

An adaptive regularization algorithm for unconstrained optimization with inexact function and derivatives values

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Abstract

An adaptive regularization algorithm for unconstrained nonconvex optimization is proposed that is capable of handling inexact objective-function and derivative values, and also of providing approximate minimizer of arbitrary order. In comparison with a similar algorithm proposed in Cartis, Gould, Toint (2021), its distinguishing feature is that it is based on controlling the relative error between the model and objective values. A sharp evaluation complexity bound is derived for the new algorithm.

Keywords: nonconvex optimization, inexact functions and derivative values, evaluation complexity, adaptive regularization.

1 Introduction: motivation, context, definitions

We consider the unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \tag{1.1}$$

where f is a p -times continuously differentiable function from \mathbb{R}^n to \mathbb{R} . In practice it sometimes happens that the value of the objective function f and/or those of its derivatives may only be computed inexactly, for a variety of reasons. It might be that the derivatives are not available, and are estimated using, for example, finite differences. Or perhaps because the evaluation is subject to intentional noise. For example, the values in question might be computed by some kind of experimental process whose accuracy can be adjusted, with the understanding that more accurate values may be, sometimes substantially, more expensive in terms of computational effort. A related case is when objective-function or derivative values result from some (one hopes, convergent) iteration—obtaining more accuracy is possible by letting the iteration converge further, but again at the price of possibly significant additional computing. A third possibility, currently much in vogue in the context of machine learning, is when the values of the objective function and/or its derivatives are obtained by sampling—say from among the terms of a sum

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of functions involving an enormous number of them. Again, using a larger sample size results in probabilistically better accuracy, but at a cost. In this report, we are particularly interested in a fourth case of growing relevance in high-performance computing, which is the emerging field of “variable accuracy” or “multi-precision” optimization [1, 12, 2, 10], where the computation of the objective function and its derivative(s) are intentionally truncated to significantly fewer digits than would be required for an accurate calculation. Doing so allows the use of specialised processing units whose chip surface, and hence power consumption, is much less than is needed for standard double precision arithmetic [9].

It is therefore important to design and analyze algorithms which are tolerant to inexactness or noise. This subject is not new. The main ideas developed in this report have their origin in the paper [3], that considers the complexity of finding weak approximate minimizers using an algorithm similar, at least in spirit, to the algorithm discussed here. Our new presentation and its associated analysis merge the elaborate approximation techniques described in that paper with techniques of [7, Chapter 12] for computing (strong) approximate minimizers and has been explored at length in [7, Chapter 13]. It also avoids using a dynamic relative accuracy threshold present in [3]. First-order trust-region methods with inexact evaluations and explicit dynamic accuracy have been described in [4] and in Section 10.6 of [8]. The complexity analysis for the first-order case was also discussed in [10].

The purpose of the report in hand is to present an alternative to one of the algorithms described in [7, Chapter 13], namely AR_{qp}EDA, the adaptive regularisation algorithm with explicit dynamic accuracy, and to provide a sharp upper bound on its evaluation complexity. The variant we will consider here, which we call AR_{qp}EDA2, uses the same Explicit Dynamic Accuracy (EDA) framework as AR_{qp}EDA, but enforces its associated controls in a different way. Obviously, these sentences require some clarification as to what these terms convey.

By “explicit dynamic accuracy”, we mean that, during the optimization process, *the required values* (objective-function or derivatives) *can always be computed with an accuracy that is explicitly⁽¹⁾ specified, before the calculation, by the algorithm itself*. It is also understood in what follows that *the algorithm should require high accuracy only if necessary*, but nonetheless *guarantee final results to full requested accuracy*. In this situation, it is hoped that many function or derivative evaluations can be carried out with a fairly loose accuracy (we will refer to these as “inexact values”), thereby resulting in a significantly cheaper optimization process.

Different kinds of optimization method can be designed to work within this framework. We focus here an “adaptive regularization” algorithms(see [11, 13, 5] among many other contributions), where, at a given iteration, a regularized local Taylor model of the objective function is minimized to define the next trial iterate. The strength of the regularization is then adaptively updated to guarantee convergence to approximate minimizers. In addition, we will consider a variant of the adaptive regularization technique whose purpose is to find such *approximate minimizers of arbitrary*, but given, *order*. If an approximate minimizer of order $q \geq 1$ is sought, this requires the Taylor model of the objective function to have degree $p \geq q$. Such a model is of the form

$$T_{f,j}(x, s) \stackrel{\text{def}}{=} f(x) + \sum_{i=1}^j \frac{1}{i!} \nabla_x^i f(x)[s]^i \equiv T_{f,j}(x, 0) + \sum_{i=1}^j \frac{1}{i!} [\nabla_v^i T_{f,j}(x, v)]_{v=0}[s]^i \quad (1.2)$$

for perturbations s around x , where $\nabla_x^j f(x)$ is the j -th order tensor giving the the j -th derivative

⁽¹⁾See [7, Chap. 13] for the description and analysis of an *implicit* dynamic accuracy framework.

of f at x , and where the notation $\nabla_x^j f(x)[s]^j$ means that it is applied on j copies of s

Given a Taylor model (1.2), we also have to define what we mean by an ϵ -approximate minimizer of order q . Following [6] (and [7]), we say that x is an ϵ -approximate minimizer of order q whenever, for some $\delta \in (0, 1]^q$ and $\epsilon \in (0, 1]^q$,

$$\phi_{f,j}^{\delta_j}(x) \leq \epsilon_j \frac{\delta_j^j}{j!} \quad \text{for } j \in \{1, \dots, q\}, \quad (1.3)$$

where

$$\phi_{f,j}^{\delta_j}(x) = f(x) - \min_{d \in \mathbb{R}^n, \|d\| \leq \delta_j} T_{f,j}(x, d). \quad (1.4)$$

In other words, this is the case when the none of the Taylor approximations $T_{f,j}$ ($j \leq q$) at x can be decreased by more than a scaled multiple of ϵ_j (the right-hand side of (1.3)) in a neighbourhood of x of radius δ_j . (The δ_j are called the *optimality radii*.) We again refer the reader to [6] or [7] for discussion of this optimality concept and of why it is a suitable generalization of the standard low order optimality conditions to arbitrary order.

In our new EDA framework, we have to be content with an inexact equivalent of (1.2) given by

$$\bar{T}_{f,j}(x, s) = \bar{f}(x) + \sum_{i=1}^j \frac{1}{i!} \overline{\nabla_x^i f}(x)[s]^i \equiv \bar{T}_{f,j}(x, 0) + \sum_{i=1}^j \frac{1}{i!} [\nabla_v^i \bar{T}_{f,j}(x, v)]_{v=0}[s]^i. \quad (1.5)$$

where, here and hereafter, we denote inexact quantities and approximations with an overbar. It is therefore pertinent to investigate the effect of inexact derivatives on (1.5) and its uses. More specifically, we will be concerned with the Taylor *decrement* $\Delta T_{f,j}(x, s)$ at x and for a step s , defined as

$$\Delta T_{f,j}(x, s) \stackrel{\text{def}}{=} T_{f,j}(x, 0) - T_{f,j}(x, s) = - \sum_{i=1}^j \frac{1}{i!} [\nabla_v^i T_{f,j}(x, v)]_{v=0}[s]^i \equiv - \sum_{i=1}^j \frac{1}{i!} \nabla_x^i f(x)[s]^i. \quad (1.6)$$

While our traditional algorithms depend on this quantity, it is of course out of the question to use it in the present context, as we only have approximate values. But an obvious alternative is instead to consider the *inexact* Taylor decrement

$$\overline{\Delta T}_{f,j}(x, s) \stackrel{\text{def}}{=} \bar{T}_{f,j}(x, 0) - \bar{T}_{f,j}(x, s) = - \sum_{i=1}^j \frac{1}{i!} [\nabla_v^i \bar{T}_{f,j}(x, v)]_{v=0}[s]^i, \quad (1.7)$$

itself resulting in an inexact version of (1.4) given by

$$\overline{\phi}_{f,j}^{\delta_j}(x) = - \max_{d \in \mathbb{R}^n, \|d\| \leq \delta_j} \overline{\Delta T}_{f,j}(x, d). \quad (1.8)$$

In the ‘‘Explicit Dynamic Accuracy’’ (EDA) framework, we assume that the conditions

$$\|\overline{\nabla_x^i f}(x) - \nabla_x^i f(x)\| \leq \varphi_i \quad (1.9)$$

for degrees $i \geq 0$ of interest *are enforced* when demanded by the algorithm, where φ_i is the required absolute error bound on the i -th derivative. It then follows that the error between the exact and inexact Taylor expansions satisfies the bound

$$|\overline{\Delta T}_{f,j}(x, s) - \Delta T_{f,j}(x, s)| \leq \sum_{i=1}^j \|\overline{\nabla_x^i f}(x) - \nabla_x^i f(x)\| \frac{\|s\|^i}{i!} \leq \sum_{i=1}^j \varphi_{k,i} \frac{\|s\|^i}{i!}, \quad (1.10)$$

using the triangle inequality and the requirement (1.9).

The distinguishing feature of the AR qp EDA2 algorithm—compared with AR qp EDA of [7, Chapter 13]—is that it enforces convergence by controlling a (scaling independent) relative error bound

$$|\overline{\Delta T}_{f,p}(x, s) - \Delta T_{f,p}(x, s)| \leq \omega \overline{\Delta T}_{f,p}(x, s), \quad (1.11)$$

for some fixed relative accuracy parameter $\omega \in (0, 1)$. Most of the difficulty in the forthcoming analysis results from the need to impose this relative error bound and it is not obvious at this point how it can be enforced using the absolute bounds (1.9). We now consider how this can be achieved.

2 Enforcing accuracy of the Taylor decrements

For clarity, we temporarily neglect the iteration index k . While there may be circumstances in which (1.11) can be enforced directly, we consider here that the only control the user has on the accuracy of $\overline{\Delta T}_{f,j}(x, s)$ is by imposing the bounds (1.9) on the *absolute* errors of the derivative tensors $\{\nabla_x^i f(x)\}_{i=1}^j$. As one may anticipate by examining (1.11), a suitable relative accuracy requirement can be achieved so long as $\overline{\Delta T}_{f,j}(x, s)$ remains safely away from zero. However, if exact computations are to be avoided, we may have to accept a simpler absolute accuracy guarantee when $\overline{\Delta T}_{f,j}(x, s)$ is small, but one that still guarantees our final optimality conditions.

Of course, not all derivatives need to be inexact. If derivatives of order $i \in \mathcal{E} \subseteq \{1, \dots, q\}$ are exact, then the left-hand side of (1.9) vanishes for $i \in \mathcal{E}$ and the choice $\varphi_i = 0$ for $i \in \mathcal{E}$ is perfectly adequate. However, we avoid the notational complication that making this distinction would entail.

We start by describing a crucial tool that we use to achieve (1.11). This is the CHECK algorithm, stated as Algorithm 2.1 on the following page. We use this to assess the relative model accuracy whenever needed in the algorithms we describe later in this section.

To put our exposition in a general context, we suppose that we have an r -th degree Taylor series $T_r(x, v)$ of a given function about x in the direction v , along with an approximation $\overline{T}_r(x, v)$ and its decrement $\overline{\Delta T}_r(x, v)$. Additionally, we suppose that a bound $\delta \geq \|v\|$ is given, and that *required* relative and absolute accuracies ω and $\xi > 0$ are on hand; the relative accuracy constant $\omega \in (0, 1)$ will be fixed throughout the forthcoming algorithms, and we assume that it is given when needed in CHECK. Finally, we assume that the *current* upper bounds $\{\varphi_j\}_{j=1}^r$ on absolute accuracies of the derivatives of $\overline{T}_r(x, v)$ with respect to v at $v = 0$ are provided. Because it will always be the case when we need it, for simplicity we will assume that $\overline{\Delta T}_r(x, v) \geq 0$.

Algorithm 2.1: Verify the accuracy of $\overline{\Delta T}_r(x, v)$ (CHECK)

$$\text{accuracy} = \text{CHECK}\left(\delta, \overline{\Delta T}_r(x, v), \{\varphi_i\}_{i=1}^r, \xi\right).$$

If $\overline{\Delta T}_r(x, v) > 0$ and $\sum_{i=1}^r \varphi_i \frac{\delta^i}{i!} \leq \omega \overline{\Delta T}_r(x, v)$, (2.1)
 set accuracy to relative.

Otherwise, if $\sum_{i=1}^r \varphi_i \frac{\delta^i}{i!} \leq \omega \xi \frac{\delta^r}{r!}$, (2.2)
 set accuracy to absolute.

Otherwise set accuracy to insufficient.

It will be convenient to say informally that **accuracy** is **sufficient**, if it is either **absolute** or **relative**. We may formalise the accuracy guarantees that result from applying the CHECK algorithm as follows.

Lemma 2.1 Let $\omega \in (0, 1]$ and δ, ξ and $\{\varphi_i\}_{i=1}^r > 0$. Suppose that $\overline{\Delta T}_r(x, v) \geq 0$, that

$$\text{accuracy} = \text{CHECK}\left(\delta, \overline{\Delta T}_r(x, v), \{\varphi_i\}_{i=1}^r, \xi\right),$$

and that

$$\left\| \left[\nabla_v^i \overline{T}_r(x, v) \right]_{v=0} - \left[\nabla_v^i T_r(x, v) \right]_{v=0} \right\| \leq \varphi_i \text{ for } i \in \{1, \dots, r\}. \quad (2.3)$$

Then

(i) accuracy is **sufficient** whenever

$$\sum_{i=1}^r \varphi_i \frac{\delta^i}{i!} \leq \omega \xi \frac{\delta^r}{r!}. \quad (2.4)$$

(ii) if accuracy is **absolute**,

$$\max \left[\overline{\Delta T}_r(x, w), \left| \overline{\Delta T}_r(x, w) - \Delta T_r(x, w) \right| \right] \leq \xi \frac{\delta^r}{r!} \quad (2.5)$$

for all w with $\|w\| \leq \delta$.

(iii) if accuracy is **relative**, $\overline{\Delta T}_r(x, v) > 0$ and

$$\left| \overline{\Delta T}_r(x, w) - \Delta T_r(x, w) \right| \leq \omega \overline{\Delta T}_r(x, w), \text{ for all } w \text{ with } \|w\| \leq \delta. \quad (2.6)$$

Proof. We first prove proposition (i), and assume that (2.4) holds, which clearly ensures that (2.2) is satisfied. Thus either (2.1) or (2.2) must hold and termination occurs, proving the first proposition.

It follows by definition of the decrements (1.6) and (1.7), the triangle inequality and (2.3) that

$$\begin{aligned} \left| \overline{\Delta T}_r(x, w) - \Delta T_r(x, w) \right| &= \left| \sum_{i=1}^r \frac{([\nabla_v^i \overline{T}_r(x, v)]_{v=0} - [\nabla_v^i T_r(x, v)]_{v=0}) [w]^i}{i!} \right| \\ &\leq \sum_{i=1}^r \frac{\|[\nabla_v^i \overline{T}_r(x, v)]_{v=0} - [\nabla_v^i T_r(x, v)]_{v=0}\| \|w\|^i}{i!} \\ &\leq \sum_{i=1}^r \varphi_i \frac{\|w\|^i}{i!}. \end{aligned} \quad (2.7)$$

Consider now the possible **sufficient** termination cases for the algorithm and suppose first that termination occurs with **accuracy as absolute**. Then, using (2.7), (2.2) and $\omega < 1$, we have that, for any w with $\|w\| \leq \delta$,

$$\left| \overline{\Delta T}_r(x, w) - \Delta T_r(x, w) \right| \leq \sum_{i=1}^r \varphi_i \frac{\delta^i}{i!} \leq \omega \xi \frac{\delta^r}{r!} \leq \xi \frac{\delta^r}{r!}. \quad (2.8)$$

If $\overline{\Delta T}_r(x, v) = 0$, we may combine this with (2.8) to derive (2.5). By contrast, if $\overline{\Delta T}_r(x, v) > 0$, then since (2.1) failed but (2.2) holds,

$$\omega \overline{\Delta T}_r(x, w) < \sum_{i=1}^r \varphi_i \frac{\delta^i}{i!} \leq \omega \xi \frac{\delta^r}{r!}.$$

Combining this inequality with (2.8) yields (2.5). Suppose now that **accuracy is relative**. Then (2.1) holds, and combining it with (2.7) gives that

$$\left| \overline{\Delta T}_r(x, w) - \Delta T_r(x, w) \right| \leq \sum_{i=1}^r \varphi_i \frac{\delta^i}{i!} \leq \omega \overline{\Delta T}_r(x, w),$$

for any w with $\|w\| \leq \delta$, which is (2.6). \square

Clearly, the outcome corresponding to our initial aim to obtain a relative error at most ω corresponds to the case where **accuracy is relative**. As we will shortly discover, the two other cases are also needed.

3 The ARqpEDA2 algorithm

We now define our adaptive regularization algorithm, which approximately minimizes the regularized, inexact Taylor-series model defined

$$\overline{m}_k(s) \stackrel{\text{def}}{=} \overline{T}_{f,p}(x_k, s) + \frac{\sigma_k}{(p+1)!} \|s\|^{p+1} \quad (3.1)$$

at iteration k , where $\overline{T}_{f,p}(x, s)$ is described in (1.5). Thus this model uses $p > 0$ inexact derivatives, each of which is required to satisfy bounds of the form (1.9) for $j \in \{1, \dots, p\}$. Success or failure is assessed by comparing the reduction in the Taylor series this gives compared to the inexact function value at the resulting trial point. Complications arise since optimality can only be assessed using inexact problem data, and because of the need for the inexact function values and derivatives to maintain appropriate coherence with their unknown true values. This makes both the algorithm and its analysis significantly involved, particularly since we need to add explicit dynamic-accuracy control to the mix.

Without further ado, and with no further apologies, here is the detailed algorithm.

Algorithm 3.1: Adaptive-regularization algorithm with explicit dynamic accuracy (AR_{qp}EDA2)

Step 0: Initialisation. A criticality order q , a degree $p \geq q$, an initial point $x_0 \in \mathbb{R}^n$ and an initial regularization parameter $\sigma_0 > 0$ are given, as well as accuracy levels $\epsilon \in (0, 1)^q$ and an initial set of absolute derivative accuracies $\{\varphi_{i,0}\}_{i=1}^p$. The constants ω , φ_{\max} , ς , δ_0 , θ , η_1 , η_2 , γ_1 , γ_2 , γ_3 and σ_{\min} are also given and satisfy $\varphi_{\max} \geq 0$, $\varsigma \in (0, 1]$, $\theta > 0$, $\delta_0 \in (\epsilon, 1]^q$, $\sigma_{\min} \in (0, \sigma_0]$,

$$0 < \eta_1 \leq \eta_2 < 1, \quad 0 < \gamma_1 < 1 < \gamma_2 < \gamma_3, \quad (3.2)$$

$$\omega \in (0, \min[\frac{1}{2}\eta_1, \frac{1}{4}(1 - \eta_2)]) \quad \text{and} \quad \varphi_{i,0} \leq \varphi_{\max} \quad (i \in \{1, \dots, p\}). \quad (3.3)$$

Set $k = 0$ and $k_\varphi = 0$.

Step 1: Compute the optimality measures and check for termination. Set $\delta_k^{(0)} = \delta_k$ and evaluate any unavailable $\{\overline{\nabla_x^i f}(x_k)\}_{i=1}^p$ to satisfy

$$\|\overline{\nabla_x^i f}(x_k) - \nabla_x^i f(x_k)\| \leq \varphi_{i,k_\varphi} \quad \text{for } i \in \{1, \dots, p\}. \quad (3.4)$$

For $j = 1, \dots, q$:

Step 1.1: Compute a displacement $d_{k,j} \in \mathcal{B}_{\delta_{k,j}}$ such that the corresponding Taylor decrement $\overline{\Delta T}_{f,j}(x_k, d_{k,j})$ satisfies

$$\varsigma \overline{\phi}_{f,j}^{\delta_{k,j}}(x_k) \leq \overline{\Delta T}_{f,j}(x_k, d_{k,j}). \quad (3.5)$$

If the call

$$\text{CHECK}\left(\delta_{k,j}, \overline{\Delta T}_{f,j}(x_k, d_{k,j}), \{\varphi_{i,k_\varphi}\}_{i=1}^j, \frac{1}{2}\epsilon_j\right). \quad (3.6)$$

returns **insufficient**, go to Step 5.

Step 1.2: If

$$\overline{\Delta T}_{f,j}(x_k, d_{k,j}) \leq \frac{\varsigma \epsilon_j}{(1 + \omega)} \frac{\delta_{k,j}^j}{j!}, \quad (3.7)$$

consider the next value of j .

Step 1.3: Otherwise, if

$$\overline{\Delta m}_k(d_{k,j}) \geq \frac{\varsigma \epsilon_j}{2(1 + \omega)} \frac{\delta_{k,j}^j}{j!}, \quad (3.8)$$

go to Step 2 with the index $j_k = j$, and $\delta_k^{(1)} = \delta_k$ and $d_k = d_{k,j}$.

Step 1.4: Otherwise set $\delta_{k,j} = \frac{1}{2}\delta_{k,j}$ and return to Step 1.1.

Terminate with $x_\epsilon = x_k$ and $\delta_\epsilon = \delta_k$.

Step 2: Step calculation.

Step 2.1: Compute a step s_k and optimality radii $\delta_{s_k} \in (0, 1]^q$ by approximately minimizing the model $\bar{m}_k(s)$ from (3.1) in the sense that

$$\Delta \bar{m}_k(s_k) \geq \Delta \bar{m}_k(d_k), \quad (3.9)$$

and either

$$\|s_k\| \geq 1 \quad (3.10)$$

or

$$\varsigma \phi_{\bar{m}_k, \ell}^{\delta_{s_k, \ell}}(s_k) \leq \Delta T_{\bar{m}_k, \ell}(s_k, d_{s_k, \ell}^m) \leq \frac{\varsigma \theta (1 - \omega)}{(1 + \omega) 2} \epsilon_\ell \frac{\delta_{s_k, \ell}^\ell}{\ell!} \quad \text{for } \ell \in \{1, \dots, q\} \quad (3.11)$$

for some radii $\delta_{s_k} \in (0, 1]^q$ and displacements $d_{s_k, \ell}^m \in \mathcal{B}_{\delta_{s_k, \ell}}$.

Step 2.2: If the call

$$\text{CHECK} \left(\|s_k\|, \overline{\Delta T}_{f, p}(x_k, s_k), \{\varphi_{i, k, \varphi}\}_{i=1}^p, \frac{\varsigma \epsilon_{j_k}}{2(1 + \omega)} \frac{p! \left(\delta_{k, j_k}^{(1)}\right)^{j_k}}{j_k! \max[\delta_{k, j_k}^{(1)}, \|s_k\|]^p} \right)$$

or $\|s_k\| < 1$ and any of the calls

$$\text{CHECK} \left(\delta_{s_k, \ell}, \Delta T_{\bar{m}_k, \ell}(s_k, d_{s_k, \ell}^m), \{3 \max_{t \in \{\ell, \dots, p\}} \varphi_{t, k, \varphi}\}_{i=1}^\ell, \frac{\varsigma \theta (1 - \omega) \epsilon_\ell}{2(1 + \omega)^2} \right) \quad (\ell \in \{1, \dots, q\})$$

returns **insufficient**, then go to Step 5.

Step 3: Acceptance of the trial point. Compute $\bar{f}(x_k + s_k)$ ensuring that

$$|\bar{f}(x_k + s_k) - f(x_k + s_k)| \leq \omega \overline{\Delta T}_{f, p}(x_k, s_k). \quad (3.12)$$

Also ensure, either by setting $\bar{f}(x_k) = \bar{f}(x_{k-1} + s_{k-1})$ or recomputing $\bar{f}(x_k)$, that

$$|\bar{f}(x_k) - f(x_k)| \leq \omega \overline{\Delta T}_{f, p}(x_k, s_k). \quad (3.13)$$

Then set

$$\rho_k = \frac{\bar{f}(x_k) - \bar{f}(x_k + s_k)}{\overline{\Delta T}_{f, p}(x_k, s_k)}. \quad (3.14)$$

If $\rho_k \geq \eta_1$, then define $x_{k+1} = x_k + s_k$ and $\delta_{k+1} = \delta_{s_k}$ if $\|s_k\| < 1$ or $\delta_{k+1} = \delta_k^{(1)}$ otherwise. If $\rho_k < \eta_1$, define $x_{k+1} = x_k$ and $\delta_{k+1} = \delta_k^{(1)}$.

Step 4: Regularization parameter update. Set

$$\sigma_{k+1} \in \begin{cases} [\max(\sigma_{\min}, \gamma_1 \sigma_k), \sigma_k] & \text{if } \rho_k \geq \eta_2, \\ [\sigma_k, \gamma_2 \sigma_k] & \text{if } \rho_k \in [\eta_1, \eta_2), \\ [\gamma_2 \sigma_k, \gamma_3 \sigma_k] & \text{if } \rho_k < \eta_1. \end{cases} \quad (3.15)$$

Increment k by one and go to Step 1.

Step 5: Improve accuracy. For $i \in \{1, \dots, p\}$, set

$$\varphi_{i,k_\varphi+1} = \gamma_\varphi \varphi_{i,k_\varphi},$$

increment k and k_φ by one and return to Step 1 with $x_{k+1} = x_k$, $\delta_{k+1} = \delta_k^{(0)}$ and $\sigma_{k+1} = \sigma_k$.

Note that extensive use is made of the CHECK algorithm we developed above to ensure that derivative approximations are sufficiently accurate. Nonetheless, a number of comments are in order to clarify and motivate this extensive description.

- Starting with Step 0, notice that that we postpone the evaluation of the inexact objective function $\bar{f}(x_0)$ until Step 3 since Steps 1 and 2 do not depend on its value.
- Next, examining Step 1, we see that the initialization of Step 1 and Step 1.1 aim at computing, while (3.8) ensures a lower bound on the model decrease, which then guarantees both that x_k is not a global model minimizer and also that the first call to CHECK in Step 2 cannot return **absolute** (see Lemma 4.7 below). Of course, we need to show that the loop within Step 1 in which $\delta_{k,j}$ is reduced terminates finitely, and that the resulting value of $\delta_{k,j}$ is not unduly small (as per Lemma 4.2 below); because of condition (3.8), this entails showing that the model decrease at $d_{k,j}$, the optimality displacement associated with the violated optimality condition (3.7), is large enough. We also note that, unless termination occurs, Step 1 specifies the first index $j \in \{1, \dots, q\}$ for which j -th-order approximate-criticality test (3.7) fails. It is then helpful to distinguish the vectors of radii

$$\delta_k^{(0)} = \delta_k \text{ at the start of Step 1,} \tag{3.16}$$

and inherited from iteration $k - 1$, from

$$\delta_k^{(1)} = \delta_k \text{ at the end of Step 1,} \tag{3.17}$$

after possible reductions within that step. Clearly, component-wise $\delta_k^{(1)} \leq \delta_k^{(0)}$.

- Complications arise in Step 2.2 where the step itself and the optimality displacements associated with model's approximate optimality have to be checked for sufficient accuracy. Just as was the case in Step 1, this entails possible accuracy improvements, re-evaluation of the derivatives and the need to recompute the step for the improved model. The absolute accuracy thresholds passed as the last argument to the two calls to CHECK undoubtedly appear rather mysterious at this stage: the first is designed to ensure that an **absolute** return from CHECK is impossible, and the second to ensure an optimality level for the exact problem that is comparable to that revealed in Step 1. More details will obviously be given in due course.
- Step 3 ensures coherence between accuracy on the function values and accuracy of the model. We stress that the requirements (3.12) and (3.13) do not imply that we need to know the true f , we only need some mechanism to ensure that x_k and $x_k + s_k$ satisfy the

required bounds and that are needed to guarantee convergence. Again, a new value of $\bar{f}(x_k)$ has to be computed to ensure (3.13) in Step 3 only when $k > 0$ and $\overline{\Delta T}_{f,p}(x_{k-1}, s_{k-1}) > \overline{\Delta T}_{f,p}(x_k, s_k)$, in which case the (inexact) function value is computed twice rather than once during that iteration. As is standard, iteration k is said to be successful when $\rho_k \geq \eta_1$ and $x_{k+1} = x_k + s_k$, and we define \mathcal{S} , \mathcal{U} , \mathcal{A} , the sets of *successful*, *unsuccessful* and *accuracy-improving* iterations, respectively, as well as \mathcal{T} , \mathcal{S}_k , \mathcal{U}_k , \mathcal{A}_k and \mathcal{T}_k by

$$\begin{aligned}\mathcal{S} &\stackrel{\text{def}}{=} \{k \in \mathbf{N} \mid \text{Step 5 is not executed and } \rho_k \geq \eta_1\}, \\ \mathcal{U} &\stackrel{\text{def}}{=} \{k \in \mathbf{N} \mid \text{Step 5 is not executed and } \rho_k < \eta_1\} \quad \text{and} \\ \mathcal{A} &\stackrel{\text{def}}{=} \{k \in \mathbf{N} \mid \text{Step 5 is executed}\}.\end{aligned}\tag{3.18}$$

If $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{S} \cup \mathcal{U}$, we also define

$$\begin{aligned}\mathcal{S}_k &\stackrel{\text{def}}{=} \mathcal{S} \cap \{0, \dots, k\}, & \mathcal{U}_k &\stackrel{\text{def}}{=} \mathcal{U} \cap \{0, \dots, k\}, \\ \mathcal{A}_k &\stackrel{\text{def}}{=} \mathcal{A} \cap \{0, \dots, k\} \quad \text{and} & \mathcal{T}_k &\stackrel{\text{def}}{=} \mathcal{T} \cap \{0, \dots, k\},\end{aligned}\tag{3.19}$$

the corresponding sets up to iteration k . Notice that $x_{k+1} = x_k + s_k$ for $k \in \mathcal{S}$, while $x_{k+1} = x_k$ for $k \in \mathcal{U} \cup \mathcal{A}$. Note also that the objective function is evaluated at most twice per successful or unsuccessful iteration (i.e. once for every $k \in \mathcal{T}$), and derivatives are evaluated once per successful or accuracy improving iteration (i.e. once for every $k \in \mathcal{S} \cup \mathcal{A}$).

- Step 4 is the standard regularization parameter update.
- Finally, Step 5 describes the accuracy improvement mechanism.

Given the definitions (3.19), we are now able to show that the number of iterations in \mathcal{T}_k is at most a multiple of that in \mathcal{S}_k .

Lemma 3.1 Suppose that the AR_{qp}EDA2 algorithm is used and that $\sigma_k \leq \sigma_{\max}$ for some $\sigma_{\max} > 0$. Then

$$|\mathcal{T}_k| \leq |\mathcal{S}_k| \left(1 + \frac{|\log \gamma_1|}{\log \gamma_2} \right) + \frac{1}{\log \gamma_2} \log \left(\frac{\sigma_{\max}}{\sigma_0} \right).\tag{3.20}$$

Proof. Observe that $\sigma_{k+1} = \sigma_k$ for $k \in \mathcal{A}$. The regularization parameter update (3.15) now gives that, for each k ,

$$\gamma_1 \sigma_j \leq \max[\gamma_1 \sigma_j, \sigma_{\min}] \leq \sigma_{j+1}, \quad j \in \mathcal{S}_k, \quad \text{and} \quad \gamma_2 \sigma_j \leq \sigma_{j+1}, \quad j \in \mathcal{U}_k.$$

Thus we deduce inductively that

$$\sigma_0 \gamma_1^{|\mathcal{S}_k|} \gamma_2^{|\mathcal{U}_k|} \leq \sigma_k.$$

Therefore, using our assumption that $\sigma_k \leq \sigma_{\max}$, we deduce that

$$|\mathcal{S}_k| \log \gamma_1 + |\mathcal{U}_k| \log \gamma_2 \leq \log \left(\frac{\sigma_{\max}}{\sigma_0} \right),$$

which then implies that

$$|\mathcal{U}_k| \leq -|\mathcal{S}_k| \frac{\log \gamma_1}{\log \gamma_2} + \frac{1}{\log \gamma_2} \log \left(\frac{\sigma_{\max}}{\sigma_0} \right),$$

since $\gamma_2 > 1$. The desired result (3.20) then follows from the equality $|\mathcal{T}_k| = |\mathcal{S}_k| + |\mathcal{U}_k|$ and the inequality $\gamma_1 < 1$ given by (3.2). \square

4 Evaluation complexity for the AR_{qp}EDA2 algorithm

Our analysis of the AR_{qp}EDA2 algorithm will be carried out under the following assumptions.

f.D0qL: f is q times continuously differentiable in \mathbb{R}^n and, for all $j \in \{0, \dots, q\}$, the j -th derivative of f is Lipschitz continuous with Lipschitz constant $L_{f,j}$, that is there exist constants $L_{f,j} \geq 1$ such that

$$\|\nabla_x^j f(x) - \nabla_x^j f(y)\| \leq L_{f,j} \|x - y\|$$

for all $x, y \in \mathbb{R}^n$ and all $j \in \{0, \dots, q\}$.

f.Bb: There exists a constant f_{low} such that $f(x) \geq f_{\text{low}}$ for all $x \in \mathbb{R}^n$.

We then define

$$L_f \stackrel{\text{def}}{=} \max_{j \in \{0, \dots, p\}} L_{f,j} \geq 1. \quad (4.1)$$

We recall that the derivatives of the objective function at the iterates x_k remain bounded under **f.D0pL**. Moreover, the absolute accuracies $\{\varphi_i\}_{i=1}^p$ never increase in the course of the AR_{qp}EDA2 algorithm, and are initialised so that $\varphi_i \leq \varphi_{\max}$ for all $i \in \{1, \dots, p\}$. As a consequence, **f.D0pL**, standard error bounds for Lipschitz functions (see [7, Corollary A.8.4], for instance), (4.1) and (3.4) imply that, for each $\ell \in \{1, \dots, p\}$,

$$\|\overline{\nabla_x^\ell f}(x_k)\| \leq \|\nabla_x^\ell f(x_k)\| + \|\overline{\nabla_x^\ell f}(x_k) - \nabla_x^\ell f(x_k)\| \leq L_f + \varphi_{\max} \stackrel{\text{def}}{=} \bar{L}_f. \quad (4.2)$$

4.1 The outcomes of Step 1

We start by considering the result of performing Step 1 and show that the loop reducing $\delta_{k,j}$ generated by the possible return to Step 1.1 from Step 1.4 is finite. This is done in two stages: we start by expressing a general property of the value of the model compared to the truncated Taylor series, which we will subsequently apply to obtain the desired conclusion.

Lemma 4.1 Suppose that **f.D0pL** holds, that $\overline{m}_k(s)$ is the inexact model (3.1) corresponding to some approximate derivatives values $\{\overline{\nabla}_x^\ell f(x_k)\}_{\ell=1}^p$, and that \overline{L}_f is given by (4.2). Given $\alpha > 0$ and

$$\delta \in \left(0, \min\left(1, \frac{\alpha}{4 \max[\overline{L}_f, \sigma_k]}\right)\right], \quad (4.3)$$

suppose that there is a displacement $d \in \mathcal{B}_\delta$ for which

$$\overline{\Delta T}_{f,j}(x_k, d) \geq \alpha \frac{\delta^j}{j!} \quad (4.4)$$

for some $j \in \{1, \dots, q\}$. Then

$$\Delta \overline{m}_k(d) \geq \frac{1}{2} \alpha \frac{\delta^j}{j!}. \quad (4.5)$$

Proof. The proof is built on the sequence of inequalities

$$\begin{aligned} \Delta \overline{m}_k(d_k) &= \overline{\Delta T}_{f,j}(x_k, d) + \sum_{\ell=j+1}^p \frac{1}{\ell!} \overline{\nabla}_x^\ell f(x_k) [d]^\ell + \frac{\sigma_k}{(p+1)!} \|d\|^{p+1} \\ &\geq \overline{\Delta T}_{f,j}(x_k, d) - \sum_{\ell=j+1}^p \frac{\delta^\ell}{\ell!} \|\overline{\nabla}_x^\ell f(x_k)\| - \frac{\sigma_k}{(p+1)!} \delta^{p+1} \\ &\geq \alpha \frac{\delta^j}{j!} - \max[\overline{L}_f, \sigma_k] \sum_{\ell=j+1}^{p+1} \frac{\delta^\ell}{\ell!} \\ &\geq \alpha \frac{\delta^j}{j!} - 2 \max[\overline{L}_f, \sigma_k] \frac{\delta^{j+1}}{j!} \\ &= \frac{\delta^j}{j!} (\alpha - 2 \max[\overline{L}_f, \sigma_k] \delta) \end{aligned}$$

that arise from the triangle inequality and because of the assumptions made. The required bound (4.5) then follows because of (4.3). \square

We now show that looping inside Step 1 is impossible.

Lemma 4.2 Suppose that **f.D0pL** holds and that algorithm AR qp EDA2 has reached the test (3.8) in Step 1.3. Suppose also that

$$\delta_{k,j} \leq \frac{\varsigma \epsilon_j}{4(1+\omega) \max[\overline{L}_f, \sigma_k]} \quad (4.6)$$

with \overline{L}_f is given by (4.2). Then (3.8) holds, and thus no return from Step 1.5 to Step 1.2 is possible.

Proof. That the algorithm has reached the test (3.8) in Step 1.3 implies that (3.7) failed

and thus

$$\overline{\Delta T}_{f,j}(x_k, d_k) > \left(\frac{\varsigma \epsilon_j}{1 + \omega} \right) \frac{\delta_{k,j}^j}{j!}.$$

Hence, using Lemma 4.1 with and the fact that (4.6) implies (4.3) for $\delta = \delta_{k,j}$, we deduce that (3.8) must hold from (4.5). \square

Thus the loop within Step 1 is finite, (3.8) eventually holds and we may therefore analyze the possible outputs of Step 1.

Lemma 4.3 Suppose that **f.D1pL** holds, and that the AR_{qp}EDA2 algorithm is applied. Then one of three situations may occur at Step 1 of iteration k :

- (i) the AR_{qp}EDA2 algorithm terminates with x_ϵ an (ϵ, δ) -approximate q -th-order minimizer, or
- (ii) control is passed to Step 5, or
- (iii) the call (3.6) returns **relative** for some $j \in \{1, \dots, q\}$ for which (3.8) holds for a $\delta_{k,j}^{(1)}$ satisfying

$$\delta_{k,j}^{(0)} \geq \delta_{k,j} \geq \delta_{k,j}^{(1)} \geq \min \left[\frac{\varsigma \epsilon_j}{8(1 + \omega) \max[\overline{L}_f, \sigma_k]}, \delta_{k,j}^{(0)} \right]. \quad (4.7)$$

and control is then passed to Step 2 with

$$(1 - \omega) \overline{\Delta T}_{f,j}(x_k, d_{k,j}) \leq \phi_{f,j}^{\delta_{k,j}^{(0)}}(x_k) \leq \left(\frac{1 + \omega}{\varsigma} \right) \overline{\Delta T}_{f,j}(x_k, d_{k,j}) \quad (4.8)$$

being satisfied.

Moreover, outcome (ii) is impossible whenever

$$\max_{i \in \{1, \dots, j\}} \varphi_i \leq \frac{\varsigma \omega}{4} \min \left[\frac{\varsigma \epsilon_j}{8(1 + \omega) \max[\overline{L}_f, \sigma_k]}, \delta_{k,j}^{(0)} \right]^{j-1} \frac{\epsilon_j}{j!}. \quad (4.9)$$

Proof.

Suppose first that branching to Step 5 does not occur in Step 1.2 for any $j \in \{1, \dots, q\}$ (i.e., outcome (ii) does not occur). This ensures that the call (3.6) always returns either **relative** or **absolute**. Consider any $j \in \{1, \dots, q\}$ and notice that Step 1.1 yields (2.3) with $T_r = T_{f,j}$ and $r = j$, so that the assumptions of Lemma 2.1 are satisfied. Moreover, because of (3.5) and the fact that $\overline{\phi}_{f,j}^{\delta_k}(x_k) \geq 0$ by definition, we have that $\overline{\Delta T}_{f,j}(x_k, d_{k,j}) \geq 0$.

If the call (3.6) returns **absolute**, then Lemma 2.1(ii) with $\xi = \frac{1}{2}\varsigma\epsilon_j$, ensures that

$$\overline{\Delta T}_{f,j}(x_k, d_{k,j}) \leq \frac{1}{2}\varsigma\epsilon_j \frac{\delta_k^j}{j!} \leq \frac{\varsigma\epsilon_j}{1 + \omega} \frac{\delta_k^j}{j!}, \quad (4.10)$$

using the requirement that $\omega < 1$. Moreover, if $d_{k,j}^* \in \mathcal{B}_{\delta_k}$ is a global maximizer of $T_{f,j}(x_k, d)$ over all $d \in \mathcal{B}_{\delta_k}$, we may again invoke (2.5) with $\xi = \frac{1}{2}\varsigma\epsilon_j$ together with the triangle inequality

to see that

$$\begin{aligned}\phi_{f,j}^{\delta_k}(x_k) &= \Delta T_{f,j}(x_k, d_{k,j}^*) \\ &\leq \overline{\Delta T}_{f,j}(x_k, d_{k,j}^*) + |\Delta T_{f,j}(x_k, d_{k,j}^*) - \overline{\Delta T}_{f,j}(x_k, d_{k,j}^*)| \\ &\leq \epsilon_j \frac{\delta_k^j}{j!}.\end{aligned}\tag{4.11}$$

By contrast, if the call (3.6) returns **relative**, observe that

$$\varsigma \overline{\Delta T}_{f,j}(x_k, d) \leq \varsigma \overline{\phi}_{f,j}^{\delta_k}(x_k) \leq \overline{\Delta T}_{f,j}(x_k, d_{k,j})\tag{4.12}$$

for any $d \in \mathcal{B}_{\delta_k}$ because of (3.5), and thus we have that

$$\begin{aligned}\varsigma \Delta T_{f,j}(x_k, d) &\leq \varsigma \left[\overline{\Delta T}_{f,j}(x_k, d) + \left| \overline{\Delta T}_{f,j}(x_k, d) - \Delta T_{f,j}(x_k, d) \right| \right] \\ &\leq \varsigma(1 + \omega) \overline{\Delta T}_{f,j}(x_k, d) \\ &\leq (1 + \omega) \overline{\Delta T}_{f,j}(x_k, d_{k,j})\end{aligned}$$

using Lemma 2.1(iii), and the second inequality in (4.8) follows by picking $d = d_{k,j}^*$. Similarly

$$\Delta T_{f,j}(x_k, d) \geq \overline{\Delta T}_{f,j}(x_k, d) - \left| \overline{\Delta T}_{f,j}(x_k, d) - \Delta T_{f,j}(x_k, d) \right| \geq (1 - \omega) \overline{\Delta T}_{f,j}(x_k, d)$$

for any $d \in \mathcal{B}_{\delta_k}$, again using Lemma 2.1(iii). Hence

$$\max_{\|d\| \leq \delta_k} \Delta T_{f,j}(x_k, d) \geq (1 - \omega) \max_{\|d\| \leq \delta_k} \overline{\Delta T}_{f,j}(x_k, d) \geq (1 - \omega) \overline{\Delta T}_{f,j}(x_k, d_{k,j}),$$

which is the first inequality in (4.8). Thus we obtain that, for any $j \in \{1, \dots, q\}$, either both (4.10) and (4.11) hold, or (4.8) holds.

If the j loop continues to termination, then for each $j \in \{1, \dots, q\}$, we must have that either (4.11) holds or

$$\phi_{f,j}^{\delta_k}(x_k) \leq \left(\frac{1 + \omega}{\varsigma} \right) \overline{\Delta T}_{f,j}(x_k, d_{k,j}) \leq \epsilon_j \frac{\delta_k^j}{j!},$$

where we used (4.8) and the fact that (3.8) must be violated for the loop to continue. Thus in either event (4.11) holds for all $j \in \{1, \dots, q\}$, and thus, as (1.3) holds with $x_\epsilon = x_k$ and $\delta_\epsilon = \delta_k$, outcome (i) will occur.

If, instead, control passes to Step 2, we must now show that the conclusions of outcome (iii) hold. Firstly, observe that the mechanism of Step 1 ensures that the first inequality in (4.7) is satisfied. Moreover, if the loop on j does not finish and branching to Step 5 does not happen, there must be a j such that (3.7) is violated and the test (3.8) is reached. But we have shown in Lemma 4.2 that the number of times that (3.8) is violated—and hence $\delta_{k,j}$ is halved—is finite since (3.8) must hold as soon as (4.6) holds. At this stage, if one or more halvings happened, the resulting value of $\delta_{k,j}$ cannot be smaller than half of the right-hand side of (4.6), and so (4.7) also holds. If no halving of $\delta_{k,j}$ occurred, $\delta_{k,j}^{(1)} = \delta_{k,j}^{(0)}$ and (4.7) obviously holds. In addition, since (3.8) ultimately holds for some $j_k \in \{1, \dots, q\}$, (3.6) cannot have returned **insufficient** (by assumption) or **absolute** (because, in view of (3.1), (4.10) would have prevented (3.8)), and thus it returned **relative**. Hence outcome (iii) occurs.

Finally, Lemma 2.1 (i) shows that outcome (ii) cannot happen if (4.9) holds with δ_k being the smallest $\delta_{k,j}$ that can occur during the execution of Step 1, which is $\delta_{k,j}^{(1)}$. In other words, outcome (ii) cannot happen if

$$\max_{i \in \{1, \dots, j\}} \varphi_i \leq \frac{\varsigma \omega}{4} \epsilon_j \frac{\left(\delta_{k,j}^{(1)}\right)^{j-1}}{j!}.$$

Substituting (4.7) into this bound then reveals (4.9). \square

4.2 The outcomes of Step 2

Our next task is to investigate Step 2 in more detail. Our aim is to show that it is possible to compute a step s_k that satisfies (3.9) and either (3.10) or (3.11). We start our analysis by stating a suitable bound on model decrease, which now involves the inexact Taylor series $\overline{T}_{f,p}(x_k, s_k)$ of degree p .

Lemma 4.4 The mechanism of the AR_{qp}EDA2 algorithm guarantees that

$$\overline{\Delta T}_{f,p}(x_k, s_k) \geq \frac{\sigma_k}{(p+1)!} \|s_k\|^{p+1} \quad (4.13)$$

for all $k \in \mathcal{T}$, and (3.14) is well-defined.

Proof. Since $k \in \mathcal{T}$, both (3.8) and (3.9) must hold at iteration k for some $j \in \{1, \dots, q\}$. We then have that

$$0 < \frac{\varsigma \epsilon_j}{4} \frac{\delta_{k,j}^j}{j!} \leq \Delta \overline{m}_k(d_{k,j}) \leq \Delta \overline{m}_k(s_k) = \overline{\Delta T}_{f,p}(x_k, s_k) - \frac{\sigma_k}{(p+1)!} \|s_k\|^{p+1},$$

using (3.9) which ensures both that $s_k \neq 0$ and (4.13) holds. \square

Because the second set of calls to CHECK in Step 2.2 aim to check the accuracy of the Taylor expansion of the *model*, we need to consider $\{\nabla_d^j T_{\overline{m}_k,j}(s_k, 0)\}_{j=1}^p$ rather than the $\{\nabla_x^j f(x_k)\}_{j=1}^p$ that we have used so far. It is easy to verify that these (approximate) derivatives are given by

$$\nabla_d^j T_{\overline{m}_k,j}(s_k, 0) = \sum_{\ell=j}^p \frac{\overline{\nabla_x^\ell f}(x_k) [s_k]^{\ell-j}}{(\ell-j)!} + \frac{\sigma_k}{(p+1)!} [\nabla_s^j \|s\|^{p+1}]_{s=s_k}, \quad (4.14)$$

where the last term of the right-hand side is computed exactly. This yields the following error bound.

Lemma 4.5 Suppose that $\|s_k\| \leq 1$. Then, for all $j \in \{1, \dots, p\}$,

$$\begin{aligned} \left| \nabla_d^j T_{\overline{m}_k,j}(s_k, 0) - \nabla_d^j T_{m_k,j}(s_k, 0) \right| &\leq 3 \max_{\ell \in \{j, \dots, p\}} \|\overline{\nabla_x^\ell f}(x_k) - \nabla_x^\ell f(x_k)\| \\ &\leq 3 \max_{\ell \in \{j, \dots, p\}} \varphi_{\ell, k_\varphi}. \end{aligned} \quad (4.15)$$

Proof. Using the triangle inequality, (4.14), the inequality $\|s_k\| \leq 1$, we have that

$$\begin{aligned} \left| \nabla_d^j T_{\bar{m}_k, j}(s_k, 0) - \nabla_d^j T_{m_k, j}(s_k, 0) \right| &\leq \sum_{\ell=j}^p \left| \overline{\nabla_x^j f}(x_k) - \nabla_x^j f(x_k) \right| \frac{\|s_k\|^{\ell-j}}{(\ell-j)!} \\ &\leq \max_{\ell \in \{j, \dots, p\}} \varphi^{\ell, k, \varphi} \sum_{\ell=j}^p \frac{1}{(\ell-j)!} \end{aligned}$$

for all $j \in \{1, \dots, p\}$, and (4.15) then follows from the fact that

$$\sum_{\ell=j}^p \frac{1}{(\ell-j)!} \leq 1 + \sum_{\ell=1}^{p-j} \frac{1}{\ell!} \leq 1 + e.$$

□

As a consequence, to be safe, we must require three times more accuracy for the derivatives of the model at s_k than what would be required at $s = 0$.

Next, we provide an upper bound on the norm of the step.

Lemma 4.6 Suppose that **f.D0pL** holds, that \bar{L}_f is given by (4.2), and that a step s_k has been found such that (3.9) holds. Then we have that

$$\|s_k\| \leq \max \left[\frac{2\bar{L}_f(p+1)!}{\sigma_{\min}}, \left(\frac{2\bar{L}_f(p+1)!}{\sigma_{\min}} \right)^{\frac{1}{p}} \right] \stackrel{\text{def}}{=} \kappa_s. \quad (4.16)$$

Proof. Using (3.15), (4.13), the definition (1.6), the Cauchy-Schwarz inequality, and the bounds (4.2) and $\sum_{i=1}^p 1/i! < e$, we have that

$$\begin{aligned} \frac{\sigma_{\min}}{(p+1)!} \|s_k\|^{p+1} &\leq \frac{\sigma_k}{(p+1)!} \|s_k\|^{p+1} < \overline{\Delta T}_{f,p}(x_k, s_k) \\ &\leq \sum_{i=1}^p \frac{1}{i!} \|\overline{\nabla_x^i f}(x)\| \|s\|^i \leq \bar{L}_f \max[\|s_k\|, \|s_k\|^p] \sum_{i=1}^p \frac{1}{i!} \\ &\leq 2\bar{L}_f \max[\|s_k\|, \|s_k\|^p], \end{aligned}$$

which then leads directly to (4.16). □

We are now in position to elucidate the possible outcomes of Step 2.

Lemma 4.7 Suppose that **f.DOPL** holds. Then, Step 2 of the AR_{qp}EDA2 algorithm is well-defined, and either branches to Step 5 or produces a pair (s_k, δ_{s_k}) such that

$$\left| \overline{\Delta T}_{f,p}(x_k, s_k) - \Delta T_{f,p}(x_k, s_k) \right| \leq \omega \overline{\Delta T}_{f,p}(x_k, s_k), \quad (4.17)$$

and, either

(i) $\|s_k\| \geq 1$, or

(ii) $\|s_k\| < 1$ and

$$\phi_{m_k, j}^{\delta_{s_k, j}}(s_k) \leq \frac{\theta(1-\omega)}{1+\omega} \epsilon_j \frac{\delta_{s_k, j}^j}{j!} \quad \text{for } j \in \{1, \dots, q\} \quad (4.18)$$

for some δ_{s_k} for which

$$\delta_{s_k, j} = 1 \quad \text{for } j \in \{1, \dots, \min(2, q)\} \quad (4.19)$$

and

$$\delta_{s_k, j} \geq \min \left[\kappa_\delta(\sigma_k) \epsilon_j, \delta_{k, j}^{(0)} \right] \quad \text{for } j \in \{3, \dots, q\}, \quad (4.20)$$

where

$$\kappa_\delta(\sigma) \stackrel{\text{def}}{=} \frac{\varsigma \theta (1-\omega)}{8(1+\omega)(3\overline{L}_f + \sigma)} < 1 \quad (4.21)$$

and \overline{L}_f is given by (4.2). Moreover,

$$\delta_{k, j}^{(1)} \geq \min \left[\kappa_\delta(\sigma_{k, \max}) \epsilon_j, \delta_{k, j}^{(0)} \right] \quad (4.22)$$

for all $k \geq 1$ and $j \in \{1, \dots, q\}$, where $\sigma_{k, \max} = \max_{i \in \{0, \dots, k\}} \sigma_i$. Finally, $k \in \mathcal{T}$ whenever

$$\max_{i \in \{1, \dots, j\}} \varphi_{i, k, \varphi} \leq \epsilon_{\min} \min \left[\kappa_{\text{step2}}(\sigma_{k, \max}) \epsilon_{\min}, (\delta_{k, \min}^{(0)})^q \right] \quad (4.23)$$

where $\kappa_{\text{step2}}(\sigma)$ is a continuous non-increasing function of σ , depending only on $\theta, \varsigma, \omega, \sigma_{\min}$ and problem constants, and where

$$\delta_{k, \min}^{(0)} \stackrel{\text{def}}{=} \min_{\ell \in \{1, \dots, q\}} \delta_{k, \ell}^{(0)}. \quad (4.24)$$

and $\epsilon_{\min} \stackrel{\text{def}}{=} \min_{j \in \{1, \dots, q\}} \epsilon_j$.

Proof. The proof proceeds in several stages.

- We first verify that Step 2.1 is well defined. As it turns out, this conclusion follows from an almost identical situation arising in the analysis of the adaptive regularization algorithm AR_{qp} (using exact function and derivatives values) when ensuring that a step s_k could be found for this algorithm. The only significant difference is that the former this proof used exact derivatives and required that they were bounded, but now we use approximate ones. Fortunately (4.2) provides a substitute bound, and as a consequence a straightforward variant

of [7, Lemma 12.2.9] still holds with \bar{L}_f from (4.2) replacing L_f . In particular, this lemma (with ϵ_j replaced by $\epsilon_j(1-\omega)/(1+\omega)^2$) ensures that (4.18) is possible with δ_{s_k} satisfying (4.19) and (4.20) with (4.21). Thus Step 2.1 is well defined, (3.8) and (3.9) ensure that

$$\Delta\bar{m}_k(s_k) \geq \Delta\bar{m}_k(d_k) \geq \frac{\varsigma\epsilon_{j_k}}{2(1+\omega)} \frac{\left(\delta_{k,j_k}^{(1)}\right)^{j_k}}{j_k!}, \quad (4.25)$$

and thus that $s_k \neq 0$, and control passes to Step 2.2.

- Consider now the evolution of the vector of δ_k as the algorithm proceeds. In particular, flow is from Step 1 either to Step 4 (via Steps 2 and 3) or Step 5 (either directly or via Step 2). The radii are unaltered on iterations that move via Step 5, and on others, they obey either (4.7) or (4.19) and (4.20). Therefore,

$$\begin{aligned} \delta_{k,j}^{(1)} &\geq \min \left[\frac{\varsigma\theta(1-\omega)\epsilon_j}{8(1+\omega)(3\bar{L}_f + \sigma_{k,\max})}, \frac{\varsigma\epsilon_j}{8(1+\omega)\max[\bar{L}_f, \sigma_{k,\max}]}, \delta_{k,j}^{(0)} \right] \\ &= \min \left[\frac{\varsigma\theta(1-\omega)\epsilon_j}{8(1+\omega)(3\bar{L}_f + \sigma_{k,\max})}, \delta_{k,j}^{(0)} \right] \end{aligned}$$

since $\omega \in (0, 1)$, which is (4.22).

- The next step in the proof is to consider the outcome of the first call to CHECK in Step 2.2, and we claim that it can only return **insufficient** or **relative**. Indeed, suppose it returns **absolute**. Lemma 2.1(ii) then implies that

$$\overline{\Delta T}_{f,p}(x_k, s_k) \leq \frac{\varsigma\epsilon_{j_k}}{2(1+\omega)} \left(\frac{p! \left(\delta_{k,j_k}^{(1)}\right)^{j_k}}{j_k! \max[\delta_{k,j_k}^{(1)}, \|s_k\|]^p} \right) \frac{\|s_k\|^p}{p!} \leq \frac{\varsigma\epsilon_{j_k}}{2(1+\omega)} \frac{\left(\delta_{k,j_k}^{(1)}\right)^{j_k}}{j_k!}$$

But the definition of \bar{m}_k and the fact that $s_k \neq 0$ give that $\overline{\Delta T}_{f,p}(x_k, s_k) > \Delta\bar{m}_k(s_k)$ and hence

$$\Delta\bar{m}_k(s_k) < \frac{\varsigma\epsilon_{j_k}}{2(1+\omega)} \frac{\left(\delta_{k,j_k}^{(1)}\right)^{j_k}}{j_k!}$$

which contradicts (4.25). Our assumption that CHECK returned **absolute** is thus impossible. As a consequence, and unless branching to Step 5 occurs because it has returned **insufficient**, the first call to CHECK in Step 2.2 must return **relative**, which then ensures (4.17) because of Lemma 2.1(iii)

- We now determine a value of the absolute accuracy below which the first call to CHECK in Step 2.2 cannot return **insufficient**. Recall that the absolute accuracies $\{\varphi_i\}_{i=1}^p$ are reduced by a factor γ_φ in Step 5 each time Step 2 passes there. But (2.1) implies that this will be impossible as soon as the accuracies are small enough that

$$\sum_{i=1}^p \varphi_{i,k,\varphi} \frac{\|s_k\|^i}{i!} \leq \omega \overline{\Delta T}_{f,p}(x_k, s_k). \quad (4.26)$$

But

$$\overline{\Delta T}_{f,p}(x_k, s_k) > \Delta\bar{m}_k(s_k) \geq \Delta\bar{m}_k(d_k) = \Delta\bar{m}_k(d_{k,j_k}) \geq \frac{\varsigma\epsilon_{j_k}}{2(1+\omega)} \frac{\left(\delta_{k,j_k}^{(1)}\right)^{j_k}}{j_k!}$$

where we successively used the definition of \bar{m}_k (3.1), (3.9) and (3.8). Moreover, the inequality $\delta_{k,j} \leq 1$ and (4.22) imply that

$$\frac{\varsigma \epsilon_j}{2(1+\omega)} \frac{(\delta_{k,j}^{(1)})^j}{j!} \geq \frac{\varsigma \epsilon_j}{2(1+\omega)} \frac{(\delta_{k,j}^{(1)})^q}{q!} \geq \frac{\varsigma \epsilon_j}{2(1+\omega)} \frac{\min[\kappa_\delta(\sigma_{k,\max})\epsilon_j, \delta_{k,j}^{(0)}]^q}{q!}$$

so that

$$\overline{\Delta T}_{f,p}(x_k, s_k) \geq \frac{\varsigma \epsilon_{j_k}}{2(1+\omega)} \frac{\min[\kappa_\delta(\sigma_{k,\max})\epsilon_{j_k}, \delta_{k,j_k}^{(0)}]^q}{q!}. \quad (4.27)$$

In addition, we know from Lemma 4.6 and the sum bound $\sum_{i=1}^p 1/i! < e$ that

$$\sum_{i=1}^p \varphi_{i,k,\varphi} \frac{\|s_k\|^i}{i!} \leq 2 \max[1, \kappa_s^p] \max_{i \in \{1, \dots, p\}} \varphi_{i,k,\varphi},$$

Combining this inequality with (4.27), we deduce that (4.26) holds, and consequently the first call to CHECK in Step 2.2 returns **relative** (as we have just verified it cannot return **absolute**), as soon as

$$\max_{i \in \{1, \dots, p\}} \varphi_{i,k,\varphi} \leq \omega \frac{\varsigma \epsilon_{j_k}}{4(1+\omega)} \frac{1}{\max[1, \kappa_s^p]} \frac{\min[\kappa_\delta(\sigma_{k,\max})\epsilon_{j_k}, \delta_{k,j_k}^{(0)}]^q}{q!}. \quad (4.28)$$

Note that this conclusion is independent of $\|s_k\|$.

• We next consider what can be said of the optimality conditions on the model when $\|s_k\| < 1$, and start by finding values of the absolute accuracy that are acceptably small to prevent any of the second set of calls to CHECK in Step 2.2 from returning **insufficient**. Lemmas 2.1(i) and 4.5—notice that the latter ensures that (2.3) holds when we invoke the former—and the argument list for CHECK suggest that this is impossible if

$$3 \max_{t \in \{\ell, \dots, p\}} \varphi_{t,k,\varphi} \sum_{i=1}^{\ell} \frac{\delta_{s_k,\ell}^i}{i!} \leq \varsigma \omega \frac{\theta(1-\omega)}{2(1+\omega)^2} \epsilon_\ell \frac{\delta_{s_k,\ell}^\ell}{\ell!} \quad (4.29)$$

for all $\ell \in \{1, \dots, q\}$. But since $\delta_{s_k} \leq 1$, the sum bound $\sum_{i=1}^p 1/i! < e$ again implies that

$$3 \max_{t \in \{\ell, \dots, p\}} \varphi_{t,k,\varphi} \sum_{i=1}^{\ell} \frac{\delta_{s_k,\ell}^i}{i!} \leq 6 \max_{t \in \{1, \dots, p\}} \varphi_{t,k,\varphi}, \quad (4.30)$$

while (4.19) and (4.20), the requirement that $\epsilon_j \leq 1$ and the bounds $\kappa_\delta(\sigma_{k,\max}) < 1$ and $\ell \leq q$ give that

$$\begin{aligned} \varsigma \omega \frac{\theta(1-\omega)}{2(1+\omega)^2} \epsilon_\ell \frac{\delta_{s_k,\ell}^\ell}{\ell!} &\geq \varsigma \omega \frac{\theta(1-\omega)\epsilon_\ell}{2(1+\omega)^2} \frac{\min[\kappa_\delta(\sigma_k)\epsilon_\ell, \delta_{k,\ell}^{(0)}]^\ell}{\ell!} \\ &\geq \varsigma \omega \frac{\theta(1-\omega)\epsilon_\ell}{2(1+\omega)^2} \frac{\min[\kappa_\delta(\sigma_{k,\max})\epsilon_\ell, \delta_{k,\ell}^{(0)}]^\ell}{\ell!} \end{aligned}$$

Therefore, in view of (4.30), (4.29) holds, and CHECK returns either **absolute** or **relative**, whenever

$$\max_{t \in \{1, \dots, p\}} \varphi_{t,k,\varphi} \leq \varsigma \omega \frac{\theta(1-\omega)\epsilon_{\min}}{12(1+\omega)^2} \frac{\min[\kappa_\delta(\sigma_{k,\max})\epsilon_{\min}, \delta_{k,\min}^{(0)}]^q}{q!} \quad (4.31)$$

where $\delta_{k,\min}^{(0)}$ is given by (4.24).

Consider the ℓ -th such call, and note that

$$\Delta T_{m_k,\ell}(s_k, d) \leq \Delta T_{\bar{m}_k,\ell}(s_k, d) + |\Delta T_{m_k,\ell}(s_k, d) - \Delta T_{\bar{m}_k,\ell}(s_k, d)| \quad (4.32)$$

for any $d \in \mathcal{B}_{\delta_{s_k,\ell}}$ because of the triangle inequality. If CHECK returns **absolute**, then (4.32), Lemma 2.1 (ii) and the fact that $\varsigma \leq 1$ yield that

$$\phi_{m_k,\ell}^{\delta_{s_k,\ell}}(s_k) = \max_{\|d\| \leq \delta_{s_k,\ell}} \Delta T_{m_k,\ell}(s_k, d) \leq 2\varsigma \frac{\theta(1-\omega)\epsilon_\ell}{2(1+\omega)^2} \frac{\delta_{s_k,\ell}^\ell}{\ell!}$$

and (4.18) holds. By contrast, if CHECK returns **relative**, then (4.32), Lemma 2.1(iii) and (3.11) ensure that, for some $d_{k,\ell}^* \in \mathcal{B}_{\delta_{s_k,\ell}}$,

$$\begin{aligned} \varsigma \phi_{m_k,\ell}^{\delta_{s_k,\ell}}(s_k) &\leq \varsigma(1+\omega) \Delta T_{\bar{m}_k,\ell}(s_k, d_{k,\ell}^*) \\ &\leq (1+\omega) \varsigma \phi_{\bar{m}_k,\ell}^{\delta_{s_k,\ell}}(s_k) \\ &\leq \varsigma \frac{\theta(1-\omega)\epsilon_\ell}{(1+\omega)} \frac{\delta_{s_k,\ell}^\ell}{\ell!} \end{aligned}$$

and (4.18) holds again. Thus (4.18) holds in both cases.

• We conclude from the above discussion, in particular from (4.28) and (4.31), that Step 2 terminates with a pair (s_k, δ_{s_k}) , for which (4.18) holds if $\|s_k\| \leq 1$, whenever

$$\max_{i \in \{1, \dots, p\}} \varphi_{i,k,\varphi} \leq \min [\kappa_{\text{step2}}(\sigma_{k,\max}) \epsilon_{\min}, \delta_{k,\min}^{(0)}]^q \epsilon_{\min}$$

where

$$\kappa_{\text{step2}}(\sigma) \stackrel{\text{def}}{=} \frac{\varsigma\omega (\kappa_\delta(\sigma))^q}{4q!(1+\omega)} \min \left[\frac{1}{\max[1, \kappa_s^p]}, \frac{\theta(1-\omega)}{3(1+\omega)} \right].$$

The algorithm then proceeds to Step 3, and thus $k \in \mathcal{T}$. □

We bring together two results obtained so far regarding bounds on $\delta_{k,j}$, namely those in Lemma 4.3 and 4.7.

Lemma 4.8 Suppose that **f.D0pL** holds, that iteration k of the AR qp EDA2 algorithm is successful, and that $\|s_k\| < 1$. Then

$$\delta_{k+1,j}^{(0)} = 1 \quad \text{for } j \in \{1, \dots, \min[2, q]\}$$

and

$$\delta_{k+1,j}^{(0)} \geq \min [\kappa_\delta(\sigma_k) \epsilon_j, \delta_{k,j}^{(0)}] \quad \text{for } j \in \{3, \dots, q\},$$

where $\kappa_\delta(\sigma)$ is defined in (4.21).

Proof. If iteration k is successful, Step 3 sets $\delta_{k+1,j}^{(0)} = \delta_{s_k}$. The stated bound then follows from (4.19) and (4.20). □

Lemma 4.9 Suppose that **f.D0pL** holds, and that the AR_{qp}EDA2 algorithm does not terminate at (or before) iteration k . Then

$$\delta_{k+1,j}^{(0)} \geq \kappa_\delta(\sigma_{k,\max}) \epsilon_j \quad \text{for } j \in \{1, \dots, q\}, \quad (4.33)$$

where $\kappa_\delta(\sigma)$ is defined in (4.21) and $\sigma_{k,\max} = \max_{i \in \{0, \dots, k\}} \sigma_i$. Moreover, $k \in \mathcal{T}$ whenever

$$\min_{i \in \{1, \dots, p\}} \varphi_{i,k,\varphi} \leq \kappa_{\text{step2}}(\sigma_{k,\max}) \epsilon_{\min}^{q+1}. \quad (4.34)$$

Proof. We prove this by induction over k for each $j \in \{1, \dots, q\}$. By assumption, the j -th initial radius satisfies

$$\delta_{0,j}^{(0)} = \delta_{0,j} \geq \epsilon_j > \kappa_\delta(\sigma_0) \epsilon_j = \kappa_\delta(\sigma_{0,\max}) \epsilon_j.$$

Suppose that $k = 0$. Then, as the algorithm does not terminate during this iteration, Lemma 4.3 indicates that either control is passed to Step 5, in which case

$$\delta_{1,j}^{(0)} = \delta_{0,j}^{(0)} \geq \kappa_\delta(\sigma_{0,\max}) \epsilon_j \quad (4.35)$$

as above, or (4.7) holds, i.e.,

$$\delta_{0,j}^{(1)} \geq \min \left[\frac{\varsigma \epsilon_j}{8(1+\omega) \max[\bar{L}_f, \sigma_0]}, \delta_{0,j}^{(0)} \right] \geq \kappa_\delta(\sigma_{0,\max}) \epsilon_j. \quad (4.36)$$

Step 2 may then pass control to Step 5 (with the same outcome (4.35) as for Step 1), but if not Step 3 will either result in

$$\delta_{1,j}^{(0)} \geq \min [\kappa_\delta(\sigma_0) \epsilon_j, \delta_{0,j}] \geq \min [\kappa_\delta(\sigma_{0,\max}) \epsilon_j, \delta_{0,j}] \geq \kappa_\delta(\sigma_{0,\max}) \epsilon_j$$

as per Lemma 4.8 when the iteration is successful and $\|s_0\| < 1$, or otherwise

$$\delta_{1,j}^{(0)} = \delta_{0,j}^{(1)} \geq \kappa_\delta(\sigma_{0,\max}) \epsilon_j$$

because of (4.36). Thus in all cases (4.33) holds for $k = 0$.

Now suppose that (4.33) holds up to iteration $k - 1$, i.e.,

$$\delta_{k,j}^{(0)} \geq \kappa_\delta(\sigma_{k-1,\max}) \epsilon_j \geq \kappa_\delta(\sigma_{k,\max}) \epsilon_j. \quad (4.37)$$

We show it also holds for iteration k . The proof is essentially identical to the $k = 0$ case. Once again, as the algorithm does not terminate during iteration k , Lemma 4.3 shows that either control is passed to Step 5, in which case

$$\delta_{k+1,j}^{(0)} = \delta_{k,j}^{(0)} \geq \kappa_\delta(\sigma_{k,\max}) \epsilon_j \quad (4.38)$$

from (4.37), or (4.7) holds, i.e.,

$$\begin{aligned} \delta_{k,j}^{(1)} &\geq \min \left[\frac{\varsigma \epsilon_j}{8(1+\omega) \max[\bar{L}_f, \sigma_k]}, \delta_{k,j}^{(0)} \right] \\ &\geq \min \left[\frac{\varsigma}{8(1+\omega) \max[\bar{L}_f, \sigma_k]}, \kappa_\delta(\sigma_{k,\max}) \right] \epsilon_j = \kappa_\delta(\sigma_{k,\max}) \epsilon_j \end{aligned} \quad (4.39)$$

using (4.37). As before, Step 2 may then pass control to Step 5 (and thus (4.38) holds), but if not Step 3 will either result in

$$\delta_{k+1,j}^{(0)} \geq \min \left[\kappa_\delta(\sigma_k) \epsilon_j, \delta_{k,j}^{(0)} \right] \geq \kappa_\delta(\sigma_{k,\max}) \epsilon_j$$

when the iteration is successful and $\|s_k\| < 1$, using Lemma 4.8 and (4.37), or otherwise

$$\delta_{k+1,j}^{(0)} = \delta_{k,j}^{(1)} \geq \kappa_\delta(\sigma_{k,\max}) \epsilon_j$$

because of (4.39). This completes the induction proving (4.33). The inequality (4.34) then follows by using (4.33) at iteration k (instead of $k+1$) in (4.23). \square

4.3 The final complexity bound

We are now poised to consider the evaluation complexity analysis of the fully-formed ARqpEDA2 algorithm. Having stated the necessary inexact variant of the standard bound on model decrease in (4.13), we next show that the regularization parameter σ_k generated by the algorithm remains bounded, even when f and its derivatives are computed inexactly.

Lemma 4.10 Suppose that **f.D0pL** holds. Then algorithm ARqpEDA2 ensures that

$$\sigma_k \leq \sigma_{\max} \stackrel{\text{def}}{=} \max \left[\sigma_0, \gamma_3 \frac{4L_{f,p}}{1-\eta_2} \right] \quad (4.40)$$

for all $k \geq 0$.

Proof. Suppose that $k \in \mathcal{T}$, and that

$$\sigma_k \geq \frac{4L_{f,p}}{1-\eta_2} \quad (4.41)$$

Then the triangle inequality, (4.17) that is guaranteed by Lemma 4.7 as iteration k proceeds to Step 3, and (3.13) together show that

$$\begin{aligned} |\overline{T}_{f,\ell}(x_k, s_k) - T_{f,\ell}(x_k, s_k)| &\leq |\overline{f}(x_k) - f(x_k)| + |\overline{\Delta T}_{f,\ell}(x_k, s_k) - \Delta T_{f,\ell}(x_k, s_k)| \\ &\leq 2\omega |\overline{\Delta T}_{f,\ell}(x_k, s_k)|. \end{aligned}$$

Therefore, again using the triangle inequality, (3.12), standard error bounds for Lipschitz

functions (see [7, Corollary A.8.4]), (4.13), (3.3) and (4.41), we deduce that

$$\begin{aligned}
|\rho_k - 1| &\leq \frac{|\bar{f}(x_k + s_k) - \bar{T}_{f,p}(x_k, s_k)|}{\Delta T_{f,p}(x_k, s_k)} \\
&\leq \frac{1}{\Delta T_{f,p}(x_k, s_k)} \left[|\bar{f}(x_k + s_k) - f(x_k + s_k)| \right. \\
&\quad \left. + |f(x_k + s_k) - T_{f,p}(x_k, s_k)| + |\bar{T}_{f,p}(x_k, s_k) - T_{f,p}(x_k, s_k)| \right] \\
&\leq \frac{1}{\Delta T_{f,p}(x_k, s_k)} \left[|f(x_k + s_k) - T_{f,p}(x_k, s_k)| + 3\omega \Delta \bar{T}_{f,p}(x_k, s_k) \right] \\
&\leq \frac{1}{\Delta T_{f,p}(x_k, s_k)} \left[\frac{L_{f,p}}{(p+1)!} \|s_k\|^{p+1} \right] + 3\omega \\
&< \frac{L_{f,p}}{\sigma_k} + \frac{3(1-\eta_2)}{4} \\
&\leq 1 - \eta_2
\end{aligned}$$

and thus that $\rho_k \geq \eta_2$. Then iteration k is very successful in that $\rho_k \geq \eta_2$ and, because of (3.15), $\sigma_{k+1} \leq \sigma_k$. As a consequence, the mechanism of the algorithm ensures that (4.40) holds for all $k \in \mathcal{T}$, while if $k \in \mathcal{A}$, Step 5 fixes $\sigma_{k+1} = \sigma_k$. \square

We now state a useful technical inequality.

Lemma 4.11 Suppose that **f.D0pL** holds. Suppose also that iteration k of the AR qp EDA2 algorithm is successful, that $\|s_k\| < 1$, but that iteration k_+ is the first iteration after k that proceeds to Step 2 rather than Step 5. Then there exists $j \in \{1, \dots, q\}$ such that

$$\frac{\varsigma(1-\theta)(1-\omega)}{1+\omega} \epsilon_j \frac{\left(\delta_{k_+,j}^{(0)}\right)^j}{j!} \leq (L_{f,p} + \sigma_{\max}) \sum_{\ell=1}^j \frac{\left(\delta_{k_+,j}^{(0)}\right)^\ell}{\ell!} \|s_k\|^{p-\ell+1}, \quad (4.42)$$

where σ_{\max} is defined by (4.40).

Proof. Observe that our assumption that iteration k_+ passes to Step 2 means that it does not terminate in Step 1, nor does it pass to Step 5. The latter implies that

$$\Delta \bar{T}_{f,j}(x_{k_+}, d_{k_+,j}) > \left(\frac{\varsigma \epsilon_j}{1+\omega} \right) \frac{\left(\delta_{k_+,j}^{(0)}\right)^j}{j!}$$

must hold for some $j \in \{1, \dots, q\}$. Furthermore, since any iteration that might occur between k and k_+ must have passed through Step 5, $\delta_{k_+,j}^{(0)} = \delta_{s_k,j}$ as the radii update rule in Step 5 does not change the input value of δ_k . Hence (4.8) from Lemma 4.3 ensures that

$$\phi_{f,j}^{\delta_{k_+,j}^{(0)}}(x_{k_+}) > \varsigma \left(\frac{1-\omega}{1+\omega} \right) \epsilon_j \frac{\left(\delta_{k_+,j}^{(0)}\right)^j}{j!}. \quad (4.43)$$

and therefore that

$$\begin{aligned}
\varsigma \left(\frac{1-\omega}{1+\omega} \right) \epsilon_j \frac{(\delta_{+,j}^{(0)})^j}{j!} &< \phi_{f,j}^{\delta_{+,j}^{(0)}}(x_{k+1}) = - \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_x^\ell f(x_{k+1}) [d]^\ell \\
&= - \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_x^\ell f(x_{k+1}) [d]^\ell + \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell T_{f,p}(x_k, s_k) [d]^\ell \\
&\quad - \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell T_{f,p}(x_k, s_k) [d]^\ell - \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell (\|s_k\|^{p+1}) [d]^\ell \\
&\quad + \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell (\|s_k\|^{p+1}) [d]^\ell.
\end{aligned} \tag{4.44}$$

To bound the terms on the right-hand side of (4.44), firstly, using standard approximation properties for Lipschitz function (once more, see [7, Corollary A.8.4]),

$$\begin{aligned}
& - \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_x^\ell f(x_{k+1}) [d]^\ell + \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell T_{f,p}(x_k, s_k) [d]^\ell \\
& \leq \sum_{\ell=1}^j \frac{\delta_{k+1,j}^\ell}{\ell!} \|\nabla_x^\ell f(x_{k+1}) - \nabla_s^\ell T_{f,p}(x_k, s_k)\| \\
& \leq L_{f,p} \sum_{\ell=1}^j \frac{\delta_{k+1,j}^\ell}{\ell!(p-\ell+1)!} \|s_k\|^{p-\ell+1},
\end{aligned} \tag{4.45}$$

where $L_{f,p}$ is defined in **f.D1pL**. Furthermore, because of (3.1) and (3.11),

$$\begin{aligned}
& - \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell T_{f,p}(x_k, s_k) [d]^\ell - \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell (\|s_k\|^{p+1}) [d]^\ell \\
& \leq \phi_{m_k,j}^{\delta_{s_k,j}}(s_k) \leq \theta \epsilon_j \frac{\delta_{s_k,j}^j}{j!} = \theta \epsilon_j \frac{\delta_{k+1,j}^j}{j!},
\end{aligned} \tag{4.46}$$

where the last equality follows as $\delta_{s_k,j} = \delta_{+,j}^{(0)}$ if iteration k is successful. Moreover, in view of the form of the derivatives of the regularization term $\|s\|^p$ (see [7, Lemma B.4.1] with $\beta = 1$) and Lemma 4.10, we also have that

$$\begin{aligned}
\frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell (\|s_k\|^{p+1}) [d]^\ell &\leq \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{1}{\ell!} \left\| \nabla_s^\ell (\|s_k\|^{p+1}) \right\| \|d\|^\ell \\
&= \sigma_k \sum_{\ell=1}^j \frac{\|s_k\|^{p-j+1} \|d\|^\ell}{\ell!(p-j+1)!} \\
&\leq \sigma_{\max} \sum_{\ell=1}^j \frac{\delta_{k+1,j}^\ell}{\ell!(p-j+1)!} \|s_k\|^{p-j+1}.
\end{aligned} \tag{4.47}$$

We then observe that,

$$\phi_{m_k,j}^{\delta_{k+1,j}^{(0)}}(s_k) = \phi_{m_k,j}^{\delta_{s_k,j}}(s_k) \leq \varsigma \theta \left(\frac{1-\omega}{1+\omega} \right) \epsilon_j \frac{\delta_{s_k,j}^j}{j!}, \tag{4.48}$$

where we have used the fact that $\delta_{k+,j}^{(0)} = \delta_{s_k,j}$ for $k \in \mathcal{S}$ to derive the first inequality and the fact that $\|s_k\| < 1$ to apply the bound (4.18) of Lemma 4.7. The proof is then concluded by replacing the second inequality in (4.46) by (4.48) and combining the result with (4.44), (4.45) and (4.47). \square

Our next step is to provide a lower bound on the step at iterations before termination.

Lemma 4.12 Suppose that **F.D0pL** holds, iteration k is successful, and that the ARqpEDA2 algorithm does not terminate at first iteration after k that proceeds to Step 2 rather than Step 5. Suppose also that the algorithm chooses δ_k for each k such that the conclusions of Lemma 4.8 hold. Then there exists a $j \in \{1, \dots, q\}$ such that

$$\|s_k\| \geq \begin{cases} \left(\frac{\varsigma(1-\theta)(1-\omega)}{2j!(L_{f,p} + \sigma_{\max})(1+\omega)} \right)^{\frac{1}{p-j+1}} \epsilon_j^{\frac{1}{p-j+1}} & \text{if } q \in \{1, 2\}, \\ \left(\frac{\varsigma(1-\theta)(1-\omega)\kappa_{\delta,\min}^{j-1}}{2j!(L_{f,p} + \sigma_{\max})(1+\omega)} \right)^{\frac{1}{p}} \epsilon_j^{\frac{j}{p}} & \text{if } q > 2, \end{cases} \quad (4.49)$$

where $\kappa_{\delta,\min} \stackrel{\text{def}}{=} \kappa_{\delta}(\sigma_{\max})$ and σ_{\max} is defined by (4.40).

Proof. Either $\|s_k\| \geq 1$ (case (i) in Lemma 4.7) and (4.49) holds automatically because the two lower bounds on its right-hand side are less than one. So suppose instead that $\|s_k\| \leq 1$. Because the algorithm does not terminate, Lemma 4.11 ensures that (4.42) holds for some $j \in \{1, \dots, q\}$. It is easy to verify that this inequality is equivalent to

$$\tau \epsilon_j (\delta_{+,j}^{(0)})^j \leq \|s_k\|^{p+1} \chi_j \left(\frac{\delta_{+,j}^{(0)}}{\|s_k\|} \right) \quad (4.50)$$

where the function χ_j is defined by

$$\chi_j(t) = \sum_{\ell=1}^j \frac{t^\ell}{\ell!} \quad (4.51)$$

and where we have set

$$\tau = \frac{(1-\omega)(1-\theta)}{j!(1+\omega)(L_{f,p} + \sigma_{\max})}.$$

In particular, since $\chi_j(t) \leq 2t^j$ for $t \geq 1$, we have that

$$\tau \epsilon_j \leq 2\|s_k\|^{p+1} \left(\frac{1}{\|s_k\|} \right)^j = 2\|s_k\|^{p-j+1} \quad (4.52)$$

whenever $\|s_k\| \leq \delta_{k+,j}$.

If j is 1 or 2, by assumption $\delta_{+,j}^{(0)} = 1$ and $\|s_k\| \leq 1 = \delta_{+,j}^{(0)}$. Thus (4.52) yields the first case of (4.49). Otherwise, if $j \geq 3$, by assumption (4.33) holds. In this case, if $\|s_k\| \leq \delta_{+,j}^{(0)}$, we may

again deduce from (4.52) that the first case of (4.49) holds, and this implies that the second case also holds since $\kappa_{\delta, \min} < 1$ and⁽²⁾ $1/(p-j+1) \leq j/p$ for $p \geq j \geq 1$. Consider therefore the alternative for which $\|s_k\| > \delta_{+,j}^{(0)}$. Then (4.50), and noting that $\chi_j(t) < 2t$ for $t \in [0, 1]$, we deduce that

$$\tau \epsilon_j (\delta_{+,j}^{(0)})^j \leq 2 \|s_k\|^{p+1} \left(\frac{\delta_{+,j}^{(0)}}{\|s_k\|} \right),$$

which, with (4.33), implies the second case of (4.49). \square

We now consolidate the above results by stating lower bounds on the minimal model and function decreases at successful iterations.

Lemma 4.13 Suppose that **f.D1qL** holds. Then

$$\overline{\Delta T}_{f,p}(x_k, s_k) \geq \kappa_{\Delta m} \min_{j \in \{1, \dots, q\}} \epsilon_j^{\pi_j}, \quad (4.53)$$

for every iteration of algorithm AR_{qp}EDA2 before termination, where

$$\kappa_{\Delta m} \stackrel{\text{def}}{=} \begin{cases} \frac{\sigma_{\min}}{(p+1)!} \left(\frac{\varsigma(1-\theta)(1-\omega)}{2q!(L_{f,p} + \sigma_{\max})(1+\omega)} \right)^{\frac{p+1}{p-q+1}} & \text{if } q \in \{1, 2\}, \\ \frac{\sigma_{\min}}{(p+1)!} \left(\frac{\varsigma(1-\theta)(1-\omega)\kappa_{\delta, \min}^{q-1}}{2q!(L_{f,p} + \sigma_{\max})(1+\omega)} \right)^{\frac{q(p+1)}{p}} & \text{if } q > 2, \end{cases}$$

whith $\kappa_{\delta, \min} \stackrel{\text{def}}{=} \kappa_{\delta}(\sigma_{\max})$, σ_{\max} by (4.40), and

$$\pi_j \stackrel{\text{def}}{=} \begin{cases} \frac{p+1}{p-j+1} & \text{if } q \in \{1, 2\}, \\ \frac{j(p+1)}{p} & \text{if } q > 2. \end{cases}$$

If, in addition, $k \in \mathcal{S}$,

$$f(x_k) - f(x_{k+1}) \geq (\eta_1 - 2\omega) \kappa_{\Delta m} \min_{j \in \{1, \dots, q\}} \epsilon_j^{\pi_j}. \quad (4.54)$$

Proof. The bound (4.13), (3.15) and Lemma 4.12 together imply that for every $k \in \mathcal{S}$, there

⁽²⁾This is easily verified by noting that, if $\lambda(t) = t/(j(t-j+1))$, then $\lambda(j) = 1$ and $\lambda^{(1)}(t) \leq 0$ for all $t \geq j$.

exists a $j \in \{1, \dots, q\}$ such that

$$\overline{\Delta T}_{f,p}(x_k, s_k) \geq \begin{cases} \frac{\sigma_{\min}}{(p+1)!} \left(\frac{\varsigma(1-\theta)(1-\omega)}{2j!(L_{f,p} + \sigma_{\max})(1+\omega)} \right)^{\frac{p+1}{p-j+1}} \epsilon_j^{\frac{p+1}{p-j+1}} & \text{if } q \in \{1, 2\}, \\ \frac{\sigma_{\min}}{(p+1)!} \left(\frac{\varsigma(1-\theta)(1-\omega)\kappa_{\delta, \min}^{q-1}}{2j!(L_{f,p} + \sigma_{\max})(1+\omega)} \right)^{\frac{j(p+1)}{p}} \epsilon_j^{\frac{j(p+1)}{p}} & \text{if } q > 2, \end{cases}$$

and (4.53) follows. Suppose now that k is the index of a successful iteration before termination. Then, using (3.12) and (3.13), the implication that $\rho_k \geq \eta_1$ with (3.14), and (4.13) and (3.15),

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq [\bar{f}(x_k) - \bar{f}(x_{k+1})] - 2\omega \overline{\Delta T}_{f,p}(x_k, s_k) \\ &\geq (\eta_1 - 2\omega) \Delta T_{f,p}(x_k, s_k) \\ &\geq (\eta_1 - 2\omega) \kappa_{\Delta m} \min_{j \in \{1, \dots, q\}} \epsilon_j^{\pi_j}, \end{aligned}$$

where we note that $\eta_1 - 2\omega > 0$ from (3.3). This proves (4.54). \square

We are now in position to state formally a bound on the evaluation complexity of the ARqpEDA2 algorithm.

Theorem 4.14 Suppose that **f.Bb** and **f.D0pL** hold. Suppose moreover that the ARqpEDA2 algorithm chooses δ_k for each k so that the conclusions of Lemma 4.8 hold.

1. If $q \in \{1, 2\}$, then there exist positive constants $\kappa_{\text{ARqpEDA2},1}^{\text{S}}$, $\kappa_{\text{ARqpEDA2},1}^{\text{A}}$, $\kappa_{\text{ARqpEDA2},1}^{\text{C}}$, $\kappa_{\text{ARqpEDA2},1}^{\text{E}}$ and $\kappa_{\text{ARqpEDA2},1}^{\text{F}}$ such that, for any $\epsilon \in (0, 1]^q$, the ARqpEDA2 algorithm requires at most

$$\begin{aligned} \#_{\text{ARqpEDA2},1}^{\text{F}} &\stackrel{\text{def}}{=} \kappa_{\text{ARqpEDA2},1}^{\text{A}} \frac{f(x_0) - f_{\text{low}}}{\epsilon_j^{\frac{p+1}{p-j+1}}} + \kappa_{\text{ARqpEDA2},1}^{\text{C}} \\ &= \mathcal{O} \left(\max_{j \in \{1, \dots, q\}} \epsilon_j^{-\frac{p+1}{p-j+1}} \right) \end{aligned} \quad (4.55)$$

evaluations of f , and at most

$$\begin{aligned} \#_{\text{ARqpEDA2},1}^{\text{D}} &\stackrel{\text{def}}{=} \kappa_{\text{ARqpEDA2},1}^{\text{S}} \frac{f(x_0) - f_{\text{low}}}{\epsilon_j^{\frac{p+1}{p-j+1}}} + \kappa_{\text{ARqpEDA2},1}^{\text{E}} \left| \log \left(\min_{j \in \{1, \dots, q\}} \epsilon_j \right) \right| + \kappa_{\text{ARqpEDA2},1}^{\text{F}} \\ &= \mathcal{O} \left(\max_{j \in \{1, \dots, q\}} \epsilon_j^{-\frac{p+1}{p-j+1}} \right) \end{aligned} \quad (4.56)$$

evaluations of the derivatives of f of orders one to p to produce an iterate x_ϵ for which $\phi_{f,j}^1(x_\epsilon) \leq \epsilon_j/j!$ for all $j \in \{1, \dots, q\}$.

2. If $q > 2$, then there exist positive constants $\kappa_{\text{ARqpEDA2,2}}^{\text{S}}$, $\kappa_{\text{ARqpEDA2,2}}^{\text{A}}$, $\kappa_{\text{ARqpEDA2,2}}^{\text{C}}$, $\kappa_{\text{ARqpEDA2,2}}^{\text{E}}$ and $\kappa_{\text{ARqpEDA2,2}}^{\text{F}}$ such that, for any $\epsilon \in (0, 1]^q$, the ARqpEDA2 algorithm requires at most

$$\begin{aligned} \#_{\text{ARqEDA},2}^{\text{F}} &\stackrel{\text{def}}{=} \kappa_{\text{ARqpEDA2,2}}^{\text{A}} \frac{f(x_0) - f_{\text{low}}}{\frac{j(p+1)}{p}} + \kappa_{\text{ARqpEDA2,2}}^{\text{C}} \\ &= \mathcal{O} \left(\max_{j \in \{1, \dots, q\}} \epsilon_j \frac{-j(p+1)}{p} \right) \end{aligned}$$

evaluations of f , and at most

$$\begin{aligned} \#_{\text{ARqEDA},2}^{\text{D}} &\stackrel{\text{def}}{=} \kappa_{\text{ARqpEDA2,2}}^{\text{S}} \frac{f(x_0) - f_{\text{low}}}{\frac{j(p+1)}{p}} + \kappa_{\text{ARqpEDA2,2}}^{\text{E}} \left| \log \left(\min_{j \in \{1, \dots, q\}} \epsilon_j \right) \right| + \kappa_{\text{ARqpEDA2,2}}^{\text{F}} \\ &= \mathcal{O} \left(\max_{j \in \{1, \dots, q\}} \epsilon_j \frac{-j(p+1)}{p} \right) \end{aligned}$$

evaluations of the derivatives of f of orders one to p to produce an iterate x_ϵ for which $\phi_{f,j}^{\delta_\epsilon, j}(x_\epsilon) \leq \epsilon_j \delta_{\epsilon,j}^j / j!$ for some $\delta_\epsilon \in (0, 1]^q$ and all $j \in \{1, \dots, q\}$.

Proof. Consider first the case where $q \in \{1, 2\}$. Using (4.54) in Lemma 4.13, **f.Bb** and the fact that the sequence $\{f(x_k)\}$ is non-increasing, we deduce that the algorithm needs at most

$$\kappa_{\text{ARqpEDA2,1}}^{\text{S}} \frac{f(x_0) - f_{\text{low}}}{\frac{p+1}{\epsilon_{\min}^{p-q+1}}} + 1 \quad (4.57)$$

successful iterations to produce a point x_ϵ for which $\phi_{f,j}^1(x_\epsilon) \leq \epsilon_j / j!$ for all $j \in \{1, \dots, q\}$, where

$$\kappa_{\text{ARqpEDA2,1}}^{\text{S}} \stackrel{\text{def}}{=} \frac{(p+1)!}{(\eta_1 - 2\omega)\sigma_{\min}} \left(\frac{2q!(L_{f,p} + \varphi_{\max} + \sigma_{\max})(1 + \omega)}{(1 - \theta)(1 - \omega)} \right). \quad (4.58)$$

We may then invoke Lemma 3.1 to deduce that the total number of iterations required is bounded by

$$|\mathcal{S}_k| \left(1 + \frac{|\log \gamma_1|}{\log \gamma_2} \right) + \frac{1}{\log \gamma_2} \log \left(\frac{\sigma_{\max}}{\sigma_0} \right) + 1,$$

where σ_{\max} is given by (4.40), and hence the total number of approximate function evaluations is at most twice this number, which yields (4.55) with the coefficients

$$\kappa_{\text{ARqpEDA2,1}}^{\text{A}} = 2\kappa_{\text{ARqpEDA2,1}}^{\text{S}} \left(1 + \frac{|\log \gamma_1|}{\log \gamma_2} \right) \quad \text{and} \quad \kappa_{\text{ARqpEDA2,1}}^{\text{C}} = \frac{2}{\log \gamma_2} \log \left(\frac{\sigma_{\max}}{\sigma_0} \right) + 2.$$

In order to derive an upper bound on the the number of derivative evaluations, we now have to count the number of additional evaluations caused by the need to approximate the derivatives to the desired accuracy. Again, repeated evaluations at a given iterate x_k are only needed when the current values of the required absolute errors are smaller than used previously at x_k . Recall that these required absolute errors are initialised in Step 0 of the ARqpDA algorithm, and by construction decrease linearly with rate γ_φ on every pass to Step 5. We may now use

Lemmas 4.3, 4.7, 4.9 and 4.10 to deduce that the maximal accuracy bound $\max_{i \in \{1, \dots, p\}} \varphi_{i, k_\varphi}$ will not be reduced below

$$\begin{aligned} \kappa_{\text{acc}} \epsilon_{\min}^{q+1} &\stackrel{\text{def}}{=} \min \left[\frac{\zeta \omega}{4q!} [\kappa_\delta(\sigma_{\max})]^{q-1}, \kappa_{\text{step}2}(\sigma_{\max}) \right] \epsilon_{\min}^{q+1} \\ &\leq \min \left[\frac{\omega}{4} \min \left[\frac{\zeta \epsilon_j}{8(1+\omega) \max[\bar{L}_f, \sigma_k]}, \delta_{k,j}^{(0)} \right]^{j-1} \frac{\epsilon_j}{j!}, \kappa_{\text{step}2}(\sigma_{k, \max}) \epsilon_{\min}^{q+1} \right] \end{aligned}$$

at iteration k . As κ_{acc} is independent of k , it follows that no further evaluations of $\{\overline{\nabla}_x^i f(x_k)\}_{i=1}^p$ can possibly be required during iteration k or beyond once the largest initial absolute error $\max_{j \in \{1, \dots, p\}} \varphi_{j,0}$ has been reduced by successive multiplications by γ_φ in Step 5 sufficiently often to ensure that

$$\gamma_\varphi^{k_\varphi} \left[\max_{i \in \{1, \dots, p\}} \varphi_{i,0} \right] \leq \kappa_{\text{acc}} \epsilon_{\min}^{q+1}. \quad (4.59)$$

Since the $\varphi_{i,0}$ are initialised in the algorithm so that $\max_{i \in \{1, \dots, p\}} \varphi_{i,0} \leq \varphi_{\max}$, the bound (4.59) is achieved once k_φ , the number of decreases in $\{\varphi_{i, k_\varphi}\}_{j=1}^p$, is large enough to guarantee that

$$\gamma_\varphi^{k_\varphi} \varphi_{\max} \leq \kappa_{\text{acc}} \epsilon_{\min}^{q+1}. \quad (4.60)$$

Thus we obtain that the number of evaluations of the derivatives of the objective function that occur during the course of the ARqpEDA2 algorithm is at most

$$|\mathcal{S}_k| + |\mathcal{A}_k| = |\mathcal{S}_k| + k_{\varphi, \min},$$

i.e., the number successful iterations in (4.57) plus

$$\begin{aligned} k_{\varphi, \min} &\stackrel{\text{def}}{=} \left\lceil \frac{1}{\log(\gamma_\varphi)} \left\{ (q+1) \log(\epsilon_{\min}) + \log\left(\frac{\kappa_{\text{acc}}}{\varphi_{\max}}\right) \right\} \right\rceil \\ &\leq \frac{q+1}{|\log(\gamma_\varphi)|} |\log(\epsilon_{\min})| + \frac{1}{|\log(\gamma_\varphi)|} \left| \log\left(\frac{\kappa_{\text{acc}}}{\varphi_{\max}}\right) \right| + 1, \end{aligned}$$

the smallest value of k_φ that ensures (4.60). This leads to the desired evaluation bound (4.56) with the coefficients

$$\kappa_{\text{ARqpEDA}2,1}^{\text{E}} \stackrel{\text{def}}{=} \frac{q+1}{|\log \gamma_\varphi|} \quad \text{and} \quad \kappa_{\text{ARqpEDA}2,1}^{\text{F}} \stackrel{\text{def}}{=} \frac{1}{|\log(\gamma_\varphi)|} \left| \log\left(\frac{\kappa_{\text{acc}}}{\varphi_{\max}}\right) \right| + 2.$$

The reasoning is essentially the same for the case where $q > 2$, except that, in view of (4.49), we use

$$\kappa_{\text{ARqpEDA}2,2}^{\text{S}} = \frac{(p+1)!}{(\eta_1 - 2\omega)\sigma_{\min}} \left(\frac{2j!(L_{f,p} + \sigma_{\max})(1+\omega)}{(1-\theta)(1-\omega)\kappa_{\delta, \min}^{j-1}} \right)^{\frac{p+1}{p}}$$

instead of (4.58). This then yields

$$\kappa_{\text{ARqpEDA}2,2}^{\text{A}} \stackrel{\text{def}}{=} \kappa_{\text{ARqpEDA}2,2}^{\text{S}} \left(1 + \frac{|\log \gamma_1|}{\log \gamma_2} \right),$$

all other constants being unchanged, that is

$$\kappa_{\text{ARqpEDA}2,2}^{\text{C}} = \kappa_{\text{ARqpEDA}2,1}^{\text{C}}, \quad \kappa_{\text{ARqpEDA}2,2}^{\text{E}} = \kappa_{\text{ARqpEDA}2,1}^{\text{E}} \quad \text{and} \quad \kappa_{\text{ARqpEDA}2,2}^{\text{F}} = \kappa_{\text{ARqpEDA}2,1}^{\text{F}}.$$

□

Since the orders in ϵ_{\min} are the same as those derived for the AR qp algorithm using exact evaluations (as defined in [7, Chapter 12]) and because these were proved to be sharp (see Section 12.2.2.4 in this reference), the same conclusion obviously holds for the AR qp EDA2 algorithm using inexact evaluations.

5 Conclusions

Given the significant complexity of the theory advanced above, the reader will undoubtedly understand why a simpler version of the AR qp EDA has been developed and analyzed in [7, Chapter 13]. However, AR qp EDA2 is not without merits. In particular, its distinguishing feature, the requirement (1.11), may be of interest as it is independent of variable scaling, a sometimes very desirable property.

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