Efficient Formulations for Multiple Allocation Hub Network Interdiction Problems

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Abstract

In this paper, we study a network interdiction problem on a multiple allocation, uncapacitated hub network. The problem is formulated as a bilevel Stackelberg game between an attacker and a defender, where the attacker identifies r out of p hubs to interdict so as to maximize the worst-case post-interdiction performance of the system with the surviving hubs. We study three variants of the problem, namely, the r-hub median interdiction problem, the r-hub center interdiction problem, and the r-hub maximal covering interdiction problem. The bilevel problems are reduced to single-level mixed integer programs (MIP) using dual and penalty-based formulations. We exploit the properties of the models to present tighter single-level MIP formulations. We compare the linear programming relaxations of dual and penalty-based formulations to establish the dominance relations between them. Our theoretical analysis shows that the single-level dual formulations of all the three problems are stronger than their corresponding penalty-based formulations. We validate these theoretical results using extensive computational experiments on moderate to large-scale instances. Our computational results on networks with up to 200 nodes and 15 hubs confirm the strength of the proposed formulations.

Keywords: Location, Hub Interdiction, Hub Location, Network Interdiction, Bilevel Programming .

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1. Introduction

Hub networks have wide applications ranging from airline transportation, less-than-truckload freight transportation, rail freight transportation, liner shipping, urban traffic, postal delivery, express package delivery, telecommunications to supply chains. These networks use transshipment, consolidation, or sorting points for commodities, called hub facilities, to connect a large number of origin/destination (O/D) pairs by using a small number of links. Commodities having the same origin but different destinations are consolidated at the hubs and are then combined with other commodities having different origins but the same destination. In a typical hub operation, a flow between a origin node to the destination node passes through at most two intermediate hubs. When the flow reaches the first hub from the origin, it is collected along with the flows from other nodes connected to the hub. The collected flows are then sorted based on their respective destinations. If the destination node is also connected to the same hub, the flow is routed directly, else it is transshipped to the hub to which the destination node is connected. In the second intermediate hub, the incoming flows are again sorted and routed to their respective destination spoke nodes. The use of hub facilities helps centralize commodity handling and sorting operations, reduce set-up costs, and achieve economies of scale on routing costs through the consolidation of flows. Hub networks can be seen as hierarchical networks, which in their most basic form, contain two levels: an access-level network connecting O/D nodes to hubs, and a hub-level network connecting hub nodes among them. A Hub network typically results in fewer links, which makes it very attractive for use in telecommunication sector where there is a fixed cost for constructing links. Sparseness of the hub network also aids in effective monitoring and maintenance of the network. For a detailed discussion on hub networks, the readers are referred to the review papers by Alumur and Kara (2008), Campbell and O'Kelly (2012), Farahani et al. (2013), Contreras (2015), Alumur et al. (2021), and Contreras (2021).

In literature, different variants of hub location problem have been studied. They are: p-hub median (minimizing the demand weighted transportation cost by locating p hubs) (O'kelly, 1987; Campbell, 1994; Skorin-Kapov et al., 1996; Ernst and Krishnamoorthy, 1996, 1998; Ebery et al., 2000; Boland et al., 2004), fixed charge (minimizing the sum of demand weighted transportation cost and the fixed cost of locating hubs), p-hub center (minimize the maximum distance between any source-destination pair by locating p hubs) (Kara and Tansel, 2000; Tan and Kara, 2007; Alumur and Kara, 2009; Ernst et al., 2009) and hub covering problems (minimizing the number of located hubs under the constraint that demand has to be met within a given threshold path length β) (Kara and Tansel, 2003; Ernst et al., 2005). These problems are further classified as single allocation and multiple allocation hub location problems. In a single allocation case, a nonhub node is connected to only one hub, while in multiple allocation, the nonhub node can be connected to either one or more than one hubs. Further, these problems are classified based on the consideration of hub capacities (capacitated or uncapacitated) in the model formulation. For a more detailed discussion on the hub location research, the readers are referred to Alumur and Kara (2008), Campbell and O'Kelly (2012), Farahani et al. (2013), Alumur et al. (2021), and Contreras (2021).

Recently, researchers have studied an anti-thesis of the hub location problem, known as the hub interdiction problem, which takes an adversarial approach to identify critical hubs in a hub network. This problem is practically very useful since hub networks are employed in industrial sectors like, power distribution, telecommunication, passenger transportation and goods logistics, some of which fall under the category of critical infrastructure. In a hub network, the average degree of a hub node is much larger than an individual spoke node, resulting in the hub nodes forming the backbone of the entire network. Thus, a failure on any of the hubs can significantly impact the network operations. The identification of critical hubs also can help the decision maker to judiciously deploy the scarce protection resources on such hubs to ensure reliable hub-and-spoke network operations.

In this paper, we study three variants of the multiple allocation uncapacitated hub interdiction problem (MUHIP), namely *r-hub median interdiction problem, r-hub center interdiction problem, and r-hub maximal covering interdiction problem.* The problems are formulated as bilevel Stackelberg games between an attacker and a defender, where the attacker identifies r (out of p) hubs to interdict so as to maximize the worst-case post-interdiction performance of the system by routing the flows through the remaining $(p - r)$ hubs in the network. Then, we present single-level mixed-integer programming reformulations of these variants of HIP using the dual and penalty-based reformulations. We exploit the properties of the models to develop tighter dual and penalty-based reformulations of the problems. By comparing the linear programming relaxations of dual and penalty formulations, we establish the dominance between them. We show that our dual reformulations yield tighter LP relaxations the penalty-based reformulations. These theoretical results are validated using extensive computational experiments on moderate to large-scale instances. Our computational results on the Australian Postal Service (AP) data set with up to 200 nodes and 15 hubs confirm the strength of the proposed reformulations.

The remainder of the paper is organized as follows. In the following section, we present a brief review of the relevant literature. Section 3 presents the bilevel formulations of the three variants of the HIP. In Section 4, we present dual reformulations of the three variants of the bilevel problem. In Section 5, we present penalty-based reformulations of the three variants. In section 6, we present theoretical comparisons between both the dual and penalty based formulations to identify the best formulation for each of the three variants of HIPs. We validate the theoretical results through extensive computational experiments described in Section 7. Using the stronger formulation, we solve large instances of the HIPs. Conclusions and future research directions are outlined in Section 8.

2. Literature Review

Network interdiction problems seek to identify critical nodes or arcs in the network. In a network interdiction problem, often modeled as a stackelberg game between a leader (also called as interdictor) and a follower (also called evader), a leader partially or fully destroys some arcs or nodes of the network in order to block the follower's flows, delay the delivery length of a supply, detect a stealth traverse, or minimize the follower's profit function. Network interdiction problems studied in the literature include shortest path network interdiction (maximize the shortest path of the defender) (Corley and Sha, 1982; Israeli and Wood, 2002; Cappanera and Scaparra, 2011), maximal flow network interdiction (minimize the maximal flow passing through the network) (Wood, 1993; Cormican et al., 1998; Akgün et al., 2011), clique interdiction (minimize the maximal clique in the graph) (Furini et al., 2019, 2021). Some of the applications of these problems include: national defense and military logistics (McMasters and Mustin, 1970), infectious disease control (Assimakopoulos, 1987; Furini et al., 2019), counter-terrorism (Farley, 2003; Furini et al., 2019), the interception of contraband and illegal items such as drugs (Washburn and Wood, 1995) and interception of nuclear material smuggling (Pan and Morton, 2008; Gutfraind et al., 2009). Interested readers in the network interdiction literature can refer to the recent review by Smith and Song (2020).

Facility interdiction problems have been well studied in the literature. Church et al. (2004) presented r-interdiction median problem (r-IMP) and r- interdiction covering problem (r-ICP) to study the interdiction of facilities in a supply system. In r-IMP, the attacker in interdicts r (out of p) facilities to maximise the post-interdiction disruption cost, whereas in r-ICP, the attacker interdicts r (out of p) facilities to minimize the coverage of the defender after interdiction. Several variants of r-IMP are studied in the literature (Church and Scaparra, 2007a; Losada et al., 2012; Aksen et al., 2014). Church and Scaparra (2007b) studied an extension of r-IMP known as the r-interdiction median problem with fortification (r-IMPF). In this problem, before the worst case attack of r facilities by the attacker, the defender has an option to protect q of the p facilities ($q + r < p$) so as to minimize the impact of the worst case attack. This problem is modeled as a trilevel stackleberg game. Several variations of r-IMPF have been studied in the literature by Scaparra and Church (2008a,b); Aksen et al. (2010); Aksen and Aras (2012); Aksen et al. (2013); Liberatore et al. (2012).

The literature on the hub interdiction is scarce. To the best of our knowledge, Lei (2013) is one of the earliest papers to study the identification of critical hubs in a hub network. The author studied the hub interdiction median problem (HIM) and presented a bilevel model where the attacker makes the first move by choosing to interdict r out of the p hubs so as the defender's minimum routing cost through the remaining hubs post-interdiction is maximum. The bilevel problem was reduced to a single level problem using a set of closest assignment constraint (CAC) which makes it easier to solve. The author introduced a hub protection problem (HPP) in which the defender has an option to protect q hubs to mitigate the worst case attack by an attacker. The paper presents no results for the HPP owing to its computational difficulty. Recent papers on hub interdiction problems include Parvaresh et al. (2013); Ghaffarinasab and Motallebzadeh (2017); Ghaffarinasab and Atayi (2017); Ramamoorthy et al. (2018); Ullmert et al. (2020), among others.

Parvaresh et al. (2013) formulated the multiple allocation p-hub median problem under intentional disruptions as a bilevel model in which the follower's objective is to identify and interdict those hubs that would cause the maximum deterioration in the system's efficiency. For solving the problem, they propose two heuristic algorithms based on simulated annealing. Ghaffarinasab and Motallebzadeh (2017) studied HIP and presented an enumeration based algorithm for solving it. Ghaffarinasab and Atayi (2017) studied the hub median, hub covering and hub center interdiction problems. The authors presented alternate set of CACs for the problem and solved it using a simulated annealing based metaheuristic approach. They solved smaller to medium size problem instances using 25-node CAB dataset and 81-node Turkish data set. Recently, Ramamoorthy et al. (2018) studied r-HMIP and presented several CACs by exploiting the properties of the model. These CACs were used for reducing the bilevel model to single level MIP model. The authors studied the dominance relationship between various CACs to identify the best among them. In addition to CAC based reduction, they also studied a dual based approach to reduce the problem to single level. The authors showed that some of the proposed CACs are more efficient in solving HIP than the dual based reduction method. They also solved large-scale instances of r-HMIPs using Benders decomposition approach. Ullmert et al. (2020) studied p-hub r-median location problem under the risk of interdiction where the decision maker has to locate p hubs knowing that r of the located hubs will be interdicted. The objective is to minimize the post-interdiction routing cost. The authors also present an exact algorithm to solve the problem.

In this paper, we study hub interdiction problems on uncapacitated p-hub median network, p-hub center network and p-hub covering network. We present bilevel formulations, where the attacker identifies r out of p hubs to interdict so as to maximize the worst-case post-interdiction performance of the system with the remaining hubs. We reduce the bilevel formulations to single levels through dual and penalty based reformulations. We identify the best formulation for each of the three variants and validate the theoretical results through computational results. We solve large scale instances of the problem for each of the variants and present their results.

3. Mathematical Formulations

Consider a hub network with a set of nodes $|N|$ and a set $\mathrm{H} \subseteq \mathrm{N}$ of p hubs. Let W_{ij} denote the amount of flow that the follower (defender) routes between origin node $i \in N$ and destination node $j \in N$ through one or at most two of the hubs from the set H. We use k and m as indices to denote the hubs that are connected to the origin node, $i \in N$ and the destination node, $j \in N$ respectively. Let d_{ijkm} represent the cost per unit flow of traversing from the origin i to destination j, through hubs k and m, in that order. Then, $d_{ijkm} = \alpha c_{ik} + \delta c_{km} + \gamma c_{mj}$, where α, δ, and γ are the discount factors on collection, transshipment, and distribution links, respectively and c_{ik}, c_{km}, and c_{mj} represent the cost of traversing from node i to k, k to m, and m to j, respectively. Typically, $\delta < \alpha$ and $\delta < \gamma$ due to economies of scale arising from consolidation of flows on inter-hub links. Let r denote the number of hubs from the existing set H of p hubs to be interdicted.

We model the r-hub interdiction problem as a Stackelberg game in which the leader (attacker) makes the first move by interdicting a subset of r hubs from the existing set H of p hubs with the objective to maximize the follower's (defender's) optimal routing/transportation cost through the $p - r$ surviving hubs in the network post-interdiction. We assume $r < p$ since the attacker usually has limited resources to interdict the hubs. We also assume that an interdicted hub is completely disabled, i.e., partial flows through an interdicted hub is not permitted. We formulate the problem as a bilevel program. We use X_{ijkm} as a decision variable to denote the fraction of flows from the origin i to the destination j through hubs k and m post-interdiction. Let z_k be a binary decision variable that equals 1 if hub k survives interdiction (is not interdicted), 0 otherwise.

3.1. *r-hub median interdiction problem*

In the r-hub median interdiction problem (r-HMIP), the attacker makes the first move by interdicting a subset of r hubs from the existing set H of p hubs with the objective to maximize the defender's optimal routing/transportation cost through the $p - r$ surviving hubs in the network post-interdiction. The bilevel formulation of r-HMIP is as follows:

$$(r\text{-HMIP}_{2L}) : \max_{\mathbf{z}} \; T_1 \tag{1}$$

$$\text{s.t.} \sum_{k \in H} z_k = p - r \tag{2}$$

$$z_k \in \{0, 1\} \qquad \forall k \in H \tag{3}$$

$$T_1 = \min_{\mathbf{X}} \; \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} \sum_{m \in H} W_{ij} d_{ijkm} X_{ijkm} \tag{4}$$

$$\text{s.t.} \; \sum_{k \in H} \sum_{m \in H} X_{ijkm} = 1 \qquad \forall i, j \in N \tag{5}$$

$$\sum_{m \in H} X_{ijkm} + \sum_{m \in H \setminus \{k\}} X_{ijmk} \leqslant z_k \qquad \forall i, j \in N; k \in H \tag{6}$$

$$X_{ijkm} \geqslant 0 \qquad \forall i, j \in N; k, m \in H \tag{7}$$

The attacker's objective function (1) maximizes the defender's optimal total transportation cost post-interdiction, which the follower wants to minimize in its objective function (4). The constraint (2) ensures that $p - r$ hubs remain open post-interdiction. Problem T_1 from (4) to (7) form the follower's problem at the lower level. The constraint set (5) ensures that the demand between every O-D pair (i, j) is satisfied using paths containing at most two hubs, while constraint set (6) ensures that this demand is routed only via surviving hubs post-interdiction.

3.2. *r-hub center interdiction problem*

In the r-hub center interdiction problem (r-HCIP), the attacker makes the first move by interdicting a subset of r hubs from the existing set H of p hubs with the objective of maximizing the defender's objective. The defender's objective at the second level is to minimize the maximal transportation cost between all O-D pairs. The bilevel formulation of r-HCIP is as follows:

$$(r\text{-HCIP}_{2L}) : \max_{\mathbf{z}} \; T_2 \tag{8}$$

$$\text{s.t.} \; (2), (3)$$

$$T_2 = \min \; \sum_{i \in N} \sum_{j \in N} Z_{ij} \tag{9}$$

$$\text{s.t.} \; (5) - (7)$$

$$Z_{ij} \geqslant \sum_{k \in H} \sum_{m \in H} D_{ijkm} X_{ijkm} \qquad \forall i, j \in N \tag{10}$$

$$Z_{ij} \geqslant 0 \qquad \forall i, j \in N \tag{11}$$

The attacker's objective function (8) maximizes the defender's objective. The defender's objective (9) along with constraints (10) and (11) minimizes the maximal distance between any source destination pair.

3.3. r-hub maximal covering interdiction problem

In the r-hub maximal covering interdiction problem (r-HMXCIP), the attacker makes the first move by interdicting a subset of r hubs from the existing set H of p hubs with the objective of minimizing the defender's total covered flows post-interdiction, which the defender wants to maximize. To model, we define V_{ijkm} as a binary parameter that indicates if a source-destination pair (i, j) is covered by hub pair (k, m) or not. For this purpose, we define a coverage radius β and if $D_{ijkm} \leqslant \beta$, the source-destination pair (i, j) is considered covered by hubs k and m ($V_{ijkm} = 1$), and if they are not covered then V_{ijkm} is set to 0. The bilevel formulation for r-HMXCIP is as follows:

$$(r\text{-HMXCIP}_{2L}) : \min_{\mathbf{z}} \; T_3 \tag{12}$$

$$\text{s.t. } (2), (3)$$

$$T_3 = \max \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} \sum_{m \in H} W_{ij} V_{ijkm} X_{ijkm} \tag{13}$$

$$\text{s.t. } (5), (7)$$

The attacker's objective function (12) minimizes the objective function of the defender. The defender's objective (13) maximizes the demand covered through the remaining hubs after interdiction of r hubs by the attacker.

In the following section, we present single level dual based reformulations of the three bilevel hub interdiction problems. Here, we take dual of the lower level linear program to construct a single level reformulation. The resultant single level reformulation is bilinear, which is then linearized. We also study several ways to strengthen the dual based reformulation.

4. Dual Based Reformulations

Dual reformulations of max-min or min-max bilevel programs are possible in cases where the objective function of both the levels are the same and the lower level problem is a linear program. Israeli and Wood (2002), Lim and Smith (2007), Ramamoorthy et al. (2018), among others, have employed this technique to reduce the bilevel network interdiction problems to single level. In the bilevel formulations of the hub interdiction problems described above, the lower level problem in all the variants is an LP and the objective functions at both the levels are the same. Therefore, the bilevel formulations can be directly reduced to single level formulations by taking the dual of the lower level problem. We present the single-level reformulations in the following subsections.

4.1. Dual based reformulation of r-hub median interdiction problem

In the bilevel formulation r-HMIP$_{2L}$, the lower level problem, represented by (4)-(7), is a linear program. By associating dual variables ϕ^1_{ij} and δ^1_{ijk} with the constraint sets (5) and (6) respectively, we get the following single-level bilinear formulation:

$$(r\text{-HMIP}_{ND}): \quad \max \sum_i \sum_j \phi^1_{ij} - \sum_i \sum_j \sum_k \delta^1_{ijk} q_k \tag{14}$$

$$\text{s.t. } (2) - (3)$$

$$\phi^1_{ij} - \delta^1_{ijk} \leqslant W_{ij} d_{ijkm} \qquad \forall i,j \in N; k,m \in H', k = m \tag{15}$$

$$\phi^1_{ij} - \delta^1_{ijk} - \delta^1_{ijm} \leqslant W_{ij} d_{ijkm} \qquad \forall i,j \in N; k,m \in H', k \neq m \tag{16}$$

$$-\infty \leqslant \phi^1_{ij} \leqslant \infty, \qquad \forall i,j \in N \tag{17}$$

$$\delta^1_{ijk} \geqslant 0 \qquad \forall i,j \in N; k \in H' \tag{18}$$

The bilinear terms in (14) can be linearized using auxiliary variables V^1_{ijk} as follows:

$$(r\text{-HMIP}_{DD}): \quad \max \sum_i \sum_j \phi^1_{ij} - \sum_i \sum_j \sum_k V^1_{ijk} \tag{19}$$

$$\text{s.t. } (2) - (3)$$

$$(15) - (18)$$

$$V^1_{ijk} \leqslant M^1_{ijk} q_k \qquad \forall i,j \in N, \forall k \in H' \tag{20}$$

$$V^1_{ijk} \geqslant \delta^1_{ijk} - M^1_{ijk}(1 - q_k) \qquad \forall i,j \in N, \forall k \in H' \tag{21}$$

$$V^1_{ijk} \geqslant 0 \qquad \forall i,j \in N, \forall k \in H' \tag{22}$$

where M^1_{ijk} is a sufficiently large number.

Note that Ramamoorthy et al. (2018) presented a linear formulation of r-HMIP$_{ND}$ which is an aggregated version of the above formulation r-HMIP$_{DD}$. However, the disaggregated version presented above yields a better LP relaxation than the aggregated version presented in Ramamoorthy et al. (2018). Hence, we use the above disaggregated formulation.

To ensure that the formulation r-HMIP$_{DD}$ is valid, M^1_{ijk} has to be sufficiently large. However, a very large value of M^1_{ijk} can lead to a weaker LP relaxation. In the following propositions, we present different values for M^1_{ijk} and study their strengths in improving the LP relaxation of r-HMIP$_{DD}$. First, we present $\tilde{M^1}_{ijk}$ which is a disaggregated version of the big M studied in Ramamoorthy et al. (2018).

Proposition 1. *For a given O-D pair (i,j), let $d_{ijk_1m_1} = max_{k,m}\{d_{ijkm}\}$ and $d_{ijkm_2} = min_m\{d_{ijkm}\}$. Then, $\tilde{M^1}_{ijk} = W_{ij}\left(d_{ijk_1m_1} - d_{ijkm_2}\right)$ is a valid value of M^1_{ijk} for the formulation r-HMIP$_{DF}$.*

Proof. Equations (20)-(22) $\implies$ $M^1_{ijk} \geqslant \delta_{ijk}$ and $\delta_{ijk} \leqslant W_{ij} d_{ijk_1m_1} - W_{ij} d_{ijkm_2}$ (since δ_{ijk} is the shadow price of the constraint (6), which is obtained by observing the maximum possible change in the objective function value (4) when changing the right hand side of the constraint (6) by a unit). Hence, $\tilde{M^1}_{ijk} = \delta_{ijk} = (d_{ijk_1m_1} - W_{ij}d_{ijkm_2})$ is a valid value of $\tilde{M^1}_{ijk}$ for the formulation r-HMIP$_{DF}$. □

Next, we explore an alternate value of M^1_{ijk}. First, we observe that for a given O-D pair (i, j) and p hubs, there are p^2 possible paths. We define an ordered set $A_{ij} = \{a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}, \ldots, a^f_{ij}, \ldots, a^{p^2}_{ij} \mid a^1_{ij} \leqslant a^2_{ij} \leqslant a^3_{ij} \leqslant \ldots \leqslant a^{p^2}_{ij}\}$, where $a^f_{ij} = d_{ijk_fm_f}(k_f, m_f \in H')$ denote the f^{th} least routing cost for the O-D pair (i, j).

Proposition 2. *For a given O-D pair (i, j), let $d_{ijkm_2} = min_m\{d_{ijkm}\}$. Then, $\overline{M}_{ijk} = W_{ij}\left(a^{p^2-(p-r)^2+1}_{ij} - d_{ijkm_2}\right)$ is a valid value of M^1_{ijk} for the formulation r-HMIP$_{DD}$.*

Proof. For this proof, we provide the following economic interpretation of the dual variables ϕ^1_{ij} and δ^1_{ijk}:

ϕ^1_{ij}: Minimum cost of routing the flows W_{ij} (since it is the dual variable associated with constraint (5))

δ^1_{ijk}: Penalty for routing the flows between origin i to destination j through an interdicted hub k (follows from (6))

By strong duality, $\phi^1_{ij} = W_{ij} \min\{d_{ijkm} \mid q_k = q_m = 1\}$.
Further, from (15) - (16), $\delta^1_{ijk} = \max\{0,\ \phi^1_{ij} - W_{ij}d_{ijkk},\ \phi^1_{ij} - W_{ij}d_{ijkm} - \delta^1_{ijm}\ \ \forall m_{y_m=0}, m \neq k \in H',\ \phi^1_{ij} - W_{ij}d_{ijkm}\ \ \forall m_{y_m=1}, m \neq k \in H'\}$, and to ensure validity of the formulation, (20) - (22), $M^1_{ijk} \geqslant \delta^1_{ijk}$.

To get a valid upper bound on the variable δ^1_{ijk}, the minuend term (ϕ^1_{ij}) should be as large as possible while the subtrahend ($W_{ij}d_{ijkk},\ W_{ij}d_{ijkm} + \delta_{ijm_1}\ \forall m_{y_m=0}, m \neq k \in H',\ W_{ij}d_{ijkm}\ \forall m_{y_m=1}, m \neq k \in H'$) should be as small as possible. The largest possible value for ϕ^1_{ij} in r-HMIP$_{DD}$ is $W_{ij}a^{p^2-(p-r)^2+1}_{ij}$.

If r hubs are interdicted, then there are $(p-r)^2$ remaining possible paths between the O-D pair (i, j). In the worst case for the defender, the set of $(p-r)^2$ remaining paths post-interdiction is given by $A'_{ij} = \{a^{p^2-(p-r)^2+1}_{ij}, \ldots, a^{p^2}_{ij}\}$ (i.e., the last $(p-r)^2$ elements from the set A_{ij}). Since $\phi^1_{ij} = \min\{\mathrm{a}^{p^2-(p-r)^2+1}_{ij}, \ldots, a^{p^2}_{ij}\}$.

$$\phi^1_{ij} = W_{ij}a^{p^2-(p-r)^2+1}_{ij} \qquad\qquad (a^{p^2-(p-r)^2+1}_{ij} \text{is the first element in the set} A'_{ij}).$$

Similarly, the smallest subtrahend term is:

$$\min_m W_{ij}d_{ijkm} \qquad\qquad (\text{since } \delta_{ijm} >= 0)$$

Therefore, $\overline{M}^1_{ijk} = W_{ij}\left(a^{p^2-(p-r)^2+1}_{ij} - d_{ijkm_2}\right)$ is a valid value of M^1_{ijk} and the proof follows. □

In the following proposition, We show that $\overline{M}^1_{ijk}$ is good enough to produce a tighter LP relaxation for r-HMIP$_{DD}$.

Proposition 3. *For the formulation r-HMIP$_{DD}$, $\overline{M}^1_{ijk} = \delta^1_{ijk}$ for some O-D pair (i,j) and hub k.*

Proof. The proof is straightforward. For any O-D pair (i,j) such that $A'_{ij} = \{a_{ij}^{p^2-(p-r)^2+1}, \ldots, a_{ij}^{p^2}\}$, $\overline{M}^1_{ijk} = \delta^1_{ijk}$. □

In the following proposition, we compare $\overline{M}^1_{ijk}$ with $\tilde{M}^1_{ijk}$ and show that $\overline{M}^1_{ijk} \leqslant \tilde{M}^1_{ijk} = W_{ij}d_{ijk_1m_1} - W_{ij}d_{ijkm_2}$.

Proposition 4. $\overline{M}^1_{ijk} \leqslant \tilde{M}^1_{ijk}$

Proof. By definition, we have

$$\overline{M}_{ijk} = W_{ij}\left(a_{ij}^{p^2-(p-r)^2+1} - \min_m\{d_{ijkm}\}\right)$$
$$\tilde{M}^1_{ijk} = W_{ij}d_{ijk_1m_1} - W_{ij}d_{ijkm_2}$$

where,

$$d_{ijk_1m_1} = max_{k,m\in H}d_{ijkm}$$

$$d_{ijkm_2} = min_{m\in H}d_{ijkm}$$

In other words,

$$\tilde{M}^1_{ijk} = W_{ij}\left(a_{ij}^{p^2} - d_{ijkm_2}\right)$$

Comparing the minuends,

$$W_{ij}a_{ij}^{p^2-(p-r)^2+1} \leqslant W_{ij}a_{ij}^{p^2}$$

Since the subtrahends are the same, $\overline{M}^1_{ijk} \leqslant \tilde{M}^1_{ijk}$ follows. □

For the single level reformulation r-HMIP$_{DD}$ we use $\overline{M}_{ijk}$ since it gives tighter LP relaxation than the one presented in Ramamoorthy et al. (2018).

4.2. Dual based reformulation of r-hub center interdiction problem

Associating dual variables ϕ^2_{ij}, δ^2_{ijk} and α^2_{ij} with constraint sets (5), (6) and (10) respectively, we get the following dual formulation of r-HCIP$_{2L}$:

$$(r\text{-HCIP}_{ND}): \quad \max \sum_{i\in N}\sum_{j\in N}\phi^2_{ij} - \sum_{i\in N}\sum_{j\in N}\sum_{k\in H}\delta^2_{ijk}q_k \tag{23}$$

$$\text{s.t. } (2)-(3)$$

$$\phi^2_{ij} - \delta^2_{ijk} \leqslant d_{ijkm}\alpha^2_{ij} \qquad \forall i,j \in N; k,m \in H', k=m \tag{24}$$

$$\phi^2_{ij} - \delta^2_{ijk} - \delta^2_{ijm} \leqslant d_{ijkm}\alpha^2_{ij} \qquad \forall i,j \in N; k,m \in H', k\neq m \tag{25}$$

$$\alpha^2_{ij} \leqslant 1 \qquad \forall i,j \in N \tag{26}$$

$$-\infty \leqslant \phi^2_{ij} \leqslant \infty \qquad \forall i,j \in N \tag{27}$$

$$\delta^2_{ijk} \geqslant 0 \qquad \forall i,j \in N; k \in H' \tag{28}$$

$$\alpha^2_{ij} \geqslant 0 \qquad \forall i,j \in N \tag{29}$$

The objective function (23) is bilinear and can be linearized by using a set of auxiliary variables V^2_{ijk} as follows:

$$
\begin{aligned}
(r\text{-HCIP}_{DD}): \quad & \max \sum_{i \in N} \sum_{j \in N} \phi^2_{ij} - \sum_{i \in N} \sum_{j \in N} \sum_{k \in H} V^2_{ijk} && (30)\\
\text{s.t.} \quad & (2) - (3) \\
& (24) - (29) \\
& V^2_{ijk} \leqslant M^2_{ijk} q_k && \forall i, j \in N, k \in H' \quad (31)\\
& V^2_{ijk} \geqslant \delta^2_{ijk} - M_{ijk}(1 - q_k) && \forall i, j \in N, k \in H' \quad (32)\\
& V^2_{ijk} \geqslant 0 && \forall i, j \in N, k \in H \quad (33)
\end{aligned}
$$

where M^2_{ijk} is a sufficiently large number. In the following proposition, we present a possible value of M^2_{ijk}.

Proposition 5. *For a given O-D pair (i, j) and $0 \leqslant \alpha^2_{ij} \leqslant 1$, $\overline{M^2}_{ijk} = \left(a^{p^2}_{ij} - a^{(p-r)^2}_{ij}\right)$ is a valid value of M^2_{ijk} for the formulation (r-HCIP$_{DD}$).*

Proof. The proof to the proposition is similar to the proof for the proposition 4.1. Maximum possible shadow price for δ_{ijk} is the difference between it's worst case value and the best case value. The worst case value for δ^2_{ijk} is $a^{p^2}_{ij}$ and similarly the best case value is $a^{(p-r)^2}_{ij}$ and the proof follows. Since, to ensure validity of r-HCIP$_{DD}$, $M^2_{ijk} > \delta^2_{ijk}$, $(a^{p^2}_{ij} - a^{(p-r)^2}_{ij})$ is a valid value for M^2_{ijk}. □

Proposition 6. *For the formulation r-HCIP$_{DD}$, $\overline{M}^2_{ijk} = \delta^2_{ijk}$ for some O-D pair (i, j) and hub k.*

Proof. For any O-D pair (i, j) such that $A'_{ij} = \{a^{p^2-(p-r)^2+1}_{ij}, \ldots, a^{p^2}_{ij}\}$, $\overline{M}^2_{ijk} = \delta^2_{ijk}$. □

4.3. Dual based reformulation of r-hub maximal covering interdiction problem

Associating dual variables ϕ^3_{ij} and δ^3_{ijk} with constraints (5) and (6) respectively, we get the following bilinear formulation.

$$
\begin{aligned}
(r\text{-HMXCIP}_{ND}): \quad & \min \sum_{i} \sum_{j} \phi^3_{ij} + \sum_{i} \sum_{j} \sum_{k} \delta^3_{ijk} q_k && (34)\\
\text{s.t.} \quad & (2) - (3) \\
& \phi^3_{ij} + \delta^3_{ijk} \geqslant W_{ij} V_{ijkm} && \forall i, j \in N; k, m \in H', k = m \quad (35)\\
& \phi^3_{ij} + \delta^3_{ijk} + \delta^3_{ijm} \geqslant W_{ij} V_{ijkm} && \forall i, j \in N; k, m \in H', k \neq m \quad (36)\\
& -\infty \leqslant \phi^3_{ij} \leqslant \infty && \forall i, j \in N \quad (37)\\
& \delta^3_{ijk} \geqslant 0 && \forall i, j \in N; k \in H' \quad (38)
\end{aligned}
$$

The objective function (34) is bilinear which is linearized as follows:

$$
\begin{aligned}
(r\text{-HMXCIP}_{DD}): \quad & \min \sum_{i\in N}\sum_{j\in N} \phi^3_{ij} + \sum_{i\in N}\sum_{j\in N}\sum_{k\in H} V^3_{ijk} && (39)\\
\text{s.t.}\ & (2)-(3) \\
& (35)-(38) \\
& V^3_{ijk} \leqslant M^3_{ijk} q_k \qquad \forall i,j \in N, k \in H' && (40)\\
& V^3_{ijk} \geqslant \delta^3_{ijk} - M^3_{ijk}(1-q_k) \qquad \forall i,j \in N, k \in H' && (41)\\
& V^3_{ijk} \geqslant 0 \qquad \forall i,j \in N, k \in H && (42)
\end{aligned}
$$

In the following proposition, we present a possible value of M^3_{ijk}.

Proposition 7. *For a given O-D pair, $\overline{M^3}_{ijk} = W_{ij}$ is a valid value of M^3_{ijk} for the formulation r-HMCIP$_{DD}$.*

Proof. The maximum possible shadow price for δ^3_{ijk} is the difference between it's worst case value and the best case value. The worst case value for δ^3_{ijk} is W_{ij} when $V^{co}_{ijkm} = 1$ and similarly the best case value is 0 when $V^{co}_{ijkm} = 0$ and the proof follows. Since, to ensure validity of $r\text{-}HMCIP_{DD}$, $M^3_{ijk} > \delta^3_{ijk}$, therefore W_{ij} is a valid value for M^3_{ijk}. □

5. Penalty based Reformulations

Note that the dual based reformulations discussed in the previous section involves bilinear terms in the objective functions (14), (23), and (34) due to the upper level binary variable in the constraint set (6) of the lower level problem. Linearizing these bilinear terms required additional variables V^1_{ijk}, V^2_{ijk} and V^3_{ijk}. We now present alternate reformulations of bilevel HIPs that obviate such bilinear terms, and hence the need for additional variables for their linearization. For this, we relax the complicating constraint set (6). However, this relaxation may result in flows through interdicted hubs. To prevent these flows, we penalize such flows in the objective function. We present the penalty based reformulations for the median, center and maximal covering versions of HIPs.

5.1. Penalty based reformulation of r-hub median interdiction problem

Let M^1_{ij} be a sufficiently large penalty associated with the flows through the interdicted hubs. The resulting formulation, r-HMIP$_{PF}$, is given as:

$$
\begin{aligned}
(r\text{-HMIP}_{PF}): \max_{\mathbf{q}} \quad & Z && (43)\\
\text{s.t.}\ & (2),(3) \\
Z = \min_{Y} & \sum_{i\in N}\sum_{j\in N}\sum_{k\in H}\sum_{m\in H} \Big\{ W_{ij} d_{ijkm} + (1-q_k)M^1_{ij} + (1-q_m)M^1_{ij} \Big\} Y_{ijkm} && (44)\\
\text{s.t.}\ & (5),(7)
\end{aligned}
$$

The lower level problem of the bilevel model r-HMIP$_{PF}$ is an LP. Hence, by taking the dual of the lower level problem, we get the following single level formulation r-HMIP$_P$:

$$(r\text{-HMIP}_P) : \max \sum_{i \in N} \sum_{j \in N} \eta_{ij} \tag{45}$$

$$\text{s.t. } (2), (3)$$

$$\eta_{ij} \leqslant W_{ij} d_{ijkm} + (2 - q_k - q_m) M^1_{ij} \qquad \forall i, j \in N; k, m \in H' \tag{46}$$

$$-\infty \leqslant \eta_{ij} \leqslant \infty; \quad q_k \in \{0, 1\} \qquad \forall i, j \in N; k \in H' \tag{47}$$

where η is the vector of dual variables corresponding to the constraint set (5).

We propose a valid value of M^1_{ij} in proposition 9. To state the proposition, we first present the following lemma.

Lemma 8. *For the formulation r-HMIP$_{PF}$, $Y_{ijkm} > 0$ only if $q_k = q_m = 1$.*

Proof. The lower level defender's problem can be decomposed into an independent (minimum cost network flow) problem for each O-D pair (i, j). Consider the network flow problem for an O-D pair (i, j) with $q_k = q_{m_1} = 1$ and $q_{m_2} = 0$. Let $d_{ijkm_1} = d_{ijkm_2} = \min_{k,m}\{d_{ijkm}\}$. Then, in (44),

$$W_{ij} d_{ijkm_1} + (1 - q_k) M_{ij} + (1 - q_{m_1}) M^1_{ij} < W_{ij} d_{ijkm_2} + (1 - q_k) M^1_{ij} + W(1 - q_{m_2}) M^1_{ij}$$

Hence, $Y_{ijkm_1} = 1$ and $Y_{ijkm_2} = 0$ from (5), (7). □

Proposition 9. *For the formulation r-HMIP$_{PF}$, $\tilde{M}^1{}_{ij} = W_{ij} a_{ij}^{p^2-(p-r)^2+1} - W_{ij} a^1_{ij} + \epsilon$, where ϵ is an infinitesimal quantity, is the tightest value of M^1_{ij}.*

Proof. Proof: To prove the above proposition, we first prove that $\tilde{M}_{ij}$ is a valid value for r-HMIP$_{PF}$ following which we also prove that it is the tightest M_{ij} value for the formulation.
When r hubs are interdicted, $a_{ij}^{p^2-(p-r)^2+1}$ is the worst case routing cost for the defender for a given O-D pair (i, j). This follows directly from the discussion of the proof for proposition 4.1. Similarly, a^1_{ij} is the best case routing cost for the defender for the O-D pair (i, j). To prevent flows through any interdicted path, the value of M_{ij} has to be chosen such that the cost of such a path should be greater than the cost of any available path a^x_{ij}, such that $1 \leqslant x \leqslant p^2 - (p - r)^2$. The best possible cost for a given O-D pair (i, j) for the defender is a^1_{ij}, while the worst cost for an available path is $a_{ij}^{p^2-(p-r)^2+1}$. Therefore, a valid value for M_{ij} is $\tilde{M}_{ij} = W_{ij} a_{ij}^{p^2-(p-r)^2+1} - W_{ij} a^1_{ij} + \epsilon$.
To prove that $\tilde{M}_{ij}$ is the tightest value for r-HMIP$_{PF}$, we show that subtracting an infinitesimal quantity ϵ from $\tilde{M}_{ij}$ makes it invalid as a value of M_{ij}. Let us denote that value after subtraction as $\dot{M}_{ij} = W_{ij} a_{ij}^{p^2-(p-r)^2+1} - W_{ij} a^1_{ij}$. Let us consider the defender's minimum cost network flow problem corresponding to O-D pair (i, j). Further, assume the worst case routing path be

available post-interdiction, while the best routing cost path be unavailable due to one of the hubs (either k_1 or m_1) on this path being interdicted. In that case, we have

$$W_{ij}a_{ij}^1 + M_{ij} = W_{ij}a_{ij}^{p^2-(p-r)^2+1},$$

which implies

$$M_{ij}^1 > W_{ij}a_{ij}^{p^2-(p-r)^2+1} - W_{ij}a_{ij}^1. \text{ (to ensure validity of the formulation.)}$$

However, $\dot{M}_{ij}$ does not satisfy the above inequality, and therefore, it is an invalid value of M_{ij}. We have already shown that $\tilde{M}_{ij}$, which is infinitesimally greater than $\dot{M}_{ij}$, is a valid value of M_{ij}. Therefore, $\tilde{M}_{ij}$ is the tightest value of M_{ij}. □

5.2. Penalty based reformulation of r-hub center interdiction problem

We present the penalty counterpart of r-HCIP$_{2L}$. Defining M_{ij}^2 as a very large value, the bilevel penalty counterpart of r-HCIP$_{2L}$ can be written as:

$$(r\text{-HCIP}_{PF}) : \max_{\mathbf{q}} \quad T_2 \tag{48}$$

$$\text{s.t. } (2), (3)$$

$$T_2 = \min_X \sum_{i \in N} \sum_{j \in N} Z_{ij}^2 + (\sum_{k \in H} \sum_{m \in H} \Big\{(1 - q_k)M_{ij}^2 + (1 - q_m)M_{ij}^2)\Big\}X_{ijkm} \tag{49}$$

$$\text{s.t. } (5), (7), (10), (11)$$

Assigning dual variable η_{ij} and α for constraint sets we present the single-level penalty formulation of r-HCIP$_{2L}$.

$$(r\text{-HCIP}_P) : \max \sum_{i \in N} \sum_{j \in N} \eta_{ij} \tag{50}$$

$$\text{s.t. } (2), (3)$$

$$\eta_{ij} \leqslant d_{ijkm}\alpha_{ij} + (2 - q_k - q_m)M_{ij}^2 \qquad \forall i, j \in N; k, m \in H' \tag{51}$$

$$\alpha_{ij} \leqslant 1 \qquad \forall i, j \in N \tag{52}$$

$$-\infty \leqslant \eta_{ij} \leqslant \infty; \quad q_k \in \{0, 1\}; \quad \alpha_{ij} \geqslant 0 \qquad \forall i, j \in N; k \in H' \tag{53}$$

We propose a valid value of M_{ij}^2 in proposition 9. To state the proposition, we first present the following lemma.

Lemma 10. *For the formulation r-HCIP$_{PF}$, $X_{ijkm} > 0$ only if $q_k = q_m = 1$.*

Proof. Consider the network flow problem for an O-D pair (i, j) with $q_k = q_{m_1} = 1$ and $q_{m_2} = 0$. Let $d_{ijkm_1} = d_{ijkm_2} = \min_{k,m}\{d_{ijkm}\}$. Then, in (49),

$$d_{ijkm_1} + (1 - q_k)M_{ij}^2 + (1 - q_{m_1})M_{ij}^2 < d_{ijkm_2} + (1 - q_k)M_{ij}^2 + (1 - q_{m_2})M_{ij}^2$$

Hence, $X_{ijkm_1} = 1$ and $X_{ijkm_2} = 0$ from (5), (7). □

Proposition 11. *For a given O-D pair (i,j) and $0 \leqslant \alpha_{ij}^2 \leqslant 1$, $\tilde{M}_{ij}^2 = \left(a_{ij}^{p^2} - a_{ij}^{(p-r)^2} + \epsilon\right)$ is the tightest valid value of M_{ij}^2 for the formulation r-HCIP$_{PD}$.*

Proof. When r hubs are interdicted, $a_{ij}^{p^2}$ and $a_{ij}^{(p-r)^2}$ are the worst and best case objective for any O-D pair (i,j), which implies the value of the optimal objective function value is either $a_{ij}^{p^2}$ or $a_{ij}^{(p-r)^2}$ any a^x between them. Therefore adding a penalty that equals to $\left(a_{ij}^{p^2} - a_{ij}^{(p-r)^2} + \epsilon\right)$ prohibits choosing an interdicted path.

To prove that $\tilde{M^2}_{ij}$ is the tightest value for r-HCIP$_{PF}$, we show that subtracting an infinitesimal quantity ϵ from $\tilde{M^2}_{ij}$ makes it invalid as a value of M_{ij}^2. Let us denote that value after subtraction as $\dot{M^2}_{ij} = a_{ij}^{p^2} - a_{ij}^{(p-r)^2}$. Let us consider the defender's problem corresponding to O-D pair (i,j). Further, assume the maximum of the available path is $a_{ij}^{p^2}$ post-interdiction, while there is an unavailable path $a_{ij}^{(p-r)^2}$ such that

$$a_{ij}^x + \dot{M}_{ij}^2 = a_{ij}^{p^2}$$

but,

$$M_{ij}^2 > a_{ij}^{p^2} - a_{ij}^{(p-r)^2}. \text{ (to ensure validity of the formulation.)}$$

Therefore, $\dot{M}_{ij}^2$ does not satisfy the above inequality, and therefore, it is an invalid value of M_{ij}^2. We have already shown that $\tilde{M}_{ij}^2$, which is infinitesimally greater than $\dot{M}_{ij}^2$, is a valid value of M_{ij}^2. Therefore, $\tilde{M}_{ij}^2$ is the tightest value of M_{ij}^2. □

5.3. Penalty based reformulation of r-hub maximal covering interdiction problem

Next, we present penalty counterpart of r-HMXCIP$_{2L}$. Defining M_{ij} as a very large value the bilevel penalty counterpart of r-HMXCIP$_{2L}$ is given below.

$$(r\text{-HMXCIP}_{PF}) : \min_{\mathbf{q}} \quad Z \tag{54}$$

$$\text{s.t. } (2)-(3)$$

$$Z = \max_{Y} \sum_{i\in N}\sum_{j\in N}\sum_{k\in H}\sum_{m\in H} \Big\{W_{ij}V_{ijkm} - (1-q_k)M_{ij}^3 - (1-q_m)M_{ij}^3\Big\}Y_{ijkm} \tag{55}$$

$$\text{s.t. } (5),(7)$$

Associating dual variables η_{ij} for constraint set (5), the single-level dual equivalent of the bilevel penalty formulation is as follows:

$$(r\text{-HMXCIP}_{P}) : \min \sum_{i\in N}\sum_{j\in N} \eta_{ij} \tag{56}$$

$$\text{s.t. } (2),(3) \tag{57}$$

$$\eta_{ij} \geqslant W_{ij}V_{ijkm} - (2-q_k-q_m)M_{ij}^3 \qquad \forall i,j \in N; k,m \in H' \tag{58}$$

$$-\infty \leqslant \eta_{ij} \leqslant \infty; \qquad \forall i,j \in N; k \in H' \tag{59}$$

We propose a valid value of M_{ij}^3 in proposition 13. To state the proposition, we first present the following lemma.

Lemma 12. *For the formulation r-HMXCIP$_{PF}$, $Y_{ijkm} > 0$ only if $q_k = q_m = 1$.*

Proof. Consider the network flow problem for an O-D pair (i, j) with $q_k = q_{m_1} = 1$ and $q_{m_2} = 0$. Let $V_{ijkm_1} = V_{ijkm_2} = 1$. Then, in (55),

$$W_{ij}V_{ijkm_1} - (2 - q_k - q_{m_1})M_{ij}^3 > W_{ij}V_{ijkm_2} - (2 - q_k - q_{m_2})M_{ij}^3$$

Hence, $Y_{ijkm_1} = 1$ and $Y_{ijkm_2} = 0$ from (5), (7). □

Proposition 13. *For the formulation r-HMXCIP$_{PF}$, $\tilde{M}_{ij}^3 = W_{ij}$, is the tightest value of M_{ij}.*

Proof. To prove the above proposition, we first prove that $\tilde{M}_{ij}^3$ is a valid value for r-HMXCIP$_{PF}$ following which we also prove that it is the tightest M_{ij}^3 value for the formulation.
Consider a hub pair (k, m) such that $V_{ijkm} = 1$, $q_k = 1$ and $q_m = 0$, it is easy to say that the objective function term for the corresponding hub pair is zero and since the objective is to maximize the optimal solution will not include such a path. Hence, the above value of $\tilde{M}_{ij}$ is valid for r-HMCXIP$_{PD}$.
To prove that $\tilde{M}_{ij}$ is the tightest value for r-HMCXIP$_{PF}$, we show that subtracting an infinitesimal quantity ϵ from $\tilde{M}_{ij}$ makes it invalid as a value of M_{ij}. Let us denote that value after subtraction as $\dot{M}_{ij} = W_{ij} - \epsilon$. Consider a hub pair (k, m^1) such that $V_{ijkm^1} = 1$, $q_k = 0$ and $q_m^1 = 1$. The demand between the O-D pair (i, j) is not covered by this hub pair (k, m^1) due to the unavailability of hub m^1. The corresponding objective function term is :

$$W_{ij}V_{ijkm_1} - \dot{M}_{ij} = W_{ij} - W_{ij} + \epsilon$$

For this case, $Y_{ijkm^1} = 1$ since the presence of a positive demand ϵ in the objective, which is a contradiction. Hence, $\tilde{M}_{ij}$ is the tightest value of M_{ij} for r-HMXCIP$_{PF}$. □

6. Dominance Relationship

We compare the linear programming relaxations of the dual and penalty formulations of r-HMIP, r-HCIP and r-HMXCIP and identify the stronger formulation between them. The comparison shows that the dual versions are stronger than the penalty formulations for all the three variants. We also validate our results through extensive computational experiments.

In the following proposition, we prove that the value of big M proposed for r-HMIP$_{DD}$ is strictly lesser than the big M proposed for r-HMIP$_{PF}$.

Proposition 14. *$\tilde{M}_{ij}^1 > \overline{M}_{ijk}^1$, where $\tilde{M}_{ij} = W_{ij}a_{ij}^{p^2-(p-r)^2+1} - W_{ij}a_{ij}^1 + \epsilon$, is the tightest value of M_{ij}^1 for the formulation r-HMIP$_{PF}$ and $\overline{M}_{ijk} = \max\{0, \hat{\phi}_{ij} - W_{ij}d_{ijkk}, \hat{\phi}_{ij} - W_{ij}d_{ijkm} - \hat{\delta}_{ijm}\}$ is a tight value of M_{ijk}^1 for the formulation r-HMIP$_{DD}$.*

Proof. By definition,

$$\tilde{M}_{ij}^1 = W_{ij}a_{ij}^{p^2-(p-r)^2+1} - W_{ij}a_{ij}^1 + \epsilon$$
$$\overline{M}_{ijk}^1 = \max\{0, \hat{\phi}_{ij} - W_{ij}d_{ijkk}, \hat{\phi}_{ij} - W_{ij}d_{ijkm} - \hat{\delta}_{ijm}\}$$

Trivially,

$$W_{ij}a_{ij}^{p^2-(p-r)^2+1} - W_{ij}a_{ij}^1 + \epsilon > 0$$

Also,

$$\hat{\phi}_{ij} = W_{ij}a_{ij}^{p^2-(p-r)^2+1}$$

and

$$\min\{W_{ij}d_{ijkk}, W_{ij}d_{ijkm} + \hat{\delta}_{ijm}\} > W_{ij}a_{ij}^1 - \epsilon = W_{ij}d_{ijk_1m_1} - \epsilon$$

Therefore, $\tilde{M}_{ij}^1 > \overline{M}_{ijk}^1$. □

In the following proposition, we state that dominance relationship between the dual and penalty reformulations of r-HMIP.

Proposition 15. *For a given value of $\tilde{M}_{ij}$, LP_R(r-HMIP$_{DD}$) $\leqslant$ LP_R(r-HMIP$_P$), where LP_R(r-HMIP$_{DD}$) and LP_R(r-HMIP$_P$) denote the optimal objective function values of LP relaxations of reformulations r-HMIP$_{DD}$ and r-HMIP$_P$ respectively.*

Proof. For any O-D pair (i, j) and hub pairs (k, m) such that, $0 < q_k < 1$ and $0 < q_m < 1$, let O_{DD} and O_P denote the objective function values of LP relaxations of r-HMIP$_{DD}$ and r-HMIP$_P$ respectively.

$$O_{DD} = \phi_{ij} - V_{ijk} - V_{ijm} \tag{60}$$

$$O_P = \nu_{ij} \tag{61}$$

From (15) and (16), $\phi_{ij} = W_{ij}D_{ijkm} + \delta_{ijk} + \delta_{ijm}$ or $W_{ij}D_{ijkk} + \delta_{ijk}$ or $W_{ij}D_{ijmm} + \delta_{ijm}$ and from (21), $V_{ijk} = \delta_{ijk} - M_{ijk}(1 - q_k)$.
Substituting the values of ϕ_{ij}, V_{ijk} and V_{ijm} in (19) we get,

$$\begin{aligned}
O_{DD_{km}} &= W_{ij}D_{ijkm} + M_{ijk}(1 - q_k) + M_{ijm}(1 - q_m)\\
O_{DD_k} &= W_{ij}D_{ijkk} + M_{ijk}(1 - q_k)\\
O_{DD_m} &= W_{ij}D_{ijmm} + M_{ijm}(1 - q_m)\\
O_{DD} &= max(O_{DD_{km}}, O_{DD_k}, O_{DD_m})
\end{aligned}$$

From (45), $\nu_{ij} = max(W_{ij}D_{ijkm} + (2 - q_k - q_m)M_{ij}, W_{ij}D_{ijkk} + (2 - 2q_k)M_{ij}, W_{ij}D_{ijmm} + (2 - 2q_m)M_{ij})$. In other words, $O_P = max(O_{P_{k,m}}, O_{P_k}, O_{P_m})$ where,

$$\begin{aligned}
O_{P_{k,m}} &= W_{ij}D_{ijkm} + M_{ij}(1 - q_k) + M_{ij}(1 - q_m)\\
O_{P_k} &= W_{ij}D_{ijkk} + M_{ij}(1 - q_k) + M_{ij}(1 - q_k)\\
O_{P_m} &= W_{ij}D_{ijmm} + M_{ij}(1 - q_m) + M_{ij}(1 - q_m)
\end{aligned}$$

From above it is clear that, $O_{DD_{k,m}} < O_{P_{k,m}}$, $O_{DD_k} < O_{P_k}$ and $O_{DD_m} < O_{P_m}$ since $M_{ijk} < M_{ij}$ and $M_{ijm} < M_{ij}$.

For the case where, $O_{DD} = O_{DD_{k,m}}, O_P = O_{P_{k,m}}$ or $O_{DD} = O_{DD_k}, O_P = O_{P_k}$ or $O_{DD} = O_{DD_m}, O_P = O_{P_m}$ it is evident that $LP_R(r\text{-HMIP}_{DD}) \leqslant LP_R(r\text{-HMIP}_P)$.

We now prove for the cases where optimal O_{DD} and O_P are dissimilar. For example where, $O_{DD} = O_{DD_{k,m}}$ and $O_P = O_{P_k}$. In this case, $O_{P_k} \geqslant O_{DD_{k,m}}$ since $O_{P_k} \geqslant O_{P_{k,m}}$ and $O_{P_{k,m}} \geqslant O_{DD_{k,m}}$. Therefore, $LP_R(r\text{-HMIP}_{DD}) \leqslant LP_R(r\text{-HMIP}_P)$. Using similar arguments, one can prove for other dissimilar cases of O_{DD} and O_P. □

In the following proposition, we state that dominance relationship between the dual and penalty reformulations of r-HCIP.

Proposition 16. *For a given value of $\tilde{M}_{ij}$, LP_R(r-HCIP$_{DD}$) dominates LP_R(r-HCIP$_P$) or LP_R(r-HCIP$_{DD}$) $\leqslant$ LP_R(r-HCIP$_P$), where LP_R(r-HCIP$_{DD}$) and LP_R(r-HCIP$_P$) denote the optimal objective function values of LP relaxations of r-HCIP$_{DD}$ and r-HCIP$_P$ respectively.*

Proof. For any O-D pair (i, j) and hub pairs (k, m) such that, $0 < q_k < 1$ and $0 < q_m < 1$, Let O_{DD} and O_P denote the objective function values of LP relaxations of $HCIP_{DD}$ and $HCIP_{PD}$ respectively.

$$O_{DD} = \phi_{ij} - V_{ijk} - V_{ijm} \tag{62}$$

$$O_P = \nu_{ij} \tag{63}$$

From (24) and (25), $\phi_{ij} = \alpha_{ij}D_{ijkm} + \delta_{ijk} + \delta_{ijm}$ or $\phi_{ij} = \alpha_{ij}D_{ijkm} + \delta_{ijm}$ or $\phi_{ij} = \alpha_{ij}D_{ijkm} + \delta_{ijk}$ and from (32), $V_{ijk} = \delta_{ijk} - M_{ijk}(1 - q_k)$. Substituting the values of ϕ_{ij}, V_{ijk} and V_{ijm} in (62) we get,

$$\begin{aligned}
O_{DD_{k,m}} &= \alpha_{ij}D_{ijkm} + M_{ijk}(1 - q_k) + M_{ijm}(1 - q_m)\\
O_{DD_k} &= \alpha_{ij}D_{ijkk} + M_{ijk}(1 - q_k)\\
O_{DD_m} &= \alpha_{ij}D_{ijmm} + M_{ijm}(1 - q_m)\\
O_{DD} &= max(O_{DD_{km}}, O_{DD_k}, O_{DD_m})
\end{aligned}$$

From (50), $\eta_{ij} = max(\alpha_{ij}D_{ijkm} + (2 - q_k - q_m)M_{ij}, \alpha_{ij}D_{ijkk} + (2 - 2q_k)M_{ij}, \alpha_{ij}D_{ijmm} + (2 - 2q_m)M_{ij})$. In other words, $O_P = max(O_{P_{k,m}}, O_{P_k}, O_{P_m})$ where,

$$\begin{aligned}
O_{P_{k,m}} &= \alpha_{ij}D_{ijkm} + M_{ij}(1 - q_k) + M_{ij}(1 - q_m)\\
O_{P_k} &= \alpha_{ij}D_{ijkk} + M_{ij}(1 - q_k) + M_{ij}(1 - q_k)\\
O_{P_m} &= \alpha_{ij}D_{ijmm} + M_{ij}(1 - q_m) + M_{ij}(1 - q_m)
\end{aligned}$$

From the above relations it is clear that, $O_{DD_{k,m}} < O_{P_{k,m}}$, $O_{DD_k} < O_{P_k}$ and $O_{DD_m} < O_{P_m}$ since $M_{ijk} < M_{ij}$ and $M_{ijm} < M_{ij}$.

For the case where, $O_{DD} = O_{DD_{k,m}}, O_P = O_{P_{k,m}}$ or $O_{DD} = O_{DD_k}, O_P = O_{P_k}$ or $O_{DD} = O_{DD_m}, O_P = O_{P_m}$ it is evident that $LP_R(r\text{-HCIP}_{DD}) \leqslant LP_R(r\text{-HCIP}_P)$.

We now prove for the cases where optimal O_{DD} and O_P are dissimilar. For example where, $O_{DD} = O_{DD_{k,m}}$ and $O_P = O_{P_k}$. For this case, $O_{P_k} \geqslant O_{DD_{k,m}}$ since $O_{P_k} \geqslant O_{P_{k,m}}$ and $O_{P_{k,m}} \geqslant$

$O_{DD_{k,m}}$. Therefore, LP_R(r-HCIP$_{DD}$) $\leqslant LP_R$(r-HCIP$_P$). By using similar arguments, one can prove, for other dissimilar cases of O_{DD} and O_P that LP_R(r-HCIP$_{DD}$) $\leqslant LP_R$(r-HCIP$_P$).

In the following proposition, we state that dominance relationship between the dual and penalty reformulations of r-HMXCIP.

Proposition 17. *For a given value of $\tilde{M}_{ij}$, LP_R(r-HMXCIP$_{DD}$) dominates LP_R(r-HMXCIP$_P$), where LP_R(r-HMXCIP$_{DD}$) and LP_R(r-HMXCIP$_P$) denote the optimal objective function values of LP relaxations of r-HMXCIP$_{DD}$ and r-HMXCIP$_P$ respectively. In other words, LP_R(r-HMXCIP$_{DD}$) $\geqslant LP_R$(r-HMXCIP$_P$) (since r-HMXCIP$_{DD}$ and r-HMXCIP$_P$ are both minimization problems).*

Proof. For any O-D pair (i, j) and hub pairs (k, m) such that, $0 < q_k < 1$ and $0 < q_m < 1$, let O_{DD} and O_P denote the objective function values of LP relaxations of r-HMXCIP$_{DD}$ and r-HMXCIP$_P$ respectively.

$$O_{DD} = \phi_{ij} + V_{ijk} + V_{ijm} \tag{64}$$

$$O_P = \nu_{ij} \tag{65}$$

From (35) and (36), $\phi_{ij} = W_{ij}V_{ijkm} - \delta_{ijk} - \delta_{ijm}$ and from (41), $V_{ijk} = \delta_{ijk} - M_{ijk}(1 - q_k)$. Substituting the values of ϕ_{ij}, V_{ijk} and V_{ijm} in (64) we get,

$$\begin{aligned}
O_{DD_{k,m}} &= W_{ij}V_{ijkm} - M_{ij}(1 - q_k) - M_{ij}(1 - q_m) \\
O_{DD_k} &= W_{ij}V_{ijkk} - M_{ij}(1 - q_k) \\
O_{DD_m} &= W_{ij}V_{ijmm} - M_{ij}(1 - q_m) \\
O_{DD} &= min(O_{DD_{km}}, O_{DD_k}, O_{DD_m})
\end{aligned}$$

From (58), $\nu_{ij} = W_{ij}V_{ijkm} + (2 - q_k - q_m)M_{ij}$
Substituting ν_{ij} in (65) we get,

$$\begin{aligned}
O_{P_{k,m}} &= W_{ij}V_{ijkm} - (2 - q_k - q_m)M_{ij} \\
O_{P_k} &= W_{ij}V_{ijkm} - (2 - 2q_k)M_{ij} \\
O_{P_m} &= W_{ij}V_{ijkm} - (2 - q_m)M_{ij} \\
O_P &= min(O_{P_{km}}, O_{P_k}, O_{P_m})
\end{aligned}$$

From the above relations it is clear that, $O_{DD_{k,m}} = O_{P_{k,m}}$, $O_{DD_k} > O_{P_k}$ and $O_{DD_m} > O_{P_m}$. For the case where, $O_{DD} = O_{DD_{k,m}}, O_P = O_{P_{k,m}}$ or $O_{DD} = O_{DD_k}, O_P = O_{P_k}$ or $O_{DD} = O_{DD_m}, O_P = O_{P_m}$ it is evident that LP_R(r-HMXCIP$_{DD}$) $\geqslant LP_R$(r-HMXCIP$_P$) .

We now prove for the cases where optimal O_{DD} and O_P are dissimilar. We take an example where, $O_{DD} = O_{DD_{k,m}}$ and $O_P = O_{P_k}$. For this case, $O_{P_k} \leqslant O_{DD_{k,m}}$ since $O_{P_k} \leqslant O_{P_{k,m}}$ and $O_{P_{k,m}} = O_{DD_{k,m}}$. Therefore, LP_R(r-HMXCIP$_{DD}$) $\geqslant LP_R$(r-HMXCIP$_P$). Using similar arguments, one can prove for other dissimilar cases of O_{DD} and O_{PD}. □

7. Computational Results

We conduct computational experiments to compare the performance of the penalty and dual based reformulations of the three variants of the hub interdiction problem. All the formulations are coded in C++ and run on a Dell workstation with a 2.60GHz Intel Xeon-e5 processor and 64 gigabytes of RAM. The models are solved using the branch-and-cut solver of CPLEX 12.8 with its default settings using only one thread.

Our experiments are performed using the Australian Post (AP) data set obtained from the OR library (http://mscmga.ms.ic.ac.uk/jeb/orlib/phubinfo.html). These instances comprise the postal flow and Euclidean distances between 200 postal districts in the metro Sydney area. In our experiments, we select instances with $|N| = 50$, 100, and 200 nodes. The number of hubs in the network is set to $p = 10$ and 15. For each of the variants, the hub network configuration,.i.e. the optimal location of the hubs is an input. This hub network configuration is set as per the optimal solution to the uncapacitated multiple allocation r-hub median location problem, r-hub covering problem and p-maximal covering problem for r-HMIP, r-HCIP and r-HMXCIP respectively. The problem instances are generated by varying the number of hubs to interdict (r) in the set $r \in \{3, 4, 5, 6, 7\}$ for $p = 10$ and in the set $r \in \{5, 6, 7, 8, 9, 10, 11, 12\}$ for $p = 15$. The discount factor for the flows on hub arcs (δ) is varied from 0.25 (high), 0.50 (moderate), to 0.75 (low), while it is set to 3 for collection arcs (α) and 2 for distribution arcs (γ). For r-HMXCIP, we set the radius (β) to 15. The time limit is set to 36,000 seconds.

Table 1 presents a summary of the results of our experiments, whereas the detailed results are presented in Tables 2 to 4. In Table 1, for every variant of the problem, we report the number of instances solved to optimality and the minimum, average and the maximum computation time (in seconds). Using the dual based reformulation, we are able to solve 137 (out of 156) instances of r-HMIP to optimality compared to 96 instances using the penalty based reformulation. Using the dual based reformulation, we are able to solve 109 (out of 156) instances of r-HCIP to optimality compared to 94 instances using the penalty based reformulation. Similarly, using the dual based reformulation, we are able to solve 137 (out of 156) instances of r-HMXCIP to optimality compared to 117 instances using the penalty formulation. Furthermore, we observe that there is a significant reduction in computational time using the dual based reformulation for all the three problem variants. Overall, our results depict that the dual based reformulation is computationally efficient compared to the penalty based reformulation. The superior performance of dual based reformulation is attributed to its better LP relaxation compared to penalty based reformulation.

Tables 2 to 4 present the detailed results of our experiments. The first three columns in these tables list the problem parameters such as the number of hubs (pre-interdiction) p, the number of hubs to be interdicted r, and discount factor δ for each instance. The top row indicates the number of nodes $|N|$ of the hub network. The column "time(s)" report the computation time (in seconds) for the corresponding formulation. The column marked "% red." refers to the percentage reduction in the CPU time of the dual formulation over penalty formulation. It is computed as follows: $\{\text{time}(r\text{-HMIP}_P)$ - $\text{time}(r\text{-HMIP}_{DD})$ $\times 100\%$ $\}$ $/(r\text{-HMIP}_P)$. The column "gap(%)" reports the optimality gap for the instances that could not be solved to optimality within the time limit.

7.1. Results for r-hub median interdiction problems

Table 2 presents the results of the the penalty and dual based reformulations of r-hub median interdiction problem. Results show that we are able to solve all the thirty nine instances for $|N| = 50$ and 100 nodes each to optimality using both the reformulations. In the case of 150-node instances, dual based reformulation solves all instances, while the penalty based reformulation solves only 18 of the 39 instances. For the 200-node instances, note that the penalty based reformulation could not solve any of the 39 instances to optimality within the time limit. Therefore, for 200-node instances, we report the results for the dual based formulation only. For 200-node instances, dual based reformulation is able to solve 20 out of 39 (51%) instances to optimality within the time limit.

For the 50-node instances, the average computation time using the penalty formulation is 176 seconds compared to 53 seconds for $p = 10$, and 695 seconds compared to 165 for $p = 15$ using the dual formulation. The computation time of the penalty formulation ranges from 139 to 213 seconds for $p = 10$, whereas the range for the dual formulation is 41 to 68 seconds. Similarly, the range for $p = 15$ is 479 - 990 seconds for the penalty formulation and 110 -229 seconds for the dual formulation. On average, the dual formulation is approximately four times faster or yields a 73% reduction in computation time. As expected, for the moderate-size 100-node instances, the computation times are comparatively higher than 50-node instances. For example, the computation time of the penalty formulation ranges from 1,941 to 4,530 seconds for $p = 10$ and 8,532 to 27,785 seconds for $p = 15$, whereas the similar figures for dual formulation is 405 to 1,684 and 1,713 to 7498 seconds respectively. The average computation time using the penalty formulation is 3,030 seconds compared to 817 seconds for $p = 10$ and 14,135 compared to 3,593 for $p = 15$, using the dual formulation, i.e. dual formulation is approximately 4 times faster. This accounts for 73% reduction in computation time, on average. For the instances with $|N| = 150$ and $p = 10$, the range of penalty formulation is from 6,118 to 20,535 seconds while similar figures for dual formulation is 1,349 to 5,104 seconds. The average time comparison of 12,502 seconds for penalty formulation over 2,663 seconds shows again that dual is faster than penalty formulation by approximately five times.

For the large-size instances with $|N| = 150$ and $p = 15$, the penalty based reformulation could solve only 3 of the 24 instances within the time limit, while the dual based reformulation could solve all problem instances. For the large-size instances with $|N| = 200$ nodes, it is worth pointing that the penalty formulation could not solve any of the 39 instances to optimality within the time limit. Hence, we report the optimality gap and computation times using the dual formulation only. Even with dual formulation, we were able to solve 20 out of 39 (i.e. 51%) instances to optimality within the time limit. Out of these 20 instances that were solved to optimality, 15 instances belong to $p = 10$, and 5 (out of 24) instances belong to $p = 15$. For the remainder of the 19 instances, the optimality gap is in the range of 60% to 175%, with an average gap of 123%. These results and observations confirm the strength of the dual based reformulation.

Table 1: Summary of results for the variants of the hub interdiction problem

Problem	Formulation	Instance		No. of Ins. Opt.	Min. time (s)	Avg. time (s)	Max. time (s)
r-HMIP	Penalty	$\|N\|$=50, p=10		15/15	139	176	213
		$\|N\|$=50, p= 15		24/24	479	695	990
		$\|N\|$=100, p=10		15/15	1,941	3,030	4,530
		$\|N\|$=100, p= 15		24/24	8,532	14,135	27,785
		$\|N\|$=150, p=10		15/15	6,118	12,502	20,535
		$\|N\|$=150, p=15		3/24	29,546	30,441	31,264
		$\|N\|$=200, p=10		0/15	-	-	-
		$\|N\|$=200, p= 15		0/24	-	-	-
			Total	96/156			
	Dual	$\|N\|$=50, p=10		15/15	41	53	68
		$\|N\|$=50, p= 15		24/24	110	165	229
		$\|N\|$=100, p=10		15/15	405	817	1,684
		$\|N\|$=100, p= 15		24/24	1,713	3,593	7,498
		$\|N\|$=150, p=10		15/15	1,349	2,663	5,104
		$\|N\|$=150, p=15		24/24	6,549	13,033	25,116
		$\|N\|$=200, p=10		15/15	7,042	13,554	28,064
		$\|N\|$=200, p= 15		5/24	-	-	-
			Total	137/156			
r-HCIP	Penalty	$\|N\|$=50, p=10		15/15	118	240	316
		$\|N\|$=50, p= 15		24/24	654	1,194	4,265
		$\|N\|$=100, p=10		15/15	2,687	4,475	7,116
		$\|N\|$=100, p= 15		24/24	7,345	12,939	29,539
		$\|N\|$=150, p=10		15/15	11,333	18,385	27,828
		$\|N\|$=150, p=15		1/24	33,685	33,685	33,685
		$\|N\|$=200, p=10		0/15	-	-	-
		$\|N\|$=200, p= 15		0/24	-	-	-
			Total	94/156			
	Dual	$\|N\|$=50, p=10		15/15	88	111	148
		$\|N\|$=50, p= 15		24/24	395	694	1,186
		$\|N\|$=100, p=10		15/15	1,128	2,467	3,830
		$\|N\|$=100, p= 15		24/24	7,165	12,208	27,568
		$\|N\|$=150, p=10		15/15	6,635	14,469	30,212
		$\|N\|$=150, p=15		1/24	29,863	29,863	29,863
		$\|N\|$=200, p=10		15/15	36,276	36,297	36,304
		$\|N\|$=200, p= 15		0/24	-	-	-
			Total	109/156			
r-HMXCIP	Penalty	$\|N\|$=50, p=10		15/15	27	43	63
		$\|N\|$=50, p= 15		24/24	92	138	376
		$\|N\|$=100, p=10		15/15	266	357	525
		$\|N\|$=100, p= 15		24/24	740	1,199	1,744
		$\|N\|$=150, p=10		15/15	347	670	1,047
		$\|N\|$=150, p=15		3/24	1,743	4,420	15,986
		$\|N\|$=200, p=10		15/15	780	1210	1876
		$\|N\|$=200, p= 15		24/24	2,266	4,622	7,367
			Total	156/156			
	Dual	$\|N\|$=50, p=10		15/15	8	14	18
		$\|N\|$=50, p= 15		24/24	20	31	46
		$\|N\|$=100, p=10		15/15	66	137	292
		$\|N\|$=100, p= 15		24/24	224	320	402
		$\|N\|$=150, p=10		15/15	155	281	482
		$\|N\|$=150, p=15		24/24	384	768	1,972
		$\|N\|$=200, p=10		15/15	360	1,007	1,544
		$\|N\|$=200, p= 15		24/24	654	1,179	1,502
			Total	156/156			

Table 2: Results for the penalty and dual formulations of r-hub median interdiction problem.

Parameters			$\|N\| = 50$			$\|N\| = 100$				$\|N\| = 150$				$\|N\| = 200$
			r-HMIP$_P$	r-HMIP$_{DD}$		r-HMIP$_P$	r-HMIP$_{DD}$			r-HMIP$_P$	r-HMIP$_{DD}$			r-HMIP$_{DD}$
p	r	δ	time(s)	time(s)	% red.	time(s)	time(s)	% red.	gap (%)	time(s)	time(s)	% red.	gap (%)	time(s)
10	3	0.25	182	61	66	2,652	1,660	37	0	6118	2,596	57	0	19,605
		0.50	213	50	77	2,402	626	74	0	9,958	1,669	83	0	28,064
		0.75	204	42	79	3,062	795	74	0	15,423	1,349	91	0	14,773
	4	0.25	202	68	66	2,916	1,684	42	0	9,350	3,821	59	0	21,387
		0.50	200	53	74	3,185	644	80	0	18,268	1,863	90	0	10,351
		0.75	215	58	73	3,169	676	79	0	20,535	1,856	91	0	7,455
	5	0.25	185	61	67	2,911	1,268	56	0	9,788	4,851	50	0	27,497
		0.50	169	46	73	3,050	740	76	0	11,242	2,081	81	0	10,600
		0.75	166	53	68	4,530	656	86	0	17,705	1,750	90	0	8,767
	6	0.25	156	57	63	2,255	742	67	0	16,518	5,104	69	0	11,875
		0.50	154	55	64	4,153	589	86	0	13,739	2,658	80	0	10,139
		0.75	139	42	70	2,960	533	82	0	8,455	3,004	64	0	7,235
	7	0.25	162	57	65	1,941	696	64	0	10,984	3,674	66	0	9,762
		0.50	152	52	66	3,796	536	86	0	10,278	2,322	77	0	8,767
		0.75	142	41	71	2,479	405	84	0	9178	2,155	76	0	7,042
		Min	139	41	63	1,941	405	37	0	6,118	1,349	57	0	7,042
		Avg.	176	53	70	3,030	817	71	0	12,502	2,663	75	0	13,554
		Max.	213	68	79	4,530	1,684	86	0	20,535	5,104	91	0	28,064
15	5	0.25	754	229	70	25,030	6,694	73	102	limit	14,949	-	144	limit
		0.50	976	177	82	19,854	5,403	73	109	limit	12,046	-	128	limit
		0.75	838	148	82	13,908	4,470	68	121	limit	6,549	-	85	limit
	6	0.25	981	212	78	17,653	7,223	59	109	limit	14,890	-	171	limit
		0.50	859	146	83	12,670	5,036	60	135	limit	12,403	-	167	limit
		0.75	663	198	70	15,651	3,165	80	175	limit	9,704	-	101	limit
	7	0.25	990	157	84	19,552	7,498	62	140	limit	13,398	-	175	limit
		0.50	719	168	77	12,430	3,441	72	157	limit	17,465	-	155	limit
		0.75	777	177	77	14,420	3,022	79	132	limit	9,138	-	110	limit
	8	0.25	784	165	79	20,754	5,033	76	121	limit	13,070	-	155	limit
		0.50	761	147	81	11,980	2,831	76	153	limit	16,514	-	146	limit
		0.75	843	215	74	13,692	2,395	83	161	limit	8,659	-	139	limit
	9	0.25	563	216	62	13,180	4,734	64	167	limit	25,116	-	113	limit
		0.50	571	179	69	13,351	2,451	82	204	limit	15,079	-	121	limit
		0.75	608	127	79	10,558	1,860	82	140	limit	13,149	-	0	23,051
	10	0.25	578	164	72	11,334	3,432	70	170	limit	21,246	-	102	limit
		0.50	479	154	68	8,532	2,490	71	104	limit	12,758	-	107	limit
		0.75	577	110	81	10,783	1,764	84	185	limit	9,040	-	0	31,296
	11	0.25	620	168	73	9,659	2,158	78	0	31,264	12,826	59	69	limit
		0.50	584	153	74	27,785	1,813	93	85	limit	17,174	-	90	limit
		0.75	571	125	78	8,234	1,713	79	222	limit	12,156	-	0	32,766
	12	0.25	550	155	72	10,051	2,380	76	0	29,546	9,577	68	0	21,759
		0.50	566	132	77	9,295	2,714	71	0	30,515	7,307	76	0	31,951
		0.75	481	127	74	8,891	2,706	70	167	limit	8,582	-	61	limit
		Min	479	110	62	8,532	1,713	59	85	29,546	6,549	59	69	21,759
		Avg.	695	165	75	14,135	3,593	74	146	30,441	13,033	68	123	28,164
		Max.	990	229	84	27,785	7,498	93	222	31,264	25,116	76	175	32,766

7.2. Results for r-hub center interdiction problem

Table 3 presents the results for the r-HCIP. Results show that using the penalty and dual formulations, we are able to solve all the 39 instances for $|N| = 50$ to optimality. However, for the 100-node instances, we are able to solve only 34 (out of 39) instances to optimality using the penalty formulation compared to all the 39 instances using the dual formulation. For 150-node instances, penalty formulation could solve 16 while dual could only solve 14 instances to optimality. For the large-scale instances with $|N| = 200$ nodes, it is worth pointing that the penalty based reformulation could not solve any of the 39 instances to optimality within the time limit. Hence, we report the results of the dual based reformulation only. Using with the dual based reformulation, we solve 15 (out of 39) instances to optimality within the time limit. For the remainder of the 24 instances, the optimality gap is in the range of 58% to 116%, with an average gap of 91%.

The results for 50-node, $p = 10$ instances show that the computation time of the penalty formulation is in the range of 118 to 316 seconds with an average of 240 seconds, whereas for the dual based reformulation, the range is from 88 to 148 seconds with an average of 111 seconds. The reduction in computation time ranges from 3% to 71%. Similarly for $p = 15$, the computational time for penalty based formulation is in the range of 654 to 4,265 seconds with an average of 1,194 seconds while for dual based formulation the range is from 395 to 1,186 seconds with an average of 694 seconds. Hence, the dual based reformulation is approximately faster by a factor of two. This amounts to a 61% reduction in the computational time, on average. The dual based reformulation outperforms penalty based reformulation on 34 (out of 39) instances. For the $|N| = 100$ and $p = 10$ instances, both the reformulations solve all 15 instances to optimality within the time limit, whereas for the 100-node, $p = 15$ instances, penalty based reformulation could solve only 19 (out of 24) instances to optimality within the time limit. The optimality gap for the instances that are not solved to optimality is in the range of 69% to 143%. On the contrary, the dual based reformulation, is able to solve all the (39 out of 39) instances to optimality within the time limit. The dual based reformulation outperforms penalty formulation on 29 (out of 39) instances. The average computation time using the penalty based reformulation is 4,475 seconds compared to 2,467 seconds using the dual based reformulation for $p = 10$, and similarly 12,939 seconds compared to 12,208 seconds for $p = 15$. The former represents a 13% reduction while the latter, 23% reduction, in computational time on average.

For the large instances with $|N| = 150$ nodes and $p = 10$, the computational time of penalty based reformulation varies between 11,333 to 27,828 while it is between 6,635 to 30,212 with an average of 14,469 for the dual based reformulation. With $p = 15$, both formulations could solve only one instance while the optimality gap of dual based reformulation is almost always better than that of the penalty formulation except for one single instance. For the large-scale instances with $|N| = 200$ nodes, we report the results (optimality gaps and times) for the dual based reformulation only as the penalty based reformulation was unable to solve any instance to optimality within the time limit. Although the dual based reformulation could solve all (15 out of 15) instances with $p = 10$ to optimality, it could not solve any of the 24 instances with p

$= 15$ to optimality within the time limit. For the 24 instances with $p = 15$, the optimality gap is in the range from 58% to 116%, with an average gap of 91%.

Table 3: Results for the penalty and dual formulations of r-hub center interdiction problem.

Parameters			$\|N\| = 50$			$\|N\| = 100$					$\|N\| = 150$					$\|N\| = 200$	
			r-HCIP$_P$	r-HCIP$_{DD}$		r-HCIP$_P$		r-HCIP$_{DD}$			r-HCIP$_P$		r-HCIP$_{DD}$			r-HCIP$_{DD}$	
p	r	δ	time(s)	time(s)	% red.	%gap	time(s)	%gap	time(s)	% red.	%gap	time(s)	%gap	time(s)	%red.	%gap	time(s)
10	3	0.25	265	115	56	0	5,081	0	2,151	58	0	16,175	0	12,510	23	0	36,276
		0.50	246	101	59	0	4,410	0	1,128	74	0	15,309	0	8,982	41	0	36,297
		0.75	245	91	60	0	6,393	0	1,545	76	0	15,248	0	6,635	58	0	36,304
	4	0.25	316	122	61	0	7,116	0	3,236	55	0	20,572	0	21,577	*	0	36,289
		0.50	307	101	67	0	3,609	0	2,085	42	0	25,705	0	10,389	60	0	36,304
		0.75	307	88	71	0	6,367	0	1,780	72	0	26,767	0	9,240	65	0	36,302
	5	0.25	315	120	62	0	4,771	0	3,156	34	0	23,835	0	17,831	25	0	36,277
		0.50	282	112	60	0	6,645	0	2,031	69	0	27,828	0	22,564	19	0	36,300
		0.75	251	101	60	0	5,947	0	2,317	61	0	22,193	0	10,280	52	0	36,302
	6	0.25	232	148	36	0	6,818	0	3,608	47	0	21,967	39	limit	*	0	36,286
		0.50	172	108	37	0	3,650	0	2,000	45	0	21,147	0	30,212	*	0	36,299
		0.75	271	103	62	0	7,084	0	2,283	68	0	18,935	0	9,766	48	0	36,303
	7	0.25	133	138	0	0	3,606	0	3,234	10	0	18,565	73	limit	*	0	36,293
		0.50	118	113	0	0	4,362	0	3,830	12	0	11,333	0	16,016	*	0	36,304
		0.75	137	112	18	0	2,687	0	2,622	2	0	12,164	0	12,102	0	0	36,313
		Min	118	88	0	0	2,687	0	1,128	2	0	11,333	39	6,635	0	0	36,276
		Avg.	240	111	47	0	4,475	0	2,467	13	0	18,385	56	14,469	39	0	36,297
		Max.	316	148	71	0	7,116	0	3,830	76	0	27,828	73	limit	*	0	36,304
15	5	0.25	3576	757	79	69	limit	0	11,938	-	62	limit	34	limit		93	limit
		0.50	4265	1100	74	0	28,044	0	11,226	60	71	limit	29	limit		90	limit
		0.75	3738	884	76	0	7,348	0	7,165	2	71	limit	0	29,863		78	limit
	6	0.25	3894	1186	70	140	limit	0	14,886	-	75	limit	32	limit		116	limit
		0.50	2450	730	70	0	27,707	0	13,424	52	101	limit	37	limit		103	limit
		0.75	3048	564	81	0	9,171	0	8,917	3	109	limit	45	limit		58	limit
	7	0.25	3128	699	78	129	limit	0	27,568	-	87	limit	39	limit		115	limit
		0.50	2532	691	73	143	limit	0	11,457	-	121	limit	60	limit		116	limit
		0.75	2558	561	78	0	10,555	0	10,196	3	136	limit	39	limit		108	limit
	8	0.25	2337	736	69	0	35,316	0	25,525	28	100	limit	45	limit		113	limit
		0.50	1510	610	60	0	29,539	0	20,678	30	135	limit	45	limit		109	limit
		0.75	1534	587	62	121	limit	0	13,362	-	126	limit	72	limit		86	limit
	9	0.25	2496	782	69	0	20,577	0	17,562	15	122	limit	56	limit		95	limit
		0.50	2098	612	71	0	21,628	0	20,728	4	153	limit	65	limit		105	limit
		0.75	1253	395	68	0	15,841	0	12,307	22	120	limit	75	limit		73	limit
	10	0.25	654	450	31	0	34,010	0	11,889	65	155	limit	67	limit		90	limit
		0.50	1161	849	27	0	22,104	0	12,548	43	154	limit	67	limit		96	limit
		0.75	716	319	55	0	12,613	0	11,623	8	0	33685	59	limit		77	limit
	11	0.25	931	906	3	0	26,058	0	9417	64	156	limit	62	limit		85	limit
		0.50	745	716	4	0	14,025	0	10133	28	135	limit	55	limit		91	limit
		0.75	769	559	27	0	19,418	0	13887	29	121	limit	65	limit		72	limit
	12	0.25	881	872	0	0	9,341	0	7072	24	95	limit	57	limit		76	limit
		0.50	697	623	11	0	9,584	0	9927	*	131	limit	38	limit		72	limit
		0.75	880	483	45	0	8,485	0	10707	*	128	limit	40	limit		63	limit
		Min	654	395	0	0	7,345	0	7,165	*						58	limit
		Avg.	1194	694	61	15	12,939	0	12,208	23						91	limit
		Max.	4,265	1,186	81	143	limit	0	36,277	-						116	limit

* - r-HCIP$_P$ is faster than r-HCIP$_{DD}$

7.3. Results for r-hub maximal covering interdiction problem

The results for 50, 100, and 200-node instances of r-hub maximal covering interdiction problem are presented in Table 4. Unlike the other two variants, where we were not able to solve most of the 200-node instances to optimality (for $p = 15$) using the dual formulation, and any of the instances using penalty formulation, results in the Table 4 show that the both the formulations solved all the 156 (i.e. 39 instances for each of $|N| = 50$, 100 and 200) instances to optimality.

Note that for all the fifteen 50-node 10-hub instances, the computation time of the penalty based reformulation is in the range from 27 to 63 seconds with an average of 43 seconds, whereas the range for the dual based reformulation the range is between 8 to 18 seconds with an average of 13 seconds. Similarly for the 50-node 15-hub instances, the computational times for penalty based reformulation varies between 92 to 376 seconds with an average around 138 seconds, while the corresponding figures for the dual based reformulation is from 66 to 91 seconds with an average of 85 seconds. For 50-node instances, the results clearly show that the dual based reformulation outperforms penalty based reformulation by being at least one and a half times faster. For 100-node, 10-hub instances, the computational times with penalty based reformulation is in the range between 266 to 525 seconds with an average of 357 seconds, while the similar figures for dual based reformulation is 66 to 292 with an average of 137 seconds, at least being 2.5 times faster. For 100-node instances with $p = 15$, we see a similar scenario. Here penalty based reformulation's computational time varies from 740 to 1,744 seconds, while for dual it is 224 to 492 seconds. The average computational time reduction is 68%.

For instances with $|N| = 150$ and $p = 10$, dual formulation outperforms penalty formulation in all but one instance. Here, computational time of penalty formulation ranges from 347 to 1,070 seconds, while for dual it varies from 155 to 482 seconds. For $p = 15$, dual formulation outperforms penalty formulation in all instances with an average computational time reduction of 78%. The computational time comparison shows that dual is at least 6 times faster than the penalty formulation on average. Finally, for large scale instances with $|N| = 200$ and $p = 10$ and 15, our computational results show that both the dual and penalty based reformulations could solve all 39 instances. For instances with $|N| = 200$ and $p = 10$, the computational time for dual based reformulation is between 360 to 1,544 seconds, while for penalty based reformulation the range is between 780 and 1,876 seconds. Here, dual based reformulation gives an average improvement of 45%. Similarly for instances with $N = 200$ and $p = 15$, we see that the computational times for dual based reformulation is between 654 to 1,502 seconds, while for penalty based reformulation the range is between 2,266 to 7,367 seconds. The average improvement of using dual based reformulation in this case is 71%.

Table 4: Results for the penalty and dual formulations of r-hub maximal covering interdiction problem.

Parameters			|N|=50			|N|=100			|N|=150			|N|=200		
			r-HMXCIP$_P$	r-HMXCIP_DD		r-HMXCIP_P	r-HMXCIP_DD		r-HMXCIP_P	r-HMXCIP_DD		r-HMXCIP_P	r-HMXCIP_DD	
p	r	δ	time(s)	time(s)	% red.	time(s)	time(s)	% red.	time(s)	time(s)	% red.	time(s)	time(s)	% red.
10	3	0.25	27	8	70	344	72	79	499	176	65	1,181	482	59
		0.50	31	11	65	284	66	77	437	165	62	1,118	606	46
		0.75	54	9	83	411	84	80	486	155	68	958	360	62
	4	0.25	48	9	81	318	80	75	626	252	60	1,847	586	68
		0.50	34	12	65	283	112	60	606	210	65	1,120	934	17
		0.75	59	12	80	363	120	67	581	238	59	1,081	675	38
	5	0.25	63	12	81	408	200	51	662	350	47	1,677	686	59
		0.50	37	15	59	525	130	75	755	244	67	1,876	513	73
		0.75	54	15	72	312	161	48	906	241	73	1,071	566	47
	6	0.25	32	15	53	353	151	57	1034	371	64	1,176	1,245	*
		0.50	36	14	61	375	145	61	823	346	58	971	731	25
		0.75	60	14	77	319	148	54	802	257	68	968	639	34
	7	0.25	37	17	54	474	175	63	1047	377	64	1,393	600	57
		0.50	34	16	53	322	292	9	347	482	*	780	781	0
		0.75	44	18	59	266	131	51	435	348	20	937	1,544	*
		Min.	27	8	53	266	66	9	347	155	20	780	360	0
		Avg.	43	24	67	357	137	60	670	281	56	1,210	1,007	45
		Max.	63	18	83	525	292	80	1047	482	73	1,876	1,544	73
15	5	0.25	92	20	78	1,102	246	78	6227	384	94	5,436	961	82
		0.50	132	21	84	1,102	227	79	2633	502	81	2,645	739	72
		0.75	84	20	76	740	232	69	3003	400	87	2,769	654	76
	6	0.25	104	25	76	1,270	224	82	4536	398	91	5,156	1,200	77
		0.50	139	22	84	1,121	228	80	3287	477	85	3,811	954	75
		0.75	116	25	78	1,181	303	74	2905	517	82	3,342	954	71
	7	0.25	116	29	75	1,713	339	80	4955	643	87	6,295	1,341	79
		0.50	156	27	83	1,296	311	76	3886	548	86	4,674	1,260	73
		0.75	108	26	76	1,049	343	67	3632	656	79	6,047	1,207	80
	8	0.25	134	33	75	1,744	374	79	7655	774	90	7,156	1,419	80
		0.50	160	30	81	1,181	369	69	3818	579	85	5,366	1,252	77
		0.75	123	31	75	1,200	376	69	3084	858	72	3,817	1,134	70
	9	0.25	121	37	69	1,722	402	77	5413	1,001	81	7,367	1,173	84
		0.50	161	31	81	1,681	375	78	6936	885	87	5,934	1,502	75
		0.75	133	33	75	1,204	321	73	4666	1,400	70	5,921	1,336	77
	10	0.25	116	36	69	1,649	358	78	5188	1,332	74	5,960	1,278	79
		0.50	170	38	78	897	325	64	1653	1,247	24	6,358	1,275	80
		0.75	121	32	74	1,214	311	74	4391	1,972	55	5,394	1,268	76
	11	0.25	124	32	74	1,282	368	71	2599	609	76	2,266	1,492	34
		0.50	159	35	78	770	327	58	2253	573	74	2,860	1,362	52
		0.75	132	39	70	984	356	64	3013	695	77	3,344	1,071	68
	12	0.25	122	46	62	859	328	62	15986	887	94	3,280	1,303	60
		0.50	376	34	91	862	321	63	2614	575	78	3,102	1,205	61
		0.75	125	42	66	953	331	65	1743	525	70	2,735	975	64
		Min.	92	20	66	740	224	62	1743	384	24	2,266	654	34
		Avg.	138	31	85	1199	320	68	4420	768	78	4,622	1,179	71
		Max.	376	46	91	1,744	402	82	15986	1,972	94	7,367	1,502	84

8. Conclusion

In this paper, we studied three variants of the hub interdiction problem on a multiple allocation, uncapacitated hub network, namely the r-hub median interdiction problem, r-hub center interdiction problem, and the r-hub maximal covering interdiction problem. The problems were formulated as bilevel MIPs and reduced to single-level MIPs using dual and penalty based reformulations. We further exploit the properties of the models to derive tighter reformulations. We also compare the linear programming relaxations of dual and penalty based reformulations to establish the dominance relationship between them. Our theoretical analysis show that the dual based reformulations dominate the penalty based reformulations. Our computational results on instances with up to 200 nodes and 15 hubs confirm the strength and efficiency of the proposed dual based reformulations over penalty based reformulations for all the three variants of the hub interdiction problem.

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