ \forall e\in U, \\ & d(S) \geq x_S \geq 0 & \forall S\in\mathcal{S}, \\ & z_{e,\omega}\geq 0 & \forall\omega\in\Omega\ \forall e\in U. \end{array} \tag{4}$$

Let $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ be an optimal solution of (4). We define the integer parts $(\boldsymbol{x}^I, \boldsymbol{z}^I) = (\lfloor \boldsymbol{x}^* \rfloor, \lfloor \boldsymbol{z}^* \rfloor)$, where the floor of a vector $\boldsymbol{v}$ denotes the vector where the ih coordinate is the floor of v_i, and the fractional parts $(\boldsymbol{x}^F, \boldsymbol{z}^F) = (\boldsymbol{x}^* - \boldsymbol{x}^I, \boldsymbol{z}^* - \boldsymbol{z}^I)$.

Then, by fixing the integer parts of an optimal solution of (4), we consider the following LP.

$$\begin{array}{lll} \text{minimize} & \sum_{S\in\mathcal{S}} c(S)y_S + \sum_{e\in U}\pi_e \sum_{\omega\in\Omega} p_{e,\omega} w_{e,\omega} & \\ \text{subject to} & \sum_{S\in\mathcal{S}:e\in S} x^I_S + z^I_{e,\omega} + \sum_{S\in\mathcal{S}:e\in S} y_S + w_{e,\omega} \geq r_{e,\omega} & \forall\omega\in\Omega\ \forall e\in U, \\ & 1 \geq y_S \geq 0 & \forall S\in\mathcal{S}, \\ & 1 \geq w_{e,\omega}\geq 0 & \forall\omega\in\Omega\ \forall e\in U, \end{array} \tag{5}$$

where we remove constant in the objective function. We can easily see that $(\boldsymbol{x}^F, \boldsymbol{z}^F)$ is an optimal solution of (5). Also, for any feasible solution $(\boldsymbol{y}, \boldsymbol{w})$ of (5), $(\boldsymbol{x}^I + \boldsymbol{y}, \boldsymbol{z}^I + \boldsymbol{w})$ is feasible to (4).

For some minimization LP problem, if an algorithm outputs an integral solution for the problem whose objective value is within α times the optimal value and runs in polynomial time, then we call this algorithm an LP-based α-approximation algorithm for the problem. The following algorithm (Algorithm 1) outputs a solution of the original problem (3) if there exists an LP-based α-approximation algorithm for (5).

Algorithm 1

Step 1: Solve (4) and get the problem (5) by letting an optimal solution of (4) be $(\boldsymbol{x}^*, \boldsymbol{z}^*) = (\boldsymbol{x}^I + \boldsymbol{x}^F, \boldsymbol{z}^I + \boldsymbol{z}^F)$.

Step 2: Get an integral solution $(\boldsymbol{y}^A, \boldsymbol{w}^A)$ of (5) by using an LP-based α-approximation algorithm for (5).

Step 3: Output $(\boldsymbol{x}^A, \boldsymbol{z}^A) = (\boldsymbol{x}^I + \boldsymbol{y}^A, \boldsymbol{z}^I + \boldsymbol{w}^A)$.

Lemma 1. *Algorithm 1 is an α-approximation algorithm for* (3).

Proof. Let $(\boldsymbol{x}^A, \boldsymbol{z}^A)$ be an output solution by Algorithm 1. Clearly, $(\boldsymbol{x}^A, \boldsymbol{z}^A)$ is a feasible solution of (3). Denoting the optimal value of (3) as OPT, we have

$$\begin{aligned}
\sum_{S \in \mathcal{S}} c(S) x_S^A + \sum_{e \in U} \pi_e \sum_{\omega \in \Omega} p_{e,\omega} z_{e,\omega}^A &= \sum_{S \in \mathcal{S}} c(S)(x_S^I + y_S^A) + \sum_{e \in U} \pi_e \sum_{\omega \in \Omega} p_{e,\omega}(z_{e,\omega}^I + w_{e,\omega}^A) \\
&\leq \alpha \left(\sum_{S \in \mathcal{S}} c(S)(x_S^I + x_S^F) + \sum_{e \in U} \pi_e \sum_{\omega \in \Omega} p_{e,\omega}(z_{e,\omega}^I + z_{e,\omega}^F) \right) \\
&\leq \alpha OPT.
\end{aligned}$$

□

From the above lemma, Algorithm 1 is an α-approximation algorithm for Stochastic SMC if there is an LP-based α-approximation algorithm for (5). In fact, there is an LP-based $H(\Delta)$-approximation algorithm for (5), which is shown in the subsection.

Lemma 2. *There is an LP-based $H(\Delta)$-approximation algorithm for* (5).

Finally, from Lemma 1 and 2, we have the following result.

Theorem 1. *There is an $H(\Delta)$-approximation algorithm for Stochastic SMC* (3).

3. Proof of Lemma 2

We give an LP-based $H(\Delta)$-approximation algorithm for (5) by reducing (5) to SMC with binary constraints, for which a greedy algorithm is known to be $H(\Delta)$-approximation.

For any $e \in U$, we define X_e^I and X_e^F as sums of the integer and fractional parts of a vector $\boldsymbol{x}^*$ in an optimal solution of (4), respectively, as follows.

$$\begin{array}{rcl} X_e^I & = & \sum_{S \in \mathcal{S}: e \in S} x_S^I \\ X_e^F & = & \sum_{S \in \mathcal{S}: e \in S} x_S^F \end{array} \tag{6}$$

For any $e \in U$, let $\tilde{\Omega}_e$ be the set of scenarios such that its cover requirement is not satisfied, that is,

$$\tilde{\Omega}_e = \{\omega \in \Omega | z_{e,\omega}^* > 0\}. \tag{7}$$

Without loss of generality, for any $e \in U$, we assume

$$\omega \in \tilde{\Omega}_e \Rightarrow z_{e,\omega}^* = r_{e,\omega} - (X_e^I + X_e^F) \tag{8}$$

$$\omega \notin \tilde{\Omega}_e \Rightarrow z_{e,\omega}^* = 0 \text{ and } X_e^I + \lfloor X_e^F \rfloor \geq r_{e,\omega}. \tag{9}$$

From (9), we can add $w_{e,\omega}^F = 0$ $(e \in U,\ \omega \notin \tilde{\Omega}_e)$ to (5). Also, since $\lceil z_{e,\omega}^F \rceil \geq z_{e,\omega}^F \geq 0$ $(e \in U,\ \omega \in \Omega)$ holds, we can add $\lceil z_{e,\omega}^F \rceil \geq w_{e,\omega} \geq 0$ $(e \in U,\ \omega \in \Omega)$. The problem (5) is rewritten as follows.

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c(S) y_S + \sum_{e \in U} \pi_e \sum_{\omega \in \Omega} p_{e,\omega} w_{e,\omega} & \\ \text{subject to} & \sum_{S \in \mathcal{S}: e \in S} y_S \geq r_{e,\omega} - X_e^I & \forall \omega \notin \tilde{\Omega}_e \ \forall e \in U, \\ & \sum_{S \in \mathcal{S}: e \in S} y_S + w_{e,\omega} \geq r_{e,\omega} - X_e^I - z_{e,\omega}^I & \forall \omega \in \tilde{\Omega}_e \ \forall e \in U, \\ & 1 \geq y_S \geq 0 & \forall S \in \mathcal{S}, \\ & \lceil z_{e,\omega}^F \rceil \geq w_{e,\omega} \geq 0 & \forall \omega \in \tilde{\Omega}_e \ \forall e \in U. \end{array} \tag{10}$$

We show that the value $z_{e,\omega}^F (e \in U, \omega \in \tilde{\Omega}_e)$ does not depend on its scenario.

Lemma 3. *For any* $e \in U$*,*

$$\omega \in \tilde{\Omega}_e \Rightarrow z^F_{e,\omega} = Z^F_e \tag{11}$$

where $Z^F_e = \lceil X^F_e \rceil - X^F_e \in [0, 1)$*.*

Proof. For $e \in U$ and $\omega \in \tilde{\Omega}_e$, from $z^*_{e,\omega} = z^I_{e,\omega} + z^F_{e,\omega}$ and (8), we have

$$\begin{aligned} z^I_{e,\omega} + z^F_{e,\omega} &= r_{e,\omega} - X^I_e - X^F_e \\ &= r_{e,\omega} - X^I_e - X^F_e + \lceil X^F_e \rceil - \lceil X^F_e \rceil \\ &= (r_{e,\omega} - X^I_e + \lceil X^F_e \rceil) + (\lceil X^F_e \rceil - X^F_e). \end{aligned}$$

Since $r_{e,\omega} - X^I_e + \lceil X^F_e \rceil$ is integer and $(\lceil X^F_e \rceil - X^F_e) \in [0, 1)$, we obtain $z^F_{e,\omega} = \lceil X^F_e \rceil - X^F_e = Z^F_e$. □

Next, we show that the value of the right-hand side of the second constraint in (10) is independent of its scenario.

Lemma 4. *For any* $e \in U$*,*

$$\omega \in \tilde{\Omega}_e \Rightarrow r_{e,\omega} - X^I_e - z^I_{e,\omega} = \lceil X^F_e \rceil. \tag{12}$$

Proof. From (8) and Lemma 3, for any $e \in U$ and $\omega \in \tilde{\Omega}_e$, we have

$$\begin{aligned} r_{e,\omega} - X^I_e - z^I_{e,\omega} &= r_{e,\omega} - X^I_e - (z^*_{e,\omega} - Z^F_e) \\ &= r_{e,\omega} - X^I_e - (r_{e,\omega} - X^I_e - X^F_e - \lceil X^F_e \rceil + X^F_e) \\ &= \lceil X^F_e \rceil. \end{aligned}$$

□

From Lemma 3 and Lemma 4, we can rewrite (10) as follows.

$$\text{minimize} \sum_{S \in \mathcal{S}} c(S) y_S + \sum_{e \in U} \pi_e \sum_{\omega \in \tilde{\Omega}_e} p_{e,\omega} w_{e,\omega} \tag{13}$$

$$\text{subject to} \sum_{S \in \mathcal{S}: e \in S} y_S \geq r_{e,\omega} - X^I_e \qquad \forall \omega \notin \tilde{\Omega}_e \; \forall e \in U, \tag{14}$$

$$\sum_{S \in \mathcal{S}: e \in S} y_S + w_{e,\omega} \geq \lceil X^F_e \rceil \qquad \forall \omega \in \tilde{\Omega}_e \; \forall e \in U, \tag{15}$$

$$1 \geq y_S \geq 0 \qquad \forall S \in \mathcal{S}, \tag{16}$$

$$\lceil Z^F_e \rceil \geq w_{e,\omega} \geq 0 \qquad \forall \omega \in \tilde{\Omega}_e \; \forall e \in U. \tag{17}$$

This problem can be further simplified.

Lemma 5. *Constraint* (14) *is redundant, that is,*

$$(\boldsymbol{y}, \boldsymbol{w}) \text{ satisfies } (15)\text{-}(17) \Rightarrow (\boldsymbol{y}, \boldsymbol{w}) \text{ satisfies } (14). \tag{18}$$

Proof. Let $(\boldsymbol{y}, \boldsymbol{w})$ be a pair of vectors satisfying (15)-(17). From (15), (17) and $Z_e^F = \lceil X_e^F \rceil - X_e^F$, for any $e \in U, \omega \in \tilde{\Omega}_e$, we obtain

$$\sum_{S \in \mathcal{S}: e \in S} y_S \geq \lceil X_e^F \rceil - w_{e,\omega} \geq \lceil X_e^F \rceil - \lceil Z_e^F \rceil = X_e^F \geq \lfloor X_e^F \rfloor. \tag{19}$$

Thus, we have $\sum_{S \in \mathcal{S}: e \in S} y_S \geq \lfloor X_e^F \rfloor$ for $e \in U$. Since $\lfloor X_e^F \rfloor \geq r_{e,\omega} - X_e^I (\omega \notin \tilde{\Omega}_e, e \in U)$ from (9), for $e \in U$ and $\omega \notin \tilde{\Omega}_e$, we have

$$\sum_{S \in \mathcal{S}: e \in S} y_S \geq \lfloor X_e^F \rfloor \geq r_{e,\omega} - X_e^I.$$

□

Focusing on (15), we see that the optimal value does not change if we add a new variable $w_e \geq 0$ $(e \in U)$ and

$$w_e = w_{e,\omega} \ \forall \omega \in \tilde{\Omega} \ \forall e \in U.$$

For $e \in U$, by defining

$$\Pi_e = \pi_e \sum_{\omega \in \tilde{\Omega}_e} p_{e,\omega} \tag{20}$$

and substituting w_e for $w_{e,\omega}$, we finally get the following LP.

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c(S) y_S + \sum_{e \in U} \Pi_e w_e & \\ \text{subject to} & \sum_{S \in \mathcal{S}: e \in S} y_S + w_e \geq \lceil X_e^F \rceil & \forall e \in U, \\ & 1 \geq y_S \geq 0 & \forall S \in \mathcal{S}, \\ & \lceil Z_e^F \rceil \geq w_e \geq 0 & \forall e \in U. \end{array} \tag{21}$$

Finding an integer solution in (21) is known as the constrained set multicover problem, where there are additional binary constraints $x_S \in \{0, 1\}$ $(S \in \mathcal{S})$ in SMC (1). For this problem, a simple greedy algorithm is known to be an LP-based $H(\Delta)$-approximation [11]. Therefore, by using the greedy algorithm for (21) and transforming the output into a solution of the original problem (5), we get an LP-based $H(\Delta)$-approximation algorithm for (5).

4. Conclusion

In this paper, we presented an $H(\Delta)$-approximation algorithm for the stochastic SMC. Our algorithm is based on the fact that the subproblem, where integer parts of an optimal solution of its LP relaxation are fixed, can be reduced to a simple set multicover problem. Since our algorithm needs to solve LP, as future work, it can be interesting to develop combinatorial approximation algorithms without solving LP for Stochastic SMC such as greedy algorithms.

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