Robust Concave Utility Maximization over a Chance-Constraint

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This paper, for the first time, studies an expected utility problem with a chance constraint with incomplete information on a decision maker's utility function. The model maximizes the worst-case expected utility of random outcome over a set of concave functions within a novel ambiguity set, while satisfying a chance constraint with a given probability. To obtain computationally tractable formulations, we employ a discretization approach to derive a max-min chance-constrained approximation of this problem. This approximation is further reformulated as a mixed-integer second-order cone program (MISOCP). We show that the discrete approximation asymptotically converges to the true counterpart under mild assumptions. We also present a row generation algorithm for optimizing the max-min program. In particular, the algorithm considers a master problem as a chance-constrained problem and a second-order cone program (SOCP) as subproblems. A computational study for a bin-packing problem and a multi-item newsvendor problem is conducted to demonstrate the performance of the proposed framework and the computational efficiency of our algorithm. We find that the row generation algorithm can significantly reduce the computational time.

Key words: Robust expected utility, robust optimization, mixed-integer second-order cone program, chance constraint, row generation, bin packing, multi-item newsvendor

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1. Introduction

Decision-making under uncertainty frequently involves balancing the value of under and over utilization of a resource, or relative to a decision. We also need to ensure certain service performance, while balancing the implications of under and over-utilization. For example, in a newsvendor model, the implications of over and under-stocking are different. Thus, with the current business trends of shortening product life-cycles especially during the COVID-19 pandemic, it is crucial to find a robust solution that optimally trades off between under and over-stocking (Natarajan et al. 2018, Hu et al. 2019, Wang and Delage 2021). To ensure customer service we also want a bound on the stockout probability. In healthcare operations, random demand results in underage and overage costs (Kim and Mehrotra 2015). In this context, the underage and overage can be modeled using a nonlinear function. The use of the nonlinear function is important since the costs may not be linear (Davis et al. 2014). As additional examples, for a finite duration work-shift, in the case of assigning patients to a finite number of timeslots, the random duration required to fully serve a patient has implications on work-life balance as the day is planned for a finite time (Guest 2002, Azeem and Akhtar 2014). Here to achieve the work-life balance, we want to ensure that the assigned patients finish within a certain time with a desirable probability. A similar situation arises in scheduling operating rooms (ORs) with multiple surgeries (e.g., Wang et al. 2021a,b).

When considering situations in the preferences of different stakeholders (e.g., hospital, patients, doctors, and nurses), such as work-life balance or patient safety, it is difficult to estimate an exact utility function u in practice, while a crude parametric estimate might be available (see Chajewska et al. 2000, Hu and Mehrotra 2015, Hu et al. 2018, Armbruster and Delage 2015, Haskell et al. 2018, and references therein). Therefore, for such problems it is prudent to assume that the utility function u is unknown, and specify a model over an ambiguity set $\mathscr{U}$ that allows a family of utility functions based on their shape and properties. In the following, we will propose a general modeling framework to address such an important class of decision-making problems with incomplete information on decision-makers' utility function.

1.1. Modeling Framework

The decision-making framework we study in this paper maximizes the worst-case expected utility of random outcomes over a set of concave functions, with a chance constraint. We consider the functions that first increase and then decrease, which suitably models the situations described above. This framework is novel since, for the first time, it combines the concept of robust decision making and chance constraint optimization as a complementary synergistic mechanism for decision modeling under risk and uncertainty. Specifically, we consider the Robust Concave Utility Maximization Problem (RCUMP) with a chance constraint, represented as

$$\text{(RCUMP)} \quad \underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}}\ \underset{u\in\mathscr{U}}{\text{minimize}}\ \mathbb{E}[u(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}))], \tag{1}$$

where $\mathcal{X} := \left\{\boldsymbol{x} := (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}^{n_1} \times \mathbb{N}^{n_2} \mid \mathbb{P}_{\mathbb{Q}}\left(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \le t\right) \ge 1-\varepsilon,\ \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{d}\right\}$, $n := n_1 + n_2$, and $t \in \mathbb{R}$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{d} \in \mathbb{R}^m$. We assume that $\mathcal{X}$ is a non-empty compact set, $u \in [0,1]$ is a concave utility function and lies in an ambiguity set $\mathscr{U}$. We define the set $\mathscr{U}$ by using functional bounds on the utility and an additional condition that is specified by using a reference utility function. $f(\boldsymbol{x},\tilde{\boldsymbol{\xi}})$ is a measurable random function over $\boldsymbol{x} \in \mathcal{X}$. $\mathbb{Q}$ denotes a joint distribution of random vector $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}_1, \cdots, \tilde{\xi}_n)^\top$. $\varepsilon \in [0,1]$ is a confidence parameter, which measures the violated probability of the chance constraint. The objective of (RCUMP) is to maximize the worse-case expected utility.

Chance constraint requires that the constraint is satisfied with a given probability $1-\varepsilon$. In this paper, we assume that the probability distribution of $\tilde{\boldsymbol{\xi}}$ has a finite support $(\boldsymbol{\xi}^1,\cdots,\boldsymbol{\xi}^{|\Omega|})$ such that $\mathbb{P}(\tilde{\boldsymbol{\xi}}=\boldsymbol{\xi}^\omega)=p_\omega$ for $\omega\in\Omega:=\{1,\cdots,|\Omega|\}$, where $\sum_{\omega\in\Omega} p_\omega=1$ and $p_\omega\geq 0$ for $\omega\in\Omega$, $|\cdot|$ is the cardinality of a set. Let Ω denote the set of all possible scenarios for the random vector.

1.2. Mathematical Formulations of Illustrative Examples

As discussed before, the model studied in this paper is motivated by many applications that involve balancing costs resulting from random overage and underage. The number of patients assigned to a clinician for service, or the number of surgeries assigned to an operating room can be thought of as a bin-packing problem with each item having a random size. We formally describe this model below, followed by a description of a multi-item newsvendor problem.

Bin Packing with Chance and Utility. Let $\mathcal{I}:=\{1,\cdots,|\mathcal{I}|\}$ denote a set of items and $\mathcal{J}:=\{1,\cdots,|\mathcal{J}|\}$ denote a set of bins. We assign $|\mathcal{I}|$ items with random size $\tilde{\boldsymbol{\xi}}:=(\tilde{\xi}_1,\cdots,\tilde{\xi}_{|\mathcal{I}|})^\top$ to $|\mathcal{J}|$ bins. We let $\Omega:=\{1,\cdots,|\Omega|\}$ denote the set of scenarios for the random size, and assume that the probability distribution of the random size has a finite support $(\boldsymbol{\xi}^1,\cdots,\boldsymbol{\xi}^{|\Omega|})$. We use ξ_i^ω to denote the size of item $i\in\mathcal{I}$ under scenario $\omega\in\Omega$. Assume that $\mathbb{Q}$ is a joint probability of $\tilde{\boldsymbol{\xi}}$ and is characterized by a probability vector $(p_1,\cdots,p_{|\Omega|})$ such that $\sum_{\omega\in\Omega} p_\omega=1$ and $p_\omega\geq 0$ for all $\omega\in\Omega$. We use t_j to represent the capacity of bin $j\in\mathcal{J}$.

We define a binary variable x_j such that $x_j=1$ if bin $j\in\mathcal{J}$ is open and $x_j=0$ otherwise, and y_{ij} such that $y_{ij}=1$ if item $i\in\mathcal{I}$ is assigned to bin $j\in\mathcal{J}$, and $y_{ij}=0$ otherwise. The bin packing robust expected utility problem (BP_RCUMP) is formulated as follows:

$$\underset{(\boldsymbol{x},\boldsymbol{y})\in\{0,1\}^{|\mathcal{J}|}\times\{0,1\}^{|\mathcal{I}||\mathcal{J}|}}{\text{maximize}} \quad \underset{u\in\mathscr{U}}{\text{minimize}} \sum_{j\in\mathcal{J}} \mathbb{E}[u(\sum_{i\in\mathcal{I}} \tilde{\xi}_i y_{ij}-t_j)] \tag{2a}$$

$$\text{subject to } \mathbb{P}_{\mathbb{Q}}\left\{\sum_{i\in\mathcal{I}} \tilde{\xi}_i y_{ij}-t_j\leq 0\right\}\geq 1-\varepsilon, \qquad \forall j\in\mathcal{J}, \tag{2b}$$

$$\sum_{j\in\mathcal{J}} y_{ij}=1, \qquad \forall i\in\mathcal{I}, \tag{2c}$$

$$y_{ij}\leq x_j, \qquad \forall i\in\mathcal{I}, j\in\mathcal{J}. \tag{2d}$$

For the (BP_RCUMP), the objective function (2a) is to maximize the worst-case expected utility of over- and under-utilization. Constraints (2b) require that the sum of item sizes assigned to bin j is less than the capacity of bin j with a probability $1-\varepsilon$. Constraints (2c) ensure that each item is assigned to exactly one bin. Constraints (2d) allow item i to be assigned to bin j only if bin j is open.

Multi-Item Newsvendor with Chance and Utility. The multi-item newsvendor robust expected utility problem is to decide the order quantities for each item with a random demand, so as to maximize the worst-case expected utility of the under- and over-stocking. More specifically, let $\mathcal{I} := \{1, \cdots, |\mathcal{I}|\}$ denote the set of items, and $\Omega := \{1, \cdots, |\Omega|\}$ denote the set of scenarios. We let the inventory capacity $\boldsymbol{t} := \{t_1, \cdots, t_{|\mathcal{I}|}\}$, and the random demand of items $\tilde{\boldsymbol{\xi}} := \{\tilde{\xi}_1, \cdots, \tilde{\xi}_{|\mathcal{I}|}\}$. Under the scenarios $\omega \in \Omega$, the demand $\boldsymbol{\xi}^\omega := \{\xi_1^\omega, \cdots, \xi_{|\mathcal{I}|}^\omega\}$ with the probability distribution $\mathbb{Q}$ such that $p_\omega \geq 0$ and $\sum_{\omega \in \Omega} p_\omega = 1$. Each item $i \in \mathcal{I}$ has a ordering cost c_i, and d is the total budget. Decision variable $\boldsymbol{x} := \{x_1, \cdots, x_{|\mathcal{I}|}\}$ denotes the ordering quantities. The following formulates the multi-item newsvendor robust expected utility problem (MN_RCUMP):

$$\underset{\boldsymbol{x} \in \mathbb{R}_+^{|\mathcal{I}|}}{\text{maximize}} \ \underset{u \in \mathscr{U}}{\text{minimize}} \sum_{i \in \mathcal{I}} \mathbb{E}[u(x_i - \tilde{\xi}_i)] \tag{3a}$$

$$\text{subject to } \mathbb{P}_{\mathbb{Q}}\left\{x_i - \tilde{\xi}_i \leq t_i\right\} \geq 1 - \varepsilon, \qquad \forall i \in \mathcal{I}, \tag{3b}$$

$$\boldsymbol{c}^\top \boldsymbol{x} \leq d. \tag{3c}$$

For the (MN_RCUMP), the objective function (3a) is to maximize the worst-case expected utility of under- and over-stocking. Constraints (3b) ensure that the over-stocking is less than the inventory capacity with the probability $1 - \varepsilon$. Constraint (3c) ensures that the total ordering cost is no more than the budget.

1.3. Literature Review

In this section, we provide a review of the existing studies that are relevant to our work from both the methodology and application aspects. More specifically, we mainly focus on the literature about robust expected utility framework and two applications that are mentioned above.

1.3.1. Literature Review on Robust Expected Utility In the robust optimization (RO) framework, the complete knowledge of uncertain data is assumed to be unavailable and lies in an uncertainty set (e.g., Bertsimas and Sim 2004), and the decision-makers aim to identify the solutions that perform best under the worst-case realizations within an uncertainty set and are robust to estimation errors. RO has been extensively developed in terms of new methodologies and its practical applications (e.g., see recent reviews by Bertsimas et al. 2011, Gabrel et al. 2014, Gorissen et al. 2015, Rahimian and Mehrotra 2019).

In terms of robust expected utility, Hansen and Sargent (2001) described a connection between the max-min expected utility theory and robust-control theory. Schied (2005) defined the robust utility function by using a set of probability measures and reformulated the terminal wealth problem as a standard utility-maximization problem associated with a subjective probability measure. Natarajan et al. (2010) studied a robust expected utility model for portfolio optimization, where only the mean, covariance, and support information are available and the investors utility is a piecewise-linear concave function. Armbruster and Delage (2015) considered the problem of maximizing the worst-case expected utility of random outcome over a set of utility functions that are assumed to be risk-averse, S-shaped, or prudent, and finally derived a linear program (LP) reformulation. Haskell et al. (2016) further extended this work by considering ambiguity about both the decision makers risk preferences and the underlying distribution. They obtained a LP reformulation under the assumption of a polyhedral distributional ambiguity set with a finite number of vertices. For more general ambiguity sets, they proposed conservative approximations that are based on reformulation-linearization techniques. Delage et al. (2021) considered a utility-based shortfall risk measure where the true loss function is unavailable and proposed a preference robust model by constructing a set of utility-based loss functions from empirical data or subjective judgments. Haskell et al. (2018) considered the ambiguity in choice functions over a multi-attribute prospect space and developed a robust preference model by constructing an ambiguity set of choice functions through preference elicitation with pairwise comparisons of prospects. Luo and Mehrotra (2019) studied a service center location problem with ambiguous utility gains upon receiving service under a distributionally robust optimization (DRO) framework, where the elicited location-dependent utilities are assumed to be described by an expected value and variance constraint.

Perhaps, the most relevant studies to ours are Hu and Mehrotra (2015) and Hu et al. (2018). Hu and Mehrotra (2015) assumed that the utility function is increasing and concave. They specified the uncertainty set by using upper bound and lower bound on the utility and marginal utility functions, as well as auxiliary equality and inequality constraints on the utility. They used a partitioning-based approach to formulate the problem as a LP. More recently, Hu et al. (2018) assumed that the uncertainty set of the utility function is non-decreasing and satisfies additional boundary and auxiliary conditions. They developed a sample average approximation (SAA) based approach (Kleywegt et al. 2002) to solve the problem. Unlike the aforementioned two studies, our work considers the utility-dependent decisions within a chance-constrained framework and constructs a novel ambiguity set in the space of risk-averse utility centered at a reference utility function using a distance metric. This allows us to model a more general set of utility functions that are first increasing then decreasing. We also provide a convergence analysis where the proof is different from Hu et al. (2018) and finally propose an efficient row generation-based solution scheme to efficiently solve our (RCUMP) model with two practical applications.

1.3.2. Literature Review on Chance-Constrained Bin Packing Problem Chance-constrained programs (CCPs) were firstly introduced by Charnes and Cooper (1959) to address optimization problems under uncertainty, which have been widely used for various decision-making context. CCPs are generally difficult to solve (e.g., Song et al. 2014), especially when the coefficients matrix is random or the chance constraints contain integer decision variables (as is shown in our (BP_RCUMP) in Section 1.2). For the study of more general CCPs under different optimization settings, we refer the interested reader to a recent review by Küçükyavuz and Jiang (2021). In recent years, the chance-constrained bin packing problem (CCBP) has been extensively studied, especially under the context of healthcare resources allocation (e.g., ORs and surgeries) and cloud computing management (e.g., Hoogervorst et al. 2019, Cohen et al. 2019, Martinovic et al. 2021). For healthcare resources allocation, Deng and Shen (2016), Deng et al. (2019) and Zhang et al. (2020) investigated a surgery scheduling problem with chance constraints to determine ORs allocation and surgery scheduling using the stochastic programming and DRO paradigms. More recently, Wang et al. (2021a) studied a chance-constrained multiple bin packing problem with application to ORs planning. Wang et al. (2021b) further extended this work to a DRO model with joint chance constraints with partial distribution information. Instead of minimizing the total cost or the number of bins as in the above studies, we consider utility-dependent decisions within a chance-constrained framework to ensure certain service performance, while balancing the implications of under and over-utilization of resources. Although CCBP is widely studied, to our best knowledge, such applications under a robust expected utility framework are very rare.

1.3.3. Literature on Multi-Item Newsvendor Problem The newsvendor problem is a fundamental operations management problem with various applications (see a recent review by Qin et al. 2011). To determine the order quantities for multiple products, the retailers assume a specifically known distribution of the random demand (e.g., Erlebacher 2000). However, in reality, the true demand distribution is hardly ever known to the retailers. Leveraging recent advances in RO, robust multi-item newsvendor problems aim to maximize the worst-case expected operating revenue over all possible demand realizations within an uncertainty set (e.g., Ardestani-Jaafari and Delage 2016, Hu et al. 2019, Zhang et al. 2021). For most real-world applications, the solutions of RO models are generally over-conservative, thus DRO multi-item newsvendor problems have been extensively explored in recent years, where one seeks a more robust solution that performs best under the worst-case demand distribution within an ambiguity set of distributions (e.g., Hanasusanto et al. 2015, Natarajan et al. 2018, Rahimian et al. 2019, Chen et al. 2020, Wang and Delage 2021). Very few studies attempt to use alternative risk preferences within the expected utility framework (Wang et al. 2012, Choi and Ruszczyński 2011), which assume that the distributions of product demands and the utility function are exactly known in advance. However, it is very difficult to derive the exact representation of the utility function in practice. This also further motivates us to model the problem under a robust expected utility framework using a novel ambiguity set of the utility functions.

1.4. Contributions of This Paper

To summarize, this paper addresses the modeling framework and resolution method of a general chance-constrained robust expected utility problem over a set of concave utility functions that lie in a novel ambiguity set. Under mild conditions, we derive a mixed-integer second-order cone programming (MISOCP) formulation, and conduct the convergence analysis for (RCUMP) that relies on the convergence theory of optimization problems, and also develop a row generation-based solution scheme to solve the resulting problem efficiently.

More specifically, the contributions of this paper are summarized as follows:

• We construct a novel ambiguity set in the space of risk-averse utility centered at a reference utility function using a distance metric, and employ a discretization scheme to model the novel utility ambiguity set $\mathcal{U}$. In doing so, we are able to reformulate (RCUMP) as a tractable MISOCP with the help of a big-M technique. Then a convergence analysis is provided to show that the discrete approximation asymptotically converges to the true counterpart under some mild assumptions. To the best of our knowledge, this is the first attempt to study a general robust expected utility problem with chance constraints when the information of the utility function is incomplete.

• We propose an efficient row generation-based solution scheme to solve the robust expected utility model. More specifically, we represent (RCUMP) as a max-min formulation and investigate the row generation approach for solving the problem. The algorithm considers a master problem as a chance-constrained problem and a second-order cone programming (SOCP) subproblem. We show that our proposed algorithm converges in a finite number of iterations.

• We perform an extensive numerical study for the bin-packing problem using real data from surgery planning and the multi-item newsvendor problem to analyze the general structure of the decisions from the decision-making framework and show the benefits of the techniques developed in this paper for the computational improvement. We find that the row generation algorithm significantly outperforms the one that uses a commercial solver to solve the MISOCP. We also evaluate the out-of-sample performance of the solutions generated from (BP_RCUMP). We found that the solutions generated from (BP_RCUMP) with the larger number of partitions $N = 20$ or the larger ambiguity set do not improve performance in terms of out-of-sample over-utilization and under-utilization measures in the simulation for most of the instances, even though the computational cost increases rapidly with the number of partitions N. Therefore, when $N = 10$, the optimal solutions obtained from (BP_RCUMP) achieve a desirable out-of-sample performance, and this performance is not improved by increasing the number of partitions. Moreover, the average under-utilization is significantly larger than the over-utilization, and decreases when ε varies from 0.05 to 0.1. Therefore, the choice of ε could affect the value of over-utilization and the under-utilization. Similar observations can also be found for the multi-item newsvendor problem.

1.5. Organization

The remainder of this paper is organized as follows. Section 2 gives a definition of ambiguity set $\mathcal{U}$ and formulates (RCUMP) as a MISOCP using a discrete approximation of $\mathcal{U}$ and big-M techniques. We then present convergence analysis to show that the discrete approximation asymptotically converges to the true counterpart under some mild assumptions in Section 3. In Section 4, we present a row generation algorithm to solve (RCUMP). Section 5 reports the computational results of the robust expected utility model for the bin-packing problem and the multi-item newsvendor problem with the help of data from the test problems. Section 6 concludes the paper with a summary of the important findings.

2. Model Formulation

Section 2.1 introduces the definition of the ambiguity set $\mathcal{U}$ and examples for the rise-averse utility functions. Using a discretization scheme, we then reformulate the robust expected chance-constrained problem as a MISOCP in Section 2.2.

2.1. The Ambiguity Set Definition

We assume that the function $f(\boldsymbol{x},\tilde{\boldsymbol{\xi}})$ has a bounded support $\Theta := [-\theta_1, \theta_2]$ for all $\boldsymbol{x} \in \mathcal{X}$, where $\theta_1, \theta_2 \in \mathbb{R}^+$. We also assume that the function u satisfies the following conditions,

$$u(-\theta_1) = 0, \quad u(0) = 1, \quad u(\theta_2) = 0. \tag{4}$$

We use function $\bar{u}$ and $\underline{u}$ as the bounds of u, that is to say,

$$\underline{u}(a) \le u(a) \le \bar{u}(a), \qquad a \in \Theta. \tag{5}$$

We construct the following ball in the space of risk-averse utility centered at a reference utility function u_0:

$$d(u, u_0) \le \sqrt{b}, \tag{6}$$

where b is a positive constant and $d(u, u_0)$ is the distance between two functions u and u_0 which is defined as the L^2-norm of $u - u_0$, i.e.,

$$d(u, u_0) = \sqrt{\int_{-\theta_1}^{\theta_2} |u(a) - u_0(a)|^2 \, da}$$

Constraint (6) ensures that the utility functions are real-valued square-integrable in the domain using u_0 as a reference, with a pre-specified bound. Let $\mathscr{U}'$ be the set of all first increasing then decreasing concave utility functions defined on Θ. We have the following ambiguity set $\mathscr{U}$,

$$\mathscr{U} := \left\{ u \in \mathscr{U}' \,\middle|\, u \text{ satisfies the conditions in the constraints } (4) - (6) \right\}.$$

This utility set is different from the one used in Hu and Mehrotra (2015), in the use of constraint (6). The set used in Hu and Mehrotra (2015) is specified using constraints linear in u, and u is non-decreasing. Here we are allowing u to increase as well as decrease. Moreover, using a benchmark to specify the ambiguity set for u is desirable when considering the situations where a parametric function is first estimated.

2.1.1. Examples of the Ambiguity Set In the following, we provide an example of the ambiguity set $\mathscr{U}$ defined above. We let the bounds on u be derived from the power utility functions. The upper and lower bound on u are given by

$$\underline{u}(a)=\begin{cases}\left(\dfrac{a+\theta_1}{\theta_1}\right)^{1-r_2}, & a\in[-\theta_1,0],\\ \left(\dfrac{\theta_2-a}{\theta_2}\right)^{1-r_2}, & a\in(0,\theta_2],\end{cases}\qquad \overline{u}(a)=\begin{cases}\left(\dfrac{a+\theta_1}{\theta_1}\right)^{1-r_1}, & a\in[-\theta_1,0],\\ \left(\dfrac{\theta_2-a}{\theta_2}\right)^{1-r_1}, & a\in(0,\theta_2],\end{cases} \tag{7}$$

respectively, where parameters r_1 and r_2 are constant coefficients. For the power utility functions, Holt and Laury (2002) suggested that one could set $r_1, r_2 \in [0.41, 0.68]$ for a risk-averse decision-makers. We give the following upper bound $\bar{u}$ and lower bound $\underline{u}$ of the utility function in Figure 1(a):

$$\underline{u}(a)=\begin{cases}(a+1)^{0.32}, & a\in[-1,0],\\ (1-a)^{0.32}, & a\in(0,1].\end{cases}\qquad \overline{u}(a)=\begin{cases}(a+1)^{0.59}, & a\in[-1,0],\\ (1-a)^{0.59}, & a\in(0,1].\end{cases}$$

As a comparison, we set $r_1 = 0.25$ and $r_2 = 0.75$ for $\bar{u}$ and $\underline{u}$, which are shown in Figure 1(b).

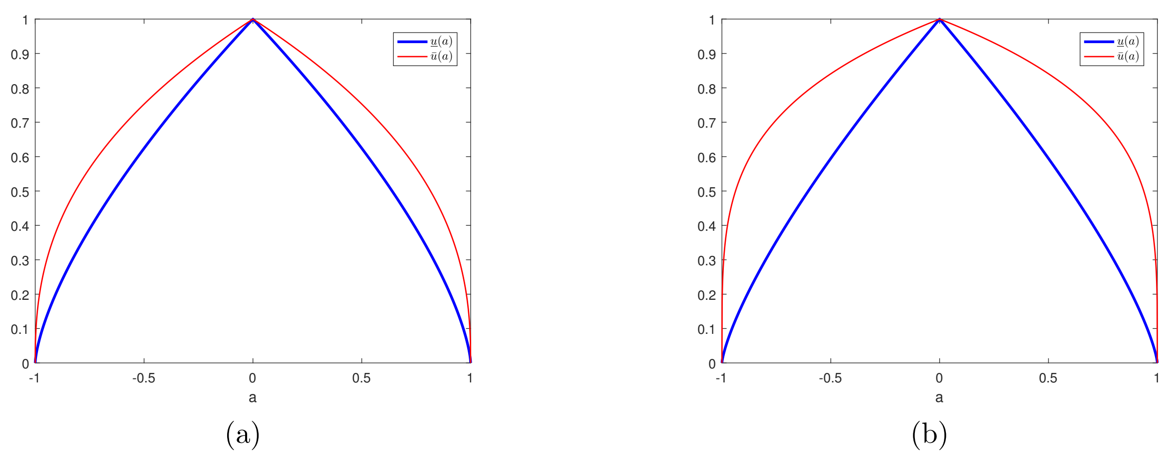

Figure 1 **The bounds of utility functions when $r_1 = 0.68$ and $r_2 = 0.41$ in (a) and $r_1 = 0.75$ and $r_2 = 0.25$ in (b)**

For the reference utility function, we let u_0 be the power utility function, defined as follows

$$u_0(a)=\begin{cases}\left(\dfrac{a+\theta_1}{\theta_1}\right)^{1-r_0}, & a\in[-\theta_1,0],\\ \left(\dfrac{\theta_2-a}{\theta_2}\right)^{1-r_0}, & a\in(0,\theta_2].\end{cases} \tag{8}$$

2.2. Reformulation of (RCUMP)

In this section, we present a reformulation for (RCUMP) associated with a set $\mathscr{U}_N$, which is defined using a discretization of the continuous problem. Let N is the number of partitions, $\mathcal{A}(N) = \{a_0, \cdots, a_l, \cdots, a_N\}$ be a set of break points such that $a_0 < \cdots < a_l < \cdots < a_N$; $a_0 = -\theta_1$, $a_l = 0$; and $a_N = \theta_2$. We assume that if $N_1 < N_2$, then $\mathcal{A}(N_1) \subset \mathcal{A}(N_2)$. We define the following piecewise linear approximation functions of $\underline{u}$ and $\bar{u}$:

$$\underline{u}_N(a) = \sum_{k=0}^{N-1} \left(\frac{\underline{u}(a_{k+1}) - \underline{u}(a_k)}{a_{k+1} - a_k} a + \frac{a_{k+1}\underline{u}(a_k) - a_k \underline{u}(a_{k+1})}{a_{k+1} - a_k} \right) \mathbf{1}(a_k \le a < a_{k+1}), \tag{9}$$

$$\bar{u}_N(a) = \sum_{k=0}^{N-1} \left(\frac{\bar{u}(a_{k+1}) - \bar{u}(a_k)}{a_{k+1} - a_k} a + \frac{a_{k+1}\bar{u}(a_k) - a_k \bar{u}(a_{k+1})}{a_{k+1} - a_k} \right) \mathbf{1}(a_k \le a < a_{k+1}), \tag{10}$$

where $\mathbf{1}(\cdot)$ represents the indicator function, which returns 1 if the clause inside is correct, and otherwise 0.

Following Figure 1(a), Figure 2 gives the approximation of bounds of utility functions when the number of partitions N=20.

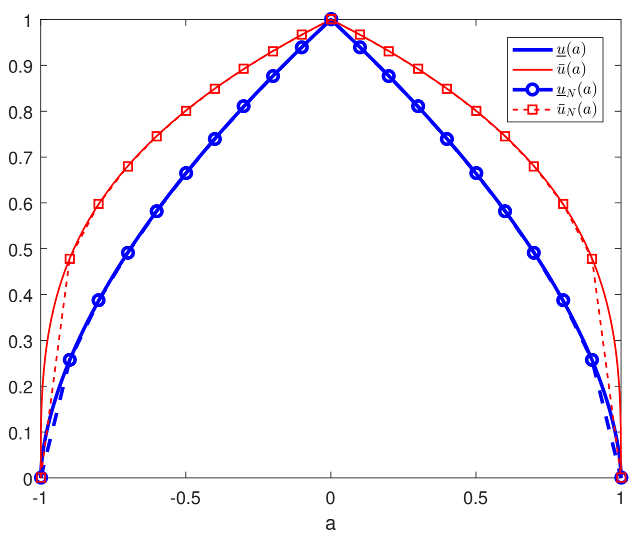

Figure 2 Approximation of bounds of utility functions when $r_1 = 0.68$**,** $r_2 = 0.41$ **and** $N = 20$

Let the ambiguity set

$$\mathscr{U}_N := \left\{ u \in \mathscr{U}' \;\middle|\; \begin{array}{l} u \text{ satisfies the conditions in constraint (4) and } \underline{u}_N \le u \le \bar{u}_N, \quad \text{and} \\ \displaystyle\sum_{k=0}^{N-1} \left(u_0(a_k) - u(a_k)\right)^2 (a_{k+1} - a_k) \le b, \end{array} \right\},$$

and

$$\pi_N(\boldsymbol{x}) = \underset{u \in \mathscr{U}_N}{\text{minimize}}\, \mathbb{E}[u(f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))], \tag{11}$$

then Lemma 1 below gives a SOCP reformulation of $\pi_N(\boldsymbol{x})$ for any given $\boldsymbol{x} \in \mathcal{X}$.

LEMMA 1. *For a given $\boldsymbol{x} \in \mathcal{X}$, problem (11) is equivalent to the following SOCP reformulation:*

$$
\begin{aligned}
\underset{\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\beta}}{\text{minimize}} \; & \sum_{\omega\in\Omega} p_\omega (f(\boldsymbol{x},\boldsymbol{\xi}^\omega)\alpha^\omega + \beta^\omega) && && \text{(12a)}\\
\text{subject to} \; & (a_{k+1}-a_{k-1})\theta_k \geq (a_{k+1}-a_k)\theta_{k-1} + (a_k - a_{k-1})\theta_{k+1}, && \forall k\in\{1,\cdots,N-1\}, && \text{(12b)}\\
& \theta_k \leq \bar{u}_N(a_k), && \forall k \in \{0,\cdots,N\}, && \text{(12c)}\\
& \theta_k \geq \underline{u}_N(a_k), && \forall k \in \{0,\cdots,N\}, && \text{(12d)}\\
& ||\boldsymbol{Q}(\boldsymbol{\theta}-\boldsymbol{u}_0)||_2 \leq 1, && && \text{(12e)}\\
& a_k\alpha^\omega + \beta^\omega - \theta_k \geq 0, && \forall k\in\{0,\cdots,N\}, \omega\in\Omega, && \text{(12f)}\\
& \beta^\omega \geq 0, && \forall \omega\in\Omega, && \text{(12g)}
\end{aligned}
$$

where $\theta_0 = 0$, $\theta_l = 1$ and $\theta_N = 0$, $\boldsymbol{Q} = diag(\sqrt{\frac{a_1-a_0}{b}},\cdots,\sqrt{\frac{a_N-a_{N-1}}{b}})$ is a diagonal matrix, and $\boldsymbol{\theta}-\boldsymbol{u}_0 := (\theta_0 - u_0(a_0),\cdots,\theta_{N-1}-u_0(a_{N-1}))^\top$.

Proof. Let $\theta_k = u(a_k)$, then $\theta_0 = u(a_0) = 0$, $\theta_l = u(a_l) = 1$ and $\theta_N = u(a_N) = 0$. Given the concavity property of $u \in \mathscr{U}_N$, we have

$$\frac{\theta_1-\theta_0}{a_1-a_0} \geq \cdots \geq \frac{\theta_l - \theta_{l-1}}{a_l - a_{1-1}} \geq 0 \geq \frac{\theta_{1+1}-\theta_l}{a_{1+1}-a_l} \geq \cdots \geq \frac{\theta_N - \theta_{N-1}}{a_N - a_{N-1}},$$

which implies that constraint (12b) holds. In addition, constraints (12c), (12d) and (12e) represent the constraints of $u(a)$.

Let $\mathscr{U}'_N$ be a subset of $\mathscr{U}_N$ which consists of all the piecewise linear function with break point $\{a_0,\cdots,a_N\}$. If u^* is an optimal solution of problem (11), we can define a piecewise linear function belonging to $\mathscr{U}'_N$ that bounds u^* from below. Hence, we can rewrite problem (11) as $\text{minimize}_{u\in\mathscr{U}'_N} \sum_{\omega\in\Omega} p_\omega u(f(\boldsymbol{x},\boldsymbol{\xi}^\omega))$. Since $u \in \mathcal{U}'_N$ is piecewise linear with the break point a_k and corresponding value θ_k, thus given a $v \in \Theta$, $u(v)$ is equivalent to

$$
\begin{aligned}
\underset{\alpha,\beta}{\text{minimize}} \; & v\alpha + \beta \\
\text{subject to} \; & a_k\alpha + \beta - \theta_k \geq 0 && \forall k \in \{0,\cdots,N\},\\
& \beta \geq 0.
\end{aligned}
$$

Therefore, when $v = f(\boldsymbol{x},\boldsymbol{\xi}^\omega)$ for $\omega \in \Omega$, problem (11) is equivalent to problem (12). □

Using the duality theory and big-M strengthening technique inspired from Song et al. (2014), we obtain a reformulation of (RCUMP) with the ambiguity set $\mathscr{U}_N$ in Theorem 1.

THEOREM 1. *The problem*

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \; \underset{u\in\mathscr{U}_N}{\text{minimize}} \, \mathbb{E}[u(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}))] \tag{13}$$

is equivalent to the following MISOCP:

$$\underset{\boldsymbol{x},\boldsymbol{z},\boldsymbol{\mu},\gamma_1,\gamma_2,\boldsymbol{\delta},\boldsymbol{\eta},\lambda}{\text{maximize}} \ \sum_{k=0}^{N}(\underline{u}(a_k)\gamma_{2k}-\bar{u}(a_k)\gamma_{1k})+\sum_{k=0}^{N-1}\eta_k u_0(a_k)-\lambda \tag{14a}$$

$$\text{subject to } \boldsymbol{A}\boldsymbol{x}\le\boldsymbol{d}, \tag{14b}$$

$$p_\omega f(\boldsymbol{x},\boldsymbol{\xi}^\omega)-\sum_{k=0}^{N}a_k\delta_{k\omega}=0, \qquad \forall\omega\in\Omega, \tag{14c}$$

$$p_\omega-\sum_{k=0}^{N}\delta_{k\omega}\ge 0, \qquad \forall\omega\in\Omega, \tag{14d}$$

$$\eta_k=-\mu_k(a_{k+1}-a_{k-1})+\mu_{k+1}(a_{k+2}-a_{k+1}) + \mu_{k-1}(a_{k-1}-a_{k-2})+\gamma_{1k}-\gamma_{2k}+\sum_{\omega\in\Omega}\delta_{k\omega}, \qquad \forall k\in\{1,\cdots,N-1\}, \tag{14e}$$

$$||\boldsymbol{Q}^{-1}\boldsymbol{\eta}||_2\le\lambda, \tag{14f}$$

$$f(\boldsymbol{x},\boldsymbol{\xi}^\omega)+(m_\omega(k_{q+1})-t)z_\omega\le m_\omega(k_{q+1}), \qquad \forall\omega\in\Omega, \tag{14g}$$

$$\sum_{\omega\in\Omega}p_\omega z_\omega\ge 1-\varepsilon, \tag{14h}$$

$$\mu_k,\gamma_{1k},\gamma_{2k},\delta_{k\omega}\ge 0, z_\omega\in\{0,1\}, \qquad \forall k\in\{0,\cdots,N\},\omega\in\Omega, \tag{14i}$$

where $\eta_0=\mu_N=\mu_0=0$, *and for* $\omega,k\in\Omega$, $m_\omega(k):=\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{maximize}}\left\{f(\boldsymbol{x},\boldsymbol{\xi}^\omega)\middle|f(\boldsymbol{x},\boldsymbol{\xi}^k)\le t,\boldsymbol{A}\boldsymbol{x}\le\boldsymbol{d}\right\}$. *We sort* $m_\omega(k)$ *in a non-decreasing order such that* $m_\omega(k_1)\le\ldots\le m_\omega(k_N)$, *and let* $q:=\max\left\{l:\sum_{j=1}^{l}p_{k_j}\le\varepsilon\right\}$.

Proof. Let $\boldsymbol{y}=\boldsymbol{Q}(\boldsymbol{\theta}-\boldsymbol{u}_0)$, then $||\boldsymbol{y}||_2\le 1$ based on constraint (12e) and $\boldsymbol{\theta}=\boldsymbol{Q}^{-1}\boldsymbol{y}+\boldsymbol{u}_0$. Let $\boldsymbol{\mu}$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\delta}$ be the dual variables of constraints (12b) to (12d), and (12f) respectively. The dual function can be formulated as

$$\begin{aligned} g(\boldsymbol{\mu},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\delta}) \ = \ & \inf_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{y}} \ L(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{y},\boldsymbol{\mu},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\delta}) \\ & \text{subject to } ||\boldsymbol{y}||_2\le 1, \\ & \beta^\omega\ge 0, \qquad \forall\omega\in\Omega, \end{aligned}$$

where

$$\begin{aligned} & L(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{y},\boldsymbol{\mu},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\delta}) \\ &=\sum_{\omega\in\Omega}(p_\omega f(\boldsymbol{x},\boldsymbol{\xi}^\omega)-\sum_{k=0}^{N}a_k\delta_{k\omega})\alpha_\omega+\sum_{\omega\in\Omega}(p_\omega-\sum_{k=0}^{N}\delta_{k\omega})\beta_\omega+\sum_{k=0}^{N}(\underline{u}(a_k)\gamma_{2k}-\bar{u}(a_k)\gamma_{1k})+\boldsymbol{\eta}^\top(\boldsymbol{Q}^{-1}\boldsymbol{y}+\boldsymbol{u}_0). \end{aligned}$$

and $\eta_k=-\mu_k(a_{k+1}-a_{k-1})+\mu_{k+1}(a_{k+2}-a_{k+1})+\mu_{k-1}(a_{k-1}-a_{k-2})+\gamma_{1k}-\gamma_{2k}+\sum_{\omega\in\Omega}\delta_{k\omega}$, for all $k=1,\cdots,N-1$.

Based on the domain of variables $\boldsymbol{y}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and the definition of dual norm, we have,

$$g(\boldsymbol{\mu},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\delta}_2)=\sum_{k=0}^{N}(\underline{u}(a_k)\gamma_{2k}-\bar{u}(a_k)\gamma_{1k})+\sum_{k=0}^{N-1}\eta_k u_0(a_k)-||\boldsymbol{Q}^{-1}\boldsymbol{\eta}||_*,$$

if $p_\omega f(\boldsymbol{x},\boldsymbol{\xi}^\omega) - \sum_{k=0}^N a_k\delta_{k\omega} = 0,\ p_\omega - \sum_{k=0}^N \delta_{k\omega} \geq 0$, for $\omega \in \Omega$. Thus, constraints (14c)-(14e) are the dual formulation of problem (12). Note that when $\theta_k = u_0(a_k)$ for all $k = 0,\cdots,N$, constraint (12e) can be reformulated as $0 \leq b$. Since b is a positive constant, $u_0 \in \textbf{relint}\ \mathscr{U}_N$ such that constraint (12e) hold with strict inequality, and problem (12) satisfies Slaters condition (Boyd et al. 2004). Strong duality holds under the Slaters conditions.

Note that the chance constraint can be rewritten as

$$f(\boldsymbol{x},\boldsymbol{\xi}^\omega) + (M_\omega - t)z_\omega \leq M_\omega, \qquad \forall \omega \in \Omega, \tag{15a}$$

$$\sum_{\omega\in\Omega} p_\omega z_\omega \geq 1-\varepsilon, \tag{15b}$$

$$z_\omega \in \{0,1\}, \qquad \forall \omega \in \Omega, \tag{15c}$$

where M_ω is a large constant such that constraint (15a) still holds when $z_\omega = 0$. We use the coefficient strengthening procedure inspired from Song et al. (2014) to obtain a tight value of M_ω. Note that for all $\omega \in \Omega$,

$$M_\omega \geq \bar{M}_\omega := \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{maximize}} \left\{ f(\boldsymbol{x},\boldsymbol{\xi}^\omega) \middle| \mathbb{P}\left\{ f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) - t \leq 0 \right\} \geq 1-\varepsilon, \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d} \right\}.$$

For any $\omega, k \in \Omega$, let

$$m_\omega(k) := \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{maximize}} \left\{ f(\boldsymbol{x},\boldsymbol{\xi}^\omega) \middle| f(\boldsymbol{x},\boldsymbol{\xi}^k) \leq t, \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d} \right\}.$$

We sort $m_\omega(k)$ in a non-decreasing order such that $m_\omega(k_1) \leq \ldots \leq m_\omega(k_N)$. Then $m_\omega(k_{q+1})$ is an upper bound for $\bar{M}_\omega$, if $q = \max\left\{ l : \sum_{j=1}^l p_{k_j} \leq \varepsilon \right\}$.

This completes our proof. □

In the following, Corollary 1 and Corollary 2 give the final reformulations of (BP_RCUMP) and (MN_RCUMP), respectively, which both are represented as the MISOCP.

Corollary 1. *Based on Theorem 1, the final MISOCP reformulation of (BP_RCUMP) under the ambiguity set $\mathscr{U}_N$ can be represented as follows:*

$$\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\mu},\gamma_1,\gamma_2,\boldsymbol{\delta},\lambda}{\text{maximize}} \sum_{k=0}^N (\underline{u}(a_k)\gamma_{2k} - \bar{u}(a_k)\gamma_{1k}) + \eta^\top u_{0N}(a) - \lambda \tag{16a}$$

$$\text{subject to (2c), (2d),} \tag{16b}$$

$$p_\omega \left(\sum_{i\in\mathcal{I}} \xi_i^\omega y_{ij} - t_j \right) - \sum_{k=0}^N a_k \delta_{kj\omega} = 0, \qquad \forall j \in \mathcal{J}, \omega \in \Omega, \tag{16c}$$

$$p_\omega - \sum_{k=0}^N \delta_{kj\omega} \geq 0, \qquad \forall j \in \mathcal{J}, \omega \in \Omega, \tag{16d}$$

$$\eta_k = -\mu_k(a_{k+1} - a_{k-1}) + \mu_{k+1}(a_{k+2} - a_{k+1})$$

$$+\mu_{k-1}(a_{k-1}-a_{k-2})+\gamma_{1k}-\gamma_{2k}+\sum_{j\in\mathcal{J}}\sum_{\omega\in\Omega}\delta_{kj\omega}, \qquad \forall k\in\{1,\cdots,N-1\}, \tag{16e}$$

$$||Q^{-1}\eta||_2\le\lambda, \tag{16f}$$

$$\sum_{i\in\mathcal{I}}\xi_i^{\omega}y_{ij}+(m_{j\omega}(k_{q+1})-t_j)z_{j\omega}\le m_{j\omega}(k_{q+1}), \qquad \forall j\in\mathcal{J},\omega\in\Omega, \tag{16g}$$

$$\sum_{\omega\in\Omega}p_{\omega}z_{j\omega}\ge 1-\varepsilon, \qquad \forall j\in\mathcal{J}, \tag{16h}$$

$$\boldsymbol{\mu},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\delta}\ge 0, x_j,y_{ij},z_{j\omega}\in\{0,1\}, \qquad \forall i\in\mathcal{I},j\in\mathcal{J},\omega\in\Omega, \tag{16i}$$

where $m_{j\omega}(k)=\underset{\boldsymbol{y}_j}{\text{maximize}}\left\{\sum_{i\in\mathcal{I}}\xi_i^{\omega}y_{ij}\,\middle|\,\sum_{i\in\mathcal{I}}\xi_i^{k}y_{ij}\le t_j,\ \boldsymbol{y}_j\in\{0,1\}^{|\mathcal{I}|}\right\}$*, for all* $j\in\mathcal{J}$ *and* $k,\omega\in\Omega$*, and* q *is defined in Theorem 1.*

Corollary 2. *Based on Theorem 1, we can reformulate (MN_RCUMP) with the ambiguity set* $\mathscr{U}_N$ *as the following problem:*

$$\underset{\boldsymbol{x},\boldsymbol{z},\boldsymbol{\mu},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\delta},\lambda}{\text{maximize}}\ \sum_{k=0}^{N}(\underline{u}(a_k)\gamma_{2k}-\bar{u}(a_k)\gamma_{1k})+\eta^{\top}u_{0N}(a)-\lambda \tag{17a}$$

$$\text{subject to (3c)}, \tag{17b}$$

$$p_{\omega}(x_i-\xi_i^{\omega})-\sum_{k=0}^{N}a_k\delta_{ki\omega}=0, \qquad \forall i\in\mathcal{I},\omega\in\Omega, \tag{17c}$$

$$p_{\omega}-\sum_{k=0}^{N}\delta_{ki\omega}\ge 0, \qquad \forall i\in\mathcal{I},\omega\in\Omega, \tag{17d}$$

$$\eta_k=-\mu_k(a_{k+1}-a_{k-1})+\mu_{k+1}(a_{k+2}-a_{k+1})$$

$$+\mu_{k-1}(a_{k-1}-a_{k-2})+\gamma_{1k}-\gamma_{2k}+\sum_{i\in\mathcal{I}}\sum_{\omega\in\Omega}\delta_{ki\omega}, \qquad \forall k\in\{1,\cdots,N-1\}, \tag{17e}$$

$$||Q^{-1}\eta||_2\le\lambda, \tag{17f}$$

$$x_i-\xi_i^{\omega}+(m_{i\omega}(k_{q+1})-t_i)z_{i\omega}\le m_{i\omega}(k_{q+1}), \qquad \forall i\in\mathcal{I},\omega\in\Omega, \tag{17g}$$

$$\sum_{\omega\in\Omega}p_{\omega}z_{i\omega}\ge 1-\varepsilon, \qquad \forall i\in\mathcal{I}, \tag{17h}$$

$$\boldsymbol{x},\boldsymbol{\mu},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\delta}\ge 0, z_{i\omega}\in\{0,1\}, \qquad \forall i\in\mathcal{I},\omega\in\Omega, \tag{17i}$$

where $m_{i\omega}(k)=\underset{x_i\in\mathbb{R}_+}{\text{maximize}}\left\{x_i-\xi_i^{\omega}\,\middle|\,x_i-\xi_i^{k}\le t_i\right\}$*, for all* $i\in\mathcal{I}$ *and* $k,\omega\in\Omega$*, and* q *is defined in Theorem 1.*

3. Convergence Analysis

In this section, we show that, the optimal solutions obtained by using the discrete approximation of the set $\mathcal{U}$ converge to the true optimal solutions in the limit. Throughout this section, we make the following assumptions, which are also commonly used in the literature.

Assumption 1. *(i)* $f(\boldsymbol{x},\tilde{\boldsymbol{\xi}})$ *is linear in an open neighborhood of* $\mathcal{X}$*; (ii)* $\mathscr{U}$ *is a non-empty set, and function* $u\in\mathscr{U}$ *has bounded derivative almost everywhere; (iii)* $\bar{u}$*,* $\underline{u}$ *and* u_0 *are continuous concave functions.*

Note that $\underline{u}_N$ and $\bar{u}_N$ are the approximation functions of $\underline{u}$ and $\bar{u}$ based on the $\mathcal{A}(N)$. Also, under the above assumption, $\underline{u}_N$ and $\bar{u}_N$ uniformly converge to $\underline{u}$ and $\bar{u}$, respectively. In the following, under Assumption 1, we first show that $\pi(\boldsymbol{x}) := \underset{u\in\mathscr{U}}{\text{minimize}}\, \mathbb{E}[u(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}))]$ and $\pi_N(\boldsymbol{x}) := \underset{u\in\mathscr{U}_N}{\text{minimize}}\, \mathbb{E}[u\left(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}})\right)]$ are continuous concave functions.

Lemma 2. *$\pi(\boldsymbol{x})$ and $\pi_N(\boldsymbol{x})$ are continuous concave functions on $\mathcal{X}$.*

Proof. Based on Assumption 1, we know that u is concave and $\xi(\cdot)$ is linear in an open neighborhood of $\mathcal{X}$ which can be denoted by $\mathcal{N}(\mathcal{X})$. Therefore, $u(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}))$ is concave in $\mathcal{N}(\mathcal{X})$ and thus $\mathbb{E}[u(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}))]$ is concave in $\mathcal{N}(\mathcal{X})$. Moreover, for $\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{X}$, we have

$$\begin{aligned}\pi(\lambda\boldsymbol{x}_1+(1-\lambda)\boldsymbol{x}_2) &= \underset{u\in\mathscr{U}}{\text{minimize}}\, \mathbb{E}[u(f(\lambda\boldsymbol{x}_1+(1-\lambda)\boldsymbol{x}_2,\tilde{\boldsymbol{\xi}}))]\\ &\geq \underset{u\in\mathscr{U}}{\text{minimize}}\, \lambda\mathbb{E}[u(f(\boldsymbol{x}_1,\tilde{\boldsymbol{\xi}}))]+(1-\lambda)\mathbb{E}[u(f(\boldsymbol{x}_2,\tilde{\boldsymbol{\xi}}))]\\ &\geq \lambda\, \underset{u\in\mathscr{U}}{\text{minimize}}\, \mathbb{E}[u(f(\boldsymbol{x}_1,\tilde{\boldsymbol{\xi}}))]+(1-\lambda)\, \underset{u\in\mathscr{U}}{\text{minimize}}\, \mathbb{E}[u(f(\boldsymbol{x}_2,\tilde{\boldsymbol{\xi}}))]\\ &= \lambda\pi(\boldsymbol{x}_1)+(1-\lambda)\pi(\boldsymbol{x}_2).\end{aligned}$$

$\pi(\boldsymbol{x})$ is concave function in $\mathcal{N}(\mathcal{X})$, thus $\pi(\boldsymbol{x})$ is continuous in $\mathcal{X}$. Similarly, $\pi_N(\boldsymbol{x})$ is continuous concave functions in $\mathcal{X}$. This completes our proof. □

The following Lemma 3 to Lemma 7 give some preliminary results, which are needed to prove the convergence of $\pi_N(\boldsymbol{x})$ to $\pi(\boldsymbol{x})$.

Lemma 3. *Function $u\in\mathscr{U}_N$ is equicontinuous.*

Proof. For $u\in\mathscr{U}_N$, since the derivative u' of u satisfies $|u'|<M$ for almost everywhere, then $u\in\mathscr{U}_N$ is Lipschitz with Lipschitz constant M. Since a set of functions with bounded Lipschitz constant forms an equicontinuous set, u are equicontinuous. □

Lemma 4. *(ArzelàAscoli Theorem in Green and Valentine (1961)) Let $\mathcal{K}$ be a compact metric space, with metric $d_K(p,p')$, and let $C(\mathcal{K})$ denote the space of real (or complex) valued continuous functions on $\mathcal{K}$. If $\{f_n\}_{n\in N}$ is a sequence in $C(\mathcal{K})$ obeying:*

- *$\{f_n\}_{n\in N}$ is pointwise bounded, and*
- *$\{f_n\}_{n\in N}$ is equicontinuous,*

then, the sequence $\{f_n\}_{n\in N}$ contains a uniformly convergent subsequence.

Lemma 5. *For any sequence $\{u_N\in\mathscr{U}_N\}$, there exists a subsequence $\{u_{N_K}\}$ that uniformly converges to $\hat{u}\in\mathscr{U}$.*

Proof. Based on Lemma 3 and Lemma 4, we have that, for any sequence $\{u_N\in\mathscr{U}_N\}$, there exists a subsequence $\{u_{N_K}\}$ that uniformly converges to $\hat{u}$. Now we show that $\hat{u}\in\mathscr{U}$. Since $\{u_{N_K}\}$ is first increasing then decreasing concave function, for any $-\theta_1\leq a_1\leq a_2\leq 0$, we have

$$\hat{u}(a_1)=\lim_{K\to\infty} u_{N_K}(a_1)\leq \lim_{K\to\infty} u_{N_K}(a_2)=\hat{u}(a_2).$$

Therefore, $\hat{u}$ is increasing at $[-\theta_1, 0]$. Similarly, $\hat{u}$ is decreasing at $[0, \theta_2]$. For $\lambda > 0$ and $-\theta_1 \leq a_1 \leq a_2 \leq \theta_2$,

$$\begin{aligned}\hat{u}(\lambda a_1 + (1-\lambda)a_2) &= \lim_{K\to\infty} u_{N_K}(\lambda a_1 + (1-\lambda)a_2) \\ &\geq \lim_{K\to\infty} \lambda u_{N_K}(a_1) + (1-\lambda)u_{N_K}(a_2) \\ &= \lambda \hat{u}(a_1) + (1-\lambda)\hat{u}(a_2).\end{aligned}$$

Hence, $\hat{u}$ is first increasing then decreasing concave function. We then consider the bound constraints. Since $\underline{u}_{N_K}$ and $\bar{u}_{N_K}$ uniformly converge to $\underline{u}$ and $\bar{u}$, respectively; we have

$$\hat{u} = \lim_{K\to\infty} u_{N_K} \geq \lim_{K\to\infty} \underline{u}_{N_K} = \underline{u}, \quad \hat{u} = \lim_{K\to\infty} u_{N_K} \leq \lim_{K\to\infty} \bar{u}_{N_K} = \bar{u}.$$

We now claim the auxiliary constraint. Let

$$h(u_{N_K}) = \sum_{k=0}^{N_K-1} \left(u_0(a_k) - u_{N_K}(a_k)\right)^2 (a_{k+1} - a_k) - b,$$

and

$$f(u_{N_K}) = \int_{-\theta_1}^{\theta_2} \left(u_0(a) - u_{N_k}(a)\right)^2 da - b, \tag{18}$$

hence

$$\lim_{K\to\infty} h(u_{N_K}) = \lim_{K\to\infty} f(u_{N_K}).$$

Given the uniform convergence of $\{u_{N_K}\}$ to $\hat{u}$, for any $\delta > 0$, there exists $\hat{K}$ such that for all $K \geq \hat{K}$:

$$|u_{N_K}(a) - \hat{u}(a)| \leq \delta.$$

Thus we have

$$\begin{aligned}\left|f(u_{N_K}) - f(\hat{u})\right| &= \Big|\int_{-\theta_1}^{\theta_2} \left(u_0(a) - u_{N_k}(a)\right)^2 - (u_0(a) - \hat{u}(a))^2 da\Big| \\ &= \left|\int_{-\theta_1}^{\theta_2} \left(\hat{u}(a) - u_{N_k}(a)\right)\left(2u_0(a) - \hat{u}(a) - u_{N_k}(a)\right) da\right| \\ &\leq \delta \int_{-\theta}^{\theta_2} \left|2u_0(a) - \hat{u}(a) - u_{N_k}(a)\right| da\end{aligned}$$

Since $2u_0(a) - \hat{u}(a) - u_{N_k}(a)$ is bounded, $f(\hat{u}) = \lim_{k\to\infty} f(u_{N_k})$. It follows that,

$$f(\hat{u}) = \lim_{k\to\infty} f(u_{N_k}) = \lim_{k\to\infty} h_{N_k}(u_{N_k}) \leq 0.$$

Therefore, we have $\hat{u} \in \mathscr{U}$. $\square$

LEMMA 6. *For any $u \in \mathscr{U}$, there exists a sequence $u_N \in \mathscr{U}_N$ such that $u = \lim_{N\to\infty} u_N$.*

Proof. For any $u \in \mathscr{U}$, since $(u_0(a) - u(a))^2$ is uniformly continuous on $[-\theta_1, \theta_2]$, $h(u)$ converges uniformly to $f(u)$. Hence, for $\delta > 0$ and $\delta < \tau/2$, there exists $\hat{N}$ such that for all $N \geq \hat{N}$:

$$|h(u) - f(u)| \leq \delta,$$

then we have $h(u) \leq f(u) + \delta \leq \delta$, which gives us that

$$\sum_{k=0}^{N-1} (u_0(a_k) - u(a_k))^2 (a_{k+1} - a_k) \leq b + \delta. \tag{19}$$

For any $\lambda \in [0,1]$, by constraint (19), it follows that

$$(1-\lambda)^2 \sum_{k=0}^{N-1} (u_0(a_k) - u(a_k))^2 (a_{k+1} - a_k) \leq (1-\lambda)^2 (b+\delta) \leq (1-\lambda)(b+\delta). \tag{20}$$

Similarly, we have

$$0 \leq \lambda(b + \delta - \tau), \tag{21}$$

for all $\tau \in \mathbb{R}^+$ and $\tau \leq b$. From constraints (20) and (21), we can obtain

$$\sum_{k=0}^{N-1} [u_0(a_k) - ((1-\lambda)u(a_k) + \lambda u_0(a_k))]^2 (a_{k+1} - a_k) \leq b + \delta - \lambda\tau$$

We let $v_\lambda = (1-\lambda)u + \lambda u_0$, thus,

$$\sum_{k=0}^{N} (u_0(a_k) - v_\lambda(a_k))^2 \leq b, \quad \forall \lambda \in [\delta/\tau, 1].$$

Hence, we have for each $\lambda \in [\delta/\tau, 1]$, $v_\lambda \in \mathcal{U}_N$ for all $N \geq \hat{N}$.

To construct the sequence u_N, we define a positive sequence $\{\delta_i\}$ such that $\delta_i \to 0$ as $i \to \infty$. Let $v_i = (1 - \frac{\delta_i}{\tau})u + \frac{\delta_i}{\tau}u_0$. Based on the above discussion, for each δ_i, there exists a positive integer number N_i such that $v_i \in \mathscr{U}_N$ for $N \geq N_i$, and $v_i \to u$ as $i \to \infty$. Let $u_N = v_i$ for $N_i \leq N < N_{i+1}$. Therefore, we have $u_N \in \mathscr{U}_N$ for all $N \geq N_1$ and $u_N \to u$ as $N \to \infty$. □

We state Theorem 2.3 in Alvarez-Mena and Hernández-Lerma (2005) in the following lemma for completeness, which gives some convergence conditions of $\pi_N(\boldsymbol{x})$.

LEMMA 7. *For $\boldsymbol{x} \in \mathcal{X}$, let $\{u_N\}$ be a sequence of $\{\pi_N(\boldsymbol{x})\}$. Suppose (i) a subsequence $\{u_{N_k}\}$ of $\{u_N\}$ converges to $u \in \mathscr{U}$; (ii) $\lim_{k\to\infty} \inf \pi_{N_k}(\boldsymbol{x}) \geq \pi(\boldsymbol{x})$; (iii) for any $u \in \mathscr{U}$, there exists a sequence $u_N \in \mathscr{U}_N$ such that $u = \lim_{N\to\infty} u_N$ and $\mathbb{E}[u(f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))] = \lim_{N\to\infty} \mathbb{E}[u_N(f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))]$. Then u is optimal for $\{\pi(\boldsymbol{x})\}$. Furthermore, $\pi_{N_k}(\boldsymbol{x})$ converges to $\{\pi(\boldsymbol{x})\}$.*

Based on Lemma 5 and 7, we shows that $\pi_N(\boldsymbol{x})$ converges to $\pi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}$ in Lemma 8 .

LEMMA 8. *$\pi_N(\boldsymbol{x}) \to \pi(\boldsymbol{x})$ as $N \to \infty$ for all $\boldsymbol{x} \in \mathcal{X}$.*

Proof. We let sequence $\{u_N\}$ be the optimal solution of $\pi_N(\boldsymbol{x})$ for given $\boldsymbol{x} \in \mathcal{X}$. By Lemma 5 we know that there exists a subsequence $\{u_{N_k}\}$ of $\{u_N\}$ converges to $u \in \mathscr{U}$. Since the limit of sequence $\{u_{N_k}\}$ in $\mathscr{U}$, hence, $\lim\limits_{k\to\infty} \inf \pi_{N_k}(\boldsymbol{x}) \geq \pi(\boldsymbol{x})$. Moreover, Lemma 6 shows that for any $u \in \mathscr{U}$, there exists a sequence $u_N \in \mathscr{U}_N$ such that $u = \lim\limits_{N\to\infty} u_N$. Since $\mathbb{E}[u_N(f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))] = \sum\limits_{\omega\in\Omega} p_\omega u_N(f(\boldsymbol{x}, \boldsymbol{\xi}^\omega))$ and $\mathbb{E}[u(f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))] = \sum\limits_{\omega\in\Omega} p_\omega u(f(\boldsymbol{x}, \boldsymbol{\xi}^\omega))$, we have $\mathbb{E}[u(f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))] = \lim\limits_{N\to\infty} \mathbb{E}[u_N(f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))]$ (Lytle 2015). Therefore, $\pi_{N_k}(\boldsymbol{x}) \to \pi(\boldsymbol{x})$ as $K \to \infty$ by Lemma 7. To prove $\pi_N(\boldsymbol{x}) \to \pi(\boldsymbol{x})$, let $\{\pi_m(\boldsymbol{x})\}$ be a subsequence of $\pi_N(\boldsymbol{x})$. By Lemma 5, there exists a subsequence $\{u_{m_i}\}$ of $\{u_m\}$ such that u_{m_i} converges to u, which with Lemma 7 implies that $\pi_{m_i}(\boldsymbol{x})$ converges to $\pi(\boldsymbol{x})$. Since $\{\pi_m(\boldsymbol{x})\}$ is an arbitrary subsequence of $\pi_N(\boldsymbol{x})$, thus, $\pi_N(\boldsymbol{x}) \to \pi(\boldsymbol{x})$ as $N \to \infty$ (Buck 1943). □

The following Lemma gives some convergence conditions of problem (13).

LEMMA 9. *(Lemma 9 in Hu and Mehrotra (2015)) Let y_N and Z_N be the optimal objective value and the set of optimal solutions of problem* (13) *and y^* and Z^* be those of (RCUMP). Denote the deviation of sets Z_N and Z^* as $D(Z_N, Z^*) := \max_{x_1\in Z_N} \min_{x_2\in Z^*} ||x_1 - x_2||$. Suppose (i) X is a non-empty compact set; (ii) the function $\pi(\cdot)$ is continuous on X; and (iii) $\pi_N(\cdot)$ uniformly converges to $\pi(\cdot)$ on X as $N \to \infty$. Then, $y_N \to y^*$ and $D(Z_N, Z^*) \to 0$ as $N \to \infty$.*

Theorem 2 shows that the optimal solutions of problem (13) converge to the true optimal solution of (RCUMP) in the limit.

THEOREM 2. *Let y_N and Z_N be the optimal objective value and the set of solutions of problem* (13)*, and y^* and Z^* be the optimal objective value and the set of solutions of (RCUMP). Then $y_N \to y^*$ and $D(Z_N, Z^*) := \underset{b_1\in Z_N}{\text{maximize}}\ \underset{b_2\in Z^*}{\text{minimize}} ||b_1 - b_2|| \to 0$ as $N \to \infty$.*

Proof. By Lemma 2, we have that $\mathcal{X}$ is compact and $\pi_N(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ are continuous, then $\{\pi_N\}$ uniformly converges to π (see Hu and Mehrotra 2015). Since $\mathscr{X}$ and $\mathscr{T}$ are non-empty compact sets, and the function $\pi(\boldsymbol{x})$ is continuous on $\mathscr{X}$ and $\mathscr{T}$, then $y_N \to y^*$ and $D(Z_N, Z^*) := \underset{b_1\in Z_N}{\text{maximize}}\ \underset{b_2\in Z^*}{\text{minimize}} ||b_1 - b_2|| \to 0$ as $N \to \infty$, based on Lemma 9. □

4. A Row Generation Solution Scheme

Note that solving the reformulation of (RCUMP) by an off-the-shelf commercial solver (e.g., CPLEX, GUROBI) directly might be time-consuming, which will be further confirmed by our numerical study in Section 5. Instead, in this section we propose a row generation algorithm as a solution method for our (RCUMP).

Based on Lemma 1, (RCUMP) can be further rewritten as the following max-min problem:

$$\underset{\boldsymbol{x}\in\mathcal{X}, \rho\in[0,1]}{\text{maximize}}\ \rho \tag{22a}$$

$$\text{subject to } \rho \le Z(\boldsymbol{x}), \tag{22b}$$

where $Z(\boldsymbol{x})$ is given as follows:

$$\begin{aligned} Z(\boldsymbol{x}) = \underset{\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\beta}}{\text{minimize}} & \sum_{\omega\in\Omega} p_\omega \left(f(\boldsymbol{x},\boldsymbol{\xi}^\omega)\alpha_j^\omega + \beta_j^\omega \right) \\ \text{subject to } & (12\text{b}) - (12\text{g}). \end{aligned}$$

We then define the master problem as follows:

$$\begin{aligned} (\text{MP}) \qquad \underset{\boldsymbol{x},\boldsymbol{z}\in\{0,1\}^{|\Omega|},\rho\in[0,1]}{\text{maximize}} & \ \rho \\ \text{subject to } & (14\text{b}), (14\text{g}), (14\text{h}), \end{aligned}$$

and the subproblem as follows:

$$\begin{aligned} (\text{SP}) \qquad \underset{\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\beta}}{\text{minimize}} & \sum_{\omega\in\Omega} p_\omega \left(f(\boldsymbol{x},\boldsymbol{\xi}^\omega)\alpha_j^\omega + \beta_j^\omega \right) \\ \text{subject to } & (12\text{b}) - (12\text{g}). \end{aligned}$$

An outline of the row generation algorithm is given in Algorithm 1. The row generation solves (MP) and (SP) iteratively until a stopping criteria is met. We let UB and LB denote the upper and lower bound, respectively. We initialize the number of iterations ℓ to 0, UB to positive infinity, and LB to negative infinity. In each iteration, we update $\ell := \ell + 1$ and solve the linear relaxation of (MP) to obtain an optimal solution $(\boldsymbol{x}^\ell, \boldsymbol{z}^\ell, \rho^\ell)$ and optimal objective value $uobj^\ell$. If the objective value $uobj^\ell$ is larger than the current lower bound: when $(\boldsymbol{x}_2^\ell, \boldsymbol{z}^\ell)$ is integer, we solve (SP) with $\boldsymbol{x}$ fixed to be $\boldsymbol{x}^\ell$ to attain an optimal solution $(\boldsymbol{\theta}^\ell, \boldsymbol{\alpha}^\ell, \boldsymbol{\beta}^\ell)$ and optimal objective value $lobj^\ell$. If $uobj^\ell$ is larger than $lobj^\ell$, we add the cut $\rho \le \sum_{\omega\in\Omega} p_\omega(f(\boldsymbol{x},\boldsymbol{\xi}^\omega)\alpha^{\omega\ell} + \beta^{\omega\ell})$ to the (MP), else update LB if necessary. When $(\boldsymbol{x}_2^\ell, \boldsymbol{z}^\ell)$ is fractional, we update UB if necessary. We terminate the algorithm when the stopping criteria is satisfied, and return the optimal value UB and the optimal solution $(\boldsymbol{x}^*, \boldsymbol{z}^*, \rho^*)$. The following theorem shows that Algorithm 1 can find a solution of (RCUMP) in a finite number of iterations under certain conditions.

Theorem 3. *Let $\boldsymbol{v} := (\boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{V} := \{\boldsymbol{v} \mid (12\text{b}) - (12\text{g})\}$. We also let $\boldsymbol{t} := (\boldsymbol{x}, \rho)$, $\mathcal{T} := \mathcal{X} \times [0,1]$, and $g(\boldsymbol{t}, \boldsymbol{v}) := \rho - \sum_{\omega\in\Omega} p_\omega(f(\boldsymbol{x},\boldsymbol{\xi}^\omega)\alpha^\omega + \beta^\omega)$. We assume that $\mathcal{T} \times \mathcal{V}$ is compact and $g(\boldsymbol{t}, \boldsymbol{v})$ is continuous on $\mathcal{T} \times \mathcal{V}$. If there exists an oracle that solves (MP) and (SP) to optimality at each iteration, then Algorithm 1 terminates within finitely many iterations. If $UB < +\infty$, then Algorithm 1 obtains a solution $(\boldsymbol{x}^*, \boldsymbol{z}^*, \rho^*)$ of a desired accuracy at termination.*

Proof. Based on the definitions, we can rewrite problem (22) as the following problem:

$$\underset{\boldsymbol{t}\in\mathcal{T}}{\text{maximize}} \ \rho \tag{23a}$$

$$\text{subject to } g(\boldsymbol{t},\boldsymbol{v}) \le 0, \qquad \forall \boldsymbol{v} \in \mathcal{V}. \tag{23b}$$

Since $\mathcal{T}\times\mathcal{V}$ is compact and $g(\boldsymbol{t},\boldsymbol{v})$ is continuous on $\mathcal{T}\times\mathcal{V}$, then $g(\boldsymbol{t},\boldsymbol{v})$ is uniformly continuous on $\mathcal{T}\times\mathcal{V}$. Thus, for the τ, there exists a $\delta>0$ such that

$$|g(\boldsymbol{t},\boldsymbol{v}) - g(\boldsymbol{t}',\boldsymbol{v}')| \le \tau/2, \quad \text{if } ||\boldsymbol{t}-\boldsymbol{t}'|| + ||\boldsymbol{v}-\boldsymbol{v}'|| \le \delta. \tag{24}$$

If Algorithm 1 generates infinite $\boldsymbol{t}$ and $\boldsymbol{v}$ values without termination. For $\boldsymbol{t}\in\mathcal{T}$, let $\mathcal{B}(\boldsymbol{v},\delta)$ be a closed ball of center $\boldsymbol{v}$ and radius δ, then we have $\boldsymbol{v}^{\ell+1} \notin \cup_{i=1}^{\ell}\mathcal{B}(\boldsymbol{v}^i,\delta)$. Otherwise, if there exists a $\boldsymbol{v}^{\ell+1}\in\mathcal{B}(\boldsymbol{v}^i,\delta)$ for some $i\le\ell$, based on condition (24), we have $g(\boldsymbol{t},\boldsymbol{v}^{\ell+1}) \le g(\boldsymbol{t},\boldsymbol{v}^i)+\tau/2\le\tau$, which implies that the termination criteria is satisfied. Therefore, $\boldsymbol{v}^{\ell+1} \notin \cup_{i=1}^{\ell}\mathcal{B}(\boldsymbol{v}^i,\delta)$ and $\mathcal{B}(\boldsymbol{v}^i,\delta)\cap\mathcal{B}(\boldsymbol{v}^j,\delta)=\emptyset$ if $\boldsymbol{v}^i\neq\boldsymbol{v}^j$, which indicates that $\cup_{i=1}^{+\infty} vol(\mathcal{B}(\boldsymbol{v}^i,\delta)) \le vol(\cup_{\boldsymbol{v}\in\mathcal{V}}\mathcal{B}(\boldsymbol{v},\delta))$, this is a contradiction. Therefore, Algorithm 1 terminates within finitely many iterations.

If Algorithm 1 terminates at iteration ℓ, the stopping criteria are met. Hence, Algorithm 1 will finally return an solution $(\boldsymbol{x}^*,\boldsymbol{z}^*,\rho^*)$ of a desired accuracy at termination. □

5. Numerical Study

In this section, we numerically evaluate the performance of our robust concave utility maximization model and the proposed row generation solution scheme with the help of the bin packing problem in Section 5.1 and the multi-item newsvendor problem in Section 5.2.

5.1. Bin Packing Robust Expected Utility Problem

In this section, we use the real data from surgery planning problem to show the performance of the algorithm proposed in Section 4 and the general structure of the decision made from (BP_RCUMP) that is presented in Section 1.2. In the context of surgery planning, the bins are ORs, items are surgeries, and capacity denotes the OR time limit. We describe implementation details in Section 5.1.1 and finally present the computational results in Section 5.1.2.

5.1.1. Implementation Details In this section, we used historical surgery duration data from a large public hospital in Beijing, China from January 2015 to October 2015, in which 5,721 historical observations of surgery duration are employed to generate our samples (see Wang et al. (2021a) for a more detailed description). More specifically, we used log-normal distribution with the mean and the standard deviation of the surgery duration to generate surgery duration samples and rounded the samples to the nearest 15 minutes. Equal probabilities are used as in the SAA method. The daily time limit t_j is set to 10 hours for $j\in\mathcal{J}$. Nine major surgery types are performed in a day and the number of surgeries and the percentage for each surgery type are used to calculate the number of surgeries for each surgery type. The bound support $\Theta=[-10,14]$, and the power utility functions (7) and (8) are used as the bounds and reference of the utility functions, respectively. We

Algorithm 1: Row Generation Algorithm

Initialize The number of iteration $\ell = 0$, UB $= +\infty$, LB $= -\infty$, a tolerance $\epsilon' > 0$, and a small number $\tau > 0$.
Initialize $\mathcal{N} = \{o\}$, where o is the root node with the LP relaxation of (MP).
while *($\mathcal{N}$ is nonempty and $UB - LB > \epsilon'$)* **do**
    Select a node $o \in \mathcal{N}$, $\mathcal{N} \leftarrow \mathcal{N}/\{o\}$.
    Solve the linear relaxation of (MP) at the node o to obtain the optimal solution $(\boldsymbol{x}^\ell, \boldsymbol{z}^\ell, \rho^\ell)$ and objective value $uobj^\ell$. $\ell = \ell + 1$.
    if $uobj^\ell > LB$ **then**
        if $(\boldsymbol{x}_2^\ell, \boldsymbol{z}^\ell)$ *is integer* **then**
            Fix $\boldsymbol{x}$ to be $\boldsymbol{x}^\ell$, and solve (SP) and obtain an optimal solution $(\boldsymbol{\theta}^\ell, \boldsymbol{\alpha}^\ell, \boldsymbol{\beta}^\ell)$ and objective value $lobj^\ell$.
            if $uobj^\ell - lobj^\ell > \tau$ **then**
                Add the following cut

$$\rho \leq \sum_{\omega \in \Omega} p_\omega (f(\boldsymbol{x}, \boldsymbol{\xi}^\omega)\alpha^{\omega\ell} + \beta^{\omega\ell}) \tag{25}$$

                to (MP).
            end
            else
                Update LB $:= \max\{\text{LB}, lobj^\ell\}$.
            end
        end
        else
            Update UB $:= \min\{\text{UB}, uobj^\ell\}$.
            Branch, resulting in nodes o^* and o^{**}, $\mathcal{N} \leftarrow \mathcal{N} \cup \{o^*, o^{**}\}$.
        end
    end
end
return LB and its corresponding optimal solution $(\boldsymbol{x}^*, \boldsymbol{z}^*, \rho^*)$.

consider the number of partitions $N \in \{2, 10, 20\}$ and $\varepsilon \in \{0.05, 0.1, 0.2\}$. We set r_0 in the reference function to 0.55. We consider two different pairs of parameters for the ambiguity set, namely, $(r_1, r_2) \in \{(0.41, 0.68), (0.25, 0.75)\}$. For each sample size, five instances were generated. Therefore, all the performance is reported over five instances on average.

All experiments are conducted on the cedar cluster of Compute Canada with a single CPU core and 32G memory. We implement the algorithm and models in the C programming language using IBM CPLEX solver, version 12.10 callable libraries. We set the runtime limit as two hours and the relative optimality gap tolerance as 0.1%. For instances that could not be solved to optimality, we give the average relative optimality gap, where the gap is calculated as $\frac{(\text{UB}-\text{LB})}{\text{UB}} * 100$, and UB and LB are the upper and lower bound, respectively. We report the average CPU solution time (in seconds) for the instances that are solved to optimality within the runtime limit.

5.1.2. Computational Results In this section, we present the computational results for (BP_RCUMP). We first provide the computational performance of the row generation algorithm (i.e., Algorithm 1) for solving (BP_RCUMP), then present the optimal solutions and objective value, and the out-of-sample performance.

Performance of row generation algorithm. We assign twelve surgeries ($|\mathcal{I}| = 12$) to four ORs ($|\mathcal{J}| = 4$) a day. The number of scenarios is set to be 30. The performance of the following two variants are compared:

- CPX: refers to using CPLEX to directly solve MISOCP reformulation of (BP_RCUMP).
- RG: refers to using the row generation algorithm (Algorithm 1) to solve (BP_RCUMP).

Table 1 reports the average CPU solution time for solving (BP_RCUMP) and the subproblem (SP), the average number of cuts (25), and the number of instances that are solved to optimality over the five generated instances.

We observe from Table 1 that, using the row generation algorithm to solve (BP_RCUMP) significantly outperforms CPX in terms of the average solution time for all the problems. Specifically, compared with CPX, using the row generation algorithm saves an average of 80% time when CPX is able to solve the problems to optimality. We also observe that for $(r_1, r_2) = (0.41, 0.68)$, CPX can only solve 15 out of the 45 instances to optimality, and solve 13 out of the 45 instances when $(r_1, r_2) = (0.25, 0.75)$, whereas, using the row generation algorithm solved all of these instances to optimality. We see from Table 1 that the average solution time for CPX increases more dramatically than the time increase in the row generation algorithm as the number of partitions N increases. In particular, the number of solved instances decreases significantly for CPX in most cases when N is increased from 2 to 10, while the increase in average solution time for RG is mild. The required solution times did not change significantly when we considered different ambiguity sets. From Table 1 we also observe that the solution time for CPX changes significantly across different instances of the same size when $\varepsilon = 0.1$ and $\varepsilon = 0.2$. For example, when $\varepsilon = 0.1$ and $N = 2$, the solution time for one of the instances is only 21 seconds however the remaining instances cannot be solved to optimality within the runtime limit. This suggests that compared with CPX, the row generation algorithm has a more stable computational performance.

Out-of-sample performance. We now discuss the solutions in the context of data from the operating room planning problem and the out-of-sample performance of the solution generated from (BP_RCUMP). For this purpose, we generated 150,000 scenarios from log-normal distribution

Table 1 **The average CPU (in seconds) solution time for solving (BP_RCUMP) (AvT) and the subproblem (SP) (AvT-Z), the average number of cuts (# of cuts), and the number of solved instances from the five instances (solved) are reported.**

(r_1, r_2)	ε	N	CPX		RG			
			AvT	solved	AvT	AvT-Z	# of cuts	solved
(0.41,0.68)	0.05	2	549	5/5	19	0.02	18	5/5
		10	4,672	5/5	2,329	255.78	10,851	5/5
		20	1,443[7.9]	1/5	2,921	709.40	12,052	5/5
	0.1	2	21[0.5]	1/5	30	0.36	466	5/5
		10	[2.0]	0/5	280	68.33	2,620	5/5
		20	[3.2]	0/5	311	177.00	2,735	5/5
	0.2	2	14[8.7]	1/5	80	0.60	937	5/5
		10	[5.7]	0/5	567	157.22	6,210	5/5
		20	29[5.6]	2/5	832	393.84	6,422	5/5
(0.25,0.75)	0.05	2	614	5/5	19	0.02	18	5/5
		10	3,210[10.7]	3/5	1,027	169.84	7,468	5/5
		20	1,704[10.4]	1/5	1,909	548.80	8,970	5/5
	0.1	2	21[0.4]	1/5	25	0.33	466	5/5
		10	[2.4]	0/5	160	50.17	2,249	5/5
		20	17[3.8]	1/5	237	125.29	2,270	5/5
	0.2	2	14[8.8]	1/5	86	0.73	937	5/5
		10	[6.7]	0/5	529	137.01	5,179	5/5
		20	46[7.1]	1/5	800	347.61	5,279	5/5

"[·]" in column of AvT means the average relative optimality gap (%) for instances that cannot be solved to optimality within the time limit.

and used the solutions obtained from (BP_RCUMP). We also share our experience in solving larger size problems generated from taking $|\mathcal{I}| = 15$, $|\mathcal{J}| = 5$ and $|\Omega| = 100$. Since both methods cannot solve any instance to optimality within the runtime limit which leave an average of about 10% optimality gap, we illustrate the out-of-sample performance of the best current solutions generated from the model with the larger problem size. Table 2 presents the average objective values and the number of opened ORs, the average over-utilization and under-utilization for (BP_RCUMP). This objective value is the value of the utility function at the optimal decision.

We see from Table 2 that as ε increases, the optimal number of opening ORs decreases at first then remains unchanged. In this example, we observe that the average objective value for the

Table 2 **The average objective value (Obj), the average number of opening ORs (# of ORs), the average over-utilization (Over)(in hours) and under-utilization (Under)(in hours) for (BP_RCUMP) are reported.**

$(\|\mathcal{I}\|, \mathcal{J}, \|\Omega\|)$			(12,4,30)				(15,5,100)			
(r_1, r_2)	ε	N	Obj	# of ORs	Over	Under	Obj	# of ORs	Over	Under
(0.68,0.41)	0.05	2	2.90	3.8	0.06	3.93	3.57	5	0.05	4.02
		10	3.29	3.8	0.05	3.92	4.08	5	0.04	4.02
		20	3.30	3.8	0.05	3.92	4.08	5	0.04	4.01
	0.1	2	3.45	3	0.13	2.00	4.24	4	0.14	2.11
		10	3.65	3	0.13	2.00	4.52	4	0.14	2.11
		20	3.65	3	0.13	2.00	4.52	4	0.14	2.11
	0.2	2	3.45	3	0.13	2.00	4.24	4	0.14	2.12
		10	3.65	3	0.13	2.00	4.52	4	0.15	2.12
		20	3.66	3	0.13	2.00	4.52	4	0.14	2.12
(0.75,0.25)	0.05	2	2.90	3.8	0.06	3.93	3.57	5	0.05	4.02
		10	3.13	3.8	0.05	3.92	3.87	5	0.05	4.02
		20	3.13	3.8	0.05	3.92	3.87	5	0.04	4.02
	0.1	2	3.45	3	0.13	2.00	4.24	4	0.14	2.11
		10	3.57	3	0.13	2.00	4.40	4	0.14	2.11
		20	3.57	3	0.13	2.00	4.41	4	0.14	2.11
	0.2	2	3.45	3	0.13	2.00	4.24	4	0.14	2.11
		10	3.57	3	0.13	2.00	4.41	4	0.14	2.11
		20	3.57	3	0.13	2.00	4.41	4	0.14	2.11

instances with $N = 10$ is better than the ones with $N = 2$, and the improvement of the average objective value is not significant when we increase the number of partitions to $N = 20$, given that the instances with $N = 10$ have the same number of opened ORs as the instances with other N values. Moreover, when $(r_1, r_2) = (0.41, 0.68)$, the average objective values are marginally larger than the ones with $(r_1, r_2) = (0.25, 0.75)$. This can be explained by the fact that the robust models are more conservative when the size of the ambiguity set increases. The results in Table 2 also show that the out-of-sample performance with the larger number of partitions $N = 20$ or the larger ambiguity set $(r_1, r_2) = (0.25, 0.75)$ in terms of over-utilization and under-utilization measures in the simulation does not improve for most of the instances, even though as observed from Table 1, the computational cost increases rapidly with the number of partitions N. Therefore, when $N = 10$, the optimal solutions obtained from (BP_RCUMP) achieve a desirable out-of-sample performance, and this performance is not improved by increasing the number of partitions. Moreover, the average under-utilization is significantly larger than the over-utilization, and decreases when ε varies from 0.05 to 0.1. When $\varepsilon = 0.05$, it has about four hours under-utilization, two hours under-utilization for $\varepsilon = 1$, whereas the over-utilization is only about 0.04 and 0.14 hour, respectively. This can be explained by the fact that in order to obtain a desirable over-utilization probability, one needs to open more ORs and as a result, it will have a larger under-utilization. And when ε increases, it allows a larger probability that violated the overtime chance constraints, thus under-utilization decreases and over-utilization increases with an increase in ε value. When ε varies from 0.1 to 0.2, it still opens 3 ORs on average, therefore the under-utilization and over-utilization are unchanged.

5.2. Multi-Item Newsvendor Robust Expected Utility Problem

In this section, we present computational results of the (MN_RCUMP) problem that is presented in Section 1.2. We follow a similar parameters setting for the multi-item newsvendor problem in a recent work by Chen et al. (2020). For each scenario ω and item i, we randomly generate ξ_i^ω from a uniform distribution on $[0, \bar{u}_i]$, where $\bar{u}_i$ is randomly generated from a uniform distribution on [0,100]. The inventory capacity t_i is randomly generated from a uniform distribution on $[10, 20]$. We let $d = 50|\mathcal{I}|$ and $c_i = 1$, for $i \in \mathcal{I}$. The number of items $|\mathcal{I}| \in \{100, 150\}$ and the number of scenarios $|\Omega| \in \{1000, 2000\}$. All experiments for (MN_RCUMP) are conducted on a 64-bit computer using Windows operating system with Intel(R) 3.10 GHz processor and 128 GB RAM with a 64-bit computer using Windows operating system. For other implementation details, we use the same settings as in Section 5.1.1.

5.2.1. Computational Results We first show the performance of the row generation algorithm (Algorithm 1) for solving (MN_RCUMP), then we discuss the out-of-sample performance for (MN_RCUMP) in this section.

Performance of row generation algorithm. We present the performance of the methods described in Section 5.1.2. Table 3 reports the average CPU solution time for solving (MN_RCUMP) and the subproblem (SP), the average number of cuts (25), and the number of instances that are solved to optimality over the five generated instances.

The results from Table 3 further demonstrate the efficiency of the proposed row generation algorithm. More specifically, in comparison to CPX, the average solution time is decreased by more than 80% by using the row generation algorithm. When $|\mathcal{I}| = 150$, $|\Omega| = 2000$, and $N = 20$, CPX cannot solve any instance to optimality, whereas, using the row generation algorithm could solve all the instances to optimality within the runtime limit. We also observe from Table 3 that as the number of partitions N increases, the average solution time increases significantly, while when ε varies from 0.05 to 0.2 or (r_1, r_2) varies from $(0.41, 0.68)$ to $(0.25, 0.75)$, the average solution time is not significantly different.

Table 3 The average CPU (in seconds) solution time for solving (MN_RCUMP) (AvT) and the subproblem (SP) (AvT-Z), the average number of cuts (# of cuts), and the number of solved instances over the five instances (solved) are reported.

$(\lvert\mathcal{I}\rvert, \lvert\Omega\rvert)$			(100,1000)						(150,2000)					
			CPX		RG				CPX		RG			
(r_1, r_2)	ε	N	AvT	solved	AvT	AvT-Z	# of cuts	solved	AvT	solved	AvT	AvT-Z	# of cuts	solved
(0.41,0.68)	0.05	2	27	5/5	7	1	2	5/5	117	5/5	47	3	2	5/5
		10	595	5/5	79	73	2	5/5	2,285	5/5	282	238	2	5/5
		20	3,077	5/5	192	186	2	5/5	–	0/5	626	582	2	5/5
	0.1	2	31	5/5	7	1	2	5/5	166	5/5	49	4	2	5/5
		10	593	5/5	79	73	2	5/5	2,351	5/5	286	240	2	5/5
		20	3,197	5/5	191	185	2	5/5	–	0/5	635	589	2	5/5
	0.2	2	40	5/5	10	2	3	5/5	226	5/5	86	6	2.8	5/5
		10	586	5/5	117	107	3	5/5	2,582	5/5	459	392	3	5/5
		20	3,084	5/5	274	265	3	5/5	–	0/5	995	941	3	5/5
(0.25,0.75)	0.05	2	27	5/5	7	1	2	5/5	119	5/5	47	3	2	5/5
		10	623	5/5	80	75	2	5/5	2,351	5/5	287	243	2	5/5
		20	3,116	5/5	190	185	2	5/5	–	0/5	659	614	2	5/5
	0.1	2	32	5/5	7	1	2	5/5	166	5/5	49	3	2	5/5
		10	641	5/5	81	75	2	5/5	2,594	5/5	283	238	2	5/5
		20	3,183	5/5	191	186	2	5/5	–	0/5	655	609	2	5/5
	0.2	2	41	5/5	11	2	3	5/5	230	5/5	85	6	2.8	5/5
		10	598	5/5	131	123	3	5/5	2,709	5/5	425	363	3	5/5
		20	3,084	5/5	291	284	3	5/5	–	0/5	1,008	944	3	5/5

"[–]" in column of AvT means that we cannot find any feasible solution for instances within the time limit.

Out-of-sample performance. We now discuss the out-of-sample performance of the solution generated from (MN_RCUMP). We generated 150,000 scenarios from the uniform distribution $(0, \bar{u})$. Table 4 presents the average objective values, the average over- and under-stocking for (MN_RCUMP). This objective value is the value of the utility function at the optimal decision.

Conclusions from the results in Table 4 are similar to those observed from Table 2 with some differences. For example, when $(r_1, r_2) = (0.41, 0.68)$, the average objective values are marginally larger than the ones with $(r_1, r_2) = (0.25, 0.75)$. Furthermore, the out-of-sample performance with the larger number of partitions $N = 20$, or the larger ambiguity set in terms of over-stocking and under-stocking measures does not improve for most of the instances, even though it has the costlier solutions with an increasing N. Besides, the over-stocking increases and the under-stocking decreases as ε increases, and the average under-stocking is significantly larger than the over-stocking for most of the instances. Different from Table 2, when $(\lvert\mathcal{I}\rvert, \lvert\Omega\rvert) = (150, 2000)$ and $\varepsilon = 0.2$, (MN_RCUMP) has a comparable number of under- and over-stocking. In this case, as ε increases,

Table 4 The average objective value (Obj), the average over-stocking (Over)(in hours) and under-stocking (Under)(in hours) for (MN_RCUMP) are reported.

$(\lvert\mathcal{I}\rvert, \lvert\Omega\rvert)$			(100,1000)			(150,2000)		
(r_1, r_2)	ε	N	Obj	Over	Under	Obj	Over	Under
(0.41,0.68)	0.05	2	90.38	3.726	15.721	135.36	4.319	9.732
		10	93.95	3.725	15.724	140.79	4.319	9.732
		20	94.02	3.725	15.724	140.89	4.319	9.732
	0.1	2	90.94	4.757	14.726	136.22	5.451	8.773
		10	94.33	4.756	14.728	141.36	5.451	8.773
		20	94.39	4.756	14.728	141.46	5.451	8.773
	0.2	2	91.69	6.993	13.121	137.34	7.872	7.298
		10	94.83	6.989	13.122	142.12	7.870	7.298
		20	94.90	6.987	13.122	142.22	7.870	7.298
(0.25,0.75)	0.05	2	90.38	3.726	15.721	135.36	4.319	9.732
		10	92.50	3.725	15.724	138.58	4.319	9.732
		20	92.55	3.725	15.724	138.66	4.319	9.730
	0.1	2	90.94	4.757	14.726	136.22	5.451	8.773
		10	92.96	4.757	14.728	139.28	5.451	8.773
		20	93.01	4.756	14.728	139.35	5.451	8.773
	0.2	2	91.69	6.993	13.121	137.34	7.872	7.298
		10	93.56	6.991	13.121	140.19	7.872	7.298
		20	93.61	6.987	13.122	140.27	7.870	7.298

the over-stocking also increases to reach the value of the under-stocking, while for (BP_RCUMP), the under-utilization is still larger than over-utilization when $\varepsilon = 0.2$.

6. Concluding Remarks

In this work, we study a general robust expected utility maximization problem with a chance-constraint over a set of a concave utility function that lies in an ambiguity set, which maximizes the worst-case expected utility of random outcome over a set of concave functions, while satisfying a constraint with a given probability. Methodologically speaking, we first apply a discrete approximation approach to formulate the ambiguity set $\mathcal{U}$ and reformulate (RCUMP) as a mixed-integer program with the help of the approximated ambiguity set and big-M techniques. We then conduct a detailed convergence analysis to show that the discrete approximation asymptotically converges to the true counterpart under some mild assumptions. On the computational side, we propose a row generation-based solution scheme to solve our chance-constrained robust expected utility model efficiently, and show that our proposed algorithm can be converged in a finite number of iterations. Finally, from a practical application viewpoint, we perform an extensive numerical study for the bin packing problem and the multi-item newsvendor problem to analyze the general structure of the decisions from the decision-making framework and show the benefits of the techniques developed in this paper for computational improvement. The numerical results show that the row generation algorithm can significantly reduce the computational time for a certain problem size when compared with CPLEX solver, and the solutions that are obtained from (RCUMP) achieve a desirable out-of-sample performance.

To the best of our knowledge, this is the first attempt to combine the concept of robust decision-making with the utility-dependent decisions and chance constraint optimization as a complementary synergistic mechanism for decision modeling under risk and uncertainty, especially when the information of the utility function is incomplete. For future research, we suspect that the modeling framework and resolution methods that are presented in this paper should also benefit several other practical applications of interest, e.g., those in facility location (e.g., Luo and Mehrotra 2019), and cloud computing (e.g., Cohen et al. 2019, Martinovic et al. 2021), etc. It is also very interesting to further explore the current framework under a DRO context when the distribution of the pay-off function $f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}})$ is ambiguous but resides in an ambiguity set (e.g., moment-based set and Wasserstein ambiguity set).

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