

# Semi-infinite models for equilibrium selection

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February 23, 2022

## Abstract

In their seminal work ‘A General Theory of Equilibrium Selection in Games’ (The MIT Press, 1988) Harsanyi and Selten introduce the notion of payoff dominance to explain how players select some solution of a Nash equilibrium problem from a set of nonunique equilibria. We formulate this concept for generalized Nash equilibrium problems, relax payoff dominance to the more widely applicable requirement of payoff nondominatedness, and show how different characterizations of generalized Nash equilibria yield different semi-infinite optimization problems for the computation of payoff nondominated equilibria. Since all these problems violate a standard constraint qualification, we also formulate regularized versions of the optimization problems. Under additional assumptions we state a nonlinear cutting algorithm and provide numerical results for a multi-agent portfolio optimization problem.

**Keywords:** Equilibrium selection, Nash game, payoff dominance, semi-infinite optimization, cutting algorithm.

**Preliminary citation:** Optimization Online, Preprint ID 2022-02-8805.

## 1 Introduction

In generalized Nash equilibrium problems (GNEPs) a collection of  $N$  players each control a decision variable  $x^\nu \in \mathbb{R}^{n_\nu}$ ,  $\nu \in \{1, \dots, N\}$ . The decision vector of all players is denoted by  $x = (x^1, \dots, x^N) \in \mathbb{R}^n$  with  $n = n_1 + \dots + n_N$ , and the notation  $x = (x^\nu, x^{-\nu})$  emphasizes the role of player  $\nu$ 's variable  $x^\nu$  within the vector  $x$ . For each player  $\nu$  a continuous cost function  $\theta_\nu(\cdot, x^{-\nu})$  and a closed strategy set  $X_\nu(x^{-\nu})$  define the optimization problem

$$P_\nu(x^{-\nu}) : \quad \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \quad \text{s.t.} \quad x^\nu \in X_\nu(x^{-\nu})$$

with a set of optimal points  $S_\nu(x^{-\nu})$ . The generalized Nash equilibrium problem then is stated as

$$\text{GNEP:} \quad \text{find some } x \in \mathbb{R}^n \text{ with } x^\nu \in S_\nu(x^{-\nu}), \nu = 1, \dots, N.$$

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A solution point  $x^*$  of *GNEP* is called a *generalized Nash equilibrium (GNE)*. In a GNE  $x^*$ , no player  $\nu$  possesses a rational incentive to deviate from the decision  $x^{\nu,*}$ , since it is a minimal point of  $P_\nu(x^{-\nu,*})$ .

The difference to a standard Nash equilibrium problem (NEP) lies in the  $x^{-\nu}$ -dependence of the strategy spaces  $X_\nu(x^{-\nu})$ , that is, in a standard NEP, each player has a fixed strategy set  $X_\nu$ , and only the cost function depends on the parameter vector  $x^{-\nu}$ . While NEPs were introduced in [24], GNEPs go back to [1, 3]. For a survey on theory, applications, and algorithms for the solution of GNEPs, we refer to [8, 10].

With the graph  $\text{gph } S_\nu = \{x \in \mathbb{R}^n \mid x^\nu \in S(x^{-\nu})\}$  of the set-valued mapping  $S_\nu$  the set

$$E := \bigcap_{\nu=1}^N \text{gph } S_\nu$$

forms the set of all GNEs of the problem *GNEP*. The set  $E$  may be empty, a singleton, or a non-singleton set. Sufficient conditions for unique solvability of GNEPs can be found in [9].

**Example 1.1.** With  $N = 2$  and  $n_1 = n_2 = 1$  let  $q_1, q_2 : \mathbb{R}^1 \rightarrow \mathbb{R}^1$  be convex quadratic functions,  $\theta_1(x) = x_1$ ,  $\theta_2(x) = x_2$ , and consider the player problems

$$\begin{aligned} P_1(x_2) : \quad & \min_{x_1} x_1 \quad \text{s.t.} \quad q_1(x_2) \leq x_1, \\ P_2(x_1) : \quad & \min_{x_2} x_2 \quad \text{s.t.} \quad q_2(x_1) \leq x_2. \end{aligned}$$

This yields  $S_1(x_2) = \{q_1(x_2)\}$ ,  $S_2(x_1) = \{q_2(x_1)\}$  and

$$E = \text{gph } S_1 \cap \text{gph } S_2 = \{x \in \mathbb{R}^2 \mid x_1 = q_1(x_2), x_2 = q_2(x_1)\}.$$

It is easy to specify functions  $q_1$  and  $q_2$  such that the equilibrium set is empty or a singleton. On the other hand, Figure 1 illustrates a situation with a non-singleton set  $E = \{x^1, x^2, x^3, x^4\}$ .

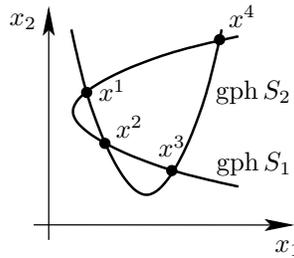


Figure 1: Nonunique equilibria in Example 1.1

The present paper focuses on the case of nonunique GNEs (i.e.,  $|E| > 1$  as in Fig. 1) where the players may prefer some equilibria over others. Their preferences may be explained by refined equilibrium concepts like the ones introduced in [14]. Such considerations are known as equilibrium selection or Nash refinement. We will relax Harsanyi

and Selten's selection concept of payoff dominance, introduced in Section 2, to payoff non-dominatedness. This yields a parametrized objective function  $f(\lambda, \cdot)$  which lets one select equilibria by computing optimal points of the equilibrium selection problems

$$ES(\lambda) : \quad \min_x f(\lambda, x) \quad \text{s.t.} \quad x \in E$$

with some parameter vector  $\lambda \in \mathbb{R}^N$  (cf. Sec. 2). Section 3 reviews different possibilities for functional descriptions of the set  $E$ , yielding the semi-infinite models studied in Section 4. Since in all these models a standard constraint qualification is violated at each feasible point, Section 5 introduces regularized problems. Section 6 formulates a cutting algorithm for the special case of player convex standard Nash equilibrium problems with polyhedral strategy sets and provides numerical results, before Section 7 concludes the article with some final remarks.

## 2 Multicriteria optimization and payoff dominance

In multicriteria optimization a vector-valued function  $\theta : \mathbb{R}^n \rightarrow \mathbb{R}^N$  is to be minimized over a feasible set  $E \subseteq \mathbb{R}^n$ . Since in many applications the entries  $\theta_1, \dots, \theta_N$  of  $\theta$  model conflicting objectives (i.e., a decrease in one of them leads to an increase in another), and the set  $\mathbb{R}^N$  is not totally ordered for  $N \geq 2$ , an appropriate concept of global minimality is not straightforward. For details and motivation of the following notions we refer to [6, 17, 23]. We mention that throughout this paper we use multiobjective optimality notions based on the natural ordering cone  $\mathbb{R}_+^N$ , and that all inequalities between vectors are meant componentwise.

The following definition works for any set  $Y$  in the image space  $\mathbb{R}^N$  of  $\theta$ , but will subsequently be applied to the set  $Y = \theta(E)$  of attainable points.

**Definition 2.1.** *For some set  $Y \subseteq \mathbb{R}^N$  let  $\bar{y} \in Y$ .*

- a) *The point  $\bar{y}$  is called a nondominated point of  $Y$  if there exists no  $y \in Y$  with  $y \leq \bar{y}$  and  $y \neq \bar{y}$ .*
- b) *The point  $\bar{y}$  is called a weakly nondominated point of  $Y$  if there exists no  $y \in Y$  with  $y < \bar{y}$ .*
- c) *The point  $\bar{y}$  is called a strongly nondominated point of  $Y$  if all  $y \in Y$  satisfy  $\bar{y} \leq y$ .*

The next definition collects the according notions in the pre-image space  $\mathbb{R}^n$ .

**Definition 2.2.** *For some set  $E \subseteq \mathbb{R}^n$  let  $\bar{x} \in E$  and  $\theta : E \rightarrow \mathbb{R}^N$ .*

- a) *The point  $\bar{x}$  is called Pareto-optimal (or efficient) if  $\theta(\bar{x})$  is a nondominated point of  $\theta(E)$ .*
- b) *The point  $\bar{x}$  is called weakly Pareto-optimal (or weakly efficient) if  $\theta(\bar{x})$  is a weakly nondominated point of  $\theta(E)$ .*

c) The point  $\bar{x}$  is called *strongly Pareto-optimal* (or *strongly efficient*) if  $\theta(\bar{x})$  is a *strongly nondominated point* of  $\theta(E)$ .

The pre-image space concepts from Definition 2.2 generalize the concept of globally minimal points from the single-objective case, while the image space concepts from Definition 2.1 generalize the concept of the globally minimal value. In particular, the latter unique scalar is generalized to some subset of  $\mathbb{R}^N$ , the so-called nondominated set.

While multiobjective optimization problems possess Pareto-optimal points under mild assumptions [6, 17, 23], the existence of strongly Pareto-optimal points is often ruled out in the presence of conflicting objectives (cf. Ex. 2.3).

Harsanyi and Selten's concept of payoff dominance, applied to GNEPs, states that among nonunique equilibria players choose a strongly Pareto-optimal point of the multi-objective problem

$$MOP : \quad \min \theta(x) \quad \text{s.t.} \quad x \in E$$

if such a point exists. Then in comparison to other equilibria no player is worse off (Harsanyi and Selten even strengthen this and require that all players are better off, i.e., the nonstrict inequality in Def. 2.1c is replaced by a strict one). Here the 'payoff' terminology originates from the formulation of the player problems as maximization problems in [14], rather than minimization. In the setting of the present paper we could use a more appropriate term like 'cost dominance', but we stick to the original terminology to avoid confusion.

**Example 2.3.** *In the situation of Example 1.1 we have  $\theta(x) = x$ , so that the sets of (strongly) Pareto-optimal points and (strongly) nondominated points of MOP coincide. While  $x^1$ ,  $x^2$  and  $x^3$  are Pareto-optimal points, no strongly Pareto-optimal point exists (the point  $x^4$ , though, is an example of a strongly Pareto-optimal point for the maximization of  $\theta$  over  $E$ ).*

Since the problem  $MOP$  can only be expected to possess strongly Pareto-optimal points under strong assumptions, the concept to select payoff *dominant* equilibria does not seem to be widely applicable. Instead one may rather select Pareto-optimal equilibria, that is, payoff *nondominated* ones. The disadvantage of the latter approach lies in the fact that, while such points exist under mild assumptions, typically they are not unique (cf. Ex. 2.3). Hence this approach does not identify a unique equilibrium as desired by Harsanyi and Selten. However, at least it filters out equilibria which are not interesting in the following sense: For an equilibrium which is not Pareto-optimal there exists another equilibrium for which the objective function  $\theta_\nu$  of at least one player improves while the objectives of all other players do not deteriorate.

A popular method to compute Pareto-optimal points of multiobjective problems is weighted sum scalarization. In fact, it is not hard to see that for any  $\lambda \in \mathbb{R}^N$  with  $\lambda \geq 0$ ,  $\lambda \neq 0$ , each globally minimal point of the single-objective optimization problem

$$ES(\lambda) : \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x \in E$$

is a weakly Pareto-optimal point of  $MOP$ , and that for  $\lambda > 0$  the globally minimal points of  $ES(\lambda)$  are Pareto-optimal points of  $MOP$  [6]. A major drawback of this approach is that it

can only find (weak) Pareto-optimal points  $\bar{x} \in E$  for which  $\theta(\bar{x})$  lies in the boundary of the convex hull of  $\theta(E) + \mathbb{R}_+^N$ . For more sophisticated methods which approximate the complete set of Pareto-optimal points we refer to [6, 17, 23] and to the recent branch-and-bound method for nonconvex mixed-integer multiobjective problems in [7].

In the present paper we focus our attention on making the computation of minimal points of the problem  $ES(\lambda)$  algorithmically tractable. This amounts to finding an appropriate functional description of its feasible set  $E$ . Such a description may also be employed in the other above-mentioned methods for the computation of Pareto-optimal points, but this would go beyond the scope of the present analysis.

Given the desired algorithmic tractability of  $ES(\lambda)$ , equilibrium selection amounts to choosing some weight vector  $\lambda > 0$  and some globally minimal point of the equilibrium selection problem  $ES(\lambda)$ . We emphasize that

- we will not discuss the choice of  $\lambda$ ,
- for given  $\lambda > 0$  globally minimal points of  $ES(\lambda)$  need not be unique,
- some Pareto-optimal equilibria may not be found by this weighted-sum-based approach since they are not globally minimal points of  $ES(\lambda)$  for any  $\lambda \geq 0$ ,  $\lambda \neq 0$ .

### 3 Functional descriptions of the equilibrium set

Since for any  $\lambda$  the constraint of the equilibrium selection problem  $ES(\lambda)$  requires  $x$  to be an equilibrium of  $GNEP$ , the problem  $ES(\lambda)$  falls into the class *mathematical programs with equilibrium constraints (MPECs)* [22]. The present section reviews four possibilities for functional descriptions of  $E$  which allow the data of the MPEC to be communicated to some optimization algorithm.

As a prerequisite we assume that the players' strategy sets possess functional descriptions

$$X_\nu(x^{-\nu}) = \{y^\nu \in \mathbb{R}^{n_\nu} \mid g^\nu(y^\nu, x^{-\nu}) \leq 0\}$$

with  $g^\nu : \mathbb{R}^n \rightarrow \mathbb{R}^{m_\nu}$ . The players' optimization problems then read

$$P_\nu(x^{-\nu}) : \quad \min_{y^\nu} \theta_\nu(y^\nu, x^{-\nu}) \quad \text{s.t.} \quad g^\nu(y^\nu, x^{-\nu}) \leq 0.$$

#### 3.1 A direct reformulation

A point  $x$  lies in  $E = \bigcap_{\nu=1}^N \text{gph } S_\nu$  if and only if  $x^\nu \in X(x^{-\nu})$  and  $\theta_\nu(x^\nu, x^{-\nu}) \leq \theta_\nu(y^\nu, x^{-\nu})$  hold for all  $y^\nu \in X_\nu(x^{-\nu})$ ,  $\nu = 1, \dots, N$ . With the joint strategy set

$$Z(x) := X_1(x^{-1}) \times \dots \times X_N(x^{-N})$$

the player-wise feasibility conditions  $x^\nu \in X(x^{-\nu})$ ,  $\nu = 1, \dots, N$ , can be aggregated to the condition  $x \in Z(x)$ . This means that  $x$  is a fixed point of the set-valued mapping  $Z : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ , briefly

$$x \in \text{fix } Z = \{x \in \mathbb{R}^n \mid x \in Z(x)\} = \{x \in \mathbb{R}^n \mid g^\nu(x) \leq 0, \nu = 1, \dots, N\}.$$

This yields the functional description

$$E = \{x \in \text{fix } Z \mid \theta_\nu(x^\nu, x^{-\nu}) \leq \theta_\nu(y^\nu, x^{-\nu}) \forall y^\nu \in X_\nu(x^{-\nu}), \nu = 1, \dots, N\} \quad (1)$$

of the equilibrium set. Thus it is not described by finitely many constraints, but the indices  $y^\nu$  of the inequalities are taken from the infinite sets  $X_\nu(x^{-\nu})$ ,  $\nu = 1, \dots, N$ . Such inequalities are called semi-infinite [15, 16, 26, 27]. Since the index sets even depend on the decision variable  $x$ , we are actually faced with  $N$  generalized semi-infinite inequalities [31].

In the special case of a standard Nash equilibrium problem joint feasibility does not need to be modeled by a set-valued mapping, but the fixed set

$$Z = X_1 \times \dots \times X_N$$

suffices. It satisfies  $\text{fix } Z = Z$ , and we obtain the description

$$E = \{x \in Z \mid \theta_\nu(x^\nu, x^{-\nu}) \leq \theta_\nu(y^\nu, x^{-\nu}) \forall y^\nu \in X_\nu, \nu = 1, \dots, N\} \quad (2)$$

of the equilibrium set by  $N$  standard semi-infinite constraints (i.e., semi-infinite constraints with fixed index sets).

### 3.2 A Nikaido-Isoda reformulation

While in Section 3.1 the feasibility requirements of the single players were aggregated to the condition  $x \in \text{fix } Z$ , the optimality requirements remained disaggregated. Nikaido and Isodas's approach from [25] allows such an aggregation along the following lines.

For each  $\nu \in \{1, \dots, N\}$  we define the optimal value function

$$\varphi_\nu(x^{-\nu}) := \inf_{y^\nu \in X_\nu(x^{-\nu})} \theta_\nu(y^\nu, x^{-\nu})$$

of the player problem  $P_\nu(x^{-\nu})$ . Then  $x^\nu$  is a minimal point of  $P_\nu(x^{-\nu})$  if and only if  $x^\nu \in X_\nu(x^{-\nu})$  and  $\theta_\nu(x^\nu, x^{-\nu}) = \varphi_\nu(x^{-\nu})$  hold. These conditions may be aggregated to  $x \in \text{fix } Z$  and

$$V(x) := \sum_{\nu=1}^N |\theta_\nu(x^\nu, x^{-\nu}) - \varphi_\nu(x^{-\nu})| = 0,$$

where  $V$  is called gap function. Since for  $x \in \text{fix } Z$  none of the problems  $P_\nu(x^{-\nu})$  is inconsistent, we have  $\varphi_\nu(x^{-\nu}) < +\infty$  and  $\theta_\nu(x^\nu, x^{-\nu}) - \varphi_\nu(x^{-\nu}) \geq 0$ , which yields the nonnegativity of  $V$  on  $\text{fix } Z$  even if the absolute values in its definition are dropped and we write

$$V(x) = \sum_{\nu=1}^N (\theta_\nu(x^\nu, x^{-\nu}) - \varphi_\nu(x^{-\nu})).$$

While the resulting functional description

$$E = \{x \in \text{fix } Z \mid V(x) = 0\}$$

is concise, it is not algorithmically useful since the definition of  $V$  involves the optimal value functions  $\varphi_\nu(x^{-\nu})$ ,  $\nu = 1, \dots, N$ .

On the other hand, with the Nikaido-Isoda function (also known as Ky-Fan function)

$$\psi(x, y) := \sum_{\nu=1}^N (\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu})) \quad (3)$$

one may rewrite the gap function as

$$V(x) = \sup_{y \in Z(x)} \psi(x, y).$$

Since by the nonnegativity of  $V$  on  $\text{fix } Z$  the condition  $V(x) = 0$  is equivalent to  $V(x) \leq 0$ , and  $\sup_{y \in Z(x)} \psi(x, y) \leq 0$  is equivalent to the generalized semi-infinite constraint  $\psi(x, y) \leq 0$  for all  $y \in Z(x)$ , we obtain the aggregated generalized semi-infinite description

$$E = \{x \in \text{fix } Z \mid \psi(x, y) \leq 0 \ \forall y \in Z(x)\}. \quad (4)$$

For standard NEPs the aggregated description is standard semi-infinite:

$$E = \{x \in Z \mid \psi(x, y) \leq 0 \ \forall y \in Z\}. \quad (5)$$

### 3.3 A quasi-variational inequality reformulation

A natural assumption to make GNEPs algorithmically tractable is the convexity of the problems  $P_\nu(x^{-\nu})$ ,  $\nu = 1, \dots, N$ , in the respective player variable  $x^\nu$ .

**Assumption 3.1** (Player convexity). *For each  $\nu \in \{1, \dots, N\}$  and each  $x^{-\nu} \in \mathbb{R}^{n-n_\nu}$  such that  $(x^\nu, x^{-\nu}) \in \text{fix } Z$  holds for some  $x^\nu \in \mathbb{R}^{n_\nu}$ , the defining functions  $\theta_\nu(\cdot, x^{-\nu})$ ,  $g_i^\nu(\cdot, x^{-\nu})$ ,  $i = 1, \dots, m_\nu$ , of  $P_\nu(x^{-\nu})$  are convex on  $\mathbb{R}^{n_\nu}$ .*

GNEPs satisfying Assumption 3.1 are called *player convex*. Since the GNEP from Example 1.1 is player convex, this example illustrates that player convexity cannot be expected to yield convex, let alone unique, equilibrium sets  $E$ . We remark that the GNEP from Example 1.1 exhibits an even stronger convexity property, namely its defining functions are convex simultaneously in *all* variables (which is called complete convexity). So even equilibrium sets of completely convex GNEPs need not be convex. Examples for player convex GNEPs are the noncooperative transportation problem [33] and the multiportfolio problem studied in Section 6.

Besides player convexity, the reformulation in this section needs the objective functions  $\theta_\nu$  to be continuously differentiable in the player variable  $x^\nu$ ,  $\nu = 1, \dots, N$ . Given feasibility of the point  $x^\nu$ , under these assumptions its optimality is equivalent to the variational condition

$$\langle \nabla_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), y^\nu - x^\nu \rangle \geq 0 \quad \forall y^\nu \in X_\nu(x^{-\nu}).$$

For  $x \in \text{fix } Z$  these conditions for the single players  $\nu \in \{1, \dots, N\}$  can be written equivalently in the aggregated form

$$\langle F(x), y - x \rangle \geq 0 \quad \forall y \in Z(x)$$

with

$$F(x) := \begin{pmatrix} \nabla_{x^1} \theta_1(x^1, x^{-1}) \\ \vdots \\ \nabla_{x^N} \theta_N(x^N, x^{-N}) \end{pmatrix},$$

that is, as a quasi-variational inequality. This yields the functional description

$$E = \{x \in \text{fix } Z \mid \langle F(x), y - x \rangle \geq 0 \quad \forall y \in Z(x)\} \quad (6)$$

of the equilibrium set. For details we refer to [9].

For standard NEPs the corresponding functional description uses a standard variational inequality:

$$E = \{x \in Z \mid \langle F(x), y - x \rangle \geq 0 \quad \forall y \in Z\}. \quad (7)$$

In contrast to the descriptions (4) and (5), the descriptions (6) and (7) use *linear* (generalized) semi-infinite constraints. This will turn out to be useful algorithmically in Section 6.

### 3.4 A Karush-Kuhn-Tucker reformulation

If not only all  $\theta_\nu$  but also all functions  $g^\nu$ ,  $\nu = 1, \dots, N$ , are differentiable in the player variable, one may define the Lagrangian

$$L_\nu(x^\nu, x^{-\nu}, \gamma^\nu) = L_\nu(x, \gamma^\nu) = \theta_\nu(x) + (\gamma^\nu)^\top g^\nu(x)$$

of  $P_\nu(x^{-\nu})$  and consider the Karush-Kuhn-Tucker system

$$\begin{aligned} \nabla_{x^\nu} L_\nu(x, \gamma^\nu) &= 0, \quad g^\nu(x) \leq 0, \quad \gamma^\nu \geq 0, \\ \gamma_i^\nu g_i^\nu(x) &= 0, \quad i = 1, \dots, m_\nu. \end{aligned}$$

Under player convexity (Ass. 3.1) and some constraint qualification like Slater's condition for each appearing set  $X_\nu(x^{-\nu})$ , optimality of  $x^\nu$  for  $P_\nu(x^{-\nu})$  is characterized by the solvability of the Karush-Kuhn-Tucker system with some  $\gamma^\nu$ . Therefore we obtain

$$E = \{x \mid \exists \gamma : \nabla_{x^\nu} L_\nu(x, \gamma^\nu) = 0, \quad g^\nu(x) \leq 0, \quad \gamma^\nu \geq 0, \\ \gamma_i^\nu g_i^\nu(x) = 0, \quad i = 1, \dots, m_\nu, \quad \nu = 1, \dots, N\}, \quad (8)$$

that is, a functional description of  $E$  employing complementarity constraints. In the case of a standard NEP this description does not simplify significantly.

Unfortunately the assumption of Slater's condition for each appearing set  $X_\nu(x^{-\nu})$  is rather strong since, for continuity reasons, it is necessarily violated at the boundaries of the effective domains  $\text{dom } X_\nu = \{x^{-\nu} \in \mathbb{R}^{n-n_\nu} \mid X_\nu(x^{-\nu}) \neq \emptyset\}$  of the set-valued mappings  $X_\nu$ ,  $\nu = 1, \dots, N$ . Thus the Karush-Kuhn-Tucker reformulation is possible in the case  $\text{dom } X_\nu = \mathbb{R}^{n-n_\nu}$ ,  $\nu = 1, \dots, N$ , which holds, for example, for standard NEPs. If the latter condition is violated, the Karush-Kuhn-Tucker reformulation may still be possible if weaker constraint qualifications, like the one of Abadie, hold at the optimal points of  $P_\nu(x^{-\nu})$  for  $x^{-\nu}$  from the boundary of  $\text{dom } X_\nu$ .

## 4 Three semi-infinite and one MPCC model

Plugging the functional descriptions of  $E$  from Section 3 into the equilibrium selection model

$$ES(\lambda) : \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x \in E$$

from Section 2 yields the following alternative optimization models. Equilibrium selection for GNEPs may be performed by choosing some  $\lambda > 0$  and

- based on (1), computing a minimal point of the generalized semi-infinite program

$$GSIP_{direct}(\lambda) : \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x) \leq 0, \\ \theta_\nu(x^\nu, x^{-\nu}) \leq \theta_\nu(y^\nu, x^{-\nu}) \quad \forall y^\nu \in X_\nu(x^{-\nu}), \quad \nu = 1, \dots, N,$$

- or, based on (4), by computing a minimal point of the generalized semi-infinite program

$$GSIP_{NI}(\lambda) : \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x) \leq 0, \quad \nu = 1, \dots, N, \\ \psi(x, y) \leq 0 \quad \forall y \in Z(x),$$

- or, based on (6), by computing a minimal point of the generalized semi-infinite program

$$GSIP_{QVI}(\lambda) : \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x) \leq 0, \quad \nu = 1, \dots, N, \\ \langle F(x), y - x \rangle \geq 0 \quad \forall y \in Z(x),$$

(provided that Ass. 3.1 holds and all functions  $\theta_\nu$  are continuously differentiable in  $x^\nu$ ),

- or, based on (8), by computing a minimal point of the mathematical program with complementarity constraints

$$MPCC(\lambda) : \min_{x, \gamma} \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad \nabla_{x^\nu} L_\nu(x, \gamma^\nu) = 0, \\ g^\nu(x) \leq 0, \\ \gamma^\nu \geq 0, \\ \gamma_i^\nu g_i^\nu(x) = 0, \quad i = 1, \dots, m_\nu, \quad \nu = 1, \dots, N,$$

(provided that Ass. 3.1 holds, all functions  $\theta_\nu$ ,  $g^\nu$  are continuously differentiable in  $x^\nu$ , and Slater's condition holds in all appearing sets  $X_\nu(x^{-\nu})$ ).

For standard NEPs, the descriptions (2), (5) and (7) yield the corresponding standard semi-infinite programs  $SIP_{direct}(\lambda)$ ,  $SIP_{NI}(\lambda)$ , and  $SIP_{VI}(\lambda)$ , while  $MPCC(\lambda)$  does not simplify significantly. A survey of state-of-the-art solution algorithms for nonconvex semi-infinite optimization problems is given in [4], and for MPCC solution methods we

refer to [22]. Problems of the type  $SIP_{VI}(\lambda)$  are also known as optimization problems with variational inequality constraints (OPVICs) [11, 18, 30].

Most of the algorithms for semi-infinite optimization problems explicitly or implicitly assume that a standard constraint qualification like the Mangasarian-Fromovitz constraint qualification (MFCQ) holds at least in the points to which they converge. Unfortunately, all of the above optimization models violate MFCQ at *each* feasible point, so that algorithmically they need to be handled with care. The violation of MFCQ is not surprising, given that equilibrium sets of GNEPs (in the absence of shared constraints) and of NEPs generically consist of isolated points [5].

For mathematical programs with equilibrium constraints like  $MPCC(\lambda)$  it is well-known that MFCQ is violated at each feasible point [28]. To see why MFCQ is violated in the above semi-infinite models, we use the following result.

**Lemma 4.1.** *Let  $\lambda > 0$  be given. Then at each feasible point of any of the problems  $GSIP_{direct}(\lambda)$ ,  $GSIP_{NI}(\lambda)$ ,  $GSIP_{QVI}(\lambda)$ ,  $SIP_{direct}(\lambda)$ ,  $SIP_{NI}(\lambda)$  and  $SIP_{VI}(\lambda)$ , all appearing semi-infinite constraints possess active indices.*

*Proof.* We give the proof only for the problem  $GSIP_{NI}(\lambda)$ , since the proofs for the other problems run along the same lines. Let  $x$  be a feasible point of  $GSIP_{NI}(\lambda)$ . Since the constraints  $g^\nu(x) \leq 0$ ,  $\nu = 1, \dots, N$ , characterize  $x \in \text{fix } Z$ , we have  $x \in Z(x)$ . The Nikaido-Isoda function from (3) satisfies  $\psi(x, x) = 0$ , so that the semi-infinite constraint  $\psi(x, y) \leq 0 \forall y \in Z(x)$  is active at the index  $y := x$ .  $\square$

With the notation for problem  $GSIP_{NI}(\lambda)$ , Lemma 4.1 implies  $\sup_{y \in Z(x)} \psi(x, y) = 0$  for all feasible points  $x$ . This means particularly that the gap function  $V(x) = \sup_{y \in Z(x)} \psi(x, y)$  is constant on the feasible set.

On the other hand, the Mangasarian-Fromovitz constraint qualification holds at some feasible point  $x$  of an optimization problem if at  $x$  a certain inner linearization cone to the feasible set is nonempty. For semi-infinite optimization problems a natural definition of the inner linearization cone is provided in [32]. In the notation for problem  $GSIP_{NI}(\lambda)$  it says that a direction  $d$  belongs to the inner linearization cone if the Hadamard upper directional derivative

$$V'_+(x, d) := \limsup_{t \searrow 0, \delta \rightarrow d} \frac{V(x + t\delta) - V(x)}{t}$$

satisfies  $V'_+(x, d) < 0$ . For a constant function  $V$ , however, this is not possible, so that the inner linearization cone is empty and MFCQ is violated. With analogous arguments for the other semi-infinite models we have thus shown the following result.

**Proposition 4.2.** *Let  $\lambda > 0$  be given. Then at each feasible point of any of the problems  $GSIP_{direct}(\lambda)$ ,  $GSIP_{NI}(\lambda)$ ,  $GSIP_{QVI}(\lambda)$ ,  $SIP_{direct}(\lambda)$ ,  $SIP_{NI}(\lambda)$ ,  $SIP_{VI}(\lambda)$  and  $MPCC(\lambda)$  the MFCQ is violated.*

## 5 Regularized problems

A possible remedy to make the semi-infinite problems and the MPCC algorithmically tractable is a relaxation which allows  $\varepsilon$ -feasible points of the semi-infinite and complemen-

tarity constraints, respectively, for some tolerance  $\varepsilon > 0$ . More precisely, the problems  $GSIP_{direct}(\lambda)$ ,  $GSIP_{NI}(\lambda)$ ,  $GSIP_{QVI}(\lambda)$  and  $MPCC(\lambda)$  are relaxed to

$$GSIP_{direct}^\varepsilon(\lambda) : \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x) \leq 0, \\ \theta_\nu(x^\nu, x^{-\nu}) \leq \theta_\nu(y^\nu, x^{-\nu}) + \varepsilon \quad \forall y^\nu \in X_\nu(x^{-\nu}), \quad \nu = 1, \dots, N,$$

$$GSIP_{NI}^\varepsilon(\lambda) : \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x) \leq 0, \quad \nu = 1, \dots, N, \\ \psi(x, y) \leq \varepsilon \quad \forall y \in Z(x),$$

$$GSIP_{QVI}^\varepsilon(\lambda) : \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x) \leq 0, \quad \nu = 1, \dots, N, \\ \langle F(x), y - x \rangle \geq -\varepsilon \quad \forall y \in Z(x),$$

and

$$MPCC^\varepsilon(\lambda) : \quad \min_{x, \gamma} \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad \nabla_{x^\nu} L_\nu(x, \gamma^\nu) = 0, \\ g^\nu(x) \leq 0, \\ \gamma^\nu \geq 0, \\ \gamma_i^\nu g_i^\nu(x) \geq -\varepsilon/m_\nu, \quad i = 1, \dots, m_\nu, \quad \nu = 1, \dots, N.$$

The latter relaxation for MPCCs was introduced by Scholtes in [29]. The problems  $SIP_{direct}(\lambda)$ ,  $SIP_{NI}(\lambda)$  and  $SIP_{VI}(\lambda)$  are regularized analogously.

For the regularized problems MFCQ is at least not ruled out at their feasible points. For the problem  $GSIP_{NI}^\varepsilon(\lambda)$  this can be seen by reformulating the semi-infinite constraint in terms of the (nonnegative) gap function  $V(x) = \sup_{y \in Z(x)} \psi(x, y)$ . The relaxation replaces the original constraint  $V(x) = 0$  by  $0 \leq V(x) \leq \varepsilon$ , so that  $V$  is not constant on the relaxed feasible set and can satisfy  $V'_+(x, d) < 0$  at feasible points  $x$  for appropriate directions  $d$ .

The effect of the regularization on the approximation of the underlying generalized Nash equilibria is clarified by the following result. For its statement recall that  $x^\nu$  is called an  $\varepsilon$ -minimal point of  $P_\nu(x^{-\nu})$  if  $x^\nu \in X(x^{-\nu})$  and  $\theta_\nu(x^\nu, x^{-\nu}) \leq \varphi_\nu(x^{-\nu}) + \varepsilon$  hold. In the following,

$$E_\varepsilon := \{x \in \mathbb{R}^n \mid x^\nu \text{ is } \varepsilon\text{-minimal for } P_\nu(x^{-\nu}), \quad \nu = 1, \dots, N\}$$

is briefly called the set of  $\varepsilon$ -equilibria of  $GNEP$ .

**Theorem 5.1.** *For some  $\varepsilon > 0$  and  $\lambda > 0$  let  $x$  be a feasible point of any of the problems  $GSIP_{direct}^\varepsilon(\lambda)$ ,  $GSIP_{NI}^\varepsilon(\lambda)$  and  $GSIP_{QVI}^\varepsilon(\lambda)$ , or let  $(x, \gamma)$  be a feasible point of  $MPCC^\varepsilon(\lambda)$ . Then  $x$  is an  $\varepsilon$ -equilibrium of  $GNEP$  (i.e.,  $x \in E_\varepsilon$  holds).*

*Proof.* For all  $\nu \in \{1, \dots, N\}$  the constraints of any of the four optimization problems include the condition  $g^\nu(x) \leq 0$ , that is,  $x^\nu \in X_\nu(x^{-\nu})$ . All subsequent arguments in this proof are based on this feasibility of  $x^\nu$  for  $P_\nu(x^{-\nu})$ .

In problem  $GSIP_{direct}^\varepsilon(\lambda)$  the constraints  $\theta_\nu(x^\nu, x^{-\nu}) \leq \theta_\nu(y^\nu, x^{-\nu}) + \varepsilon \quad \forall y^\nu \in X_\nu(x^{-\nu})$  are equivalent to  $\theta_\nu(x^\nu, x^{-\nu}) \leq \varphi_\nu(x^{-\nu}) + \varepsilon$ , that is, to  $\varepsilon$ -minimality of  $x^\nu$  for  $P_\nu(x^{-\nu})$ .

In problem  $GSIP_{NI}^\varepsilon(\lambda)$  the semi-infinite constraint implies

$$\varepsilon \geq \sup_{y \in Z(x)} \psi(x, y) = V(x) = \sum_{\lambda=1}^N (\theta_\lambda(x^\lambda, x^{-\lambda}) - \varphi_\lambda(x^{-\lambda})) \geq \theta_\nu(x^\nu, x^{-\nu}) - \varphi_\nu(x^{-\nu}),$$

that is,  $\varepsilon$ -minimality of  $x^\nu$  for  $P_\nu(x^{-\nu})$ . The last inequality follows from the nonnegativity of all summands in the sum.

In problem  $GSIP_{QVI}^\varepsilon(\lambda)$  the semi-infinite constraint yields for the particular index vectors  $(y^\nu, x^{-\nu}) \in Z(x)$  player  $\nu$ 's semi-infinite constraint

$$\langle \nabla_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), y^\nu - x^\nu \rangle \geq -\varepsilon \quad \forall y^\nu \in X_\nu(x^{-\nu}).$$

The  $C^1$ -characterization of convexity of the function  $\theta_\nu(\cdot, x^{-\nu})$  on  $X_\nu(x^{-\nu})$ ,

$$\theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu}) \geq \langle \nabla_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), y^\nu - x^\nu \rangle \quad \forall y^\nu \in X_\nu(x^{-\nu}),$$

thus implies

$$\theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu}) \geq -\varepsilon \quad \forall y^\nu \in X_\nu(x^{-\nu}),$$

that is,  $\varepsilon$ -minimality of  $x^\nu$  for  $P_\nu(x^{-\nu})$ .

In problem  $MPCCE^\varepsilon(\lambda)$  we consider the Wolfe dual

$$D_\nu(x^{-\nu}) : \max_{y^\nu, \gamma^\nu} L_\nu(y^\nu, x^{-\nu}, \gamma^\nu) \quad \text{s.t.} \quad \nabla_{x^\nu} L_\nu(y^\nu, x^{-\nu}, \gamma^\nu) = 0, \quad \gamma^\nu \geq 0$$

of  $P_\nu(x^{-\nu})$ . Then for the feasible point  $(x, \gamma)$  of  $MPCCE^\varepsilon(\lambda)$  the point  $(x^\nu, \gamma^\nu)$  is feasible for  $D_\nu(x^{-\nu})$ . Weak duality between  $P_\nu(x^{-\nu})$  and  $D_\nu(x^{-\nu})$  thus implies

$$\varphi_\nu(x^{-\nu}) \geq L_\nu(x^\nu, x^{-\nu}, \gamma^\nu) = \theta_\nu(x^\nu, x^{-\nu}) + \sum_{i=1}^{m_\nu} \gamma_i^\nu g_i^\nu(x^\nu, x^{-\nu}) \geq \theta_\nu(x^\nu, x^{-\nu}) - \varepsilon,$$

that is,  $\varepsilon$ -minimality of  $x^\nu$  for  $P_\nu(x^{-\nu})$ . □

Theorem 5.1 also covers the problems  $SIP_{direct}^\varepsilon(\lambda)$ ,  $SIP_{NI}^\varepsilon(\lambda)$  and  $SIP_{VI}^\varepsilon(\lambda)$ . We remark that regularization by replacing optimality with  $\varepsilon$ -optimality is also a successful technique in bilevel optimization, in particular for the pessimistic point of view [20, 21].

Note that the following assumption was *not* required for the analysis of Sections 4 and 5.

**Assumption 5.2** (player solvability). *For each  $x \in \text{fix } Z$  all player problems  $P_\nu(x^{-\nu})$ ,  $\nu = 1, \dots, N$ , are solvable.*

Since we work under the blanket assumption of continuous cost functions  $\theta_\nu$  and closed strategy sets  $X_\nu(x^{-\nu})$ , and since  $\text{fix } Z \subseteq \text{dom } Z$  holds, Assumption 5.2 follows from the Weierstrass theorem under the additional assumption that  $X_\nu(x^{-\nu})$  is bounded (under Ass. 3.1 unbounded strategy sets can be treated, e.g., by the regularization approach from [12]). However, if Assumption 5.2 is violated and for some  $x \in \text{fix } Z$  some player problem  $P_\nu(x^{-\nu})$  is not solvable, this just means that  $x$  is infeasible for any reformulation of  $ES(\lambda)$ .

Let us also point out that Assumption 5.2 does not imply solvability of the equilibrium selection problem  $ES(\lambda)$  or any of its reformulations from this section. In particular, the set  $E$  may still be empty.

## 6 Equilibrium selection for standard NEPs with polyhedral strategy sets

For illustration of the above ideas let us focus on a standard Nash equilibrium problem satisfying Assumption 3.1 with bounded polyhedral strategy sets  $X_\nu$  and (in  $x^\nu$ ) continuously differentiable cost functions  $\theta_\nu$ ,  $\nu = 1, \dots, N$ . Choosing the semi-infinite model stemming from the variational inequality reformulation (Sec. 3.3) yields a linear semi-infinite constraint, and with a feasibility tolerance  $\varepsilon > 0$  we arrive at the relaxed reformulation

$$\begin{aligned} SIP_{VI}^\varepsilon(\lambda) : \quad & \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x^\nu) \leq 0, \quad \nu = 1, \dots, N, \\ & \langle F(x), y - x \rangle \geq -\varepsilon \quad \forall y \in Z \end{aligned}$$

of the equilibrium selection problem  $ES(\lambda)$ . Here the functions  $g^\nu$  do not depend on  $x^{-\nu}$  since we model a standard NEP, and they can be chosen to be linear by the polyhedrality of the strategy sets. Due to  $\lambda > 0$  the objective function  $\langle \lambda, \theta(x) \rangle$  is convex on the feasible set, but the function  $F$  may be nonconvex.

The present assumptions allow us to reformulate the semi-infinite problem into one with finitely many constraints. To this end, let  $\text{vert}(Z)$  denote the vertex set of the bounded polyhedral set  $Z = X_1 \times \dots \times X_N$ .

**Proposition 6.1.** *The problem  $SIP_{VI}^\varepsilon(\lambda)$  is equivalent to*

$$\begin{aligned} NLP_{VI}^\varepsilon(\lambda) : \quad & \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x^\nu) \leq 0, \quad \nu = 1, \dots, N, \\ & \langle F(x), y - x \rangle \geq -\varepsilon \quad \forall y \in \text{vert}(Z). \end{aligned}$$

*Proof.* The semi-infinite constraint of  $SIP_{VI}^\varepsilon(\lambda)$  may be rewritten as  $\min_{y \in Z} \langle F(x), y - x \rangle \geq -\varepsilon$ . Since for any given  $x$  the appearing infimum is the optimal value of a linear optimization problem, the vertex theorem of linear programming yields

$$\min_{y \in Z} \langle F(x), y - x \rangle = \min_{y \in \text{vert}(Z)} \langle F(x), y - x \rangle.$$

This shows the assertion. □

The following example shows that the size of the vertex set  $\text{vert}(Z)$  may be vast even for moderate numbers of players and dimension  $n$ . Subsequently we shall also use this example for numerical illustration.

**Example 6.2** (Multi-portfolio optimization). *Consider the following multi-portfolio optimization problem from [19]. A group of  $N$  portfolio managers wish to invest their respective budgets  $b^\nu \geq 0$  in  $K$  assets of a market. Their decision variables  $x^\nu \in X_\nu \subseteq \mathbb{R}^K$  denote the fractions of  $b^\nu$  invested in each available asset, where  $X_\nu = \{x^\nu \in \mathbb{R}^K \mid e^\top x^\nu = 1, x^\nu \geq 0\}$  is the standard simplex and  $e$  denotes the all-ones vector. Let  $r \in \mathbb{R}^K$  be the vector of random variables  $r_k$  modelling the return of asset  $k \in \{1, \dots, K\}$  over a single-period investment. Then  $\mu^\nu = \mathbb{E}^\nu(r) \in \mathbb{R}^K$  is the vector of player  $\nu$ 's expectations of the assets' returns (depending on the player's assumption on the distribution of  $r$ ), leading to the*

expected income  $b^\nu(\mu^\nu)^\top x^\nu$ . Moreover, in a Markowitz framework the covariance matrix  $\Sigma^\nu = \mathbb{E}^\nu((r - \mu^\nu)(r - \mu^\nu)^\top)$  may be used for the definition of a risk measure  $\frac{1}{2}b^\nu(x^\nu)^\top \Sigma^\nu x^\nu$ .

In deviation from a classical Markowitz framework the present model addresses the case in which trades are grouped and simultaneously executed. Then individual accounts can suffer from the market impact caused by a shortage of liquidity, which results from the fact that the joint demand of an asset can be tremendously larger than the individual demand. To take account of the transaction cost effect, the entry  $(i, j)$  of a market impact matrix models the impact of the liquidity of asset  $i$  on the liquidity of asset  $j$ . The assumed market impact matrix  $\Omega^\nu$  is different for each player  $\nu$  and not necessarily symmetric but, as motivated in [19], positive semi-definite. As a consequence, for each player  $\nu$  a different linear market impact unitary cost function  $\Omega^\nu \sum_{\lambda=1}^N b^\lambda(x^\lambda - \bar{x}^\lambda)$  occurs which depends on the invested capital from the aggregated trades from all accounts, where  $\bar{x}^\lambda \in \mathbb{R}^K$  denotes the current portfolio of player  $\lambda$ . The total transaction costs (in unit of currency) of player  $\nu$  then is  $b^\nu(x^\nu - \bar{x}^\nu)^\top \Omega^\nu \sum_{\lambda=1}^N b^\lambda(x^\lambda - \bar{x}^\lambda)$ . One arrives at the optimization problem

$$P_\nu(x^{-\nu}) : \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \quad \text{s.t.} \quad x^\nu \in X_\nu$$

for player  $\nu$  with

$$\theta_\nu(x^\nu, x^{-\nu}) = b^\nu(\mu^\nu)^\top x^\nu + \rho^\nu \frac{1}{2} b^\nu(x^\nu)^\top \Sigma^\nu x^\nu + b^\nu(x^\nu - \bar{x}^\nu)^\top \Omega^\nu \sum_{\lambda=1}^N b^\lambda(x^\lambda - \bar{x}^\lambda),$$

and with the risk aversion parameter  $\rho^\nu \geq 0$ .

In [19] it is shown that the corresponding player convex GNEP may possess nonunique equilibria, and certain equilibrium selection problems (not based on payoff dominance) are solved by a Tikhonov-like algorithm from [18]. There the numerical experiments use two data sets, one consisting of daily returns time series of  $K = 10$  assets from banking, insurance and financial companies belonging to Euro Stoxx 50 (SX5E), and the other in  $K = 29$  assets from the Dow Jones Industrial Average (DJIA) stock markets.

In this example each player's decision variable  $x^\nu$  possesses dimension  $n_\nu = K$ , and the overall dimension of the GNEP is  $n = NK$ . For  $N = 25$  players this yields the practically relevant but moderate dimensions  $n = 500$  and  $n = 725$  for the two above data sets. On the other hand, the set

$$Z = X_1 \times \dots \times X_N = \{x \in \mathbb{R}^{25K} \mid x \geq 0, e^\top x^\nu = 1, \nu = 1, \dots, 25\}$$

possesses  $K^{25}$  vertices, that is, we have  $|\text{vert}(Z)| = 10^{25}$  and  $|\text{vert}(Z)| = 29^{25}$  for the two respective data sets.

In view of the possibly vast index set  $\text{vert}(Z)$  we suggest a cutting algorithm for the solution of  $NLP_{VI}^\varepsilon(\lambda)$ . It is reminiscent of discretization methods for semi-infinite programming [15, 16, 26, 27], but does not employ adaptively refined finite subsets  $Z_d$  of the infinite set  $Z$ , but of the finite (though possibly vast) set  $\text{vert}(Z)$ .

In fact, for any set  $Z_d \subseteq \text{vert}(Z)$  the master problem

$$\begin{aligned} NLP_{VI,d}^\varepsilon(\lambda) : \quad & \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x^\nu) \leq 0, \quad \nu = 1, \dots, N, \\ & \langle F(x), y - x \rangle \geq -\varepsilon \quad \forall y \in Z_d. \end{aligned}$$

is a relaxation of  $NLP_{VI}^\varepsilon(\lambda)$ . It possesses a closed and bounded feasible set even for  $Z_d = \emptyset$ , since the constraints  $g^\nu(x^\nu) \leq 0$ ,  $\nu = 1, \dots, N$ , characterize the polyhedral bounded set  $\text{fix } Z = Z$ .

An optimal point  $\bar{x}$  of the relaxed problem  $NLP_{VI,d}^\varepsilon(\lambda)$  solves  $NLP_{VI}^\varepsilon(\lambda)$  if also the ignored constraints  $\langle F(x), y - x \rangle \geq -\varepsilon \forall y \in \text{vert}(Z) \setminus Z_d$  hold. This may be checked *without knowledge* of the complete set  $\text{vert}(Z)$  by computing an optimal vertex  $\bar{y}$  of the linear optimization problem

$$LP(\bar{x}) : \quad \min_y \langle F(\bar{x}), y - \bar{x} \rangle \quad \text{s.t.} \quad y \in Z$$

and checking whether its minimal value satisfies

$$\langle F(\bar{x}), \bar{y} - \bar{x} \rangle \geq -\varepsilon.$$

If it does, then even the semi-infinite constraint  $\langle F(x), y - x \rangle \geq -\varepsilon \forall y \in Z$  holds at  $\bar{x}$ , the more so the finitely many constraints (corresponding to  $y \in \text{vert}(Z)$ ) from  $NLP_{VI}^\varepsilon(\lambda)$ . The crucial point of this construction is that an optimal vertex of the problem  $LP(\bar{x})$  can be computed without the knowledge of the whole vertex set  $\text{vert}(Z)$  by, for example, the simplex algorithm. Such ideas are also used in Benders-type cutting plane algorithms for mixed-integer linear optimization problems [2].

In the case

$$\langle F(\bar{x}), \bar{y} - \bar{x} \rangle < -\varepsilon.$$

the point  $\bar{x}$  is infeasible for  $NLP_{VI}^\varepsilon(\lambda)$ , and the master problem is refined by a cut. In fact, since the point  $\bar{x}$  violates the inequality

$$\langle F(x), \bar{y} - x \rangle \geq -\varepsilon,$$

and since, due to  $\bar{y} \in \text{vert}(Z)$ , this inequality is valid for the feasible set of  $NLP_{VI}^\varepsilon(\lambda)$ , it may be employed as a (nonlinear) cut. This means that the old discretization  $Z_d$  is updated to  $Z_d \cup \{\bar{y}\}$ . Since any  $\bar{y} \in \text{vert}(Z)$  may only be chosen once for such an update, and  $\text{vert}(Z)$  is a finite set, in the worst case the cutting algorithm stops after generating the whole set  $Z_d = \text{vert}(Z)$ . The resulting algorithmic scheme is summarized in Algorithm 1.

**Example 6.3** (Numerical experience). *We test Algorithm 1 on the multi-portfolio optimization problem from Example 6.2 with estimates for the problem data from [19]. Since the players' objective functions  $\theta_\nu$  are convex quadratic, also the objective function of the master problems  $NLP_{VI,d}^\varepsilon(\lambda)$  in Algorithm 1 is convex quadratic. Moreover, the function  $F$  then is linear, and the constraints  $\langle F(x), y - x \rangle \geq -\varepsilon \forall y \in Z_d$  are quadratic as well, but not necessarily convex. We compute globally minimal points of the master problems by the solver GUROBI [13]. The linear subproblems  $LP(\bar{x})$  are solved by GUROBI as well. All experiments were run on an Intel i7 processor with 8 cores with 3.60 GHz and 32 GB of RAM and with version 9.1.1 of GUROBI.*

*Tables 1 and 2 provide an analysis of the output of Algorithm 1 for the two data sets SX5E ( $K = 10$ ) and DJIA ( $K = 29$ ), respectively, with  $N = 25$  players and four randomly generated weight vectors  $\lambda^1, \dots, \lambda^4$ . Rather than reporting the output points  $\bar{x}$  (of dimension 500 and 725, resp.), we provide their  $\ell_1$ -distance from their mean in order*

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**Algorithm 1:** Equilibrium selection by nonlinear cuts

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**Input:** Player convex standard NEP with bounded polyhedral strategy sets, weight vector  $\lambda > 0$ , and feasibility tolerance  $\varepsilon > 0$ .

**Output:** Approximation  $\bar{x} \in E_\varepsilon$  of a Pareto optimal Nash equilibrium.

1  $Z_d \leftarrow \emptyset$

2 **repeat**

3     Compute an optimal point  $\bar{x}$  of

$$NLP_{VI,d}^\varepsilon(\lambda) : \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad g^\nu(x^\nu) \leq 0, \nu = 1, \dots, N, \\ \langle F(x), y - x \rangle \geq -\varepsilon \quad \forall y \in Z_d.$$

4     Compute an optimal vertex  $\bar{y}$  of

$$LP(\bar{x}) : \min_y \langle F(\bar{x}), y - \bar{x} \rangle \quad \text{s.t.} \quad y \in Z.$$

5      $Z_d \leftarrow Z_d \cup \{\bar{y}\}$

6 **until**  $\langle F(\bar{x}), \bar{y} - \bar{x} \rangle \geq -\varepsilon$ ;

---

to indicate that the choice of different weight vectors yields different approximations of Pareto-optimal equilibria.

The reported number of iterations coincides with the number of computed vertices of  $Z$  before termination. In this example it turns out that for the data set SX5E at most 11 of the  $10^{25}$  vertices need to be computed, and at most 8 of the  $29^{25}$  vertices for the data set DJIA.

$\lambda$	$\ x^* - \bar{x}^*\ _1$	iterations	run time [s]
$\lambda^1$	0.666517	9	22.796807
$\lambda^2$	0.658469	10	28.127697
$\lambda^3$	0.578055	11	43.600155
$\lambda^4$	0.774364	10	24.194173

Table 1: Data set SX5E,  $N = 25$ ,  $K = 10$ ,  $\varepsilon = 10^{-4}$

$\lambda$	$\ x^* - \bar{x}^*\ _1$	iterations	run time [s]
$\lambda^1$	2.658054	7	214.447290
$\lambda^2$	3.123464	8	306.947685
$\lambda^3$	2.850787	7	209.581576
$\lambda^4$	1.941140	8	290.675761

Table 2: Data set DJIA,  $N = 25$ ,  $K = 29$ ,  $\varepsilon = 10^{-3}$

The numerical experience with Algorithm 1 reported in Example 6.3 indicates that also in general the cutting idea for computing Pareto-optimal equilibria of standard Nash equilibrium problems with polyhedral strategy sets may work well for practically relevant problem sizes, provided that the master problems can be solved to global optimality.

## 7 Final remarks

This paper provides ideas for the algorithmic selection of generalized Nash equilibria. It shows that different semi-infinite and an MPCC reformulation suffer from violation of the Mangasarian-Fromovitz constraint qualification at each point of their feasible sets, and it suggests a regularization by allowing a feasibility tolerance. Numerical results for a standard Nash equilibrium problem with polyhedral strategy sets of practically relevant size encourage future research on this topic.

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