

INEXACT RESTORATION FOR MINIMIZATION WITH INEXACT EVALUATION BOTH OF THE OBJECTIVE FUNCTION AND THE CONSTRAINTS

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ABSTRACT. In a recent paper an Inexact Restoration method for solving continuous constrained optimization problems was analyzed from the point of view of worst-case functional complexity and convergence. On the other hand, the Inexact Restoration methodology was employed, in a different research, to handle minimization problems with inexact evaluation and simple constraints. These two methodologies are combined in the present report, for constrained minimization problems in which both the objective function and the constraints, as well as their derivatives, are subject to evaluation errors. Together with a complete description of the method, complexity and convergence results will be proved.

1. INTRODUCTION

Consider an optimization problem given by

$$(1.1) \quad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & H(x) = 0 \\ & x \in \Omega, \end{array}$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}$, $H : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and Ω is a nonempty compact polytope. As usually, if inequality constraints $G(x) \leq 0$ are present, we reduce the problem to the standard form (1.1) by means of the addition of slack variables. Assume that exact evaluation of $F(x)$, $H(x)$ and their derivatives is not always possible. Instead, each evaluation of $F(x)$ (or $H(x)$) is, according to availability or convenience, replaced with $f(x, y)$ (or $h(x, y)$, respectively) where y lies in an abstract set Y and determines the degree of precision in which the objective function or the constraints are evaluated. We will assume that $g_f : Y \rightarrow \mathbb{R}_+$ is such that $f(x, y) = F(x)$ when

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$g_f(y) = 0$, $g_h : Y \rightarrow \mathbb{R}_+$ is such that $h(x, y) = H(x)$ when $g_h(y) = 0$ and that, roughly speaking, the precision in the evaluations improves when $g_f(y)$ and $g_h(y)$ decrease. If the precision of the objective function is governed by a set Y_1 and the precision of the constraints are governed by Y_2 , where both Y_1 and Y_2 are abstract sets, we may define $Y = Y_1 \times Y_2$. Writing $g(y) = \max\{g_f(y), g_h(y)\}$, problem (1.1) is equivalent to:

$$(1.2) \quad \begin{array}{ll} \text{Minimize (with respect to } x) & f(x, y) \\ \text{subject to} & h(x, y) = 0, \\ & g(y) = 0, \\ & x \in \Omega, \\ & y \in Y. \end{array}$$

A solution of (1.2) could be obtained fixing $y \in \Omega$ in such a way that $g_f(y) = g_h(y) = 0$ and handling the resulting problem as a standard constrained optimization problem. However, we are interested in problems in which such procedure is not affordable because solving (1.2) fixing $g_f(y) = g_h(y) = 0$ is overwhelmingly expensive or even impossible.

The definition (1.2) makes sense independently of the meaning of y, Y , or $g(y)$. We have especially in mind the case in which $f(x, y)$ represents $F(x)$ with an error governed by $y \in Y$, $h(x, y)$ is $H(x)$ computed with an error that depends on y , $g_f(y) = 0$ if and only if $f(x, y) = F(x)$, and $g_h(y) = 0$ if and only if $h(x, y) = H(x)$ for all $x \in \Omega$. However, the results of this paper can be read without reference to this meaning.

In this paper we extend the results of [15] and [22]. In [15] the problem (1.2) is considered without the presence of the constraints $h(x, y) = 0$. In [22] an Inexact Restoration method with worst-case complexity results is introduced for solving the classical constrained optimization problem. The techniques of [15] and [22] are merged in the present paper in order to handle the constrained optimization problem with inexactness both in the objective function and the constraints.

Let us give an example of the applicability of the present approach which, in fact, motivated the algorithmic framework and theoretical analysis developed in this paper. We are involved with real-life river simulations and the corresponding inverse problems [46] and [48]. The Saint-Venant equations

$$(1.3) \quad \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

and

$$(1.4) \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + \mathbf{g}A \frac{\partial z}{\partial x} + \frac{\mathbf{n}^2 g Q |Q|}{AR^{4/3}} = 0$$

are usually employed for river-flow simulations. In (1.3) and (1.4) $x \in [x_{\min}, x_{\max}]$ and $t \in [t_{\min}, t_{\max}]$, where $z_b(x)$ is the bed elevation, measured from a datum, $z(x, t) - z_b(x)$ is the depth of the river at (x, t) , $A(x, t) = [z(x, t) - z_b(x, t)]$, $w(x)$ is the transversal wetted area at (x, t) , $P(x, t) = w(x) + 2[z(x, t) - z_b(x, t)]$ is the wetted perimeter at (x, t) , $R(x, t) =$

$A(x, t)/P(x, t)$ is the hydraulics radius at (x, t) , $V(x, t) = Q(x, t)/A(x, t)$ is the speed of the fluid at (x, t) , and \mathbf{g} is the acceleration of gravity. Equation (1.3) describes mass conservation and equation (1.4) represents conservation of the linear momentum. Finally, \mathbf{n} is called Manning Coefficient and takes account of friction.

When the Saint-Venant equations are solved by means of a stable implicit method [41], the estimation of Manning coefficients require to solve a constrained optimization problem [5].

However real rivers are not rectilinear, their flux is not homogeneous, cross sections are not rectangular and, sometimes, are time-dependent, and the Manning coefficients are not constant. Therefore, increasing levels of problem complexity arise when we incorporate variable Manning coefficients with different dimensions, cross section variations and when we increase the number of observations or expert guesses for the flux evolution. Further difficulties arise when we refine the discretization grid for solving realistic Saint Venant equations. These considerations lead to different formulations of real-life river simulations, each corresponding to variable precisions for the computation of the objective function and the constraints.

Inexact Restoration (IR) methods for constrained continuous optimization were introduced in [43], inspired in several classical papers by Rosen [47] and Miele [45], among others. At each iteration of an IR algorithm feasibility is firstly improved and, then, optimality is improved along a tangent approximation of the feasible region. The so far generated trial point is accepted or not as new iterate according to the decrease of a merit function or using filter criteria [25, 26, 32, 36, 50]. Theoretical papers concerning Inexact Restoration methods for constrained optimization include [13, 22, 42]. Algorithmic variations are discussed in [4, 19, 20, 25–28, 42], and applications may be found in [1, 6, 7, 17, 18, 21, 29–31, 37, 38, 44, 51].

The idea of using the IR framework to deal with optimization problems in which the objective function is subject to evaluation errors comes from [40], where inexactness came from the fact that the evaluation was the result of an iterative process. Evaluating the function with additional precision was considered in [40] as a sort of inexact restoration. This basic principle was developed in [14] and [15], where complexity results were also proved. Moreover, in [16] the case in which derivatives are not available was considered. Inexactness of the objective function in optimization problems was addressed in several additional papers in recent years [8–10, 33–35, 39]. The objective of the present paper is to use the ideas of [14, 15, 40] to handle the constrained optimization problem in which the evaluation of the objective functions and the constraints is subject to error. We will show that, although the main ideas are applicable, a number of technical difficulties appear whose solution offer additional insight in the problem. From the theoretical point of view we will prove convergence to feasible points (whenever possible) and asymptotic fulfillment of optimality conditions.

This paper is organized as follows. In Section 2 we describe BIRA, the main algorithm for solving (1.2). In Section 3 we state our final goal in terms of complexity and convergence of BIRA and we highlight the general lines that will be followed in the proofs. In Section 4 we state general assumptions on the problem that will be used throughout the paper. In Section 5 we state several theoretical results with respect to the Restoration Algorithm RESTA that will be useful in forthcoming sections. In Section 6 we show that every iteration of BIRA is well defined. In Section 7 we prove convergence towards feasible points. In Section 8 we finish up proving complexity and convergence of the main algorithm. Conclusions are stated in Section 9. Proofs of the technical lemmas are presented in Appendix A.

Notation

- (1) All along this paper $\|\cdot\|$ represents the Euclidean norm.
- (2) We define

$$(1.5) \quad c(x, y) = \frac{1}{2} \|h(x, y)\|^2.$$

- (3) $P_\Omega(z)$ denotes the Euclidean projection of z onto Ω .

2. ALGORITHMS

In this section we define the Basic Inexact Restoration Algorithm (BIRA) for solving our main problem and the restoration algorithm RESTA, which is called at each iteration of BIRA.

All along the paper we will use the merit function that combines objective function and constraints defined by means of the penalty parameter θ according to:

$$(2.1) \quad \Phi(x, y, \theta) = \theta f(x, y) + (1 - \theta) [\|h(x, y)\| + g(y)]$$

for all $x \in \Omega$, $y \in Y$, and $\theta \in [0, 1]$.

2.1. Basic Inexact Restoration Algorithm (BIRA). The iterative Algorithm BIRA has three main steps. Each iteration begins with a Restoration Phase, at which, starting from the current iteration x^k and the current precision variable y^k , one computes an inexactly restored x_R^k and a better precision parameter y_R^k . At the second step, the penalty parameter that defines the merit function is conveniently updated. At Step 3 (Optimization Phase) we try to improve the merit function by approximate minimization of a quadratic approximation of the objective function with an adaptive regularization parameter. At the first iterations of the Optimization Phase we admit to relax the accuracy defined by y_R^k with the aim of reducing computational cost. If this relaxation is not successful the Optimization Phase uses the precision level y_R^k .

The description of Algorithm BIRA begins reporting all the algorithmic parameters that will be used in the calculations. The parameter $r \in (0, 1)$ is used in the Restoration Phase. At Step 2 we use $\theta_0 \in (0, 1)$, the initial

penalty parameter. Bounds for the first regularization parameter used in the Optimization Phase are given by μ_{min} and μ_{max} . The parameter $\alpha > 0$ is used at Step 3 to decide acceptance or rejection of the trial point obtained at this step, M is a bound for Hessian approximations, and N_{acce} is the maximal number of steps, at the Optimization Phase, in which relaxing precision is admitted.

Other parameters ($\alpha_R, \sigma_{max}, \sigma_{min}, \beta_c, r_{feas}, \bar{\epsilon}_{prec}, N_{prec}, \beta_{PDP}$) are used in Algorithm RESTA and will be commented later.

Algorithm 2.1 (BIRA)

Given $\alpha_R, \alpha > 0, M \geq 1, \sigma_{max} \geq \sigma_{min} > 0, \mu_{max} \geq \mu_{min} > 0, \beta_c > 0, \beta_{PDP} > 0, r \in (0, 1), r_{feas} \in (0, r), \bar{\epsilon}_{prec} \geq 0, N_{prec} \geq 0,$ and $N_{acce} \geq 0,$ choose $\mu_{-1} \in [\mu_{min}, \mu_{max}], x^0 \in \Omega, y^0 \in Y,$ set $k \leftarrow 0$ and $\theta_0 \in (0, 1).$

Step 1. Restoration Phase

Compute (x_R^k, y_R^k) using Algorithm RESTA.

If

$$(2.2) \quad \|h(x_R^k, y_R^k)\| > r \|h(x^k, y_R^k)\|$$

or

$$(2.3) \quad \|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| < \frac{1-r}{2r} [g(y^k) - g(y_R^k)],$$

stop the execution of BIRA declaring *Restoration Failure*.

Step 2. Update penalty parameter

Test the inequality

$$(2.4) \quad \leq \frac{1-r}{2} [\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k)].$$

If (2.4) holds, define $\theta_{k+1} = \theta_k$. Else, compute

$$(2.5) \quad \theta_{k+1} = \frac{(1+r)[\|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| + g(y^k) - g(y_R^k)]}{2[f(x_R^k, y_R^k) - f(x^k, y_R^k) + \|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| + g(y^k) - g(y_R^k)]}.$$

Step 3. Optimization Phase

Initialize $\ell \leftarrow 0$.

Step 3.1 Choose $y^{k+1} \in Y$ (perhaps $g(y^{k+1})$ bigger than $g(y_R^k)$). Choose $\mu \in [\mu_{min}, \mu_{max}]$ and a symmetric matrix $H_k \in \mathbb{R}^{n \times n}$ such that $\|H_k\| \leq M$.

Step 3.2 If $\ell \geq N_{acce}$, re-define $y^{k+1} = y_R^k$.

Step 3.3 Compute $x \in \Omega$ an approximate solution of

$$(2.6) \quad \begin{aligned} \text{Min.} \quad & \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) + \mu \|x - x_R^k\|^2 \\ \text{s. to} \quad & \nabla_x h(x_R^k, y^{k+1})^T (x - x_R^k) = 0 \\ & x \in \Omega. \end{aligned}$$

Step 3.4 Test the conditions

$$(2.7) \quad f(x, y^{k+1}) \leq f(x_R^k, y_R^k) - \alpha \|x - x_R^k\|^2$$

and

(2.8)

$$\begin{aligned} \Phi(x, y^{k+1}, \theta_{k+1}) &\leq \Phi(x^k, y^{k+1}, \theta_{k+1}) \\ &\quad + \frac{1-r}{2} \left[\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k) \right]. \end{aligned}$$

If (2.7) and (2.8) are fulfilled, define $\mu_k = \mu$, $x^{k+1} = x$, $k \leftarrow k + 1$, and go to Step 1. Otherwise, choose $\mu_{new} \in [2\mu, 10\mu]$, $\mu \leftarrow \mu_{new}$, set $\ell \leftarrow \ell + 1$ and go to Step 3.2.

Remark In Assumption A9 we will define in which sense, at Step 3.3, x should be an approximate solution of (2.6).

The condition (2.2) used at Step 1 in BIRA is not the natural generalization of the restoration condition used in previous IR algorithms. Such “natural” generalization should be $\|h(x_R^k, y_R^k)\| > r\|h(x^k, y^k)\|$. The reason why the traditional alternative is not adequate in the context of BIRA is the following: Suppose that, by chance, $\|h(x^k, y^k)\|$ vanishes or is very small. In this case, the restored (x_R^k, y_R^k) would be rejected almost certainly, and the algorithm would stop by Restoration Failure. However, this decision could be unreasonable because, even if $\|h(x_R^k, y_R^k)\|$ is greater than $\|h(x^k, y^k)\|$, the point x_R^k could be better than x^k when the constraints are evaluated with the same accuracy defined by y_R^k , which may be substantially better than the one defined by y^k . This is the reason why we preferred (2.2) instead of $\|h(x_R^k, y_R^k)\| > r\|h(x^k, y^k)\|$ for deciding failure of restoration. In general, the level of precision used in each of the conditions used in the algorithm needs to be carefully chosen. A technical consequence of these decisions is that the theoretical proofs in this paper are, many times, reasonably different than the corresponding proofs of other IR papers.

2.2. Algorithm for the Restoration Phase. The objective of the restoration algorithm RESTA is to find x_R^k and y_R^k such that the inequalities (2.9) below are fulfilled. In general, the fulfillment of $g_f(y_R^k) \leq r g_f(y^k)$ and $g_h(y_R^k) \leq r g_h(y^k)$ is easy to obtain as, under the usual interpretation, these inequalities merely impose that the precision with which F and H are evaluated at x_R^k should be better than the precision with which F and H were evaluated at x^k . However, the requirement $\|h(x_R^k, y_R^k)\| \leq r\|h(x^k, y_R^k)\|$ could be difficult to achieve. We try to do this by minimizing a regularized quadratic model of the sum of squares of the constraints. The regularization parameter is initialized between σ_{min} and σ_{max} and α_R is associated with the sufficient decrease criterion for acceptance of the trial point. Parameters β_c and β_{PDP} control the distance between some restored point estimates and the current iterate.

In critical cases, where the original problem is infeasible, the fulfillment of $\|h(x_R^k, y_R^k)\| \leq r\|h(x^k, y_R^k)\|$ could be even impossible. Therefore, “Restoration Failure” is a possible diagnostic that needs to appear at any algorithm

that aims to fulfill those requirements. In order to declare that we are probably in this situation, we use the parameter r_{feas} , defined to be smaller than r in BIRA, to check if the projected gradient of the sum of squares of the constraint violations is sufficiently smaller than the infeasibility measure. When we solve the problem with precision w^i and we obtain a point z^l indicating that the original problem may be infeasible, we have to decide whether we progressively try to get out of this situation by improving precision and seeking a smaller infeasibility with respect to h or if we demand more quickly a better quality in the representation of constraints and their derivatives, decreasing g_h , to accurately check the infeasibility status. The N_{prec} parameter determines a limit of attempts with an indication of infeasibility until we force the precision in the calculation of the constraints to be at $\bar{\epsilon}_{prec}$, the level required by the user.

Algorithm 2.2 (RESTA)

Assume that $x^k \in \Omega$, $y^k \in Y$, and the parameters that define BIRA are given. If $\|h(x^k, y^k)\| + g(y^k) = 0$, return defining $(x_R^k, y_R^k) = (x^k, y^k)$. Else, set $i \leftarrow 0$ and $w^0 = y^k$.

Step 1 Using an optional inexpensive problem-dependent procedure (PDP) (if available), try to compute $y_R^k \in Y$ and $x_R^k \in \Omega$ such that

$$(2.9) \quad g_f(y_R^k) \leq r g_f(y^k), \quad g_h(y_R^k) \leq r g_h(y^k), \quad \|h(x_R^k, y_R^k)\| \leq r \|h(x^k, y^k)\|,$$

and

$$(2.10) \quad \max\{\|x_R^k - x^k\|, \|y_R^k - y^k\|\} \leq \beta_{PDP} \|h(x^k, y_R^k)\|.$$

If such procedure is activated and (2.9) and (2.10) hold, return.

Step 2 If $i \leq N_{prec}$, set $\bar{g}_h \leftarrow r g_h(w^i)$, else $\bar{g}_h \leftarrow \min\{\bar{\epsilon}_{prec}, r g_h(w^i)\}$. If $g_f(w^i) = 0$ and $g_h(w^i) = 0$ define $w^{i+1} = w^i$, else choose $w^{i+1} \in Y$ such that

$$(2.11) \quad g_f(w^{i+1}) \leq r g_f(y^k) \quad \text{and} \quad g_h(w^{i+1}) \leq \bar{g}_h.$$

(This choice of w^{i+1} will be assumed to be possible and inexpensive since, in general, merely represents increasing the precision of forthcoming evaluations.)

Step 3 Compute $z^0 \in \Omega$ such that

$$(2.12) \quad c(z^0, w^{i+1}) \leq c(x^k, w^{i+1})$$

(see (1.5) for the definition of c) and

$$(2.13) \quad \|z^0 - x^k\| \leq \beta_c \|h(x^k, w^{i+1})\|.$$

(Note that the choice of z^0 satisfying (2.12) and (2.13) is always possible because the trivial choice $z^0 = x^k$ is admissible.)

Set $\ell \leftarrow 0$.

Step 4 Test the stopping criteria

$$(2.14) \quad c(z^\ell, w^{i+1}) \leq r^2 c(x^k, w^{i+1})$$

and

(2.15)

$$\|P_\Omega \left(z^\ell - \nabla_x c(z^\ell, w^{i+1}) \right) - z^\ell\| \leq r_{feas} \|h(x^k, w^{i+1})\| \text{ and } g_h(w^{i+1}) \leq \bar{\epsilon}_{prec}.$$

If (2.14) holds or (2.15) holds, return to BIRA defining $x_R^k = z^\ell$ and $y_R^k = w^{i+1}$.

(Although both (2.14) and (2.15) are reasons for returning, these inequalities have quite different meanings since (2.14) indicates success of restoration whereas (2.15) indicates possible failure. In any case, the final success restoration test is made in BIRA.)

If

$$(2.16) \quad \|P_\Omega \left(z^\ell - \nabla_x c(z^\ell, w^{i+1}) \right) - z^\ell\| \leq \epsilon_c \text{ and } g_h(w^{i+1}) > \bar{\epsilon}_{prec},$$

set $i \leftarrow i + 1$ and go to Step 2.

Step 5 Choose $\sigma \in [\sigma_{min}, \sigma_{max}]$ and $B_\ell \in \mathbb{R}^{n \times n}$ such that $B_\ell + \sigma_{min} I$ be symmetric and positive definite with $\|B_\ell\| \leq M$ and $\|(B_\ell + \sigma_{min} I)^{-1}\| \leq M$.

Step 5.1 Compute $z^{trial} \in \Omega$ as an approximate solution of

(2.17)

$$\begin{aligned} & \text{Minimize} && \nabla_x c(z^\ell, w^{i+1})^T (z - z^\ell) + \frac{1}{2} (z - z^\ell)^T (B_\ell + \sigma I) (z - z^\ell) \\ & \text{subject to} && z \in \Omega. \end{aligned}$$

Step 5.2 Test the condition

$$(2.18) \quad c(z^{trial}, w^{i+1}) \leq c(z^\ell, w^{i+1}) - \alpha_R \|z^{trial} - z^\ell\|^2.$$

If (2.18) is fulfilled, define $z^{\ell+1} = z^{trial}$, set $\ell \leftarrow \ell + 1$, and go to Step 4. Otherwise, choose

$$(2.19) \quad \sigma_{new} \in [2\sigma, 10\sigma],$$

set $\sigma \leftarrow \sigma_{new}$, and go to Step 5.1.

Remark In Assumption A2 we will specify the way in which we choose z^{trial} in (2.17).

3. PLAN OF THE PROOFS

The goal of the present research is to show that, using BIRA and under suitable assumptions, convergence to feasible and optimal solutions takes place and worst-case complexity results, that provide bounds on the evaluation computer work used by the algorithm in terms of given small tolerances, can be proved. These results will be stated in Theorems 8.1 and 8.2.

The main assumption in these theorems is that the algorithm does not stop by Restoration Failure. Note that the possibility of stopping by Restoration Failure is unavoidable in any algorithm for constrained optimization as, in some cases, feasible solutions may not exist at all. In our approach optimality will be measured by means of the Euclidean projection of the gradient of the objective function onto the tangent approximation to the constraints. This is related to using the Sequential Optimality Condition called L-AGP

in [2]. Such condition holds at a local minimizer of constrained optimization problems without invoking constraint qualifications. Under weak constraint qualifications, the fulfillment of L-AGP implies KKT conditions [3].

Let us draw, now, the general map along which the main results of the paper are proved.

- (1) The success of the method proposed in this work is associated with the decrease of the infeasibility $\|h(x, y)\| + g(y)$, that should go to zero, and the decrease of the merit function, which, ultimately, should behave as the true objective function onto the feasible region.
- (2) The iteration of the main algorithm BIRA begins calling Algorithm RESTA, which forces the improvement of similarity (precision) and feasibility of algebraic constraints $h(x, y) = 0$. However, RESTA may fail because the original problem could be infeasible. In this case BIRA stops.
- (3) At each iteration k , after success of RESTA, we update the penalty parameter θ that defines the merit function and we go to the Optimization Phase. At the first N_{acce} attempts of the Optimization Phase we try to improve optimality without necessarily increasing precision in evaluations. For example, it is interesting, in practical implementations, to try $y^{k+1} = y^k$ at the first iterations of the Optimization Phase. If we are not successful in the first N_{acce} attempts, we improve the precision taking $y^{k+1} = y_R^k$, as computed by RESTA. In any case, given the accuracy level induced by y^{k+1} , we try to improve optimality using quadratic programming, and we test if sufficient decrease of both the objective and the merit function were obtained. If this is the case, the iteration finishes. Otherwise, the regularization parameter that defines the quadratic programming problem is increased and quadratic minimization is employed again.

The description given above induces the map of the proofs presented in this paper. Firstly, we need to prove that each iteration is well defined. Looking at the steps described above, for this purpose we need to prove that RESTA is well defined and stops in finite time. This is done in Section 5. Moreover we need to prove that the Optimization Phase finishes in finite time too. This fact will be proved in Section 6.

In Section 7 we prove that the infeasibility measure tends to zero. This fact is essential to show that, ultimately, the algorithm finds solutions of the original problem.

Finally, in Section 8 we show that, not only the infeasibility measure but also a suitable optimality measure tends to zero.

In all the cases, convergence results are complemented with complexity results. That is, we will prove not only that crucial quantities produced by the algorithm tend to zero, but also that the computer work necessary to

reduce those quantities to a small tolerance is suitably bounded as a function of the tolerance.

4. GENERAL ASSUMPTIONS

The assumptions stated in this section are supposed to hold all along this paper without specific mention. These assumptions state regularity and boundedness of the functions involved in the definition of the problem.

Assumption G1 Differentiability of f : The function $f(x, y)$ is continuously differentiable with respect to x for all $x \in \Omega$ and all $y \in Y$.

Assumption G2 Boundedness: There exists $C_f > 0$ such that, for all $x \in \Omega$ and for all $y \in Y$, we have that

$$(4.1) \quad |f(x, y)| \leq C_f.$$

Assumption G3 Lipschitz-continuity: There exists $L_f \geq 0$ such that, for all $x_1, x_2 \in \Omega$ and all $y \in Y$, we have that:

$$(4.2) \quad |f(x_1, y) - f(x_2, y)| \leq L_f \|x_1 - x_2\|$$

and

$$(4.3) \quad \|\nabla_x f(x_1, y) - \nabla_x f(x_2, y)\| \leq L_f \|x_1 - x_2\|.$$

Assumption G4 Upper bound for f : For all $x_1, x_2 \in \Omega$ and all $y \in Y$ we have that

$$(4.4) \quad f(x_2, y) \leq f(x_1, y) + \nabla_x f(x_1)^T (x_2 - x_1) + L_f \|x_2 - x_1\|^2.$$

Assumption G5 Differentiability of h : The function $h(x, y)$ is continuously differentiable with respect to x for all $x \in \Omega$ and all $y \in Y$.

Assumption G6 Boundedness of $\|h\|$ and $\|\nabla_x h\|$: There exists $C_h \geq 0$ such that, for all $x \in \Omega$ and all $y \in Y$, we have that

$$(4.5) \quad \|h(x, y)\| \leq C_h$$

and

$$(4.6) \quad \|\nabla_x h(x, y)\| \leq C_h.$$

Assumption G7 Lipschitz-continuity of h and $\nabla_x h$: There exists $L_h \geq 0$ such that, for all $x_1, x_2 \in \Omega$ and all $y \in Y$, we have that:

$$(4.7) \quad \|h(x_1, y) - h(x_2, y)\| \leq L_h \|x_1 - x_2\|,$$

and

$$(4.8) \quad \|\nabla_x h(x_1, y)^T - \nabla_x h(x_2, y)^T\| \leq L_h \|x_1 - x_2\|.$$

Assumption G8 Upper bound of $\|h\|$: For all $x_1, x_2 \in \Omega$ and all $y \in Y$ we have that

$$(4.9) \quad \|h(x_2, y)\| \leq \|h(x_1, y) + \nabla_x h(x_1, y)^T (x_2 - x_1)\| + L_h \|x_2 - x_1\|^2.$$

Assumption G9 Boundedness of g_f and g_h : There exists $C_g \geq 1$ such that

$$(4.10) \quad g_f(y) \leq C_g \text{ and } g_h(y) \leq C_g$$

for all $y \in Y$.

Assumption G10 Differentiability of $c(x, y)$: The function $c(x, y)$, defined by (1.5), is continuously differentiable with respect to x for all $x \in \mathbb{R}^n$ and $y \in Y$.

Assumption G11 Lipschitz continuity of $\nabla_x c$: There exists $L_c \geq 0$ such that for all $x_1, x_2 \in \Omega$ and all $y \in Y$, we have that

$$(4.11) \quad \|\nabla_x c(x_1, y) - \nabla_x c(x_2, y)\| \leq L_c \|x_1 - x_2\|.$$

Assumption G12 Upper bound of $c(x, y)$: For all $x_1, x_2 \in \Omega$ and all $y \in Y$ we have that

$$(4.12) \quad c(x_2, y) \leq c(x_1, y) + \nabla_x c(x_1, y)^T (x_2 - x_1) + L_c \|x_2 - x_1\|^2.$$

5. THEORETICAL RESULTS CONCERNING THE RESTORATION PHASE

The Restoration Phase is the subject of Step 1 of BIRA. This phase begins acknowledging the possibility that, using some problem-dependent procedure (PDP), one may be able to compute x_R^k and y_R^k fulfilling the conditions (2.9) and (2.10).

If there is no problem-dependent procedure that computes x_R^k and y_R^k satisfying (2.9) and (2.10) we try improve feasibility executing steps 2–5 of RESTA. However, even Algorithm RESTA may fail in that purpose, and in this case we declare “Restoration Failure” and Algorithm BIRA stops. Note that every algorithm for constrained optimization may fail to find feasible points, unless special conditions are imposed to the problem. The main reason is that, in extreme cases, feasible points could not exist at all.

The idea of RESTA is to show that, using quadratic programming resources, we are able to compute a condition similar to (2.10). This means that only (2.2) may fail to occur in cases of probable infeasibility.

Assumption A1 states that finding w^{i+1} at Step 2 of RESTA is always inexpensive. The reason is that, in general, (2.11) merely represent increasing the precision in which the objective function and the constraints will be evaluated.

Assumption A1. *Step 2 of Algorithm RESTA, leading to the definition of w^{i+1} satisfying (2.11), can be computed in finite time for all k and i , without evaluations of f or h .*

At Step 5.1 of RESTA we defined z^{trial} as an approximate solution of problem (2.17). Assumption A2 states a simple condition that such approximate solution must satisfy. According to this very mild assumption, the trial point z^{trial} should not be worse than z^ℓ in terms of functional value. Note that even $z^{trial} = z^\ell$ satisfies this assumption.

Assumption A2. For all z^ℓ and w^{i+1} , the point z^{trial} found at Step 5.1 of Algorithm RESTA satisfies:

$$(5.1) \quad \nabla_x c(z^\ell, w^{i+1})^T (z^{trial} - z^\ell) + \frac{1}{2} (z^{trial} - z^\ell)^T (B_\ell + \sigma I) (z^{trial} - z^\ell) \leq 0.$$

In Lemma 5.1 we prove that, taking the regularization parameter σ large enough when solving (2.17) we obtain sufficient reduction of the sum of squares infeasibility at the approximate solution z^{trial} . In other words, the loop at Steps 5.1–5.2 of RESTA necessarily finishes with the fulfillment of (2.18).

Lemma 5.1. Suppose that Assumptions A1 and A2 hold. Define $\bar{\sigma} = 2(L_c + \frac{M}{2} + \alpha_R)$. Then, if z^{trial} is computed at Step 5.1 with $\sigma \geq \bar{\sigma}$, we have that

$$c(z^{trial}, w^{i+1}) \leq c(z^\ell, w^{i+1}) - \alpha_R \|z^{trial} - z^\ell\|^2.$$

As a consequence, for all k, i , and ℓ we have that $\sigma \leq \max\{10\bar{\sigma}, \sigma_{max}\}$.

Assumption A3 adds an additional condition that must be satisfied by the approximate solution of the subproblem (2.17). It will be required that an approximate optimality condition, expressed in terms of the projected gradient of the objective function of (2.17), should be fulfilled with tolerance proportional to $\|z^{\ell+1} - z^\ell\|$.

Assumption A3. There exists $\kappa_R > 0$ such that, whenever $z^{\ell+1}$ is defined at Step 5.1 of RESTA, we have that:

$$(5.2) \quad \begin{aligned} \|P_\Omega(z^{\ell+1} - [\nabla_x c(z^\ell, w^{i+1}) + B_\ell(z^{\ell+1} - z^\ell) + \sigma(z^{\ell+1} - z^\ell)]) - z^{\ell+1}\| \\ \leq \kappa_R \|z^{\ell+1} - z^\ell\|. \end{aligned}$$

As a consequence of the previous assumptions, Lemma 5.2 proves that the projected gradient of the linear approximation of the sum of squares at z^ℓ , computed at the subproblem solution $z^{\ell+1}$, is proportional to the norm of the difference between z^ℓ and $z^{\ell+1}$.

Lemma 5.2. Suppose that Assumptions A1, A2, and A3 hold. Define $c_{P_\Omega} = L_c + M + \kappa_R + \max\{10\bar{\sigma}, \sigma_{max}\}$, where $\bar{\sigma}$ was defined in Lemma 5.1. Then, whenever $z^{\ell+1}$ is defined at Step 5.2 of RESTA, we have:

$$(5.3) \quad \|P_\Omega(z^{\ell+1} - \nabla_x c(z^{\ell+1}, w^{i+1})) - z^{\ell+1}\| \leq c_{P_\Omega} \|z^{\ell+1} - z^\ell\|.$$

Lemma 5.3 establishes that, in a bounded finite number of steps, the sum of squares of infeasibilities is smaller than $r^2 c(x^k, w^{i+1})$ or its projected gradient at z^ℓ is smaller than $r_{feas} \|h(x^k, w^{i+1})\|$. In other words either the squared residual or its projected gradient is smaller than a multiple of the

residual norm at the current iterate.

Lemma 5.3. *Suppose that Assumptions A1, A2, and A3 hold. Define $C_{rest} = \frac{c_{P\Omega}^2(1-r^2)}{2\alpha_R r_{feas}^2} + 1$, where $c_{P\Omega}$ is defined in Lemma 5.2. Then, at every call to Algorithm RESTA, there exists $\ell \leq C_{rest}$ such that, defining*

$$(5.4) \quad c_{target} = r^2 c(x^k, w^{i+1}) \quad \text{and} \quad \epsilon_c = r_{feas} \|h(x^k, w^{i+1})\|.$$

we have that

$$(5.5) \quad c(z^\ell, w^{i+1}) \leq c_{target}$$

or

$$(5.6) \quad \|P_\Omega(z^\ell - \nabla_x c(z^\ell, w^{i+1})) - z^\ell\| \leq \epsilon_c.$$

The following is a technical assumption that involves $z^{\ell+1}$ obtained at Step 5 of RESTA. It states that, if we add to (2.17) the constraint that $z - z^\ell$ is a multiple of $z^{\ell+1} - z^\ell$, the corresponding solution is close to $z^{\ell+1}$. Clearly, this assumption holds if z^{trial} is the global solution of (2.17) and very plausibly holds for approximate solutions.

Assumption A4. *There exists $\kappa_\varphi > 0$ such that, whenever $z^{\ell+1}$ is the approximate solution of (2.17) obtained in RESTA and $z_*^{\ell+1}$ is an exact solution to the problem that has the same objective function and constraints as (2.17) and, in addition a constraint saying that $z - z^\ell$ is a multiple of $z^{\ell+1} - z^\ell$, we have:*

$$(5.7) \quad \|z^{\ell+1} - z^\ell\| \leq \kappa_\varphi \|z_*^{\ell+1} - z^\ell\|.$$

In the following lemma we prove that the difference between consecutive internal iterations in RESTA is proportional to the infeasibility at x^k .

Lemma 5.4. *Suppose that Assumptions A1–A4 hold. Define*

$$(5.8) \quad C_s = \kappa_\varphi M C_h,$$

where C_h is defined in Assumption G6. Then, for all k, i and ℓ , the iterates generated in RESTA satisfy

$$(5.9) \quad \|z^{\ell+1} - z^\ell\| \leq C_s \|h(x^k, w^{i+1})\|.$$

In Lemma 5.5 we prove that, at every call of RESTA, the descent condition on the sum of squares of infeasibilities (2.18) is tested a finite number of times.

Lemma 5.5. *Suppose that Assumptions A1–A4 hold. Define $n_\sigma = \lceil \log_2(\bar{\sigma}) - \log_2(\sigma_{min}) \rceil + 1$ and $N_{RESTA} = (C_{rest} n_\sigma + 1) N_{prec}$. Then, at every call to RESTA, the number of tests of the condition (2.18) and the number of evaluations of h and $\nabla_x h$ is bounded by N_{RESTA} .*

In the following lemma we prove that the norm of the difference between the restored point x_R^k and the current point x^k is bounded by a multiple of $\|h(x^k, y_R^k)\|$.

Lemma 5.6. *Suppose that Assumptions A1–A4 hold. Define $\beta_R = \max\{\beta_{PDP}, \beta_c + N_{RESTA}C_s\}$, where C_s is defined by (5.8). Then, for every iteration k of BIRA, (x_R^k, y_R^k) satisfies*

$$(5.10) \quad \|x_R^k - x^k\| \leq \beta_R \|h(x^k, y_R^k)\|.$$

In Lemma 5.7 we prove that the deterioration in the objective function in x_R^k with respect to the objective function at x^k is bounded by quantity that is proportional to the infeasibilities h and g . For proving that result we need a final assumption that states that fixing x^k and restoring y^k the deterioration in f is smaller than a multiple of $g(y^k)$.

Assumption A5. *There exists $\beta > 0$ such that, for all iteration k ,*

$$(5.11) \quad f(x^k, y_R^k) \leq f(x^k, y^k) + \beta g(y^k).$$

Lemma 5.7. *Suppose that Assumptions A1–A5 hold. Define $\beta_f = L_f \beta_R + \beta$. Then, for every iteration k of Algorithm BIRA, the point (x_R^k, y_R^k) computed at Step 5 of the Restoration Phase, satisfies*

$$f(x_R^k, y_R^k) \leq f(x^k, y^k) + \beta_f [\|h(x^k, y_R^k)\| + g(y^k)].$$

Finally, in Assumption A6 we state the sense in which the problem-dependent restoration procedure PDP is considered to be inexpensive. Then, in Theorem 5.1, the main results of the present section are condensed.

Assumption A6. *There exists N_{PDP} , independent of $\bar{\epsilon}_{prec}$, such that if the problem-dependent restoration procedure is used at Step 1 of RESTA, it employs at most N_{PDP} evaluations of h and $\nabla_x h$ and no evaluation of f and $\nabla_x f$.*

Successful restoration procedures in IR methods usually satisfy stability conditions that say that the distance between restored points and current iterates is bounded by a constant times the infeasibility measure. Alternatively, it is generally proved that the objective function at the restored point is smaller than the objective function at the current iterate plus a constant times the infeasibility. The stability conditions obviously hold when y_R^k and x_R^k are computed by the problem-dependent procedure PDP, as stated in (2.10). In Theorem 5.1 we prove that similar results hold in the case that restoration is achieved by means of Steps 2–5 of RESTA.

Theorem 5.1. *Suppose that the General Assumptions and Assumptions A1–A6 hold. Then, there exist N_R and β_f , independent of $\bar{\epsilon}_{prec}$, such that, for every iteration k of Algorithm BIRA, the point (x_R^k, y_R^k) is computed*

employing at most N_R evaluations of h and $\nabla_x h$, no evaluation of f and $\nabla_x f$, satisfying

$$(5.12) \quad \|x_R^k - x^k\| \leq \beta_R \|h(x^k, y_R^k)\|.$$

and

$$(5.13) \quad f(x_R^k, y_R^k) \leq f(x^k, y^k) + \beta_f [\|h(x^k, y_R^k)\| + g(y^k)].$$

Proof. Conditions (5.12) and (5.13) follow from Lemmas 5.6 and 5.7, respectively. Observe that no evaluation of f and $\nabla_x f$ is made when calling Algorithm RESTA. So, defining $N_R = N_{RESTA} + N_{PDP}$, by Lemma 5.5 and Assumption A6 we have the desired result. \square

6. BIRA IS WELL DEFINED

All along this section we will assume, without specific mention, that the General Assumptions G1–G12 and the Restoration Assumptions A1–A6 are fulfilled. Assumption A7 will be added when needed to prove specific results and its fulfillment will be mentioned whenever necessary.

As the title of this section indicates, the objective will be that Algorithm BIRA is well defined, that is, that for any iteration of BIRA, either the algorithm stops or it is possible to compute the next iterate.

We begin showing that the penalty parameter is well defined and satisfies the inequality (6.1), that states that, from the point of view of the merit function, the restored point x_R^k is better than the current iterate x^k .

Lemma 6.1. *At every iteration k of BIRA, the penalty parameter θ_{k+1} is well defined, $0 < \theta_{k+1} \leq \theta_k$, and*

$$(6.1) \quad \begin{aligned} & \Phi(x_R^k, y_R^k, \theta_{k+1}) - \Phi(x^k, y_R^k, \theta_{k+1}) \\ & \leq \frac{1-r}{2} [\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k)]. \end{aligned}$$

In Lemma 6.2 we prove that the penalty parameters are bounded away from zero.

Lemma 6.2. *Define $\bar{\theta} = \min \left\{ \theta_0, \left[\frac{2}{1+r} \left(\frac{L_f \beta_R}{1-r} + 1 \right) \right]^{-1} \right\}$. Then, for every iteration k in Algorithm BIRA we have that*

$$(6.2) \quad \theta_k \geq \bar{\theta} > 0.$$

The following assumption establishes the conditions that must be satisfied by an approximate solution of (2.6).

Assumption A7. *There exists $\kappa_T > 0$ such that, at every iteration k of Algorithm BIRA, the approximate solution of the quadratic programming*

problem (2.6) satisfies

$$(6.3) \quad \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) + \mu \|x - x_R^k\|^2 \leq 0$$

and

$$(6.4) \quad \|\nabla_x h(x_R^k, y^{k+1})^T (x - x_R^k)\| \leq \kappa_T \|x - x_R^k\|^2.$$

In the following lemma we prove that, when in the Optimization Phase, for a sufficiently large regularization parameter μ , the descent conditions for the objective function and the merit function are satisfied. As a consequence, in the subsequent corollary we establish the maximal number of iterations that could be needed to fulfill those conditions.

Lemma 6.3. *Suppose that Assumption A7 holds. Define $C_\mu = M + \tilde{\alpha} + L_f$, where*

$$(6.5) \quad \tilde{\alpha} = \max \left\{ \alpha, \frac{1 - \bar{\theta}}{\bar{\theta}} (\kappa_T + L_h) \right\}.$$

Then, if $\mu \geq C_\mu$, $y^{k+1} = y_R^k$, and x is the solution of (2.6), the conditions (2.7) and (2.8) are fulfilled.

Corollary 6.1. *Suppose that Assumption A7 holds. Define $N_{reg} = \max\{\lceil \log_2(C_\mu) - \log_2(\mu_{min}) \rceil, N_{acce}\} + 1$ and $\bar{\mu} = \max\{10C_\mu, 10^{N_{acce}} \mu_{max}\}$. Then, after at most N_{reg} sub-iterations at Step 3 of BIRA, the conditions (2.7) and (2.8) are fulfilled. Moreover, $\mu_k \leq \bar{\mu}$ for all k .*

7. CONVERGENCE TO FEASIBILITY

In this section we will prove that, when executing BIRA, the infeasibility measure tends to zero. Moreover, we will prove a crucial theorem which shows that the norm of the difference between x^{k+1} and x_R^k tends to zero.

For all the proofs of this section we will assume, without specific mention, that all the General Assumptions, the Assumptions A1–A7, and the following Assumption A8 take place. Assumption A8 states that bounded deterioration of objective function and also h -feasibility occurs in a restricted way, depending on a possibly small parameter that depends on $\bar{\theta}$. This means that, in the worst case, bounded deterioration with respect to precision does not occur at all. Note that, however, the new bounded deterioration condition needs to hold only for k large enough.

Assumption A8. *Let $\bar{\theta}$ be as defined in Lemma 6.2. Then, there exist k_R , and $\gamma \in (0, 1)$ such that, for $\bar{\beta} = \frac{\bar{\theta}(1-\gamma)(1-r)^2}{2}$ and all $k \geq k_R$,*

$$(7.1) \quad f(x^k, y^{k+1}) \leq f(x^k, y^k) + \bar{\beta} g(y^k) \quad \text{and} \quad \|h(x^k, y_R^k)\| \leq \|h(x^k, y^k)\| + \bar{\beta} g(y^k).$$

Theorem 7.1 states the summability of all infeasibilities.

Theorem 7.1. *Define*

$$(7.2) \quad C_{feas} = \frac{2}{\gamma(1-r)^2} [k_R(2C_f + C_h) + C_\rho + C_h + C_g].$$

Then,

$$(7.3) \quad \sum_{j=0}^k [\|h(x^j, y_R^j)\| + g(y^j)] \leq C_{feas}.$$

Proof. Let us define

$$(7.4) \quad \rho_j = \frac{1 - \theta_j}{\theta_j} = \frac{1}{\theta_j} - 1, \text{ for all } j \leq k.$$

By Lemma 6.2, we know that $\theta_j \in (0, 1)$, $\{\theta_j\}$ is non-increasing and bounded below by $\bar{\theta}$. Then, the sequence $\{\rho_j\}$ is positive, non-decreasing and bounded above by $\bar{\rho} = \frac{1}{\bar{\theta}} - 1$. So, since $\rho_0 > 0$,

$$(7.5) \quad \sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j) = \rho_k - \rho_0 < \rho_k = \frac{1}{\theta_k} - 1 \leq \frac{1}{\bar{\theta}} - 1 < \frac{1}{\bar{\theta}} < \infty.$$

By (4.5), we have that $\|h(x^j, y^{j+1})\| \leq C_h$ for all j . Since $\rho_{j+1} - \rho_j \geq 0$, taking $C_\rho \equiv \frac{C_h}{\bar{\theta}}$, thanks to (7.5), we have that

$$(7.6) \quad \sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j) \|h(x^j, y^{j+1})\| \leq \sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j) C_h \leq \frac{C_h}{\bar{\theta}} = C_\rho < \infty.$$

We have that

$$(7.7) \quad \begin{aligned} & \Phi(x^{j+1}, y^{j+1}, \theta_{j+1}) - \Phi(x^j, y^j, \theta_{j+1}) \\ & \leq \frac{1-r}{2} \left[\|h(x_R^j, y_R^j)\| - \|h(x^j, y_R^j)\| + g(y_R^j) - g(y^j) \right] \\ & \leq -\frac{(1-r)^2}{2} \left[\|h(x^j, y_R^j)\| + g(y^j) \right], \end{aligned}$$

where the second inequality comes from $\|h(x_R^j, y_R^j)\| \leq r\|h(x^j, y_R^j)\|$ and $g(y_R^j) \leq rg(y^j)$.

By the definition of Φ , dividing (7.7) by θ_{j+1} , we have that, for all $j \leq k-1$,

$$\begin{aligned} & f(x^{j+1}, y^{j+1}) + \frac{1-\theta_{j+1}}{\theta_{j+1}} [\|h(x^{j+1}, y^{j+1})\| + g(y^{j+1})] - f(x^j, y^{j+1}) \\ & - \frac{1-\theta_{j+1}}{\theta_{j+1}} [\|h(x^j, y^{j+1})\| + g(y^{j+1})] \leq -\frac{(1-r)^2}{2\theta_{j+1}} [\|h(x^j, y_R^j)\| + g(y^j)]. \end{aligned}$$

By the definition of ρ_j in (7.4), using that $\theta_j \in (0, 1)$, we deduce that

$$(7.8) \quad \begin{aligned} & \frac{(1-r)^2}{2} [\|h(x^j, y_R^j)\| + g(y^j)] \leq \frac{(1-r)^2}{2\theta_{j+1}} [\|h(x^j, y_R^j)\| + g(y^j)] \\ & \leq f(x^j, y^{j+1}) - f(x^{j+1}, y^{j+1}) + \rho_{j+1} \|h(x^j, y^{j+1})\| - \rho_{j+1} \|h(x^{j+1}, y^{j+1})\|. \end{aligned}$$

Adding and subtracting $\rho_j \|h(x^j, y^{j+1})\|$ on the right-hand side of (7.8), and arranging terms, we have:

$$\begin{aligned} \frac{(1-r)^2}{2} [\|h(x^j, y_R^j)\| + g(y^j)] &\leq f(x^j, y^{j+1}) - f(x^{j+1}, y^{j+1}) \\ &\quad + (\rho_{j+1} - \rho_j) \|h(x^j, y^{j+1})\| \\ &\quad + \rho_j \|h(x^j, y^{j+1})\| - \rho_{j+1} \|h(x^{j+1}, y^{j+1})\|. \end{aligned}$$

Then,

$$\begin{aligned} (7.9) \quad \frac{(1-r)^2}{2} \sum_{j=0}^{k-1} [\|h(x^j, y_R^j)\| + g(y^j)] &\leq f(x^0, y^1) - f(x^k, y^k) \\ &\quad + \sum_{j=1}^{k_R-1} [f(x^j, y^{j+1}) - f(x^j, y^j)] \\ &\quad + \sum_{j=k_R}^{k-1} [f(x^j, y^{j+1}) - f(x^j, y^j)] \\ &\quad + \sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j) \|h(x^j, y^{j+1})\| \\ &\quad + \rho_0 \|h(x^0, y^1)\| - \rho_k \|h(x^k, y^k)\| \\ &\quad + \sum_{j=1}^{k_R-1} \rho_j [\|h(x^j, y^{j+1})\| - \|h(x^j, y^j)\|] \\ &\quad + \sum_{j=k_R}^{k-1} \rho_j [\|h(x^j, y^{j+1})\| - \|h(x^j, y^j)\|]. \end{aligned}$$

By (7.6), $\sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j) \|h(x^j, y^{j+1})\| \leq C_\rho$. Moreover, since $\rho_j \leq \bar{\rho}$, by Assumption A8 and disregarding the certainly non-positive terms, (7.9)

implies that

$$\begin{aligned}
\frac{(1-r)^2}{2} \sum_{j=0}^{k-1} [\|h(x^j, y_R^j)\| + g(y^j)] &\leq |f(x^0, y^1)| + |f(x^k, y^k)| \\
&+ \sum_{j=1}^{k_R-1} |f(x^j, y^{j+1}) - f(x^j, y^j)| \\
&+ \sum_{j=k_R}^{k-1} \bar{\beta} g(y^j) + C_\rho + \rho_0 \|h(x^0, y^1)\| \\
&+ \sum_{j=1}^{k_R-1} \bar{\rho} \|h(x^j, y^{j+1})\| + \sum_{j=k_R}^{k-1} \bar{\rho} \bar{\beta} g(y^j).
\end{aligned}$$

By (4.1), (4.5), and (4.10) we have that f , $\|h\|$, and g are bounded above by C_f , C_h , and C_g respectively. Then, as $\bar{\rho} + 1 = \frac{1}{\bar{\theta}}$, we obtain that

$$\frac{(1-r)^2}{2} \sum_{j=0}^{k-1} [\|h(x^j, y_R^j)\| + g(y^j)] \leq k_R(2C_f + C_h) + C_\rho + \frac{\bar{\beta}}{\bar{\theta}} \sum_{j=k_R}^{k-1} g(y^j).$$

Therefore, using that $0 \leq g(y^j) \leq g(y^j) + \|h(x^j, y_R^j)\|$ and $\frac{\bar{\beta}}{\bar{\theta}} = \frac{(1-\gamma)(1-r)^2}{2}$ we obtain:

$$\begin{aligned}
\frac{(1-r)^2}{2} \sum_{j=0}^k [\|h(x^j, y_R^j)\| + g(y^j)] &\leq k_R(2C_f + C_h) + C_\rho \\
&+ \frac{(1-\gamma)(1-r)^2}{2} \sum_{j=k_R}^{k-1} g(y^j) \\
&+ \|h(x^k, y_R^k)\| + g(y^k) \\
&\leq k_R(2C_f + C_h) + C_\rho \\
&+ \frac{(1-\gamma)(1-r)^2}{2} \sum_{j=0}^k \|h(x^j, y_R^j)\| \\
&+ \sum_{j=0}^k g(y^j) + \|h(x^k, y_R^k)\| + g(y^k).
\end{aligned}$$

Thus,

$$\frac{\gamma(1-r)^2}{2} \sum_{j=0}^k [\|h(x^j, y_R^j)\| + g(y^j)] \leq k_R(2C_f + C_h) + C_\rho + C_h + C_g.$$

So, by (7.2), we obtain (7.3), as desired. \square

The result stated in Theorem 7.2 will be used in the proof of Lemma 8.2 which, in turn, is essential for the proof of the main theorems in Section 8.

Theorem 7.2. *Define $C_d = \frac{1}{\alpha}[(\beta_f + \beta)C_{feas} + 2C_f]$. Then,*

$$(7.10) \quad \sum_{j=0}^k \|x^{j+1} - x_R^j\|^2 \leq C_d.$$

Proof. By (2.7) we have that

$$(7.11) \quad \begin{aligned} \alpha \|x^{j+1} - x_R^j\|^2 &\leq f(x_R^j, y_R^j) - f(x^{j+1}, y^{j+1}) \\ &\leq f(x_R^j, y_R^j) - f(x^j, y_R^j) + f(x^j, y_R^j) - f(x^j, y^j) \\ &\quad + f(x^j, y^j) - f(x^{j+1}, y^{j+1}). \end{aligned}$$

For all $j \leq k-1$, by (5.13), we have that $f(x_R^j, y_R^j) - f(x^j, y_R^j) \leq \beta_f [\|h(x^j, y_R^j)\| + g(y^j)]$. On the other hand, (5.11) implies that $f(x^j, y_R^j) - f(x^j, y^j) \leq \beta g(y^j)$. So,

$$(7.12) \quad \alpha \|x^{j+1} - x_R^j\|^2 \leq \beta_f [\|h(x^j, y_R^j)\| + g(y^j)] + \beta g(y^j) + f(x^j, y^j) - f(x^{j+1}, y^{j+1}).$$

Using that $\|h(x^j, y_R^j)\| \geq 0$ and adding terms from j to $k-1$, we obtain:

$$\begin{aligned} &\alpha \sum_{j=0}^{k-1} \|x^{j+1} - x_R^j\|^2 \\ &\leq (\beta_f + \beta) \sum_{j=0}^{k-1} [\|h(x^j, y_R^j)\| + g(y^j)] + \sum_{j=0}^{k-1} [f(x^j, y^j) - f(x^{j+1}, y^{j+1})]. \end{aligned}$$

Therefore, by (7.3),

$$(7.13) \quad \alpha \sum_{j=0}^{k-1} \|x^{j+1} - x_R^j\|^2 \leq (\beta_f + \beta)C_{feas} + f(x^0, y^0) - f(x^k, y^k).$$

Finally, by (4.1) and (7.13), the desired result is obtained. \square

8. COMPLEXITY AND CONVERGENCE

In this section we suppose, without specific mention, that the General Assumptions, Assumptions A1–A8, and the following Assumption A9 hold. Assumption A9 merely states the approximate optimality conditions that the approximate solutions of (2.6) must fulfill.

Assumption A9. *There exists $\kappa > 0$ such that, for every iteration k at Algorithm BIRA, the approximate solutions of (2.6) satisfy*

$$(8.1) \quad \begin{aligned} \|P_{D^{k+1}}(x^{k+1} - \nabla_x f(x_R^k, y^{k+1}) - H_k(x^{k+1} - x_R^k) - 2\mu_k(x^{k+1} - x_R^k)) - x^{k+1}\| \\ \leq \kappa \|x^{k+1} - x_R^k\|, \end{aligned}$$

where D^{k+1} is defined by

$$(8.2) \quad D^{k+1} = \{x \in \Omega \mid \nabla_x h(x_R^k, y^{k+1})^T (x - x_R^k) = 0\}.$$

In Lemma 8.1 we prove that the projected gradient of the objective function onto the tangent set to the constraints tends to zero proportionally to the norm of the difference between x^{k+1} and the restored point x_R^k .

Lemma 8.1. *Define*

$$(8.3) \quad C_p = M + \kappa + 2\bar{\mu} + 2,$$

where $\bar{\mu}$ is defined in Corollary 6.1. Then,

$$(8.4) \quad \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - x_R^k\| \leq C_p \|x^{k+1} - x_R^k\|.$$

Lemma 8.2 establishes the summability of squared norms of the projected gradients of the objective function computed as the restored iterates.

Lemma 8.2. *Define $C_{proj} = C_p^2 C_d$. Then, for every iteration k of BIRA, we have that*

$$(8.5) \quad \sum_{j=0}^k \|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\|^2 \leq C_{proj}.$$

Lemma 8.3 is a complexity result establishing that the number of iterations at which infeasibility takes place with respect to given precisions is, in the worst case, proportional to the multiplicative inverse of the precisions required. From a practical point of view, to be consistent with the Restoration Failure criterion, the accuracy with respect to g should be less demanding than the one used in RESTA. However this is not a mathematical requirement and is not used in the following lemma.

Lemma 8.3. *Let $\epsilon_{feas} > 0$ and $\epsilon_{prec} > 0$ be given. Let $N_{hinfeas}$ be the number of iterations of BIRA at which $\|h(x_R^k, y_R^k)\| > \epsilon_{feas}$, $N_{ginfeas}$ the number of iterations of BIRA at which $g(y^k) > \epsilon_{prec}$, and N_{infeas} the number of iterations of BIRA such that $\|h(x_R^k, y_R^k)\| > \epsilon_{feas}$ or $g(y_R^k) > \epsilon_{prec}$. Then,*

$$(8.6) \quad N_{hinfeas} \leq \left\lfloor \frac{rC_{feas}}{\epsilon_{feas}} \right\rfloor, \quad N_{ginfeas} \leq \left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor$$

and $N_{infeas} \leq \left\lfloor \max \left\{ \frac{rC_{feas}}{\epsilon_{feas}}, \frac{rC_{feas}}{\epsilon_{prec}} \right\} \right\rfloor.$

Lemma 8.4 is a complexity result that states that the number of iterations at which the projected gradient of the objective function at the restored points is bigger than a given tolerance ϵ_{opt} is proportional, in the worst case, to ϵ_{opt}^{-2} .

Lemma 8.4. *Suppose that $\epsilon_{opt} > 0$. Let N_{opt} be the number of iterations such that $\|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\| > \epsilon_{opt}$. Then,*

$$(8.7) \quad N_{opt} \leq \left\lfloor \frac{C_{proj}}{\epsilon_{opt}^2} \right\rfloor.$$

Theorem 8.1. *Suppose that the General Assumptions and Assumptions A1–A9 hold. Given $\epsilon_{prec} > 0$, $\epsilon_{feas} > 0$, and $\epsilon_{opt} > 0$, then:*

- *If RESTA does not stop by Restoration Failure and N_{max} is the maximum number of iterations j of BIRA such that $g(y_R^j) > \epsilon_{prec}$, or $g(y^{j+1}) > \epsilon_{prec}$, or $\|h(x_R^j, y_R^j)\| > \epsilon_{feas}$ or $\|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\| > \epsilon_{opt}$, then*

$$(8.8) \quad N_{max} \leq \left\lfloor \max \left\{ \frac{rC_{feas}}{\epsilon_{feas}}, \frac{rC_{feas}}{\epsilon_{prec}} \right\} \right\rfloor + \left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor + \left\lfloor \frac{C_{proj}}{\epsilon_{opt}^2} \right\rfloor.$$

- *The total number of evaluations of h , $\nabla_x h$, f , and $\nabla_x f$ in BIRA until declaring Restoration Failure or finding x_R^j such that*

$$(8.9) \quad \begin{aligned} \|h(x_R^j, y_R^j)\| &\leq \epsilon_{feas}, \quad g(y_R^j) \leq \epsilon_{prec}, \quad g(y^{j+1}) \leq \epsilon_{prec} \quad \text{and} \\ \|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\| &\leq \epsilon_{opt} \end{aligned}$$

is bounded by N_{av} , where

$$(8.10) \quad N_{av} = O(\min\{\epsilon_{prec}, \epsilon_{feas}\}^{-1} + \epsilon_{prec}^{-1} + \epsilon_{opt}^{-2}).$$

Proof. Assume firstly that BIRA does not stop with Restoration Failure. By Lemma 8.3, the inequalities $\|h(x_R^j, y_R^j)\| > \epsilon_{feas}$ or $g(y_R^j) > \epsilon_{prec}$ may occur at most during $\left\lfloor \max \left\{ \frac{rC_{feas}}{\epsilon_{feas}}, \frac{rC_{feas}}{\epsilon_{prec}} \right\} \right\rfloor$ iterations. Therefore, after $\left\lfloor \max \left\{ \frac{rC_{feas}}{\epsilon_{feas}}, \frac{rC_{feas}}{\epsilon_{prec}} \right\} \right\rfloor + \left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor + \left\lfloor \frac{C_{proj}}{\epsilon_{opt}^2} \right\rfloor + 1$ iterations, we know that at least at $\left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor + \left\lfloor \frac{C_{proj}}{\epsilon_{opt}^2} \right\rfloor + 1$ of these iterations, the inequalities $g(y_R^j) \leq \epsilon_{prec}$ and $h(x_R^j, y_R^j) \leq \epsilon_{feas}$ are fulfilled.

By Lemma 8.4, the inequality $\|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\| > \epsilon_{opt}$ may occur at most in $\lfloor C_{proj}/\epsilon_{opt}^2 \rfloor$ iterations. Thus, at least in $\left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor + 1$ over $\left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor + \left\lfloor \frac{C_{proj}}{\epsilon_{opt}^2} \right\rfloor + 1$ iterations we should have that $\|h(x_R^j, y_R^j)\| \leq \epsilon_{feas}$, $g(y_R^j) \leq \epsilon_{prec}$, and $\|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\| \leq \epsilon_{opt}$.

Analogously, by Lemma 8.3, the number of iterations at which $g(y^{j+1}) \leq \epsilon_{prec}$ is bounded by $\left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor$, then at least in one over the $\left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor + 1$ iterations (8.9) takes place. So, (8.8) is proved.

Thinking (8.9) as a “stopping criterion” for BIRA, the total number of iterations would be at most $N_{max} + 1$, since we would have stopped by Restoration Failure or the conditions (8.9) would be satisfied. Let us now analyze the number of functions evaluations at each iteration.

For every iteration of BIRA, by Lemma 5.5, the Restoration Phase finishes after at most N_R evaluations of h and $\nabla_x h$. Moreover, f and $\nabla_x f$ are not evaluated in the Restoration Phase. At Step 2 of BIRA, we have two evaluations of f and additional evaluations of h are not necessary, since $h(x^k, y_R^k)$ and $h(x_R^k, y_R^k)$ have been already computed in RESTA or to check (2.9). Furthermore, no derivatives are used at Step 2.

Now, let us see what happens at the Optimization Phase. Firstly, note that, for building subproblem (2.6), we only use one evaluation of $\nabla_x h(x_R^k, y^{k+1})$ and $\nabla_x f(x_R^k, y^{k+1})$ in the first N_{acce} attempts of the Optimization Phase (when y^{k+1} does not need to be y_R^k) and an extra computation of them for the remaining ones.

In the test of (2.7) there is no need to calculate $f(x_R^k, y_R^k)$, which has already been evaluated in the Restoration Phase. However, we need to compute $f(x, y^{k+1})$, which can be used for every verification of (2.8). In every loop of the Optimization Phase it is necessary an evaluation of $h(x, y^{k+1})$ too. By Corollary 6.1, the Optimization Phase finishes after at most N_{reg} calls to Step 3. Then, $f(x, y^{k+1})$ and $h(x, y^{k+1})$ are evaluated at most N_{reg} times at each call of the Optimization Phase. The values of $h(x_R^k, y_R^k)$ and $h(x^k, y_R^k)$ involved in (2.8) have already been computed at Step 2. Also to check (2.8), we need to compute $f(x^k, y^{k+1})$ and $h(x^k, y^{k+1})$ once for the N_{acce} first attempts, and no additional evaluation is needed in the other ones, since $f(x^k, y_R^k)$ and $h(x^k, y_R^k)$ have been already computed at Step 2.

Therefore, the number of evaluations of h and $\nabla_x h$, at each iteration of BIRA is, respectively, $N_R + N_{reg} + 1$ and $N_R + 2$. Moreover, f is computed at each iteration of BIRA at most $N_{reg} + 3$ times and, at most, two evaluations of $\nabla_x f$ are necessary. Since N_{reg} and N_R do not depend on ϵ_{prec} , ϵ_{feas} , and ϵ_{opt} , the total number of iterations and evaluations of h , $\nabla_x h$, f , and $\nabla_x f$ before declaring Restoration Failure or obtaining (8.9) is $O(\min\{\epsilon_{prec}, \epsilon_{feas}\}^{-1} + \epsilon_{opt}^{-2})$. \square

Our last theorem concerns the asymptotic convergence of BIRA. For this, it is natural to consider that the algorithm generates an infinite sequence, not meeting a stopping criterion. Therefore, it is reasonable to think that $\bar{\epsilon}_{prec}$ is null.

Theorem 8.2. *Suppose that the General Assumptions, Assumptions A1–A9 hold, and BIRA does not stop by Restoration Failure. Then,*

$$(8.11) \quad \begin{aligned} \lim_{k \rightarrow \infty} g(y^k) = 0, \quad \lim_{k \rightarrow \infty} g(y_R^k) = 0, \quad \lim_{k \rightarrow \infty} \|h(x_R^k, y_R^k)\| = 0, \\ \text{and} \quad \lim_{k \rightarrow \infty} \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - x_R^k\| = 0. \end{aligned}$$

Proof. Assume, by contradiction that BIRA computes infinitely many iterations and at least one of the sequences $\{\|h(x_R^k, y_R^k)\|\}$, $\{g(y^k)\}$ or $\{\|P_{D^k}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - x_R^k\|\}$ does not converge to zero. To fix ideas, suppose that $\{g(y^k)\}$ does not converge to zero. Then, there exists $\varepsilon > 0$ and infinitely

many indices K such that $g(y^j) > \varepsilon$ for all $j \in K$. Therefore, $g(y^k) > \epsilon_{feas}$ occurs infinitely many times if we define $\epsilon_{prec} = \varepsilon$. By Theorem 8.1, this is impossible and so $\lim_{k \rightarrow \infty} g(y^k) = 0$. Since $0 \leq g(y_R^k) \leq g(y^k)$, we also have that $\lim_{k \rightarrow \infty} g(y_R^k) = 0$.

The convergence to zero of the sequences $\{\|h(x_R^k, y_R^k)\|\}$ and $\{\|P_{D^k}(x_R^k - \nabla_x f^k(x_R^k, y^{k+1})) - x_R^k\|\}$ is proved in an entirely analogous way using $\epsilon_{feas} = \epsilon$ or $\epsilon_{opt} = \epsilon$, respectively. □

9. CONCLUSIONS

Many practical problems require the minimization of functions that are very difficult to evaluate with constraints with the same characteristics. In these cases, common sense indicates that one should try to minimize suitable progressive approximations with the hope that successive partial minimizers would converge to the solution of the original problem. In many cases error bounds are not available, so that we know how to get closer to the true problem but we cannot estimate distances between partial and final solutions.

The natural questions that arise are: With which precision we need to solve each partial problem? How to choose the approximate problem that should be addressed after finishing each stage of the process? For solving these questions one needs to consider two different objectives: decreasing the objective function and increasing the precision. It is natural to combine these objectives in a single merit function.

The papers [14–16, 40] suggested that a good framework to address this problem is given by the Inexact Restoration approach of classical constrained optimization. The idea is that “maximal evaluation precision” can be considered as a constraint of the problem depending on a precision variable y that lies in an abstract set Y . The tools of Inexact Restoration indicate an algorithmic path for modifying y and decreasing the objective function in such a way that, hopefully, most iterations are performed with moderate precision and the overall computational cost is affordable.

The present paper is the first contribution in which the Inexact Restoration framework is applied to the case in which, not only the objective function but also the constraints are subject to uncertainty. An interesting feature of our approach is that our method applied to the particular case in which exact evaluations are possible (Y is a singleton, $g_f(y) = 0$ and $g_h(y) = 0$) coincides with (a version of) the classical Inexact Restoration method for smooth constrained optimization. Paradoxically, this nice feature motivates a challenging open problem: Is it really necessary to use the IR approach both for the algebraic and the precision constraints? From the aesthetic point of view our “double IR” strategy seems to be attractive but it cannot be discarded that using different underlying strategies for the algebraic constraints could result in more efficient algorithms.

An important possible branch of application of the theory of this work is Stochastic Optimization. In [12, 23, 24] only the objective function is stochastic whereas the constraints are deterministic. However, some nontrivial adaptations of the main algorithm may be necessary to consider the specific contribution of stochasticity. The paper [11] presents several successful applications of the IR approach and, in particular, show the way in which functions $h(y)$ and merit functions can be defined for that type of problems. The application to noisy derivative free optimization [49], on the other hand, will be also the subject of future research.

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APPENDIX A: PROOFS OF AUXILIARY RESULTS

A.1. Proof of Lemma 5.1.

Proof. Using (4.12) for $x_2 = z^{trial}$ and $x_1 = z^\ell$ we have that

$$c(z^{trial}, w^{i+1}) \leq c(z^\ell, w^{i+1}) + \nabla_x c(z^\ell, w^{i+1})^T (z^{trial} - z^\ell) + L_c \|z^{trial} - z^\ell\|^2.$$

Then, taking

$$v = \frac{1}{2} (z^{trial} - z^\ell)^T B_\ell (z^{trial} - z^\ell) + \frac{\sigma}{2} \|z^{trial} - z^\ell\|^2,$$

we obtain:

$$\begin{aligned} c(z^{trial}, w^{i+1}) - c(z^\ell, w^{i+1}) &\leq \nabla_x c(z^\ell, w^{i+1})^T (z^{trial} - z^\ell) + L_c \|z^{trial} - z^\ell\|^2 \\ &= \nabla_x c(z^\ell, w^{i+1})^T (z^{trial} - z^\ell) + v - v \\ &\quad + L_c \|z^{trial} - z^\ell\|^2 \\ &= \nabla_x c(z^\ell, w^{i+1})^T (z^{trial} - z^\ell) + v \\ &\quad + \left(L_c - \frac{\sigma}{2} \right) \|z^{trial} - z^\ell\|^2 \\ &\quad - \frac{1}{2} (z^{trial} - z^\ell)^T B_\ell (z^{trial} - z^\ell). \end{aligned}$$

Since $\|B_\ell\| \leq M$, we have that $|(z^{trial} - z^\ell)^T B_\ell (z^{trial} - z^\ell)| \leq M \|z^{trial} - z^\ell\|^2$, so:

$$\begin{aligned} (A.1) \quad c(z^{trial}, w^{i+1}) - c(z^\ell, w^{i+1}) &\leq \left[\nabla_x c(z^\ell, w^{i+1})^T (z^{trial} - z^\ell) + v \right] \\ &\quad + \left(L_c + \frac{M}{2} - \frac{\sigma}{2} \right) \|z^{trial} - z^\ell\|^2. \end{aligned}$$

By (5.1),

$$(A.2) \quad c(z^{trial}, w^{i+1}) - c(z^\ell, w^{i+1}) \leq \left(L_c + \frac{M}{2} - \frac{\sigma}{2} \right) \|z^{trial} - z^\ell\|^2.$$

Therefore, we obtain that, if $\sigma \geq \bar{\sigma}$, (2.18) is fulfilled. \square

A.2. Proof of Lemma 5.2.

Proof. Define

$$u = z^{\ell+1} - [\nabla_x c(z^\ell, w^{i+1}) + B_\ell(z^{\ell+1} - z^\ell)] \quad \text{and} \quad w = u - \sigma(z^{\ell+1} - z^\ell).$$

By (5.2),

$$(A.3) \quad \begin{aligned} \|P_\Omega(u) - z^{\ell+1}\| &= \|P_\Omega(u) - P_\Omega(w) + P_\Omega(w) - z^{\ell+1}\| \\ &\leq \|P_\Omega(u) - P_\Omega(w)\| + \kappa_R \|z^{\ell+1} - z^\ell\|. \end{aligned}$$

By the non-expansivity projections, we have that

$$\|P_\Omega(u) - P_\Omega(w)\| \leq \|u - w\| = \sigma \|z^{\ell+1} - z^\ell\|.$$

So, (A.3) implies that

$$(A.4) \quad \|P_\Omega(u) - z^{\ell+1}\| \leq (\sigma + \kappa_R) \|z^{\ell+1} - z^\ell\|.$$

Now, define $v = z^{\ell+1} - \nabla_x c(z^{\ell+1}, w^{i+1})$. Using (A.4) and, again, the non-expansivity of projections, we obtain:

$$(A.5) \quad \begin{aligned} \|P_\Omega(v) - z^{\ell+1}\| &\leq \|P_\Omega(v) - P_\Omega(u)\| + \|P_\Omega(u) - z^{\ell+1}\| \\ &\leq \|v - u\| + (\sigma + \kappa_R) \|z^{\ell+1} - z^\ell\| \\ &\leq \|\nabla_x c(z^\ell, w^{i+1}) - \nabla_x c(z^{\ell+1}, w^{i+1}) + B_\ell(z^{\ell+1} - z^\ell)\| \\ &\quad + (\sigma + \kappa_R) \|z^{\ell+1} - z^\ell\|. \end{aligned}$$

By (4.11), we have that $\|\nabla_x c(z^{\ell+1}, w^{i+1}) - \nabla_x c(z^\ell, w^{i+1})\| \leq L_c \|z^{\ell+1} - z^\ell\|$. So, since $\|B_\ell\| \leq M$,

$$(A.6) \quad \begin{aligned} \|P_\Omega(v) - z^{\ell+1}\| &\leq \|\nabla_x c(z^\ell, w^{i+1}) - \nabla_x c(z^{\ell+1}, w^{i+1})\| + \|B_\ell(z^{\ell+1} - z^\ell)\| \\ &\quad + (\sigma + \kappa_R) \|z^{\ell+1} - z^\ell\| \\ &\leq L_c \|z^{\ell+1} - z^\ell\| + M \|z^{\ell+1} - z^\ell\| + (\sigma + \kappa_R) \|z^{\ell+1} - z^\ell\| \\ &= (L_c + M + \sigma + \kappa_R) \|z^{\ell+1} - z^\ell\|. \end{aligned}$$

Therefore, recalling that, by Lemma 5.1, we have that $\sigma \leq \max\{10\bar{\sigma}, \sigma_{max}\}$, we deduce (5.3), as desired. \square

A.3. Proof of Lemma 5.3.

Proof. If $\|h(x^k, w^{i+1})\| = 0$, then $z^0 = x^k$ satisfies (5.5) and (5.6). Assume now that $\|h(x^k, w^{i+1})\| > 0$ and (5.6) is not true for the first ℓ iterations of Step 4 of RESTA. Then, for all $j \in \{0, 1, \dots, \ell\}$ we have that

$$(A.7) \quad \|P_\Omega(z^j - \nabla_x c(z^j, w^{i+1})) - z^j\| > \epsilon_c.$$

By (5.3), for all $j \in \{0, 1, \dots, \ell - 1\}$,

$$(A.8) \quad \|P_\Omega(z^{j+1} - \nabla_x c(z^{j+1}, w^{i+1})) - z^{j+1}\| \leq c_{P_\Omega} \|z^{j+1} - z^j\|.$$

Then, by (A.7) and (A.8),

$$(A.9) \quad \begin{aligned} \ell \epsilon_c^2 &= \sum_{j=0}^{\ell-1} \epsilon_c^2 \leq \sum_{j=0}^{\ell-1} \|P_\Omega(z^{j+1} - \nabla_x c(z^{j+1}, w^{i+1})) - z^{j+1}\|^2 \\ &\leq c_{P_\Omega}^2 \sum_{j=0}^{\ell-1} \|z^{j+1} - z^j\|^2. \end{aligned}$$

On the other hand,

$$c(z^\ell, w^{i+1}) = \sum_{j=0}^{\ell-1} [c(z^{j+1}, w^{i+1}) - c(z^j, w^{i+1})] + c(z^0, w^{i+1}).$$

By (2.18) at Step 5.2 of Algorithm RESTA, we have that $c(z^{j+1}, w^{i+1}) \leq c(z^j, w^{i+1}) - \alpha_R \|z^{j+1} - z^j\|^2$, for all $j \in \{0, 1, \dots, \ell-1\}$. Therefore,

$$\begin{aligned} c(z^\ell, w^{i+1}) - c(z^0, w^{i+1}) &= \sum_{j=0}^{\ell-1} [c(z^{j+1}, w^{i+1}) - c(z^j, w^{i+1})] \\ &\leq -\alpha_R \sum_{j=0}^{\ell-1} \|z^{j+1} - z^j\|^2. \end{aligned}$$

By (A.9) and the fact that, at Step 3 of RESTA, we choose z^0 such that $c(z^0, w^{i+1}) \leq c(x^k, w^{i+1})$, we have that

$$c(z^\ell, w^{i+1}) \leq c(z^0, w^{i+1}) - \alpha_R \frac{\ell \epsilon_c^2}{c_{P_\Omega}^2} \leq c(x^k, w^{i+1}) - \alpha_R \frac{\ell \epsilon_c^2}{c_{P_\Omega}^2}.$$

Therefore, if

$$(A.10) \quad c(x^k, w^{i+1}) - \frac{\alpha_R \epsilon_c^2}{c_{P_\Omega}^2} \ell \leq c_{target}$$

we would have that $c(z^\ell, w^{i+1}) \leq c_{target}$ and the stopping condition (2.14) at Step 4 would be fulfilled.

Moreover, (A.10) occurs if and only if

$$(A.11) \quad \frac{c_{P_\Omega}^2}{\alpha_R \epsilon_c^2} [c(x^k, w^{i+1}) - c_{target}] \leq \ell.$$

Using the definitions of ϵ_c and c_{target} , given in (5.4), we obtain that (A.11) is equivalent to

$$\ell \geq \frac{c_{P_\Omega}^2}{\alpha_R [r_{feas} \|h(x^k, w^{i+1})\|]^2} [c(x^k, w^{i+1}) - r^2 c(x^k, w^{i+1})] = \frac{c_{P_\Omega}^2 (1 - r^2)}{2\alpha_R r_{feas}^2}.$$

Therefore, if $\ell \geq \frac{c_{P\Omega}^2(1-r^2)}{2\alpha_R r_{feas}^2}$ and (5.6) has not been fulfilled before, we have that $c(z^\ell, w^{i+1}) \leq c_{target}$. Therefore, we have that in at most C_{rest} sub-iterations of RESTA either (5.5) holds or (5.6) would have been obtained before. This completes the proof. \square

A.4. Proof of Lemma 5.4.

Proof. If $z^{\ell+1} = z^\ell$, (5.9) is trivial. If $z^{\ell+1} \neq z^\ell$, define $v = \frac{z^{\ell+1} - z^\ell}{\|z^{\ell+1} - z^\ell\|}$.

Since $z^{\ell+1}$ is an approximate minimizer of (2.17), then, by Assumption A2,

$$\nabla_x c(z^\ell, w^{i+1})^T (z^{\ell+1} - z^\ell) \leq -\frac{1}{2} (z^{\ell+1} - z^\ell)^T (B_\ell + \sigma I) (z^{\ell+1} - z^\ell).$$

By Step 5 of RESTA, we have that $\sigma \geq \sigma_{min}$ and $B_\ell + \sigma_{min}I$ is positive definite, so $\nabla_x c(z^\ell, w^{i+1})^T v < 0$.

Now, consider the function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$(A.12) \quad \varphi(t) = t \nabla_x c(z^\ell, w^{i+1})^T v + \frac{t^2}{2} v^T (B_\ell + \sigma I) v.$$

The unconstrained minimizer of $\varphi(t)$ is

$$t^* = -\frac{\nabla_x c(z^\ell, w^{i+1})^T v}{v^T (B_\ell + \sigma I) v} \leq -\frac{\nabla_x c(z^\ell, w^{i+1})^T v}{v^T (B_\ell + \sigma_{min} I) v} \leq \frac{\|\nabla_x c(z^\ell, w^{i+1})\| \|v\|}{\lambda_1(B_\ell + \sigma_{min} I) \|v\|^2},$$

where $\lambda_1(B_\ell + \sigma_{min} I) > 0$ is the smaller eigenvalue of $B_\ell + \sigma_{min} I$. As $\|v\| = 1$, by Step 5 of RESTA,

$$(A.13) \quad t^* \leq \|(B_\ell + \sigma_{min} I)^{-1}\| \|\nabla_x c(z^\ell, w^{i+1})\| \leq M \|\nabla_x c(z^\ell, w^{i+1})\|.$$

Let \bar{t} be the minimizer of $\varphi(t)$ subject to $z^\ell + tv \in \Omega$. By the convexity of Ω we have that $\bar{t} \leq t^*$. Moreover, by construction, $z^\ell + \bar{t}v = z_*^{\ell+1}$. So, by $\|v\| = 1$, Assumption A4, and (A.13), we have that

$$\|z^{\ell+1} - z^\ell\| \leq \kappa_\varphi \|z_*^{\ell+1} - z^\ell\| = \kappa_\varphi \bar{t} \leq \kappa_\varphi t^* \leq \kappa_\varphi M \|\nabla_x c(z^\ell, w^{i+1})\|.$$

Now, by (4.6), $\|\nabla_x h(z^\ell, w^{i+1})^T\| = \|\nabla_x h(z^\ell, w^{i+1})\| \leq C_h$, thus:

$$\begin{aligned} \|z^{\ell+1} - z^\ell\| &\leq \kappa_\varphi M \|\nabla_x h(z^\ell, w^{i+1})^T h(z^\ell, w^{i+1})\| \\ &\leq \kappa_\varphi M \|\nabla_x h(z^\ell, w^{i+1})^T\| \|h(z^\ell, w^{i+1})\| \\ &\leq \kappa_\varphi M C_h \|h(z^\ell, w^{i+1})\|. \end{aligned}$$

By (2.18) we have that $\|h(z^{\ell+1}, w^{i+1})\| \leq \|h(z^\ell, w^{i+1})\|$ and, by the choice of z^0 at Step 3 of RESTA, we have that $\|h(z^0, w^{i+1})\| \leq \|h(x^k, w^{i+1})\|$. Then, $\|h(z^\ell, w^{i+1})\| \leq \|h(x^k, w^{i+1})\|$ for all ℓ . Therefore, (5.9) holds. \square

A.5. Proof of Lemma 5.5.

Proof. For a fixed i , by Lemma 5.3, after at most C_{rest} steps we find z^ℓ satisfying (5.5) or (5.6). At each of these steps, $\nabla_x c$ is evaluated only once, therefore, the total number of evaluations of $\nabla_x c$ is bounded by C_{rest} .

By Lemma 5.1, $\sigma \geq \bar{\sigma}$ implies that (2.18) is fulfilled and, so, $z^{\ell+1}$ is well defined. By (2.19), σ is increased according to $\sigma \in [2\sigma, 10\sigma]$. Therefore, as the initial value of σ is not smaller than σ_{min} , we have that after n_σ trials we will have that $\sigma \geq 2^{n_\sigma} \sigma_{min}$. Therefore, we have that $\sigma \geq 2^{n_\sigma} \sigma_{min} \geq \bar{\sigma}$ and, so, $z^{\ell+1}$ is obtained.

Therefore, the descent condition (2.18) is tested at most n_σ times for each value of ℓ . Consequently, h is evaluated at most n_σ times for every ℓ . So, the condition (2.18) is tested at most $C_{rest}n_\sigma$ times for all fixed w^{i+1} .

Finally, observe that, by Step 2, after at most N_{prec} trials we have that $g_h(w^{i+1}) \leq \bar{\epsilon}_{prec}$. In this case, the process would ensure that conditions (2.14) or (2.15) hold and, so, the Restoration Phase would be finished. Moreover, only one additional evaluation of h is performed at each update of w^{i+1} . Then, we obtain the desired result. \square

A.6. Proof of Lemma 5.6.

Proof. If (x_R^k, y_R^k) is computed by a problem-dependent procedure, the result is true by (2.10). Now, let us consider that (x_R^k, y_R^k) is computed by RESTA. For given k , let i be such that $w^{i+1} = y_R^k$ and let N_{Rk} be the number of sub-iterations performed for the minimization of $c(z, w^{i+1})$. By Lemma 5.5 we have that $N_{Rk} \leq N_{RESTA}$. Then, by Lemma 5.4 and the choice of z^0 at Step 3 of RESTA, we have that

$$\begin{aligned} \|x_R^k - x^k\| &\leq \|z^0 - x^k\| + \sum_{l=1}^{N_{Rk}} \|z^\ell - z^{\ell-1}\| \\ &\leq \beta_c \|h(x^k, y_R^k)\| + \sum_{l=1}^{N_{Rk}} C_s \|h(x^k, y_R^k)\| \\ &\leq \beta_c \|h(x^k, y_R^k)\| + N_{RESTA} C_s \|h(x^k, y_R^k)\|. \end{aligned}$$

Therefore, we obtain the desired result. \square

A.7. Proof of Lemma 5.7.

Proof. By (4.2) we have that $|f(x_R^k, y_R^k) - f(x^k, y_R^k)| \leq L_f \|x_R^k - x^k\|$. Then, by (5.11),

$$\begin{aligned} f(x_R^k, y_R^k) - f(x^k, y^k) &\leq f(x_R^k, y_R^k) - f(x^k, y_R^k) + f(x^k, y_R^k) - f(x^k, y^k) \\ &\leq L_f \|x_R^k - x^k\| + \beta g(y^k). \end{aligned}$$

By Lemma 5.6 we have that $\|x_R^k - x^k\| \leq \beta_R \|h(x^k, y_R^k)\|$. Then,

$$\begin{aligned} f(x_R^k, y_R^k) - f(x^k, y^k) &\leq L_f \beta_R \|h(x^k, y_R^k)\| + \beta g(y^k) \\ &\leq [L_f \beta_R + \beta] [\|h(x^k, y_R^k)\| + g(y^k)]. \end{aligned}$$

Thus, we have the desired result. \square

A.8. Proof of Lemma 6.1.

Proof. At each iteration k of BIRA we have two options, according to the fulfillment of (2.4). If (2.4) holds, we define $\theta_{k+1} = \theta_k$, therefore θ_{k+1} is well defined and does not increase with respect to θ_k . Moreover, in this case (6.1) is equivalent to (2.4), so it is fulfilled.

In the second case, θ_{k+1} is defined by (2.5) at Step 2, according to:

$$\theta_{k+1} = \frac{(1+r)[\|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| + g(y^k) - g(y_R^k)]}{2[f(x_R^k, y_R^k) - f(x^k, y_R^k) + \|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| + g(y^k) - g(y_R^k)]}.$$

Let us show that both the numerator and the denominator of this expression are positive and that the quotient is smaller than θ_k .

By the restoration step and the assumptions G1–G12, we have that $g(y_R^k) \leq rg(y^k)$, so $g(y_R^k) - g(y^k) \leq 0$. Therefore, as $\frac{1-r}{2} \in (0, 1)$, we have that

$$(A.14) \quad g(y_R^k) - g(y^k) \leq \frac{1-r}{2}[g(y_R^k) - g(y^k)].$$

Moreover, if the execution of BIRA is not stopped declaring Restoration Failure, the restoration always guarantees that $\|h(x_R^k, y_R^k)\| \leq r\|h(x^k, y_R^k)\|$. Therefore,

$$(A.15) \quad \|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| \leq \frac{1-r}{2} [\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\|].$$

Now, the equalities in (A.14) and (A.15) only take place if $\|h(x_R^k, y_R^k)\| = \|h(x^k, y_R^k)\| = g(y_R^k) = g(y^k) = 0$. In this case, if (x_R^k, y_R^k) is computed by the PDP, by (2.10), we have that $(x_R^k, y_R^k) = (x^k, y^k)$. On the other hand, if RESTA is used, since $g(y^k) = 0$, by Step 2, we would have that $w^i = y^k$ for all i , implying that $y_R^k = y^k$. So, $\|h(x^k, y^k)\| = \|h(x^k, y_R^k)\| = 0$ and, by Step 1 of RESTA, we also have that $(x_R^k, y_R^k) = (x^k, y^k)$. In this case, (2.4) would be trivially fulfilled and we would have that $\theta_{k+1} = \theta_k$. Then, at least one of the conditions (A.14) or (A.15) is strictly satisfied. This proves that, when $\theta_{k+1} \neq \theta_k$, the numerator of (2.5) is positive.

Now let us analyze the expression $\Phi(x_R^k, y_R^k, \theta) - \Phi(x^k, y_R^k, \theta)$ as a function of θ . By the definition of the merit function in (2.1), we have that

$$(A.16) \quad \begin{aligned} \Phi(x_R^k, y_R^k, \theta) - \Phi(x^k, y_R^k, \theta) &= \theta \left[f(x_R^k, y_R^k) - f(x^k, y_R^k) + \|h(x^k, y_R^k)\| \right. \\ &\quad \left. - \|h(x_R^k, y_R^k)\| \right] - [\|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\|], \end{aligned}$$

which is linear with respect to θ and its slope is less or equal half the denominator of (2.5). Moreover, this slope must be positive, otherwise, by (2.3), (2.4) would hold for all non-negative θ . So the expression of θ_{k+1} is well defined and $\Phi(x_R^k, y_R^k, \theta) - \Phi(x^k, y_R^k, \theta)$ is an increasing bijection from \mathbb{R} to \mathbb{R} .

When $\theta = 0$ we have that

$$\Phi(x_R^k, y_R^k, 0) - \Phi(x^k, y_R^k, 0) = [\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\|].$$

Since one of the inequalities in (A.14) or (A.15) is strict, by (2.3), we have that

$$(A.17) \quad \begin{aligned} & \Phi(x_R^k, y_R^k, 0) - \Phi(x^k, y_R^k, 0) \\ & < \frac{1-r}{2} [\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k)]. \end{aligned}$$

However, if (2.4) does not hold, we have that

$$(A.18) \quad \begin{aligned} & \Phi(x_R^k, y_R^k, \theta_k) - \Phi(x^k, y_R^k, \theta_k) \\ & > \frac{1-r}{2} [\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k)]. \end{aligned}$$

So there exists only one value of $\theta \in (0, \theta_k)$ verifying

$$\begin{aligned} & \Phi(x_R^k, y_R^k, \theta) - \Phi(x^k, y_R^k, \theta) \\ & = \frac{1-r}{2} [\|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k)]. \end{aligned}$$

By (A.16), this value of θ is greater than or equal to θ_{k+1} computed in (2.5). Therefore we also have that (6.1) holds. So, the proof is complete. \square

A.9. Proof of Lemma 6.2.

Proof. It is enough to prove that θ_{k+1} is bounded below by $\bar{\theta}$ when it is defined by (2.5).

Equivalently, we need to show that $\frac{1}{\theta_{k+1}}$ is bounded above in this situation. In fact,

$$(A.19) \quad \begin{aligned} \frac{1}{\theta_{k+1}} &= \frac{2[f(x_R^k, y_R^k) - f(x^k, y_R^k)] + \|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| + g(y^k) - g(y_R^k)}{(1+r)[\|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| + g(y^k) - g(y_R^k)]} \\ &= \frac{2}{1+r} \left[\frac{f(x_R^k, y_R^k) - f(x^k, y_R^k)}{\|h(x^k, y_R^k)\| - \|h(x_R^k, y_R^k)\| + g(y^k) - g(y_R^k)} + 1 \right]. \end{aligned}$$

By Step 1 of RESTA or (2.9) when using a PDP,

$$-\|h(x_R^k, y_R^k)\| - g(y_R^k) \geq -r\|h(x^k, y_R^k)\| - rg(y^k),$$

therefore

$$\begin{aligned} & \|h(x^k, y_R^k)\| + g(y^k) - \|h(x_R^k, y_R^k)\| - g(y_R^k) \\ & \geq \|h(x^k, y_R^k)\| + g(y^k) - r\|h(x^k, y_R^k)\| - rg(y^k) \\ & = (1-r)(\|h(x^k, y_R^k)\| + g(y^k)) > 0. \end{aligned}$$

Positivity necessarily takes place, otherwise we would have that $(x_R^k, y_R^k) = (x^k, y^k)$ and $\theta_{k+1} = \theta_k$. Thus,

$$\begin{aligned} 0 &< \frac{1}{\|h(x^k, y_R^k)\| + g(y^k) - \|h(x_R^k, y_R^k)\| - g(y_R^k)} \\ &\leq \frac{1}{(1-r)(\|h(x^k, y_R^k)\| + g(y^k))}. \end{aligned}$$

On the other hand, by (4.2), we have that $|f(x_R^k, y_R^k) - f(x^k, y_R^k)| \leq L_f \|x_R^k - x^k\|$. Then, by (A.19),

$$\frac{1}{\theta_{k+1}} \leq \frac{2}{1+r} \left[\frac{L_f \|x_R^k - x^k\|}{(1-r)(\|h(x^k, y_R^k)\| + g(y^k))} + 1 \right].$$

By (5.12) in Theorem 5.1, there exists a positive constant $\beta_R = O(1)$ such that $\|x_R^k - x^k\| \leq \beta_R \|h(x^k, y_R^k)\|$. Then, since $g(y^k) \geq 0$,

$$\begin{aligned} \frac{1}{\theta_{k+1}} &\leq \frac{2}{1+r} \left[\frac{L_f \beta_R \|h(x^k, y_R^k)\|}{(1-r)(\|h(x^k, y_R^k)\| + g(y^k))} + 1 \right] \\ &\leq \frac{2}{1+r} \left[\frac{L_f \beta_R (\|h(x^k, y_R^k)\| + g(y^k))}{(1-r)(\|h(x^k, y_R^k)\| + g(y^k))} + 1 \right] = \frac{2}{1+r} \left[\frac{L_f \beta_R}{1-r} + 1 \right]. \end{aligned}$$

The inequality above implies that, when θ_k is updated, $\{\frac{1}{\theta_k}\}$ is bounded, so $\{\theta_k\}$ is bounded away from zero, with $\bar{\theta} > 0$ as lower bound. \square

A.10. Proof of Lemma 6.3.

Proof. By (4.4), we have that $f(x, y^{k+1}) \leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + L_f \|x - x_R^k\|^2$.

Then, since $\|H_k\| \leq M$, we have that

$$\begin{aligned} f(x, y^{k+1}) &\leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) \\ &\quad + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) - \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) \\ &\quad + \tilde{\alpha} \|x - x_R^k\|^2 - \tilde{\alpha} \|x - x_R^k\|^2 + L_f \|x - x_R^k\|^2 \\ &\leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) \\ &\quad + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) + M \|x - x_R^k\|^2 \\ &\quad + \tilde{\alpha} \|x - x_R^k\|^2 - \tilde{\alpha} \|x - x_R^k\|^2 + L_f \|x - x_R^k\|^2 \\ &\leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) \\ &\quad + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) + (M + \tilde{\alpha} + L_f) \|x - x_R^k\|^2 \\ &\quad - \tilde{\alpha} \|x - x_R^k\|^2. \end{aligned}$$

Taking $\mu \geq C_\mu$, by Assumption A7,

$$\begin{aligned} f(x, y^{k+1}) &\leq f(x_R^k, y^{k+1}) - \tilde{\alpha} \|x - x_R^k\|^2 + [\nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) \\ &\quad + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) + \mu \|x - x_R^k\|^2] \\ &\leq f(x_R^k, y^{k+1}) - \tilde{\alpha} \|x - x_R^k\|^2. \end{aligned}$$

Since $\alpha \leq \tilde{\alpha}$ and $y^{k+1} = y_R^k$, (2.7) necessarily holds. Moreover, by (6.5), we have that

$$(A.20) \quad f(x, y^{k+1}) - f(x_R^k, y^{k+1}) \leq -\frac{1-\bar{\theta}}{\bar{\theta}} (\kappa_T + L_h) \|x - x_R^k\|^2.$$

Let us prove that (2.8) also holds when $\mu \geq C_\mu$. Note that

$$(A.21) \quad \begin{aligned} &\Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x^k, y^{k+1}, \theta_{k+1}) \\ &= [\Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x_R^k, y^{k+1}, \theta_{k+1})] \\ &\quad + [\Phi(x_R^k, y^{k+1}, \theta_{k+1}) - \Phi(x^k, y^{k+1}, \theta_{k+1})]. \end{aligned}$$

Define $v = \Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x_R^k, y^{k+1}, \theta_{k+1})$. By the definition of Φ and (A.20), the first term in the right-hand side of the equality above we have that

$$\begin{aligned} v &= \theta_{k+1} [f(x, y^{k+1}) - f(x_R^k, y^{k+1})] \\ &\quad + (1 - \theta_{k+1}) \left[\|h(x, y^{k+1})\| - \|h(x_R^k, y^{k+1})\| \right] \\ &\leq \theta_{k+1} \left[-\frac{1 - \bar{\theta}}{\bar{\theta}} (\kappa_T + L_h) \|x - x_R^k\|^2 \right] \\ &\quad + (1 - \theta_{k+1}) \left[\|h(x, y^{k+1})\| - \|h(x_R^k, y^{k+1})\| \right]. \end{aligned}$$

By (4.9) and (6.4),

$$\begin{aligned} v &\leq \theta_{k+1} \left[-\frac{1 - \bar{\theta}}{\bar{\theta}} (\kappa_T + L_h) \|x - x_R^k\|^2 \right] \\ &\quad + (1 - \theta_{k+1}) \left[\|\nabla_x h(x_R^k, y^{k+1})^T (x - x_R^k)\| + L_h \|x - x_R^k\|^2 \right] \\ &\leq \theta_{k+1} \left[-\frac{1 - \bar{\theta}}{\bar{\theta}} (\kappa_T + L_h) \|x - x_R^k\|^2 \right] \\ &\quad + (1 - \theta_{k+1}) \left[\kappa_T \|x - x_R^k\|^2 + L_h \|x - x_R^k\|^2 \right]. \end{aligned}$$

Since $\{\theta_k\}$ is bounded below by $\bar{\theta}$, we have that

$$v \leq \bar{\theta} \left[-\frac{1 - \bar{\theta}}{\bar{\theta}} (\kappa_T + L_h) \|x - x_R^k\|^2 \right] + (1 - \bar{\theta}) (\kappa_T + L_h) \|x - x_R^k\|^2 = 0.$$

Then, the first term in (A.21) is not positive. On the other hand, as $y^{k+1} = y_R^k$, (6.1) is equivalent to

$$\begin{aligned} &\Phi(x_R^k, y^{k+1}, \theta_{k+1}) - \Phi(x^k, y^{k+1}, \theta_{k+1}) \\ &\leq \frac{1-r}{2} \left[\|h(x_R^k, y^{k+1})\| - \|h(x^k, y^{k+1})\| + g(y^{k+1}) - g(y^k) \right]. \end{aligned}$$

Then, by (A.21),

$$\begin{aligned} &\Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x^k, y^{k+1}, \theta_{k+1}) \\ &\leq \frac{1-r}{2} \left[\|h(x_R^k, y^{k+1})\| - \|h(x^k, y^{k+1})\| + g(y^{k+1}) - g(y^k) \right]. \end{aligned}$$

Therefore, if $\mu \geq C_\mu$, both (2.7) and (2.8) are fulfilled, guaranteeing that x^{k+1} is well defined. \square

A.11. Proof of Corollary 6.1.

Proof. If x^{k+1} is computed at the first N_{acce} iterations of the Optimization Phase, we have that μ is increased at most N_{accel} times, starting from a value limited above by μ_{max} . Therefore $\mu_k \leq 10^{N_{acce}} \mu_{max}$. On the other hand, if $y^{k+1} = y_R^k$ and $\mu \geq C_\mu$, by Lemma 6.3, the decrease conditions at Step 3.2 are satisfied. So, if the initial value of μ is greater than C_μ , $\mu_k = \mu$. Otherwise, as $\mu_{new} \in [2\mu, 10\mu]$, we would have that $\mu_k \leq 10C_\mu$. Since $10^{N_{acce}} \mu_{max} \geq \mu_{max}$, we have that $\mu_k \leq \max\{10C_\mu, 10^{N_{acce}} \mu_{max}\}$.

Moreover, as the initial value of μ is not smaller than μ_{min} , after N_{reg} updatings we have that $\mu \geq 2^{N_{reg}} \mu_{min}$. So, if $y^{k+1} = y_R^k$ and $2^{N_{reg}} \mu_{min} \geq C_\mu$, or, equivalently, $N_{reg} + \log_2(\mu_{min}) \geq \log_2(C_\mu)$, (2.7) and (2.8) are fulfilled. \square

A.12. Proof of Lemma 8.1.

Proof. Define $v = P_{D^{k+1}}(x^{k+1} - \nabla_x f(x_R^k, y^{k+1}) - H_k(x^{k+1} - x_R^k) - 2\mu_k(x^{k+1} - x_R^k))$.

By (8.1),
(A.22)

$$\begin{aligned} & \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - x_R^k\| \\ &= \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - v + v - x^{k+1} + x^{k+1} - x_R^k\| \\ &\leq \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - v\| + \|v - x^{k+1}\| + \|x^{k+1} - x_R^k\| \\ &\leq \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - v\| + (\kappa + 1)\|x^{k+1} - x_R^k\|. \end{aligned}$$

By Step 3 of BIRA, we have that $\|H_k\| \leq M$. By the non-expansive property of projections,

$$\begin{aligned} & \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - v\| \\ (A.23) \quad &\leq \|x_R^k - x^{k+1} + H_k(x^{k+1} - x_R^k) + 2\mu_k(x^{k+1} - x_R^k)\| \\ &\leq \|H_k(x^{k+1} - x_R^k)\| + (2\mu_k + 1)\|x^{k+1} - x_R^k\| \\ &\leq (M + 2\mu_k + 1)\|x^{k+1} - x_R^k\|. \end{aligned}$$

By Corollary 6.1, we have that $\mu_k \leq \bar{\mu}$. Then, by (A.22) and (A.23):

$$\begin{aligned} & \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - x_R^k\| \\ &\leq \|P_{D^{k+1}}(x_R^k - \nabla_x f(x_R^k, y^{k+1})) - v\| + (\kappa + 1)\|x^{k+1} - x_R^k\| \\ &\leq (M + 2\mu_k + 1)\|x^{k+1} - x_R^k\| + (\kappa + 1)\|x^{k+1} - x_R^k\| \\ &= (M + \kappa + 2\bar{\mu} + 2)\|x^{k+1} - x_R^k\|. \end{aligned}$$

\square

A.13. Proof of Lemma 8.2.

Proof. By Lemma 8.1,

$$(A.24) \quad \|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\| \leq C_p \|x^{j+1} - x_R^j\|,$$

for all j .

Adding the first k squared terms of (A.24), by (7.10), we have that

$$\begin{aligned} \sum_{j=0}^k \|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{j+1})) - x_R^j\|^2 &\leq \sum_{j=0}^k \left[C_p \|x^{j+1} - x_R^j\| \right]^2 \\ &= C_p^2 \sum_{j=0}^k \|x^{j+1} - x_R^j\|^2 \\ &\leq C_p^2 C_d. \end{aligned}$$

\square

A.14. Proof of Lemma 8.3.

Proof. By (7.3),

$$\sum_{j=0}^k [\|h(x^j, y_R^j)\| + g(y^j)] \leq C_{feas}.$$

Then, as $0 \leq \|h(x_R^j, y_R^j)\| \leq r\|h(x^j, y_R^j)\|$ and $g(y^j) \geq 0$,

$$\begin{aligned} rC_{feas} &\geq \sum_{j=0}^k r\|h(x^j, y_R^j)\| + rg(y^j) \geq \sum_{\substack{j=0 \\ \|h(x_R^j, y_R^j)\| > \epsilon_{feas}}}^k \|h(x_R^j, y_R^j)\| \\ &\geq N_{hinfeas}\epsilon_{feas}. \end{aligned}$$

So, $\left\lfloor \frac{rC_{feas}}{\epsilon_{feas}} \right\rfloor \geq N_{hinfeas}$. Analogously, for $g(y^j) > \epsilon_{prec}$, we have that

$$C_{feas} \geq \sum_{j=0}^k [\|h(x^j, y_R^j)\| + g(y^j)] \geq \sum_{\substack{j=0 \\ g(y^j) > \epsilon_{prec}}}^k g(y^j) \geq N_{ginfeas}\epsilon_{prec},$$

therefore $\left\lfloor \frac{C_{feas}}{\epsilon_{prec}} \right\rfloor \geq N_{ginfeas}$.

Finally, if $\|h(x_R^k, y_R^k)\| > \epsilon_{feas}$ or $g(y_R^k) > \epsilon_{prec}$ we have that $\|h(x_R^j, y_R^j)\| + g(y_R^j) > \min\{\epsilon_{feas}, \epsilon_{prec}\}$. Thus, as $\|h(x_R^j, y_R^j)\| + g(y^j) \leq r(\|h(x^j, y_R^j)\| + g(y^j))$,

$$\begin{aligned} rC_{feas} &\geq \sum_{j=0}^k r(\|h(x^j, y_R^j)\| + g(y^j)) \\ &\geq \sum_{\substack{j=0 \\ \|h(x_R^j, y_R^j)\| + g(y_R^j) > \min\{\epsilon_{feas}, \epsilon_{prec}\}}}^k [\|h(x_R^j, y_R^j)\| + g(y_R^j)] \\ &\geq N_{infeas} \min\{\epsilon_{feas}, \epsilon_{prec}\}, \end{aligned}$$

$$\text{so } N_{infeas} \leq \left\lfloor \max\left\{ \frac{rC_{feas}}{\epsilon_{feas}}, \frac{rC_{feas}}{\epsilon_{prec}} \right\} \right\rfloor. \quad \square$$

A.15. Proof of Lemma 8.4.

Proof. If during N_{opt} iterations we have $\|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{k+1})) - x_R^j\| > \epsilon_{opt}$, by (8.5), we have that

$$C_{proj} \geq \sum_{j=0}^k \|P_{D^{j+1}}(x_R^j - \nabla_x f(x_R^j, y^{k+1})) - x_R^j\|^2 \geq N_{opt}\epsilon_{opt}^2$$

Therefore, $\left\lfloor \frac{C_{proj}}{\epsilon_{opt}^2} \right\rfloor \geq N_{opt}$. □

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