Stable Recovery of Sparse Signals With Non-convex Weighted r-Norm Minus 1-Norm

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Abstract. Given the measurement matrix A and the observation signal y, the central purpose of compressed sensing is to find the most sparse solution of the underdetermined linear system $y = Ax + z$, where x is the s-sparse signal to be recovered and z is the noise vector. Zhou and Yu [1] recently proposed a novel non-convex weighted $\ell_r - \ell_1$ minimization method for effective sparse recovery. In this paper, we reveal that based on (y, A), any s-sparse signal can be robustly reconstructed via this method provided that the mutual coherence μ of A fulfills $\mu < 1/(s - 1 + 2^{1/r-1}s^{1/r})$. To our best of knowledge, this is the first mutual coherence based sufficient condition for such approach.

Key words. Compressed sensing; sparse recovery; mutual coherence; sufficient condition.

1 Introduction

Compressed sensing [2, 3] has recently triggered much interest in signal and imaging processing, statistics and applied mathematics. The crucial aim is to recover a high dimensional sparse signal from a small quantity linear measurements. Generally, one thinks about the linear model:

$$y = Ax + z, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times N}$ is the measurement matrix with $m \ll N$, $z \in \mathbb{R}^m$ is the noise vector and $x \in \mathbb{R}^N$ is an s-sparse (i.e., the number of nonzero elements of x is not more than s) vector to be recovered. Two widely utilized types of noises are the bounded ℓ_2 noise [4] and the Dantzig selector noise [5], respectively. Throughout the article suppose that the columns of A are standardized, i.e., for all i, $A_i^\top A_i = 1$, where $A_i(1 \le i \le N)$ denotes the i-the column of A.

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Because the linear model (1.1) is the underdetermined linear system, it is impossible to stably reconstruct x based on A, z and y. Fortunately, it is possible to stably reconstruct s-sparse signal x from (1.1) with a few appropriately exploiting sparse reconstruction methods under suitable assumptions regarding A and z. There exist two extensively applied frameworks to describe such assumptions concerning A, which are separately the restricted isometry property (for short RIP) and the mutual coherence determined as [6]

$$\mu = \max_{1\leq i<j\leq N} |\langle A_i, A_j\rangle| . \tag{1.2}$$

It is well known that ℓ_1 minimization method [7], viewed as a convex extension of ℓ_0 minimization method, presents an efficiency approach for recovering s-sparse signal in numerous contexts. In recent years, one alternative approach of estimating the s-sparse signal in the references [8–11] is to solve the below model

$$\min_{\tilde{x}\in\mathbb{R}^N} \|\tilde{x}\|_r^r \text{ subject to } y = A\tilde{x} + z,\ \|z\|_2 \leq \epsilon, \tag{1.3}$$

where $\|\tilde{x}\|_r^r = \sum_{i=1}^N \tilde{x}_i^r$ with $\tilde{x}_i$ being the i-th entry of $\tilde{x}$, $r \in (0,1]$ and $\|z\|_2 = (\sum_{i=1}^m z_i^2)^{1/2}$. Although compared with ℓ_1 minimization, it is more difficult to resolve model (1.3) for its noncovexity, there still exist a lot of algorithms to find the local optimal solution of (1.3). Besides, it has been showed by [12] that solving the ℓ_r minimization model with small r can significantly reduce the number of measurements.

Zhou and Yu [1] recently introduced a novel weighted $\ell_r - \ell_1$ minimization method as follows

$$\min_{\tilde{x}\in\mathbb{R}^N} \|\tilde{x}\|_r^r - \alpha\|\tilde{x}\|_1 \text{ subject to } y = A\tilde{x} + z,\ \|z\|_2 \leq \epsilon, \tag{1.4}$$

where $\alpha \in [0,1]$, $\|\tilde{x}\|_1 = \sum_{i=1}^N |\tilde{x}_i|$, and suppose that $\alpha \neq 1$ in the case of $r = 1$. It is obvious that (1.4) degenerates to the traditional ℓ_r minimization model in the case of $\alpha = 0$. Though resolving model (1.4) is more hard than to resolve model (1.3) for mixed norm in it, a great deal of algorithms also can be employed to solve it, see, e.g., [1, 13, 14]. Furthermore, it can enhance the ℓ_1 minimization in a robust pattern. Because there are effective algorithms for resolving (1.4), main objective of this paper is to exploit a sufficient condition that can guarantee the stable recovery of x through resolving (1.4) rather than discussing how to efficiently resolving (1.4).

Recently, many literatures [1, 14, 15] have studied the sufficient conditions for robust recovery of x via solving model (1.4) based on RIP. Different from former contributions utilizing the RIP to depict the sufficient condition, this paper makes use of the mutual coherence μ to describe the new sufficient condition. Actually, under the same conditions, the mutual coherence μ of a given matrix is easier to calculate than its RIP constant. In addition, it is difficult to verify the RIP condition within efficient time for a given matrix.

Since the Dantzig selector noise is also extensively investigated noise in compressed sensing, the sufficient condition of stably reconstruction x through

$$\min_{\tilde{x}\in\mathbb{R}^N} \|\tilde{x}\|_r^r - \alpha\|\tilde{x}\|_1 \text{ subject to } y = A\tilde{x} + z,\ \|A^\top z\|_\infty \leq \epsilon \tag{1.5}$$

is also discussed.

Our results will show that any s-sparse signal x from (1.1) can be stably reconstructed through solving (1.4) or (1.5) provided that the mutual coherence μ of A fulfills $\mu < 1/(s-1+2^{1/r-1}s^{1/r})$. To our best of knowledge, this is the first sufficient condition for robust reconstruction of x by solving (1.4) and (1.5) based on mutual coherence. The Gaussian noise is of special interest in signal and image processing as well as in statistics. Since the Gaussian noise is essentially bounded, the results can be generalized to it.

The content of this paper is constructed as follows. We begin by providing some notations and some lemmas that will be needed in our analysis in Section 2. The main results and its corresponding proofs are presented in Section 3 and Section 4, respectively. In Section 5, a conclusion is provided.

2 Preliminaries

We first of all explicate some necessary notations. Let S represent the support of x, that is, $S = \{i|\ x_i \neq 0\}$. For any set T, let x_T stand for a vector that keeps the entries indexed by T of x and 0 otherwise. Let S^c indicate the complement of S, that is, $S^c = \{1, 2, \cdots, N\}\backslash S$. For any matrix Φ, $\Phi^\top$ denote the transpose of Φ.

In the following, we present some auxiliary lemmas which are needed for the proofs of our main results.

Lemma 2.1. *([16–18]) For any s-sparse vector u, we get*

$$1 - (s-1)\mu \leq \frac{\|Au\|_2^2}{\|u\|_2^2} \leq 1 + (s-1)\mu. \tag{2.6}$$

Lemma 2.2. *For a general vector x (i.e., x is not s-sparse), we have*

$$\|h_{S^c}\|_r^r \leq \|h_S\|_r^r + \alpha\|h\|_1^r + 2\|x_{S^c}\|_r^r. \tag{2.7}$$

Lemma 2.3. *Let $\hat{x}$ be the solution of (1.4), and stand for the recovery error $h = \hat{x} - x$. It is assumed that A and z in (1.1) fulfill*

$$\mu < \frac{1}{s - 1 + 2^{\frac{1}{r}-1}s^{\frac{1}{r}}} \tag{2.8}$$

and $\|z\|_2 \leq \epsilon$, separately. Then,

$$\|h_S\|_2 \leq \frac{2\epsilon\sqrt{1+(s-1)\mu}}{1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu} + \frac{2^{\frac{1}{r}-1}\mu N^{\frac{1}{2}}\alpha^{\frac{1}{r}}s^{\frac{1}{2}}\|h\|_2}{1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu}. \tag{2.9}$$

Lemma 2.4. *Let $\hat{x}$ be the solution of (1.5), and indicate the recovery error $h = \hat{x} - x$. We assume A and z in (1.1) meet (2.8) and $\|A^\top z\|_\infty \leq \epsilon$, respectively. Then,*

$$\|h_S\|_2 \leq \frac{2\epsilon\sqrt{s}}{1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu} + \frac{2^{\frac{1}{r}-1}\mu N^{\frac{1}{2}}\alpha^{\frac{1}{r}}s^{\frac{1}{2}}\|h\|_2}{1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu}. \tag{2.10}$$

3 Main results

In this part, based on the mutual coherence of A, the sufficient conditions for robust reconstruction of s-sparse signals x via (1.4) and (1.5) are explored. First of all, a sufficient condition for robust reconstruction of k-sparse signals x by (1.4) is given, which is stated as follows.

Theorem 3.1. *Let $\hat{x}$ be the solution of (1.4). If A and z in (1.1) fulfill (2.8) and $\|z\|_2 \leq \epsilon$, then*

$$\|\hat{x} - x\|_2 \leq C\epsilon, \tag{3.11}$$

where C depends on μ, r, N, α and s, which is determined in the proof of Theorem 3.1.

Then, a sufficient condition for robust recovery of s-sparse signal x by (1.5) is provided.

Theorem 3.2. *Let $\hat{x}$ be the solution of (1.5). If A and z in (1.1) fulfill (2.8) and $\|A^\top z\|_\infty \leq \epsilon$, then*

$$\|\hat{x} - x\|_2 \leq D\epsilon, \tag{3.12}$$

where D relies on μ, r, N, α and s, which is defined in the proof of Theorem 3.2.

Remark 3.3. *From Theorems 3.1 and 3.2, we can observe that any s-sparse signal x corrupted by the bounded ℓ_2 noise or Dantzig selector noise can be robustly reconstructed through (1.4) and (1.5), separately, if the matrix A satisfies condition (2.8), and the associating reconstruction error can be controlled by (3.11) and (3.12), separately. This reveals the effectiveness of reconstructing s-sparse signals by methods (1.4) and (1.5) from a theoretical point of viewpoint.*

Remark 3.4. *When $r = 1$ and $\alpha \neq 1$, the mutual coherence condition (2.8) is the same as Theorems 2.1 and 2.2 in [16].*

Remark 3.5. *The literature has presented the relationship between RIP and mutual coherence. It follows from [19] that $\delta_s \leq (s-1)\mu$. Moreover, the monotone property on δ_s can be found in [20, 21]: $\delta_s \leq \delta_t$, if $s \leq t \leq N$. With the above connections, the results regarding RIP in literature can also be characterized with respect to mutual coherence. As far as we know, one of the optimal conditions in the reference [1] until now is*

$$\delta_{as} + b\delta_{(a+1)s} < b - 1$$

with $b = \left((as)^{1-\frac{r}{2}} - \alpha(as)^{\frac{r}{2}}\right) / \left(s^{1-\frac{r}{2}} + \alpha s^{\frac{r}{2}}\right) > 1$, where $a > 0$ is suitably selected so that as is an integer. Utilizing the above relationship and property as well as choosing $a = 2$, this derives the associating mutual coherence condition

$$\mu < \frac{2^{1-\frac{r}{2}} - 1}{2(2s-1)}. \tag{3.13}$$

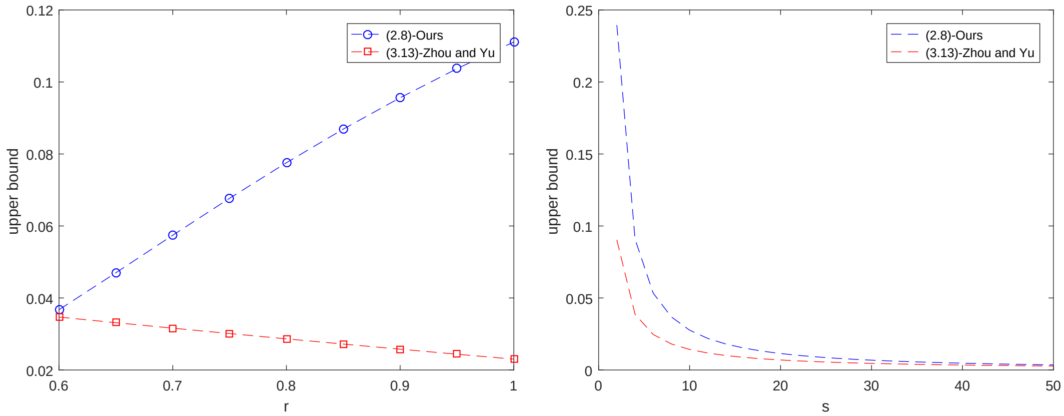
The upper bounds of mutual coherence μ are presented in Fig. 3.1. From Fig. 3.1, we observe that our condition (2.8) is weaker than (3.13) for $1 \geq r > 0.6$ or $s \leq 50$.

4 Proofs

In this section, we prove the main results. First, we give the proofs of previous lemmas.

Proof of Lemma 2.2. Here we assume that x is a general signal. Since $\hat{x}$ is the optimal solution of (1.4) or (1.5), it implies that

$$\|x\|_r^r - \alpha\|x\|_1^r \geq \|\hat{x}\|_r^r - \alpha\|\hat{x}\|_1^r.$$



(a) (b)

Fig. 3.1: Comparison conditions (2.8) and (3.13) for (a) $s = 5$, (b) $r = 3/4$.

Due to $h = \hat{x} - x$, we get

$$\|x_S\|_r^r + \|x_{S^c}\|_1^r - \alpha\|x\|_1^r \geq \|x_S + x_{S^c} + h_S + h_{S^c}\|_r^r - \alpha\|x + h\|_1^r. \tag{4.14}$$

By using the triangular inequality and the inequality $(a + b)^r \leq a^r + b^r$ for nonnegative a and b, we get

$$\begin{aligned}
&\|x_S + x_{S^c} + h_S + h_{S^c}\|_r^r - \alpha\|x + h\|_1^r \\
&\geq \|x_S + h_S\|_r^r + \|x_{S^c} + h_{S^c}\|_r^r - \alpha(\|x\|_1 + \|h\|_1)^r \\
&\geq \|x_S\|_r^r - \|h_S\|_r^r + \|h_{S^c}\|_r^r - \|x_{S^c}\|_r^r - \alpha\|x\|_1^r - \alpha\|h\|_1^r.
\end{aligned} \tag{4.15}$$

A combination of (4.14) and (4.15) leads to the desired result.

□

Proof of Lemma 2.3. Note that $\hat{x}$ is a minimizer of (1.4) and z fulfills $\|z\|_2 \leq \epsilon$, by the equality $h = \hat{x} - x$ and the triangular inequality, then

$$\|Ah\|_2 = \|A\hat{x} - x\|_2 \leq \|A\hat{x} - y\|_2 + \|Ax - y\|_2 \leq 2\epsilon. \tag{4.16}$$

Combining with Lemma 2.1, (4.16) and the inequality $|\langle u, v\rangle| \leq \|u\|_2\|v\|_2$ for $0 \neq u, v \in \mathbb{R}^m$, we get

$$\begin{aligned}
|\langle Ah, Ah_S\rangle| &\leq \|Ah\|_2\|Ah_S\|_2 \\
&\leq 2\epsilon\sqrt{1 + (s-1)\mu}\|h_S\|_2.
\end{aligned} \tag{4.17}$$

By applying (1.2), it results in

$$|A_i^\top A_j| \leq \mu,\ 1 \leq i < j \leq N, \tag{4.18}$$

which implies that

$$|\langle Ah_S, Ah_{S^c}\rangle| = \left|\left\langle \sum_{i\in S} A_i h_i, \sum_{j\in S^c} A_j h_j, \right\rangle\right|$$

$$
\begin{aligned}
&\leq \sum_{i\in S}\sum_{j\in S^c} |\langle A_i, A_j\rangle|\, |h_i||h_j| \leq \mu \left(\sum_{i\in S}|h_i|\right)\left(\sum_{j\in S^c}|h_j|\right)\\
&= \mu\|h_S\|_1\|h_{S^c}\|_1 \leq \mu s^{\frac{1}{2}}\|h_S\|_2\|h_{S^c}\|_r\\
&\overset{\text{(a)}}{\leq} \mu s^{\frac{1}{2}}\|h_S\|_2(\|h_S\|_r^r + \alpha\|h\|_1^r)^{\frac{1}{r}}\\
&\overset{\text{(b)}}{\leq} 2^{\frac{1}{r}-1}\mu s^{\frac{1}{2}}\|h_S\|_2(\|h_S\|_r + \alpha^{\frac{1}{r}}\|h\|_1)\\
&\overset{\text{(c)}}{\leq} 2^{\frac{1}{r}-1}\mu s^{\frac{1}{2}}\|h_S\|_2(s^{\frac{1}{r}-\frac{1}{2}}\|h_S\|_2 + \alpha^{\frac{1}{r}}N^{\frac{1}{2}}\|h\|_2)\\
&= 2^{\frac{1}{r}-1}\mu s^{\frac{1}{r}}\|h_S\|_2^2 + 2^{\frac{1}{r}-1}\mu N^{\frac{1}{2}}\alpha^{\frac{1}{r}}s^{\frac{1}{2}}\|h_S\|_2\|h\|_2,
\end{aligned}
\tag{4.19}
$$

where (a) follows from Lemma 2.2, (b) is due to $(a^r+b^r)^{\frac{1}{r}} \leq 2^{\frac{1}{r}-1}(a+b)$ for any $a, b \geq 0$, and (c) is because of the Hölder's inequality and the Cauchy-Schwarz inequality.

By using Lemma 2.1 together with the triangular inequality, it leads to

$$
\begin{aligned}
&|\langle Ah, Ah_S\rangle| = |\langle Ah_S + Ah_{S^c}, Ah_S\rangle|\\
&\geq |\langle Ah_S, Ah_S\rangle| - |\langle Ah_{S^c}, Ah_S\rangle|\\
&\geq [1-(s-1)\mu]\|h_S\|_2^2 - |\langle Ah_{S^c}, Ah_S\rangle|\,.
\end{aligned}
$$

Plugging (4.19) into the above inequality, we get

$$
\begin{aligned}
|\langle Ah, Ah_S\rangle| \geq &[1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu]\|h_S\|_2^2\\
&- 2^{\frac{1}{r}-1}\mu N^{\frac{1}{2}}\alpha^{\frac{1}{r}}s^{\frac{1}{2}}\|h_S\|_2\|h\|_2.
\end{aligned}
\tag{4.20}
$$

Combining with (2.8), (4.17) and (4.20), the wanted result follows.

□

Proof of Lemma 2.4. For $\hat{x}$ is a minimizer of (1.5) and z fulfills $\|A^\top z\|_\infty \leq \epsilon$, by $h = \hat{x} - x$ and the triangular inequality, we have

$$
\begin{aligned}
&\|A^\top Ah\|_\infty \leq \|A^\top A(\hat{x}-x)\|_\infty\\
&\leq \|A^\top(A\hat{x}-y)\|_\infty + \|A^\top(Ax-y)\|_\infty \leq 2\epsilon.
\end{aligned}
\tag{4.21}
$$

By applying Hölder inequality, it implies that

$$
|\langle Ah, Ah_S\rangle| \leq \left|\left\langle A^\top Ah, h_S\right\rangle\right| \leq \|h_S\|_1\|A^\top Ah\|_\infty \leq s^{\frac{1}{2}}\|h_S\|_2\|A^\top Ah\|_\infty.
$$

Putting (4.21) into the above inequality, we get

$$
|\langle Ah, Ah_S\rangle| \leq 2\epsilon s^{\frac{1}{2}}\|h_S\|_2. \tag{4.22}
$$

A combination of (4.20) and (4.22), we obtain the desired result.

□

Proof of Theorem 3.1. Observing that $\|A_i\|_2 = 1$, $i = 1, 2, \cdots, N$ and (1.2), it leads to

$$
\begin{aligned}
\langle Ah, Ah\rangle &= \sum_{i=1}^{N}\sum_{j=1}^{N}\langle A_i h_i, A_j h_j\rangle \\
&= \sum_{i=1}^{N}\langle A_i h_i, A_i h_i\rangle + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\langle A_i h_i, A_j h_j\rangle \\
&= \sum_{i=1}^{N}\|A_i\|_2^2 |h_i|^2 + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\langle A_i, A_j\rangle\, h_i h_j \\
&\geq \sum_{i=1}^{N}|h_i|^2 - \mu\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}|h_i h_j| \\
&= \|h\|_2^2 - \mu\left(\sum_{i=1}^{N}\sum_{j=1}^{N}|h_i h_j| - \sum_{i=1}^{N}|h_i|^2\right) \\
&= (1+\mu)\|h\|_2^2 - \mu\|h\|_1^2 \\
&= (1+\mu)\|h\|_2^2 - \mu(\|h_S\|_1 + \|h_{S^c}\|_1)^2 \\
&\geq (1+\mu)\|h\|_2^2 - \mu(\|h_S\|_1 + \|h_{S^c}\|_r)^2 \\
&\overset{\text{(a)}}{\geq} (1+\mu)\|h\|_2^2 - \mu\left[\|h_S\|_1 + (\|h_S\|_r^r + \alpha\|h\|_1^r)^{\frac{1}{r}}\right]^2 \\
&\overset{\text{(b)}}{\geq} (1+\mu)\|h\|_2^2 - \mu\left[s^{\frac{1}{2}}\|h_S\|_2 + 2^{\frac{1}{r}-1}(\|h_S\|_r + \alpha^{\frac{1}{r}}\|h\|_1)\right]^2 \\
&\overset{\text{(c)}}{\geq} (1+\mu)\|h\|_2^2 - \mu\left[s^{\frac{1}{2}}\|h_S\|_2 + 2^{\frac{1}{r}-1}(s^{\frac{1}{r}-\frac{1}{2}}\|h_S\|_2 + \alpha^{\frac{1}{r}}N^{\frac{1}{2}}\|h\|_2)\right]^2 \\
&= (1+\mu)\|h\|_2^2 - \mu\left[(s^{\frac{1}{2}} + 2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\|h_S\|_2 + 2^{\frac{1}{r}-1}\alpha^{\frac{1}{r}}N^{\frac{1}{2}}\|h\|_2\right]^2,
\end{aligned} \tag{4.23}
$$

where (a) follows from (2.7), (b) is because of the Cauchy-Schwarz inequality and $(a^r + b^r)^{\frac{1}{r}} \leq 2^{\frac{1}{r}-1}(a+b)$ for any $a, b \geq 0$, and (c) is thanks to the Hölder inequality.

By (2.9), (4.16) and (4.23), it implies that

$$
\begin{aligned}
(1+\mu)\|h\|_2^2 - \mu\Bigg[&\frac{2\epsilon(s^{\frac{1}{2}} + 2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\sqrt{1+(s-1)\mu}}{1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu} \\
&+ \frac{2^{\frac{1}{r}-1}(1+\mu)N^{\frac{1}{2}}\alpha^{\frac{1}{r}}}{1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu}\|h\|_2\Bigg]^2 \leq 4\epsilon^2.
\end{aligned} \tag{4.24}
$$

(4.24) can be recast to

$$
\begin{aligned}
&\left(1+\mu - \frac{2^{\frac{2}{r}-2}\mu(1+\mu)^2 N\alpha^{\frac{2}{r}}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)\|h\|_2^2 \\
&- \frac{2^{\frac{1}{r}+1}\mu(1+\mu)N^{\frac{1}{2}}\alpha^{\frac{1}{r}}(s^{\frac{1}{2}} + 2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\epsilon\sqrt{1+(s-1)\mu}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\|h\|_2 \\
&- \left(1 + \frac{\mu(s^{\frac{1}{2}} + 2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})^2(1+(s-1)\mu)}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)4\epsilon^2 \leq 0.
\end{aligned} \tag{4.25}
$$

Under the condition of (2.8), the coefficient of the quadratic term of the univariate quadratic inequality concerning $\|h\|_2$ is nonnegative.

Solving the one-variable quadratic inequality (4.25) together with $\|u\|_2 \leq \|u\|_1$ for any $u \in \mathbb{R}^2$, we get

$$\|h\|_2 \leq 2\left(1+\mu-\frac{2^{\frac{2}{r}-2}\mu(1+\mu)^2N\alpha^{\frac{2}{r}}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)^{-1}\left\{\frac{2^{\frac{1}{r}}\mu(1+\mu)N^{\frac{1}{2}}\alpha^{\frac{1}{r}}(s^{\frac{1}{2}}+2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\sqrt{1+(s-1)\mu}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right.$$
$$\left.+\left(1+\mu-\frac{2^{\frac{2}{r}-2}\mu(1+\mu)^2N\alpha^{\frac{2}{r}}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)^{\frac{1}{2}}\left(1+\frac{\mu(s^{\frac{1}{2}}+2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})^2(1+(s-1)\mu)}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)^{\frac{1}{2}}\right\}\epsilon.$$

The proof is complete.

□

Proof of Theorem 3.2. It follows from (4.23) that

$$\langle Ah, Ah\rangle \geq (1+\mu)\|h\|_2^2-\mu\left[(s^{\frac{1}{2}}+2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\|h_S\|_2+2^{\frac{1}{r}-1}\alpha^{\frac{1}{r}}N^{\frac{1}{2}}\|h\|_2\right]^2. \tag{4.26}$$

By Lemma 2.2, (4.21) and the Hölder inequality, it leads to

$$\begin{aligned}\langle Ah, Ah\rangle =& \left\langle h, A^\top Ah\right\rangle \leq \|h\|_1\|A^\top Ah\|_\infty\\ \leq& 2\epsilon(\|h_S\|_1+\|h_{S^c}\|_1)\\ \leq& 2\epsilon(s^{\frac{1}{2}}\|h_S\|_2+\|h_{S^c}\|_r)\\ \leq& 2\epsilon(s^{\frac{1}{2}}\|h_S\|_2+(\|h_S\|_r^r+\alpha\|h\|_1^r)^{\frac{1}{r}})\\ \leq& 2\epsilon(s^{\frac{1}{2}}\|h_S\|_2+2^{\frac{1}{r}-1}(\|h_S\|_r+\alpha^{\frac{1}{r}}\|h\|_1))\\ \leq& 2\epsilon[(s^{\frac{1}{2}}+2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\|h_S\|_2+2^{\frac{1}{r}-1}\alpha^{\frac{1}{r}}N^{\frac{1}{2}}\|h\|_2].\end{aligned} \tag{4.27}$$

Therefore,

$$\begin{aligned}&(1+\mu)\|h\|_2^2-\mu\left[(s^{\frac{1}{2}}+2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\|h_S\|_2+2^{\frac{1}{r}-1}\alpha^{\frac{1}{r}}N^{\frac{1}{2}}\|h\|_2\right]^2\\ &\leq 2\epsilon[(s^{\frac{1}{2}}+2^{\frac{1}{r}-1}s^{\frac{1}{r}-\frac{1}{2}})\|h_S\|_2+2^{\frac{1}{r}-1}\alpha^{\frac{1}{r}}N^{\frac{1}{2}}\|h\|_2].\end{aligned} \tag{4.28}$$

Plugging (2.10) into (4.28), by some elementary calculation, we get

$$\begin{aligned}&\left(1-\frac{2^{\frac{2}{r}-2}\mu(1+\mu)N\alpha^{\frac{2}{r}}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)\|h\|_2^2-\frac{2^{\frac{1}{r}}\mu N^{\frac{1}{2}}\alpha^{\frac{1}{r}}(s+2^{\frac{1}{r}}s^{\frac{1}{r}})\epsilon}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\|h\|_2\\ &-\frac{4\epsilon^2(s+2^{\frac{1}{r}-1}s^{\frac{1}{r}})}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\leq 0.\end{aligned} \tag{4.29}$$

By solving the above univariate quadratic inequality about $\|h\|_2$, we obtain

$$\|h\|_2 \leq \left(1-\frac{2^{\frac{2}{r}-2}\mu(1+\mu)N\alpha^{\frac{2}{r}}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)^{-1}\left\{\frac{2^{\frac{1}{r}}\mu N^{\frac{1}{2}}\alpha^{\frac{1}{r}}(s+2^{\frac{1}{r}}s^{\frac{1}{r}})}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right.$$
$$\left.+2\left(1-\frac{2^{\frac{2}{r}-2}\mu(1+\mu)N\alpha^{\frac{2}{r}}}{(1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu)^2}\right)^{\frac{1}{2}}\frac{(s+2^{\frac{1}{r}-1}s^{\frac{1}{r}})^{\frac{1}{2}}}{1-(s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}})\mu}\right\}\epsilon.$$

The proof is finished.

□

5 Conclusion

In this article, we have investigated the mutual coherence based sufficient condition for a novel non-convex weighted $\ell_r - \ell_1$ minimization method for sparse signal reconstruction. Our results have showed that any s-sparse signal x from (1.1) can be robustly reconstructed via (1.4) or (1.5), if the the mutual coherence μ of the measurement matrix A fulfills $\mu < 1/(s - 1 + 2^{1/r-1}s^{1/r})$.

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