An active signature method for constrained abs-linear minimization

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In this paper we consider the solution of optimization tasks with a piecewise linear objective function and piecewise linear constraints. First, we state optimality conditions for that class of problems using the abs-linearization approach and prove that they can be verified in polynomial time. Subsequently, we propose an algorithm called Constrained Active Signature Method that explicitly exploits the piecewise linear structure to solve such optimization problems. Convergence of the algorithm within a finite number of iterations is proven. Numerical results for various testcases including linear complementarity conditions and a bi-level problem illustrate the performance of the new algorithm.

Keywords: piecewise linear; constrained optimization; abs-linear form; linear independence kink qualification (LIKQ); optimality conditions; active signature method (ASM)

1 Introduction

Motivated by numerous applications, e.g., from machine learning, there has been a growing interest in optimization problems that lack differentiability. That is, the objective function and/or the constraints are not differentiable everywhere. One important class of such problems is given by piecewise linear functions, where corresponding optimization tasks arise, e.g., in train time tabling [FH10], as local models [LM16] or in the training of deep neural networks with the Rectified Linear Unit (ReLU) as activation function. For unconstrained optimization problems with piecewise linear objective functions, a so-called Active Signature Method (ASM) for determining local minima has been proposed in [GW19a]. In this paper, an extension of ASM will be presented, which, in addition to the piecewise linear objective function, also takes piecewise linear functions as equality and inequality constraints into account.

A piecewise linear function can always be given in its abs-linear form as introduced for the first time in [Gri13]:

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Definition 1.1 (Abs-linear form, switching vector)**.** A continuous piecewise linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is in *abs-linear form* if $y \equiv \varphi(x)$ is given by

$$y = d + a^\top x + b^\top z \;, \tag{1a}$$
$$z = c + Zx + Mz + L|z| \;, \tag{1b}$$

with $x \in \mathbb{R}^n$ the argument vector, $z \in \mathbb{R}^s$ the vector of switching variables, called *switching vector*, and constants $d \in \mathbb{R}$, $a \in \mathbb{R}^n$, $b, c \in \mathbb{R}^s$, $Z \in \mathbb{R}^{s\times n}$, $L, M \in \mathbb{R}^{s\times s}$, where the last two matrices are strictly lower triangular. Eq. (1b) is called *switching system.*

Here and throughout, $|z|$ denotes the component-wise modulus of a vector z. Without loss of generality, we can always assume that $d = 0$. Using the reformulation

$$\max(x_1, x_2) = \frac{1}{2}(x_1 + x_2 + |x_1 - x_2|) \;, \quad \min(x_1, x_2) = \frac{1}{2}(x_1 + x_2 - |x_1 - x_2|) \tag{2}$$

and Prop. 2.2.2 of [Sch12] it follows that every continuous piecewise linear function can be represented in an abs-linear form.

Using the signatures of the switching vector, it is possible to decompose $\mathbb{R}^n$ into polyhedra [GW16]. This decomposition plays an essential role in the algorithm we will present in this paper, because one of the main ideas of the algorithm will be the solution of suitable adapted smooth optimization problems on these polyhedra.

Definition 1.2 (Signature vector and signature matrix)**.** Let a piecewise linear function be given in an abs-linear form (1). For each $x \in \mathbb{R}^n$, we define the *signature vector*

$$\sigma(x) \equiv (\mathbf{sgn}(z_1(x)), \ldots, \mathbf{sgn}(z_s(x))) \in \{-1, 0, 1\}^s \equiv \mathcal{S} \;,$$

where $\mathcal{S}$ comprises the set of all possible signature vectors. The corresponding *signature matrix* is given by $\Sigma(x) = \mathbf{diag}(\sigma(x))$. A signature vector $\sigma(x)$ is called *definite*, if no component vanishes, i.e., $\sigma(x) \in \{-1, 1\}^s$. This situation is denoted by $0 \notin \sigma(x)$. Otherwise it is called *indefinite.*

Since L is assumed to be strictly lower triangular, $|z_s(x)|$ does not contribute to the value of the abs-linear objective function and hence does not impose any nonsmoothness. This fact has to be taken into account correspondingly as described below. Since we will also consider frequently fixed signature vectors, we will state the dependence on x always explicitly. Based on these signature vectors, it is possible to decompose the $\mathbb{R}^n$ into polyhedra as follows.

Definition 1.3 ((Extended) Signature domain)**.** For a fixed $\sigma \in \{-1, 0, 1\}^s$, we define

$$\mathcal{P}_\sigma \equiv \{x \in \mathbb{R}^n \mid \mathbf{sgn}(z(x)) = \sigma\} \subset \overline{\mathcal{P}}_\sigma \equiv \{x \in \mathbb{R}^n \mid \Sigma z(x) = |z(x)|\} \;.$$

The set $\mathcal{P}_\sigma$ is called *signature domain* and the set $\overline{\mathcal{P}}_\sigma$ *extended signature domain.*

The domains $\mathcal{P}_\sigma$ are given as inverse images of the corresponding σ and represent a disjoint decomposition of $\mathbb{R}^n$ into relatively open polyhedra. The boundaries of the polyhedra $\mathcal{P}_\sigma$ are exactly the sets where φ is nonsmooth. Motivated by the graphical representation in low dimensions as illustrated also in Example 3.1, these sets are called kinks. As shown in [Gri+16] it is also possible to define a partial ordering as follows

$$\sigma \prec \tilde{\sigma} \quad \Longleftrightarrow \quad \sigma_l^2 \leq \tilde{\sigma}_l \sigma_l \quad \text{for } 1 \leq l \leq s \quad \Longleftrightarrow \quad \overline{\mathcal{P}}_\sigma \subseteq \overline{\mathcal{P}}_{\tilde{\sigma}}.$$

For more details about this decomposition see for example [GW16] or [GW19a].

Next, we state the optimization problem for which we will present and analyze a solution algorithm in this paper, i.e., the *constrained abs-linear optimization problem* (CALOP). It has the following structure:

$$\begin{aligned} \min_{x \in \mathbb{R}^n, z \in \mathbb{R}^s} \quad & a^\top x + b^\top z \\ \text{s.t.} \quad & 0 = g + Ax + Bz + C|z| \; , \\ & 0 \geq h + Dx + Ez + F|z| \; , \\ & z = c + Zx + Mz + L|z| \; , \end{aligned} \tag{CALOP}$$

where $g \in \mathbb{R}^m, h \in \mathbb{R}^p, A \in \mathbb{R}^{m \times n}, B, C \in \mathbb{R}^{m \times s}, D \in \mathbb{R}^{p \times n}$ and $E, F \in \mathbb{R}^{p \times s}$. As can be seen, we assume that the objective function combined with the switching system in the last constraint is in abs-linear form, cf. Eq. (1). The first constraint in (CALOP) represents the equality constraint and the second one the inequality constraint. For later use, we define

$$\begin{aligned} & f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}, && (x, z) \mapsto a^\top x + b^\top z \; , \\ & G : \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^m, && (x, z, |z|) \mapsto g + Ax + Bz + C|z| \; , \\ \text{and} \quad & H : \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^p, && (x, z, |z|) \mapsto h + Dx + Ez + F|z| \; . \end{aligned} \tag{3}$$

The paper is organized as follows. Section 2 briefly describes the ASM algorithm published in [GW19a] to prepare the ground for the extensions derived in the present paper. Optimality conditions for constrained nonsmooth optimization problems of the form (CALOP) are derived in Section 3. For this purpose, the optimality conditions of the unconstrained case as introduced in [GW16] and discussed further in [GW20] are modified to take the feasibility with respect to the constraints into account. It is possible to derive an optimality test that can be verified in polynomial time representing one main new result of this paper. In Section 4, the ASM is extended to cover also piecewise linear constrained optimization problems of the form (CALOP). The convergence analysis of the resulting new algorithm is also given in Section 4 as another main contribution of this paper. This includes also a statement on finite convergence. Numerical results for several test problems are presented in Section 5. Finally, the paper concludes with a summary and an outlook in Section 6.

2 The active signature method for unconstrained problems

In this section, the Active Signature Method (ASM) published in [GW19a] is explained, such that it can be extended subsequently to constrained problems of the form (CALOP). It should be noted that the notation has been adjusted in comparison to [GW19a] because the abs-linear form is used in a slightly different representation.

The aim of the ASM is to determine

$$\min_{x\in\mathbb{R}^n} \varphi(x)$$

for a piecewise linear objective function $\varphi : \mathbb{R}^n \to \mathbb{R}$ given in abs-linear form (1). The basic idea is to decompose the $\mathbb{R}^n$ into polyhedra as sketched above using the signature vectors $\sigma(x)$ and optimize a penalized version of the objective function on these domains, switching from one polyhedron to the next in an appropriate way. To achieve this behavior, for each $x \in \mathbb{R}^n$ and the corresponding $z = z(x)$, information about the structure of nonsmoothness is exploited. Using the abs-linear form of the objective function φ, we obtain the following equivalent *abs-linear optimization problem*

$$\begin{aligned} \min_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad & a^\top x + b^\top z \\ \text{s.t.} \quad & z = c + Zx + Mz + L|z| \ . \end{aligned} \tag{ALOP}$$

Since piecewise linear functions may be unbounded below, we add the term $\frac{1}{2}x^\top Qx$ with a positive definite matrix $Q = Q^\top \in \mathbb{R}^{n\times n}$ to the objective. Furthermore, we fix one signature vector $\sigma \in \{-1, 0, 1\}^s$ to obtain from (ALOP) the smooth quadratic optimization problem

$$\min_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad a^\top x + b^\top z + \frac{1}{2}x^\top Qx \tag{4a}$$

$$\text{s.t.} \quad z = c + Zx + Mz + L\Sigma z \ , \tag{4b}$$

$$0 = (I_s - |\Sigma|)z \ , \tag{4c}$$

$$0 \le \Sigma z \ , \tag{4d}$$

on $\mathcal{P}_\sigma$, i.e., all $x \in \mathcal{P}_\sigma$ are feasible. Here, I_s denotes the identity matrix in $\mathbb{R}^{s\times s}$. Due to the penalty term, it is ensured that a global minimum exists on $\overline{\mathcal{P}}_\sigma$. As shown in [GW19a] applying standard KKT theory for the smooth constrained quadratic optimization problem Eq. (4) yields the following system of necessary optimality conditions

$$\begin{bmatrix} Q & 0 & Z^\top \\ 0 & I_s - |\Sigma| & \Sigma(M^\top + \Sigma L^\top - I_s) \\ Z & M + L\Sigma - I_s & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \lambda \end{bmatrix} = - \begin{bmatrix} a \\ \Sigma b \\ c \end{bmatrix} \ . \tag{5}$$

A solution of this equation is denoted by $(\hat{x}, \hat{z}, \lambda)$, where λ represents the Lagrange multiplier associated with the equality constraint (4b). As described also in [GW19a], the system (5) can be solved efficiently using the special structure of the triangular matrices L and M. However, Eq. (5) ignores the inequality constraints (4d) of the optimization problem (4). Hence, the resulting $\Sigma\hat{z}$ may have negative components. In this case, there must be a so called blocking constraint or more specifically a blocking kink on the line segment from the current iterate (x, z) to the infeasible point $(\hat{x}, \hat{z})$. With $\hat{z}$ part of the solution of the system given in Eq. (5) and z the current iterate, this situation can be easily detected by calculating the maximal step size β^z as

$$\beta^z = \inf_{1\le l\le s} \left\{ \beta_l^z \equiv \frac{-z_l}{\hat{z}_l - z_l} \,\middle|\, (\hat{z}_l - z_l)\sigma_l < 0 \right\} \in \ (0, \infty] \tag{6}$$

with $\inf \emptyset \equiv \infty$. Note that in comparison to [GW19a] we replace the factor z_l by σ_l in the test of the sign. Since only the sign is important the formulation in Eq. (6) is equivalent to the original formulation but numerically more stable.

If $\beta^z < \infty$ the first index for which the minimum is attained is denoted by j^z. For $\beta^z \le 1$, there exists at least one blocking constraint and the next iterate is given by

$$x^+ = (1 - \beta^z)x + \beta^z\hat{x} \quad \text{and} \quad z^+ = (1 - \beta^z)z + \beta^z\hat{z}\ .$$

The part x^+ lies on the boundary of $\mathcal{P}_\sigma$, i.e., at least one component of $\sigma(x^+)$ drops to zero in comparison to σ. Therefore, one updates $\sigma^+ = \sigma - \sigma_{j^z} e_{j^z}$ setting $\sigma^+_{j^z} = 0$, which amounts to the activation of a kink. In other words the feasible domain of the optimization problem (4) is effectively reduced to the face polyhedron $\mathcal{P}_{\sigma^+} \subset \overline{\mathcal{P}}_\sigma$. After finitely many such kink activations one must have $\beta^z = 1$ so that the full step reaches the unique minimizer x_σ within the current $\mathcal{P}_\sigma$.

Definition 2.1 (Signature optimal point)**.** Let an optimization problem of the form (ALOP) be given. Consider a fixed signature vector $\sigma \in \{-1, 0, 1\}^s$. A minimizer $x_\sigma \in \mathcal{P}_\sigma$ of the optimization problem

$$\min_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad a^\top x + b^\top z \tag{7a}$$

$$\text{s.t.} \quad z = c + Zx + Mz + L\Sigma z\ , \tag{7b}$$

$$0 = (I_s - |\Sigma|)z\ , \tag{7c}$$

$$0 \le \Sigma z\ , \tag{7d}$$

is called *signature optimal point* of the original, unconstrained optimization problem (ALOP).

Note that for many or even most σ the polyhedra $\mathcal{P}_\sigma$ do not contain minimizers, in which case the solutions of (7) lie on their relative boundary. The piecewise linear and convex function $\varphi : \mathbb{R}^2 \to \mathbb{R}$,

$$\varphi(x) = \max\{\max\{-100, 2x_1 + 5|x_2|\}, 3x_1 + 2|x_2|\} \tag{8}$$

considered already by Hiriart-Urruty and Lemaréchal in [HL93] will be used to illustrate this observation. The representation given in Eq. (8) results in only four switching

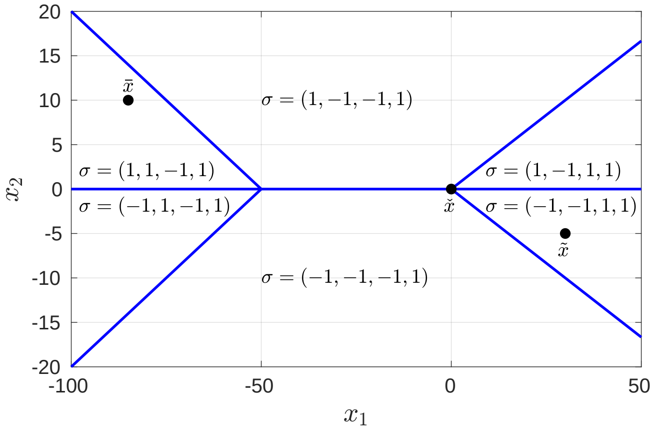


Figure 1: Signature domains and (non) signature optimal points

variables in contrast to the original formulation given in [HL93]. Figure 1 shows the decomposition of the $\mathbb{R}^2$ in the different polyhedra $\mathcal{P}_\sigma$. As can be seen, the point $\tilde{x} = (30, -5)$ has the signature vector $\sigma(\tilde{x}) = (-1, -1, 1, 1) \equiv \sigma$. When optimizing over $\overline{\mathcal{P}}_\sigma$ one obtains the minimizer $\check{x} = (0, 0)$ with the signature vector $\sigma(\check{x}) = (0, -1, 0, 1) \neq \sigma$. Hence, one has $\check{x} \notin \mathcal{P}_{(-1,-1,1,1)}$ but $\check{x}$ is signature optimal on the polyhedron $\mathcal{P}_{(0,-1,0,1)}$ that contains only $\check{x}$. The function $\varphi(.)$ is constant on the polyhedra $\mathcal{P}_{(1,1,-1,1)}$ and $\mathcal{P}_{(-1,1,-1,1)}$. Hence, $\bar{x} = (-85, 10)$ with the signature $\sigma(\bar{x}) = (1, 1, -1, 1)$ is a minimizer on the polyhedron $\mathcal{P}_{(1,1,-1,1)}$. Therefore, $\bar{x}$ is signature optimal.

There exist optimality conditions that can be used to verify in polynomial time whether a signature optimal point x_σ is a minimizer of the full optimization problem (ALOP) or not, see, e.g., [GW20]. For this purpose, let $\tilde{\sigma} \succ \sigma$ so that $\mathcal{P}_{\tilde{\sigma}} \supset \mathcal{P}_\sigma$. Any such $\tilde{\sigma}$ can be decomposed into $\sigma + \gamma$, where $|\sigma|^\top |\gamma| = 0$ holds. It was shown in [GW20] that minimality of x_σ on $\mathcal{P}_{\tilde{\sigma}}$ then requires

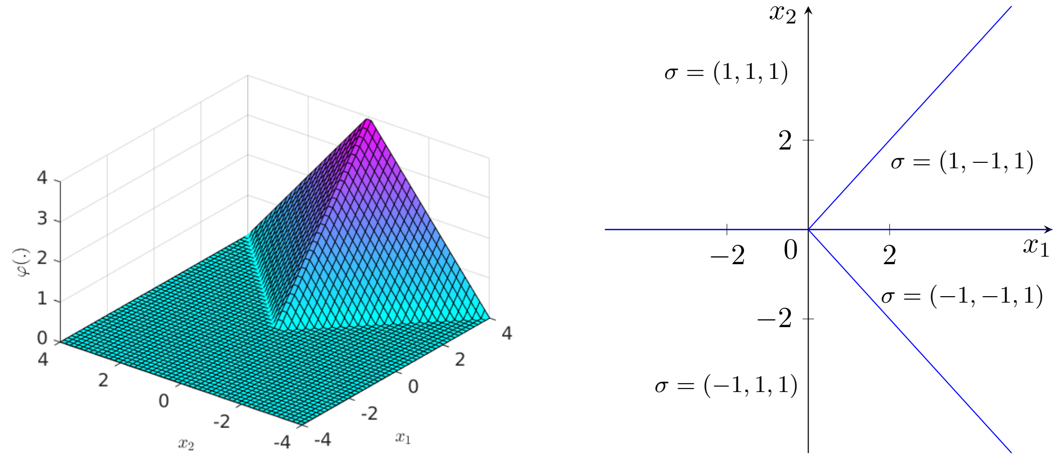
$$0 \leq \mu^\top |\gamma| \equiv b^\top \gamma + \lambda^\top L|\gamma| - \lambda^\top (I_s - M)\gamma = \Big(b^\top - \lambda^\top (I_s - M)\Big)\,\gamma + \lambda^\top L|\gamma|\,.$$

This optimality condition is violated if and only if there is at least one index $k \in \{1, \ldots, s-1\}$ such that $\gamma = e_k \mathbf{sgn}(\lambda^\top (I_s - M)e_k)$ satisfies

$$0 > -\Big|b^\top - \lambda^\top (I_s - M)\Big|\, e_k + \lambda^\top L e_k \tag{9}$$

with $\sigma_k = 0$. Here, the fact that $|z_s(x)|$ does not contribute to the value of $f(x, z)$ and therefore does not lead to a nonsmoothness has to be taken into account. Hence, $k = s$ must not be considered. If the optimality condition does not hold one possible strategy is to choose the index k for which the right-hand side of Eq. (9) is minimal. Hence, the next σ^+ has one component less that equals zero. This can be interpreted as releasing a kink in that one does not insist anymore that the corresponding absolute value is evaluated at zero.

These considerations result in the Active Signature Method that is described in much more detail in [GW19a].



The function $\varphi()$. Corresponding definite signature vectors

Figure 2: Illustration of the unconstrained case from Example 3.1

3 Optimality conditions of the constrained optimization problem

In this section, we derive optimality conditions for the constrained optimization problem (CALOP) as defined in Sec. 1 and show that they can be verified in polynomial time at a given point. To simplify notation, we assume throughout that all switching variables z_i with $i < s$ occur as arguments in an evaluation of the absolute value function. If this is not the case, the abs-linear representation of (CALOP) can be adapted correspondingly. For the constrained optimization problem (CALOP), the functions G and H may or may not depend on the value $|z_s|$ as illustrated next.

Example 3.1. *Let the function $\varphi(x_1, x_2) = \max\{0, x_1 - |x_2|\}$ be given. This nonsmooth nonconvex function is illustrated on the left hand side of Fig. 2. Using the reformulation of the* max*-function given in Eq.* (2)*, we obtain*

$$\varphi(x_1, x_2) = \frac{1}{2}\left(x_1 - |x_2| + \big| - x_1 + |x_2|\big|\right),$$

which can be converted into the following abs-linear form, see Eq. (1)*:*

$$\begin{aligned}
z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &= \begin{pmatrix} x_2 \\ -x_1 + |z_1| \\ -\frac{1}{2}|z_1| + \frac{1}{2}|z_2| \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{pmatrix} |z_1| \\ |z_2| \\ |z_3| \end{pmatrix} \\
y = 0 + \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &+ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = f(x, z)\,.
\end{aligned}$$

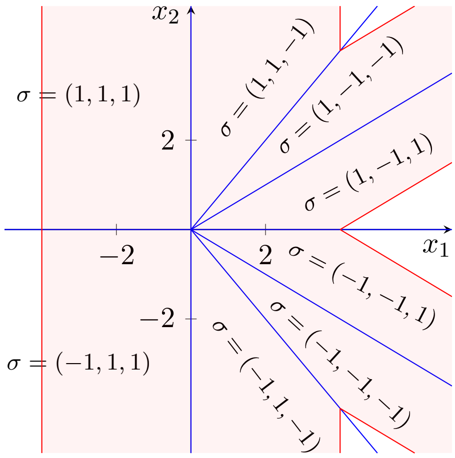


Figure 3: Definite signature vectors for the constrained case of Example 3.1.

If we consider this function as objective of an unconstrained optimization problem as in Sec. 2, the resulting definite signature vectors and the corresponding polyhedra are illustrated on the right hand side of Fig. 2, where the blue lines mark the arguments that cause the nonsmoothness. Now, we add the constraint

$$|z_3| = \left| -\frac{1}{2}|x_2| + \frac{1}{2}\left| - x_1 + |x_2| \right| \right| \leq 2$$

that can be formulated as

$$H(x, z, |z|) = -2 + \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |z_1| \\ |z_2| \\ |z_3| \end{pmatrix} \leq 0$$

to obtain a constrained optimization problem of the form (CALOP)*. Then, $|z_3|$ contributes explicitly to the evaluation of the abs-linear constraint.*

Figure 3 shows the definite signature vectors for the constrained situation. In comparison to the unconstrained case, further kinks are added resulting in more polyhedra. The red area represents the feasible set. All points that lie inside or on the edges of the red area are feasible.

Next, we define polyhedra that take the additional constraints into account:

Definition 3.2 (Feasible (extended) signature domain)**.** For a fixed signature vector $\sigma \in \{-1, 0, 1\}^s$, we define

$$\mathcal{F}_\sigma \equiv \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} G(x, z(x), \Sigma z(x)) = 0, \\ H(x, z(x), \Sigma z(x)) \leq 0, \\ \mathbf{sgn}(z(x)) = \sigma, \end{array} \right\} \subset \overline{\mathcal{F}}_\sigma \equiv \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} G(x, z(x), |z(x)|) = 0, \\ H(x, z(x), |z(x)|) \leq 0, \\ \Sigma z(x) = |z(x)| \end{array} \right\} .$$

The set $\mathcal{F}_\sigma$ is called *feasible signature domain* and $\overline{\mathcal{F}}_\sigma$ the *feasible extended signature domain.*

With these definitions, the inclusions $\mathcal{F}_\sigma \subseteq \mathcal{P}_\sigma$ and $\overline{\mathcal{F}}_\sigma \subseteq \overline{\mathcal{P}}_\sigma$ hold, where $\mathcal{F}_\sigma$ may be empty. In a similar way as we introduced the signature vector for kinks, we define a vector containing the signs of the inequality constraints.

Definition 3.3 (Signature vector and signature matrix of inequality constraints)**.** Let an x be given that fulfills the equality and inequality constraints of (CALOP). We define the *signature vector of the inequality constraints* as

$$\omega(x) \equiv \mathbf{sgn}(H(x, z, |z|)) \in \{-1, 0\}^p \ .$$

The jth inequality constraint is called *active* if $\omega_j(x) = 0$ and *inactive* otherwise. The *signature matrix of the inequality constraints* is denoted by $\Omega(x) = \mathbf{diag}(\omega(x))$. Furthermore, $\mathcal{I} \equiv \mathcal{I}(x)$ collects the indices of the active inequality constraints at x. The projection onto the active components of $H(x)$ is defined as $P_{\mathcal{I}} \equiv (e_i^\top)_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|\times p}$ with e_i denoting the ith unit vector of appropriate size.

Next, we prepare the formulation of optimality conditions that can be verified in polynomial time. For this purpose, we introduce the following notations:

Definition 3.4 (Active switching variables)**.** A switching variable z_i is called *active* at x if $z_i(x) = 0$. The active switching set $\alpha(x)$ collects all indices of active switching variables that directly depend on x, i.e.,

$$\alpha(x) \equiv \{i \in \{1, \ldots, s\} \mid z_i(x) = 0 \text{ and } e_i^\top Z \neq 0_n\}$$

with $e_i \in \mathbb{R}^s$ the ith unit vector. The projection onto the active components of $z(x)$ is defined as $P_\alpha \equiv (e_i^\top)_{i\in\alpha} \in \mathbb{R}^{|\alpha|\times s}$ with e_i denoting the ith unit vector of appropriate size.

For each fixed signature vector $\sigma \in \{-1, 0, 1\}^s$, we obtain from (CALOP) similar to Eq. (7) the smooth optimization problem

$$\min_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad a^\top x + b^\top z \tag{10a}$$

$$\text{s.t.} \quad 0 = g + Ax + Bz + C\Sigma z \ , \tag{10b}$$

$$0 \geq h + Dx + Ez + F\Sigma z \ , \tag{10c}$$

$$z = c + Zx + Mz + L\Sigma z \ , \tag{10d}$$

$$0 = (I_s - |\Sigma|)z \ , \tag{10e}$$

$$0 \leq \Sigma z \ . \tag{10f}$$

Now, we can extend the concept of signature optimality to the situation considered in this section:

Definition 3.5 (Feasible signature optimal point)**.** Let an optimization problem of the form (CALOP) be given. Consider a fixed signature vector $\sigma \in \{-1, 0, 1\}^s$. A minimizer $x_\sigma \in P_\sigma$ of the optimization problem (10) is called *feasible signature optimal point* of the original, constrained optimization problem (CALOP).

To simplify the notation, we define

$$\tilde{Z} = (I_s - M - L\Sigma)^{-1} Z \qquad \text{and} \qquad \tilde{c} = (I_s - M - L\Sigma)^{-1} c \; . \tag{11}$$

Then one can combine Eqs. (10d) and (10e) to one equality constraint and obtains the following optimization problem that is equivalent to the one stated in Eq. (10)

$$\min_{x \in \mathbb{R}^n, z \in \mathbb{R}^s} \quad a^\top x + b^\top |\Sigma| z \tag{12a}$$

$$\text{s.t.} \quad 0 = g + Ax + B|\Sigma|z + C\Sigma z \; , \tag{12b}$$

$$0 \geq h + Dx + E|\Sigma|z + F\Sigma z \; , \tag{12c}$$

$$0 = |\Sigma|z - \tilde{c} - \tilde{Z}x \; , \tag{12d}$$

$$0 \leq \Sigma z \; . \tag{12e}$$

Since we consider only linear constraints, one has for the optimization problem (12) that the set of feasible directions at x coincides with the tangent cone at x, see [NW06, Lem. 12.7]. In this case, no further constraint qualification is needed to ensure the existence of Lagrange multipliers but then their uniqueness is not guaranteed. Our goal is to derive optimality conditions that can be verified in polynomial time. Hence, any dependence on the signature vectors that would lead to a combinatorial complexity in 2^s in the worst case must be avoided. Therefore, we have to ensure that the Lagrange multipliers are unique, see also [GW19b]. For this reason, we adapt the kink qualification LIKQ that was introduced in [GW16] for the unconstrained case appropriately. In [HS20], LIKQ has already been extended for constrained nonsmooth nonlinear optimization problems. However, since we focus in this paper on the piecewise linear case, LIKQ can be specified in more detail.

In the unconstrained case, LIKQ requires the full rank of the matrix $P_\alpha \tilde{Z}$, i.e., the active Jacobian of the reformulated switching system. To derive a similar result for the constrained case, we analyze the optimization problem (12) for a feasible signature optimal point x_σ in more detail. Due to the continuity of all involved functions and the relation $\Sigma z = |z|$, the components z_i, $i \notin \alpha$, of the vector z determined by Eq. (12d) will not drop to zero in an open neighborhood $U(x_\sigma)$ of x_σ such that one has

$$0 < \Sigma(\tilde{c} + \tilde{Z}x) \qquad \forall x \in U(x_\sigma) \; .$$

In combination with the identity $\Sigma z = \Sigma|\Sigma|z$, in this neighborhood $U(x_\sigma)$ the optimization problem (12) is then equivalent to

$$\min_{x \in U(x_\sigma)} \quad a^\top x + b^\top |\Sigma|(\tilde{c} + \tilde{Z}x) \tag{13a}$$

$$\text{s.t.} \quad 0 = g + Ax + B|\Sigma|(\tilde{c} + \tilde{Z}x) + C\Sigma(\tilde{c} + \tilde{Z}x) \; , \tag{13b}$$

$$0 \geq h + Dx + E|\Sigma|(\tilde{c} + \tilde{Z}x) + F\Sigma(\tilde{c} + \tilde{Z}x) \; , \tag{13c}$$

$$0 = P_\alpha(\tilde{c} + \tilde{Z}x) \; . \tag{13d}$$

Definition 3.6 (Active Jacobian)**.** Consider for the constrained optimization problem (CALOP) and a given signature vector $\sigma \in \{-1, 0, 1\}^s$ a feasible signature optimal point x_σ. The *active Jacobian* of the equivalent problem (13) is given by

$$\mathcal{J}_\sigma \equiv \begin{bmatrix} A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z} \\ P_\mathcal{I}(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) \\ P_\alpha\tilde{Z} \end{bmatrix} \in \mathbb{R}^{(m+|\mathcal{I}|+|\alpha|)\times n} .$$

Now, the required kink qualification can be stated for the setting considered in this paper:

Definition 3.7 (LIKQ (constrained case))**.** Let a constrained optimization problem of the form (CALOP) and a signature vector $\sigma \in \{-1, 0, 1\}^s$ be given. We say that the *Linear Independence Kink Qualification (LIKQ)* holds at a feasible signature optimal point x_σ if the active Jacobian $\mathcal{J}_\sigma$ has full row rank $m + |\mathcal{I}| + |\alpha|$.

After these preparations, we are able to show that the optimality of a feasible signature optimal point can be verified in polynomial time extending the results given in [GW20] to the constrained case.

Theorem 3.8 (Necessary and sufficient optimality condition)**.** *Let a constrained optimization problem of the form* (CALOP) *and a signature vector* $\sigma \in \{-1, 0, 1\}^s$ *be given. Assume that* x_σ *is feasible signature optimal for* (10) *and that LIKQ holds at* x_σ*. Then* x_σ *is a local minimizer of* (CALOP) *if and only if there exist Lagrange multipliers* $\delta \in \mathbb{R}^m$, $0 \leq \nu \in \mathbb{R}^p$ *and* $\lambda \in \mathbb{R}^s$*, such that*

$$0 = a^\top + b^\top|\Sigma|\tilde{Z} + \delta^\top(A+B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z}) + \nu^\top(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) - \lambda^\top P_\alpha^\top P_\alpha\tilde{Z} \quad (14)$$

and

$$|P_\alpha(b + B^\top\delta + E^\top\nu + \lambda)| \leq P_\alpha(C^\top\delta + F^\top\nu - \tilde{L}^\top\lambda) \quad (15)$$

with $\tilde{L}$ *given by*

$$\tilde{L} = (I_s - M - L\Sigma)^{-1}L .$$

Proof. Since x_σ is feasible signature optimal for the given signature vector σ, x_σ is also a minimizer of the optimization problem (13). Then, we obtain from standard KKT theory that there exist unique Lagrange multipliers $\delta \in \mathbb{R}^m$, $0 \leq \nu \in \mathbb{R}^p$ and $\check{\lambda} \in \mathbb{R}^{|\alpha|}$ associated with the equality constraint (13b), the inequality constraint (13c) and the reformulated switching system (13d) such that

$$0 = a^\top + b^\top|\Sigma|\tilde{Z} + \delta^\top(A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z}) + \nu^\top(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) + \check{\lambda}^\top P_\alpha\tilde{Z} .$$

Hence, together with $\delta \in \mathbb{R}^m$ and $0 \leq \nu \in \mathbb{R}^p$, each vector $\lambda \in \mathbb{R}^s$ such that $\check{\lambda} = -P_\alpha\lambda$ fulfills Eq. (14).

As introduced before, $\omega = \omega(x_\sigma)$ denotes the signature vector of the inequality constraints. Then, it is necessary and sufficient for local minimality that $(x_\sigma, z(x_\sigma))$ is a minimizer of $f(.,.)$ as defined in Eq. (3) on all feasible extended signature domains $\overline{\mathcal{F}}_{\tilde{\sigma}}$ with definite $\tilde{\sigma} \succ \sigma$. Any such $\tilde{\sigma} \succ \sigma$ can be written as $\tilde{\sigma} = \sigma + \gamma$ with $\gamma \in \{-1, 0, 1\}^s$ structurally orthogonal to σ such that for $\Gamma \equiv \mathbf{diag}(\gamma)$ we have the matrix equations

$$\tilde{\Sigma} = \Sigma + \Gamma \quad \text{and} \quad \Sigma\Gamma = 0 = |\Sigma|\Gamma \, . \tag{16}$$

Then we can express $z(x) = z_{\tilde{\sigma}}(x)$ for $x \in \mathcal{P}_{\tilde{\sigma}}$ as

$$z_{\tilde{\sigma}}(x) = z_{\sigma+\gamma}(x) = (I_s - M - L\Sigma - L\Gamma)^{-1}(c + Zx) = (I_s - \tilde{L}\Gamma)^{-1}(\tilde{c} + \tilde{Z}x) \, . \tag{17}$$

Since x_σ must be a minimizer of the objective function also on $\overline{\mathcal{F}}_{\tilde{\sigma}}$, it solves the smooth optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad & a^\top x + b^\top z \\ \text{s.t.} \quad & 0 = g + Ax + Bz + C(\Sigma + \Gamma)z \, , \\ & 0 \geq h + Dx + Ez + F(\Sigma + \Gamma)z \, , \\ & 0 = (I_s - \tilde{L}\Gamma)z - \tilde{c} - \tilde{Z}x \, , \\ & 0 \leq P_\alpha \Gamma z \, . \end{aligned}$$

Once more, we obtain from KKT theory that there exist Lagrange multipliers $\delta \in \mathbb{R}^m$, $0 \leq \nu \in \mathbb{R}^p$, $\lambda \in \mathbb{R}^s$ and $0 \leq \mu \in \mathbb{R}^{|\alpha|}$ associated with the equality constraint, the inequality constraint, the reformulated switching system and the sign conditions such that

$$0 = a^\top + \delta^\top A + \nu^\top D - \lambda^\top \tilde{Z} \quad \text{and} \tag{18}$$

$$0 = b^\top + \delta^\top \left(B + C(\Sigma + \Gamma)\right) + \nu^\top \left(E + F(\Sigma + \Gamma)\right) + \lambda^\top \left(I_s - \tilde{L}\Gamma\right) - \mu^\top P_\alpha \Gamma \, . \tag{19}$$

Multiplying the last equation from the right by $|\Sigma|\tilde{Z}$, we obtain with the identity $\Sigma = \Sigma|\Sigma|$ and Eq. (16)

$$0 = b^\top |\Sigma|\tilde{Z} + \delta^\top \left(B|\Sigma| + C\Sigma\right)\tilde{Z} + \nu^\top \left(E|\Sigma| + F\Sigma\right)\tilde{Z} + \lambda^\top |\Sigma|\tilde{Z} \, .$$

Adding this equation to (18) and exploiting $I_s = |\Sigma| + P_\alpha^\top P_\alpha$ yields

$$0 = a^\top + b^\top |\Sigma|\tilde{Z} + \delta^\top (A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z}) + \nu^\top (D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) - \lambda^\top P_\alpha^\top P_\alpha \tilde{Z} \, .$$

Hence, it follows that the Lagrange multipliers $\delta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^p$, $\lambda \in \mathbb{R}^s$ fulfill Eq. (14) with $\check{\lambda} = -P_\alpha \lambda$. Due to the kink qualification LIKQ, one also has that the vectors $\delta \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ as well as the components $P_\alpha \lambda \in \mathbb{R}^\alpha$ are determined uniquely. The remaining components of $\lambda \in \mathbb{R}^s$ can be obtained by multiplying Eq. (19) this time only with $|\Sigma|$ from the right yielding

$$0 = b^\top |\Sigma| + \delta^\top \left(B|\Sigma| + C\Sigma\right) + \nu^\top \left(E|\Sigma| + F\Sigma\right) + \lambda^\top |\Sigma| \, .$$

To derive the second condition, we multiply Eq. (19) from the right by $\Gamma P_\alpha^\top$. Using $\Gamma\Gamma = P_\alpha^\top P_\alpha$, $P_\alpha P_\alpha^\top = I_s$ and $\mu \geq 0$, it follows that

$$\begin{aligned}-(b^\top + \delta^\top B + \nu^\top E + \lambda^\top)\Gamma P_\alpha^\top &= (\delta^\top C + \nu^\top F - \lambda^\top \tilde{L})P_\alpha^\top - \mu^\top \\ &\leq (\delta^\top C + \nu^\top F - \lambda^\top \tilde{L})P_\alpha^\top \ .\end{aligned}$$

Now the key observation is that this condition is linear in Γ and is strongest for the choice $\gamma_i = \mathbf{sgn}(\lambda^\top + b^\top + \delta^\top B + \nu^\top E)_i$ for $i \in \alpha$ yielding the inequalities

$$|(b + B^\top \delta + E^\top \nu + \lambda)_i| \leq e_i(C^\top \delta + F^\top \nu - \tilde{L}^\top \lambda) \quad \text{for} \quad i \in \alpha$$

completing the proof. □

It is important to note that for given Lagrange multipliers δ, ν, and λ, it can be verified in polynomial time whether the conditions (14)–(15) hold. Hence, this optimality test at a feasible signature optimal point is independent from the combinatorial complexity caused by all the possible values of Γ.

Furthermore, for the unconstrained case, i.e., $A = 0$, $B = 0$, $C = 0$, $D = 0$, $E = 0$, $F = 0$ in the appropriate dimensions, one rediscovers the conditions

$$0 = a^\top + b^\top |\Sigma| \tilde{Z} + \lambda^\top P_\alpha \tilde{Z} \qquad \text{and} \qquad |P_\alpha(b + \lambda)| \leq P_\alpha(-\tilde{L}^\top \lambda) \ ,$$

i.e., tangential stationarity and normal growth as introduced in [GW20].

4 The active signature method for constrained problems

In this section, we extend the algorithm ASM described in Sec. 2 to problems with abs-linear constraints of the form (CALOP). It should be mentioned that due to the additional equality and inequality constraints, the set of feasible points could be empty. Throughout, we assume that this is not the case such that the iteration can start with a feasible point. Subsequently, feasibility is maintained, i.e., the derived algorithm is a feasible point method. If no feasible starting point is given from the application context, one can calculate such a starting point with a variant of the Phase-I method known from linear optimization (cf. [NW06, Chapter 16]). If a feasible starting point exists, one can show that the resulting algorithm terminates within a finite number of iterations.

Computing a direction for given σ and ω Similar to the unconstrained case, we add a quadratic penalty term with a positive definite matrix $Q = Q^\top \in \mathbb{R}^{n \times n}$ to the target function ensuring that the problem is bounded from below. Hence, we want to solve

$$\min_{x \in \mathbb{R}^n, z \in \mathbb{R}^s} \quad a^\top x + b^\top |\Sigma| z + \frac{1}{2} x^\top Q x \tag{20a}$$

$$\text{s.t.} \quad 0 = g + Ax + B|\Sigma|z + C\Sigma z \ , \tag{20b}$$

$$0 \geq h + Dx + E|\Sigma|z + F\Sigma z \ , \tag{20c}$$

$$0 = |\Sigma|z - \tilde{c} - \tilde{Z}x \ , \tag{20d}$$

$$0 \leq \Sigma z \ , \tag{20e}$$

with $\tilde{Z}$ and $\tilde{c}$ as defined in Eq. (11). Due to the fixed signature vector, this optimization problem is smooth, has a quadratic target function and linear constraints. Hence, it could be solved with a standard QP method. However, we want to exploit the structure provided by the signature vector as additional feature.

Once more standard KKT theory can be applied. With Lagrange multipliers $\delta \in \mathbb{R}^m, \nu \in \mathbb{R}^p, \lambda \in \mathbb{R}^s$ and $\mu \in \mathbb{R}^s$, we obtain the following necessary optimality conditions

$$0 = a^\top + x^\top Q + \delta^\top A + \nu^\top D - \lambda^\top \tilde{Z} \; , \tag{21a}$$

$$0 = b^\top |\Sigma| + \delta^\top (B|\Sigma| + C\Sigma) + \nu^\top (E|\Sigma| + F\Sigma) + \lambda^\top |\Sigma| - \mu^\top \Sigma \; , \tag{21b}$$

$$0 = g + Ax + B|\Sigma|z + C\Sigma z \; , \tag{21c}$$

$$0 \geq h + Dx + E|\Sigma|z + F\Sigma z \; , \tag{21d}$$

$$0 = |\Sigma|z - \tilde{c} - \tilde{Z}x \; , \tag{21e}$$

$$0 \leq \Sigma z \; , \quad 0 \leq \mu \; , \quad 0 = \mu^\top \Sigma z \; , \tag{21f}$$

$$0 \leq \nu \; , \qquad 0 = \nu^\top (h + Dx + Ez + F\Sigma z) \; . \tag{21g}$$

Multiplying Eq. (21b) by Σ from the right and using Eq. (21f) yields

$$0 \leq \mu^\top |\Sigma| = b^\top \Sigma + \delta^\top (B\Sigma + C|\Sigma|) + \nu^\top (E\Sigma + F|\Sigma|) + \lambda^\top \Sigma \; . \tag{22}$$

Due to the complementarity condition $\mu^\top \Sigma z = 0$, this inequality must hold as an equation. Hence, it follows that

$$-b^\top \Sigma = \delta^\top (B\Sigma + C|\Sigma|) + \nu^\top (E\Sigma + F|\Sigma|) + \lambda^\top \Sigma \; .$$

Thus with $\omega = \mathbf{sgn}(H(x, |z|))$ and $\Omega = \mathbf{diag}(\omega)$ denoting as before the projection onto the active inequality constraints, we get the linear system

$$\begin{bmatrix} Q & 0 & -\tilde{Z}^\top & A^\top & D^\top \\ 0 & 0 & \Sigma & \Sigma B^\top + |\Sigma| C^\top & \Sigma E^\top + |\Sigma| F^\top \\ \tilde{Z} & -|\Sigma| & 0 & 0 & 0 \\ A & B|\Sigma| + C\Sigma & 0 & 0 & 0 \\ \bar{\Omega} D & \bar{\Omega}(E|\Sigma| + F\Sigma) & 0 & 0 & \Omega \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \\ \lambda \\ \delta \\ \nu \end{bmatrix} = - \begin{bmatrix} a \\ \Sigma b \\ \tilde{c} \\ g \\ \bar{\Omega} h \end{bmatrix} , \tag{23}$$

where $\bar{\Omega} = I_p - |\Omega|$ forces the active inequalities to vanish. The matrix Ω in the right lower corner ensures that ν is zero for the inactive inequality constraints. We denote a solution by $(\hat{x}, \hat{z}, \lambda, \delta, \nu)$ and define for the current iterate x and z

$$\Delta x := \hat{x} - x \quad \text{and} \quad \Delta z := \hat{z} - z$$

as directions towards the next iterate.

Computing a step size β Provided the system (23) is solvable yielding $(\hat{x}, \hat{z})$, we must now check whether $\sigma(\hat{x}) = \sigma$ is still valid and that the inequality constraints still hold to ensure feasibility. For this purpose, we calculate two step sizes. As in the unconstrained case, the first step size is the step length from the current iterate x in the direction Δx to a possible kink, i.e., a sign change in one component of z. Therefore, this step size is also denoted by β^z and defined as

$$\beta^z = \inf_{1 \le l \le s} \left\{ \beta_l^z \equiv \frac{-z_l}{\hat{z}_l - z_l} \middle| (\hat{z}_l - z_l)\sigma_l < 0 \right\} \in [0, \infty] \,. \tag{24}$$

Once more, if $\beta^z < \infty$ the first index for which the minimum is attained is denoted by j^z. For $\beta^z \le 1$, there exists a blocking kink with the same consequences as in the unconstrained case.

The second step size is the step length from the current iterate x in the direction Δx to a possible inequality constraint $H_l(x, z, \Sigma z), 1 \le l \le p$, that becomes active. In a similar way to the computation of β^z this step size β^H is given by

$$\beta^H = \inf_{1 \le l \le p} \left\{ \beta_l^H \equiv \frac{H_l}{H_l - \hat{H}_l} \middle| (\hat{H}_l - H_l)\omega_l < 0 \right\} \,, \tag{25}$$

where $H \equiv H(x, z, \Sigma z)$, $\hat{H} \equiv H(\hat{x}, \hat{z}, \Sigma \hat{z})$ and l denotes the lth component of H and $\hat{H}$, respectively. Similar to the first step size, we denote by j^H the smallest index for which the minimum is attained. For $\beta^H < 1$, there exists a blocking inequality constraint, i.e., the solution $\hat{x}$ is not feasible. Therefore, the new iterate x^+ should be chosen such that the j^Hth components of $H(x^+, z^+, \Sigma z^+)$ and $\omega(x^+)$ drop to zero in comparison to $H(x, z, \Sigma z)$ and ω, respectively. Setting $\omega^+_{j^H} = 0$ changes the optimality system (21) and a new solution of system (23) has to be computed. If $\beta^H \le \beta^z$ then we have $z_{j^H} \hat{z}_{j^H} \ge 0$ such that the iterate $\hat{x}$ is still contained in $\mathcal{P}_\sigma$, i.e., $\sigma(\hat{x}) = \sigma$ is still valid.

The step sizes β^z and β^H are illustrated in Fig. 4, where the blue line represents a kink and the red one an active inequality constraint. The yellow arrows indicate the corresponding step sizes, i.e., on the left hand side β^z and on the right hand side β^H.

Finally, we determine the actual step size

$$\beta = \min\{\beta^z, \beta^H, 1\} \,, \tag{26}$$

where the upper bound 1 on β ensures with the update

$$x^+ = (1 - \beta)x + \beta \hat{x} = x + \beta \Delta x$$

that the next iterate is still contained in $\overline{\mathcal{F}}_\sigma$. If $\beta = 1$, one has for the new iterate $x^+ = \hat{x}$ that $\sigma(x^+) = \sigma$ and $\omega(x^+) = \omega$. In this case, x^+ is called *signature stationary* since the two signature vectors are kept.

Updating σ **and** ω If x^+ is not signature stationary on the current polyhedron $\overline{\mathcal{F}}_\sigma$, we continue the optimization of problem (20) with one component of σ dropped to zero

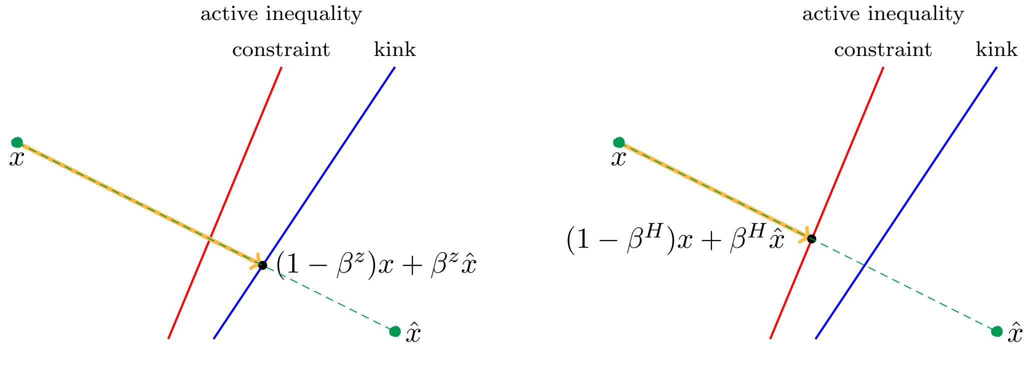


Figure 4: The two different step sizes β^z and β^H

or a change in the active set for the equality constraints as described below in detail. If x^+ is signature stationary on the current polyhedron $\mathcal{P}_\sigma$, one has to check whether x^+ is a minimizer of (CALOP). If this is the case the iteration stops. Otherwise, the optimization continues in one of the neighboring polyhedra $P_{\tilde{\sigma}}$ with $\tilde{\sigma} \succ \sigma$. Such a $\tilde{\sigma}$ can be again decomposed into $\sigma + \gamma$ where $|\sigma|^\top |\gamma| = 0$. Replacing Σ in the optimality conditions (21) by the corresponding $\Sigma + \Gamma$ and using Eq (17) we see that most of the relations are still fulfilled by the current values $\hat{x}, \hat{z}$ and λ. The only thing that changes is that Eq. (22) has as many new nontrivial component as γ which can be written as

$$\begin{aligned} 0 \le \mu^\top |\Gamma| &= b^\top \Gamma + \delta^\top (B\Gamma + C|\Gamma|) + \nu^\top (E\Gamma + F|\Gamma|) + \lambda^\top (I_s - \tilde{L}\Gamma)\Gamma \\ &= (b^\top + \delta^\top B + \nu^\top E + \lambda^\top)\Gamma + (\delta^\top C + \nu^\top F - \lambda^\top \tilde{L})|\Gamma| \, . \end{aligned}$$

This condition is violated if and only if there exists at least one index k such that $\gamma \equiv e_k \mathbf{sgn}(b^\top + \delta^\top B + \nu^\top E + \lambda^\top)$ satisfies

$$\begin{aligned} &0 > (\delta^\top C + \nu^\top F - \lambda^\top \tilde{L})e_k - \left| b^\top + \delta^\top B + \nu^\top E + \lambda^\top \right| e_k \quad \text{and} \qquad (27) \\ &\sigma_k = 0 \, . \end{aligned}$$

Note that this condition agrees exactly with the second optimality condition (15) derived in Theo. 3.8. In addition we must check whether any one of the components ν_l for $1 \le l \le p$ of the Lagrange multiplier ν associated with the inequality constraints is negative. If such a violation occurs, one possible strategy is to take the most negative component of ν and Eq. (27) as discussed below in more detail.

The overall algorithm Combining all the considerations described above, one obtains Algorithm 1 that consists of two main parts: The computation of the new iterate (cf. line 2-6 of Algorithm 1) and the optimality check (cf. line 7-14).

To compute a new iterate, the optimality system (23) is solved (cf. line 2). Then, the corresponding step sizes β^H, β^z and β are determined (cf. line 3) and the iterate is

Algorithm 1 Constrained active signature method (CASM)

Require: Feasible start point $x \in \mathbb{R}^n, n \in \mathbb{N}$, $s, m, p \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{R}^n$, $b, c \in \mathbb{R}^s$, $Z \in \mathbb{R}^{s \times n}$, $L, M \in \mathbb{R}^{s \times s}$ strictly lower triangular, $Q = Q^\top \in \mathbb{R}^{n \times n}$ positive definite, $g \in \mathbb{R}^m$, $h \in \mathbb{R}^p$, $A \in \mathbb{R}^{m \times n}$, $B, C \in \mathbb{R}^{m \times s}$, $D \in \mathbb{R}^{p \times n}$, $E, F \in \mathbb{R}^{p \times s}$, $\beta = 0$

Ensure: $z := z(x)$ via Eq. (1a), $\sigma := \sigma(x)$, $\omega := \omega(x)$ and check $G(x, \Sigma z) = 0$, $H(x, \Sigma z) \leq 0$, $\Omega H(x, \Sigma z) = 0$.

1: **loop**
2:     Compute $(\hat{x}, \hat{z}, \lambda, \delta, \nu)$ by solving Eq. (23)
3:     Compute β^z via Eq. (24), β^H via Eq. (25) and β via Eq. (26)
4:     Set $(x^+, z^+) = (1 - \beta)(x, z) + \beta(\hat{x}, \hat{z})$
5:     **if** $\beta^H = \beta$ **then** Restrict ω ▷ Add constraint
6:     **if** $\beta^z = \beta$ **then** Restrict σ ▷ Add kink
7:     **if** $\beta = 1$ **then** ▷ x^+ is feasible signature stationary
8:         **if** $\nu \ngeq 0$ **then**
9:             Relax ω, set $\beta = 0$ ▷ Drop constraint
10:         **else** ▷ x^+ is feasible signature optimal
11:             **if** Eq. (27) holds true **then**
12:                 Relax σ, set $\beta = 0$ ▷ Drop kink
13:             **else** ▷ x^+ is local optimal
14:                 **return** (x^+, z^+) ▷ Problem solved
15:     Set $(x, z) = (x^+, z^+)$

updated accordingly (cf. line 4). Subsequently, we check whether an inequality constraint or a kink becomes active and restrict ω and/or σ accordingly (cf. line 5-6).

The optimality is verified in line 7-14. If $\beta < 1$ (cf. test in line 7 of Algorithm 1), there was a change in σ and/or ω in the last iteration such that the current iterate can not be signature stationary. Hence, for x being a minimizer it is necessary that $\beta = 1$ holds. Then, we can check optimality starting with the Lagrange multiplier ν associated with the inequality constraints. If $\nu \geq 0$ does not holds, we choose the component for which $\nu \geq 0$ is most violated and drop the corresponding constraint. Hence, the associated entry of ω is set to -1 relaxing ω. Furthermore, $\beta = 0$ signals that a change has occurred in the optimality system (23) such that it must be solved again in the next iteration (cf. line 8-9). If $\nu \geq 0$ holds, the current iterate is feasible signature optimal. Therefore, we check the optimality condition given by Eq. (27). If it does not hold the current point is not a minimizer of (CALOP) and we leave a kink by relaxing σ, i.e., setting the corresponding entry to a nonzero value. Once more, we mark the change in the optimality system (23) by setting $\beta = 0$ (cf. line 10-12). If both optimality conditions are satisfied, we found an optimal point of (CALOP) and the algorithm terminates (cf. line 13-14). Otherwise, we continue with the next iterate given by $(x, z) = (x^+, z^+)$ (cf. line 15).

Convergence analysis of CASM Next, we analyze the convergence behavior of Algorithm 1. For this purpose, we first examine the question, whether CASM yields a monotone decreasing sequence of function values. In each iteration, the optimality system (23) is solved which corresponds to the computation of a Newton step for the smooth optimization problem (20) when one ignores the inequality constraints. Since the resulting optimization problem is convex, the Newton step is a descent direction if it is not equal to zero. Hence, then one can find a $\beta > 0$ to move along this direction to define a feasible next iterate due to the continuity of all involved functions. These considerations yield the following result:

Proposition 4.1. *Suppose that* $\Delta x = \hat{x} - x \neq 0$ *holds for the solution* $\hat{x}$ *of Eq. (23) at the current iterate* x*. Then, the objective function of problem (20) decreases strictly along the direction* Δx*.*

Now, we analyze the convergence of the Algorithm 1.

Theorem 4.2. *Suppose that an optimization problem of the form* (CALOP) *is given and that* $x \in \mathbb{R}^n$ *is a feasible starting point for* (CALOP)*. Let* $Q = Q^\top \in \mathbb{R}^{n\times n}$ *be a positive definite matrix. Then, Algorithm 1 terminates after finitely many iterations at a minimizer of the quadratically penalized optimization problem*

$$\begin{aligned} \min_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad & a^\top x + b^\top z + \frac{1}{2} x^\top Q x \\ s.t. \quad & 0 = g + Ax + Bz + C|z| \\ & 0 \geq h + Dx + Ez + F|z| \\ & z = c + Zx + Mz + L|z| \ . \end{aligned} \tag{28}$$

Proof. Algorithm 1 prioritizes the multiplier ω of the inequality constraints, see line 8 versus line 11 of Algo. 1. Therefore, as long as the signature vector σ does not change, the proposed approach resembles an active set method to solve QPs. Furthermore, we always ensure a decrease in the function value, see Prop. 4.1. Such an approach determines a minimizer of problems with the structure (20) in finitely many steps, see, e.g., [NW06, Chap. 16].

If the current iterate is a feasible signature optimal point of (20) on $\bar{\mathcal{F}}_\sigma$, Algo. 1 may change also the signature vector σ, see line 12, resulting in a change to a different polyhedron $\bar{\mathcal{F}}_{\tilde{\sigma}}$ before the feasible signature optimal point on $\bar{\mathcal{F}}_\sigma$ is found. However, since there are only finitely many polyhedra and the value of the common function value is consistently reduced, Algo. 1 can modify the signature vector only finitely many times leading to a finite convergence of the overall algorithm. □

The theorem considers the penalized version (28) of the original optimization task (CALOP). This ensures also that the optimization problem is bounded below such that a minimizer must exist. Hence, when Algo. 1 stops at a local minimizer of (28) in line 14, one has to check the first optimality condition (14) given in Theo. 3.8 to verify that the current point is also a minimizer of (CALOP). If this is not the case, one has to

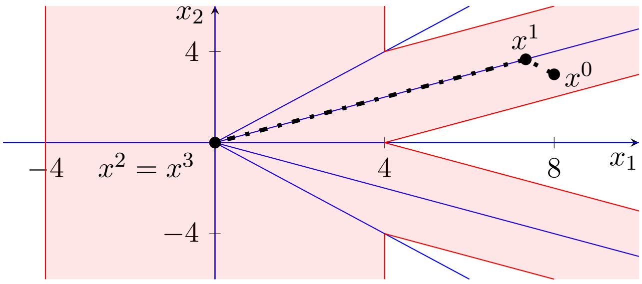


Figure 5: Iterates generated by Algorithm 1 for Example 5.1.

Iteration	x^i	σ^i	ω^i
0	(8.00, 3.00)	(1, -1, 1)	-1
1	(7.33, 3.66)	(1, -1, 0)	-1
2	(0.00, 0.00)	(1, 0, 0)	-1
3	(0.00, 0.00)	(1, 0, 0)	-1

Table 1: Optimization history of Algorithm 1 for Example 5.1.

reduce the influence of the quadratic penalty term and start Algo. 1 again. If (CALOP) has a minimizer and the influence of the penalty term is driven close to zero in finitely many steps this yields convergence to a minimizer of (CALOP). In our numerical tests preformed so far, such a reduction was not necessary.

5 Numerical results

To illustrate the algorithm proposed in this paper, we implemented Algorithm 1 in Matlab and applied it to some constrained piecewise linear test problems.

Example 5.1. *Consider again the constrained optimization problem given in Example 3.1. For the starting point $x^0 = (8, 3)$, Figure 5.1 shows the iterates generated by Algorithm 1. Once more, the resulting kinks are given by the blue lines and the feasible set is marked by the red area. Four iterations are performed. The corresponding iterates are stated in Table 1 together with the signature vector σ and the signature vector of the constraints ω.*

Example 5.2. *(Constrained HUL) As mentioned already before, Hiriart-Urruty and Lemaréchal considered the piecewise linear and convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$,*

$$\varphi(x) = \max\{\max\{-100, 2x_1 + 5|x_2|\}, 3x_1 + 2|x_2|\} .$$

To test our algorithm, we add the two constraints

$$H_1(x) = -0.25x_1 - x_2 - 10 \leq 0 \ ,$$
$$H_2(x) = 2 - 0.2|x_1 + 9| - |x_2 + 1| \leq 0 \ ,$$

and choose the feasible starting point $x^0 = (9, -2.5)$*. This optimization problem requires six switching variables, e.g., one has* $n = 2$*,* $s = 6$*,* $m = 0$ *and* $p = 2$*. Using Algorithm 1, 15 iterations are needed.*

Figure 6 shows a plot of the resulting kinks originating from the objective function (blue lines) and from the constraints (cyan blue lines). The inequality constraints are marked by the red lines and therefore the red area represents the feasible set. Finally, the iterates generated by Algorithm 1 are denoted by the black dots. In the plot only eight of the 15 iterations are marked. This is due to the fact that some of the iterations duplicate the point x *when* σ *and* ω *are restricted or relaxed, i.e., kinks or constraints are activated or deactivated.*

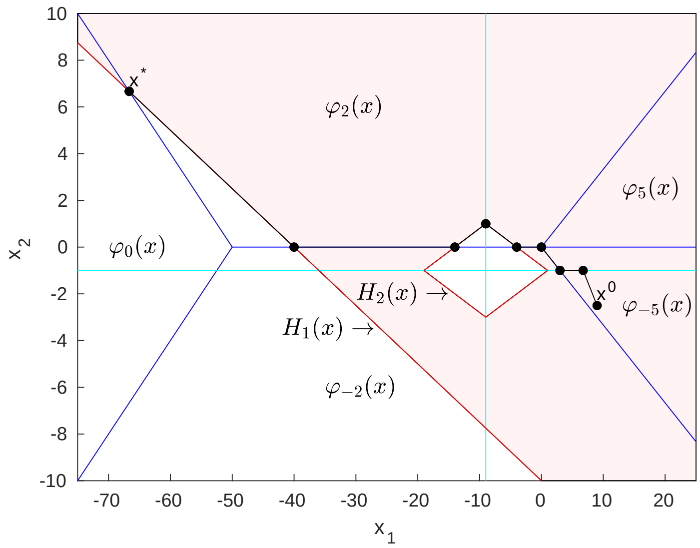


Figure 6: Iterates generated by Algorithm 1 for Example 5.2.

Example 5.3. *(Constrained Rosenbrock-Nesterov II) According to [GO12], Nesterov suggested the Rosenbrock-like test function*

$$\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \varphi(x) = \frac{1}{4}|x_1 - 1| + \sum_{i=1}^{n-1} |x_{i+1} - 2|x_i| + 1|$$

that is piecewise linear and nonconvex. It has the unique global minimizer given by $x = (1, 1, \ldots, 1) \in \mathbb{R}^n$ *and* $2^{n-1} - 1$ *other Clarke stationary points non of which is a local minimizer. For the starting point*

$$x_1^0 = -1, \quad x_i^0 = 1 \quad \textit{for} \quad 2 \leq i \leq n \ ,$$

the paper [GW19a] contains numerical results and comparisons to other solvers showing that nonsmooth optimization algorithms may get stuck at one of these stationary points that are no minimizers. Since we consider constrained problems in this paper, we add the piecewise linear constraint

$$\sum_{i=1}^{n} |x_i - 1| \geq \frac{1}{2n} \ .$$

Hence, there is an n*-dimensional rhombus around the global optimum which is cut out of the* $\mathbb{R}^n$*. The remaining* $2^{n-1}-1$ *stationary points are still feasible. To derive a abs-linear representation of this constrained optimization problem, we define* $s = 3\,n-1$ *switching variables, namely*

$$z_i = x_i \quad \textit{for} \quad 1 \leq i < n, \qquad z_{n+i} = x_{i+1} - 1 \quad \textit{for} \quad 0 \leq i < n,$$

$$z_{2n+i} = x_{i+2} - 2\,|z_{i+1}| + 1 \quad \textit{for} \quad 0 \leq i < n-1, \qquad z_{3n-1} = |z_n| + \sum_{i=0}^{n-2} |z_{2n+i}| \ .$$

Hence, we obtain the matrices and vectors

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{s \times n}, \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -2\,I_{n-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mathbf{1}^\top & 0 \end{bmatrix} \in \mathbb{R}^{s \times s}, \quad M = 0 \ ,$$

$$h = \tfrac{1}{2n}, \quad D = 0, \quad E = 0, \quad F = (\underbrace{0, \ldots, 0}_{n-1}, \underbrace{-1, \ldots, -1}_{n}, \underbrace{0, \ldots, 0}_{n}) \ ,$$

$$a = 0 \in \mathbb{R}^n, \quad b = e_{3n-1} \in \mathbb{R}^s, \quad c = (\underbrace{0, \ldots, 0}_{n-1}, \underbrace{-1, \ldots, -1}_{n}, \underbrace{1, \ldots, 1}_{n-1}, 0)$$

with $\mathbf{1} \in \mathbb{R}^{n-1}$ *as the vector with* 1 *in every component. Consider the point*

$$x_i^* = 1 - \frac{2^{i-1}}{2^n - 1} \cdot \frac{1}{2n} \in (0,1) \quad \textit{for } 1 \leq i \leq n \ .$$

Then one has

$$\sigma(x^*) = (\underbrace{1, \ldots, 1}_{n-1}, \underbrace{-1, \ldots, -1}_{n}, \underbrace{0, \ldots, 0}_{n})^\top \Rightarrow \alpha = \{2n, \ldots, 3n-2\} \quad \textit{and} \quad \omega(x^*) = 0 \ .$$

Note that the index $3n-1$ *is not contained in* α *according to Def. 3.4, since* z_{3n-1} *does*

n	1	2	3	4	5	6	7	8	9	10	11
#	2	5	10	23	40	105	250	431	928	1905	3594

n	12	13	14	15	16	17	18
#	7583	14736	29113	58382	119643	234072	470109

Table 2: Example 5.3: Number of iterations for different values of n.

not directly depend on x. Then, one obtains

$$\begin{bmatrix} P_{\mathcal{I}}(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) \\ P_\alpha \tilde{Z} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & 1 \\ -2 & 1 & 0 & & & 0 \\ 0 & -2 & 1 & 0 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & 0 \\ & & & 0 & -2 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n},$$

such that LIKQ holds. Then, the optimality conditions (14) *and* (15) *require*

$$\nu^\top = \lambda^\top P_\alpha^\top P_\alpha \tilde{Z} \qquad \text{and} \qquad |P_\alpha \lambda| \le -P_\alpha \tilde{L}^\top \lambda \, .$$

These two conditions hold for $\nu = 0 \in \mathbb{R}$ and $\lambda = 0 \in \mathbb{R}^s$. Hence, x_* is a minimizer.

For varying values of n, the required number of iterations required by Algo. 1 is shown in Table 2. The number of switches is given by $s = 3n - 1$. As can be seen from the iteration counts, the number of visited polyedra is much less than the total number of polyhedra with definite signatures given by 2^s. We also applied MPBNGC solver, cf. [MKW16] to solve this problem. MPBNGC is a multiobjective proximal bundle method for nonconvex, nonsmooth and generally constrained minimization. For $n = 1$, seven iterations are needed. Already for $n = 2$, the solver gets stuck after seven iterations at a stationary point. The same can be observed for larger values of n.

Example 5.4. *As a fourth example, we consider a linear complementarity problem (LCP) given by*

$$Mx + q \ge 0 \qquad \text{and} \qquad x^\top (Mx + q) = 0 \tag{29}$$

for $0 \le x \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. In [BG12], the LCP is formulated as a system of piecewise linear equations

$$\min(x, Mx + q) = 0 \, , \tag{30}$$

where the minimum operator acts componentwise. In the same paper, the authors present an algorithm that can be viewed as a semismooth Newton method and show nonconvergence for a special choice of the matrix M. They pointed out that the problem has a *unique solution for any $q \in \mathbb{R}^n$ if and only if M is a $\boldsymbol{P}$-matrix, i.e., M has positive principal minors* $\det M_{II} > 0$ *for all nonempty* $I \subseteq \{1, \ldots, n\}$.

To solve Eq. (30) with Algorithm 1 we reformulate the problem (29) *as*

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} |\min(x_i, (Mx+q)_i)| \ .$$

For the matrix M, we set

$$M_3 \equiv \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \textit{and} \quad M_4 \equiv \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{4}{3} \\ \frac{4}{3} & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{3} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{4}{3} & 1 \end{bmatrix}$$

and $q = \mathbf{1}$ as the vector with 1 *in every component of appropriate dimension as considered also in [BG12]. As starting point we use the first unit vector in $\mathbb{R}^n$ as proposed in [BG12]. Then, Algorithm 1 needs five iterations in both cases, i.e., for each M_3 and M_4, to reach the solution* 0 *as zero vector of the appropriate dimension. In [BG12, Proposition 3.7] it is shown that the algorithm proposed in that paper does not converge but generates a circle of three resp. four reoccurring iterates.*

Bi-level problems, i.e., problems where a lower level optimization problem has to be solved and its solution impacts the upper level optimization problem, play an important role in many real-world applications and are closely related to linear complementarity problems as in Ex. 5.4. Here, we consider a bi-level problem, where all functions appearing as objective functions of the upper and lower level, as well as all constraints are linear. For the lower level problem we use standard KKT theory to convert it into a set of equations and inequalities representing the necessary and sufficient optimality conditions for the lower level. Subsequently, these constraints substitute the lower level problem. However, the resulting complementarity condition is no longer a linear function. For the application of CASM, we can reformulate this constraint analogous to Eq. (30) as a piecewise linear function. Thus, the Lagrange multipliers from the lower problem also become optimization variables.

Example 5.5. *Consider the following linear bi-level problem taken from [TMV94, Chapter 7]:*

$$\begin{aligned} \min_{x,y} \quad & 3x_1 + 2x_2 + y_1 + y_2 \\ \text{s.t.} \quad & x_1 + x_2 + y_1 + y_2 \leq 4 \ , \\ & y \in \operatorname*{argmin}_{\tilde{y}} \quad 4\tilde{y}_1 + \tilde{y}_2 \\ & \qquad \text{s.t.} \qquad 3x_1 + 5x_2 + 6\tilde{y}_1 + 2\tilde{y}_2 \geq 15 \ , \\ & x \in \mathbb{R}^2_{\geq 0}, \ y \in \mathbb{R}^2_{\geq 0} \ . \end{aligned}$$

We use the starting point

$$x = (2.5, 1.5) \ , \quad y = (0, 0) \ , \quad \textit{and} \quad \mu = (0, 4, 1) \ ,$$

where μ represents the Lagrange multiplier resulting from the lower level problem as described above. Table 3 shows information about the iterates solving this problem with CASM. In [TMV94], a structurally quite different algorithm is used to solve the problem, making it difficult to compare the effort. Both algorithms perform some preparatory work in that a pre-solve is performed before applying the algorithm proposed in [TMV94] and a feasible starting point has to be determined for CASM. Subsequently, the algorithm presented in [TMV94] requires three iterations, each of which requires the solution of two linear programs. CASM needed six iterations, where one system of equations with a 27×27 *system matrix must be solved in each iteration. Both algorithms attain the same solution.*

i	x^i	y^i	μ^i	σ^i	ω^i
0	(2.5, 1.5)	(0, 0)	(0.0, 4.0, 1.0)	(0, 1, 1)	(0, -1, -1, 0, 0, 0, 0, -1, -1)
1	(2.5, 1.5)	(0, 0)	(0.0, 4.0, 1.0)	(0, 1, 1)	(-1, -1, -1, 0, 0, 0, 0, -1, -1)
2	(0.0, 3.0)	(0, 0)	(0.0, 4.0, 1.0)	(0, 1, 1)	(-1, 0, -1, 0, 0, 0, 0, -1, -1)
3	(0.0, 3.0)	(0, 0)	(0.0, 4.0, 1.0)	(0, 1, 1)	(-1, 0, -1, 0, -1, 0, 0, -1, -1)
4	(0.0, 3.0)	(0, 0)	(0.0, 4.0, 1.0)	(0, 1, 1)	(-1, 0, -1, 0, -1, 0, -1, -1, -1)
5	(0.0, 3.0)	(0, 0)	(0.0, 4.0, 1.0)	(1, 1, 1)	(-1, 0, -1, 0, -1, 0, -1, -1, -1)
6	(0.0, 3.0)	(0, 0)	(0.5, 4.0, 0.0)	(1, 1, 0)	(-1, 0, -1, 0, -1, 0, -1, -1, 0)

Table 3: Optimization history of Algorithm 1 for Example 5.5.

6 Summary and outlook

In this paper, we considered optimization problems with a piecewise linear target function and piecewise linear constraints as they arise for example in linear complementarity problems or certain bi-level optimization problems.

Using the approach of abs-linearization, we have shown that we can verify the optimality of a given point with polynomial effort. This is in contrast to most optimality conditions available for nonsmooth optimization.

Furthermore, starting from the already known Active Signature Method to solve unconstrained piecewise linear optimization problems, we developed an extension for the constrained case. For this purpose, we adapted the idea of decomposing the $\mathbb{R}^n$ into polyhedra such that the constraints are taken into account. On one such polyhedron, the objective function was additionally penalized by a quadratic term ensuring the existence of a minimizer on each polyhedron. This minimizer can be determined using an adapted method to solve smooth quadratic problems. Employing the optimality conditions derived before, a switching strategy between the polyhedra was derived that ensures finite convergence of the overall algorithm. Numerical results for several test cases illustrate the performance of the resulting Constrained Active Signature Method.

The optimization problems solved in this paper have been of a purely academic nature.

In the future, we want to apply the algorithm to larger problems stemming from realistic applications. For example, solution approaches for the optimization of gas networks yield constrained piecewise linear subproblems, cf. [KLS21a; KLS21b; ALS19]. First promising results in this direction were already obtained, see [Kre+21]. The optimization problems considered there are of much larger dimension than the test examples in this paper having more than 500 optimization variables, 1000 constraints and almost 2000 switches.

One remaining challenge is the determination of a feasible starting point. For some real-world applications, such as the gas networks just mentioned, there are sometimes simple ways to find such a feasible starting point, cf. [Kre+21]. However, for other problems, such as general bi-level problem considered in the section on the numerical examples, the determination of a feasible starting point has turned out to be much more complicated. There, the reformulation of the lower level problem leads to new optimization variables corresponding to the Lagrange multipliers for which there are no intuitive starting values. The development of a suitable Phase-I method could help to overcome this challenge. An already established Phase-I method, as known for linear optimization problems [NW06], is usually not easily applicable, since the linear problems can be considered only on the polyhedra. Thus it can happen that on some polyhedra no feasible point exists at all. A very costly approach would then be to examine each polyhedron during a Phase-I method.

Furthermore, the Constrained Active Signature Method proposed in this paper could be used as solver for the inner loop problem of a SALMIN approach [FWG19] extended for constrained piecewise smooth problems,where a local piecewise linear model is considered. However, similar to the smooth situation this might lead to non-feasible iterates in the outer loop dealing with the nonlinear problem. Hence, suitable strategies to handle this infeasibility have to be designed.

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