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A filter sequential adaptive cubic regularisation algorithm for nonlinear constrained optimization

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Abstract In this paper, we propose a filter sequential adaptive regularisation algorithm using cubics (ARC) for solving nonlinear equality constrained optimization. Similar to sequential quadratic programming methods, an ARC subproblem with linearized constraints is considered to obtain a trial step in each iteration. Composite step methods and reduced Hessian methods are employed to tackle the linearized constraints. As a result, a trial step is decomposed into the sum of a normal step and a tangential step which is computed by a standard ARC subproblem. Then, the new iteration is determined by filter methods and ARC framework. The global convergence of the algorithm is proved under some reasonable assumptions. Preliminary numerical experiments are reported.

Keywords nonlinear constrained optimization · cubic regularization · global convergence · filter methods · sequential quadratic programming

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1 Introduction

Nonlinear optimization algorithms are applied in many fields of science and engineering such as transportation analysis, chemical engineering, radar applications, structural engineering, modeling design, circuit design and so on [14].

Filter method was initially reported to ensure the global convergence of the algorithm which is used to solve nonlinear programming in [18]. This approach saves us from choosing an appropriate penalty parameter whose adjustment is difficult, and the numerical results with this technique is promising and encouraging, which makes this method is very popular. Many filter methods have been proposed based on line search or trust region framework. Fletcher et al. [17,33] and Wächter et al. [36] proposed filter trust-region algorithms for nonlinear programming. Filter line search methods [34,35] also attempt to solve this problem. Further, nonmonotone filter SQP methods were proposed in [2,19,32]. Pei et al.[29] presented a trust region algorithm combining line search filter technique.

Adaptive regularization algorithm using cubics (ARC) was proposed by Cartis et al. [10] for solving unconstrained optimization problem. There have been many (variations) methods [3,4,7,8,11,15,16,27,28,31,37] on using ARC for unstrained optimization, and some methods [6,20,22,23,25] are designed to solve or use ARC subproblems more efficiently. Meanwhile, some impressive algorithms based on ARC framework are also proposed to solve constrained optimization problem. Cartis et al. extended the ARC method to solve nonconvex optimization with convex constraints in [13], and presented a class of adaptive regularization methods and applied it to convex constrained optimization problem in [12]. Recently, Agarwal [1] developed a generalization of ARC methods for optimization on Riemannian manifolds. Lars Lubkoll et al. [26] proposed an affine covariant composite step method, designed for equality constrained optimization with partial differential equations. A cubic hybrid approach was constructed for inequality constrained problem in [5]. In particular, a few algorithms have been presented which may be used in [2,9].

In this paper, we study a generalization of ARC to solve equality constrained optimization by embedding filter technic in ARC framework. Inspired by sequential quadratic programming (SQP) methods to handle constrained optimization, we construct a filter sequential ARC algorithm. In each iteration, composite step approaches are employed to compute the trial step which is decomposed into the sum of the normal step and the tangential step. The normal step is used to reduce the constraint violation degree, and it is required to satisfy the linearization constraint. The tangential step is used to provide sufficient reduction of the model. It is computed by solving a standard ARC subproblem which is constructed via reduced Hessian methods. After the trial step is computed, an acceptance mechanism by using filter method is used to decide whether the step is accepted. Global convergence is proved under some suitable assumptions.

The remainder of this paper is organized as follows. In section 2, the computation of search direction is described and the filter sequential ARC algorithm for equality constrained optimization is presented. The global convergence to first-order critical point is presented in Section 3. Preliminary numerical results are reported in Section 4 and conclusion is offered in Section 5.

Notations. In this paper, $\|\cdot\|$ denotes the Euclidean norm. The inner product of vectors $x, y \in \mathbb{R}^n$ is denoted by $x^T y$.

2 Filter sequential ARC algorithm

In this section, we propose a filter sequential ARC algorithm to solve the following problem (1). First, we describe the process for computing the trial step. Then, we give the acceptance mechanism for the trial step. The whole algorithm is reported in the end of this section.

Consider the nonlinear constrained optimization problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) \tag{1a}$$

$$\text{subject to} \quad c(x) = 0, \tag{1b}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the equality constraints $c : \mathbb{R}^n \to \mathbb{R}^m$ are sufficiently smooth functions.

The Lagrangian function for this problem is

$$\mathcal{L}(x, \lambda) = f(x) - \lambda^T c(x).$$

Let $A(x)$ denote the Jacobian matrix of $c(x)$, namely,

$$A(x)^T = [\nabla c_1(x), \nabla c_2(x), \ldots, \nabla c_m(x)],$$

where $c_i(x)$ is the ith of the vector $c(x)$. The first-order conditions of the equality constrained problem (1) are that there is a point x^* and a vector of Lagrange multipliers λ^* such that

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \quad \text{and} \quad c(x^*) = 0.$$

To obtain search directions, we consider the following subproblem similar to SQP methods in iteration k,

$$\underset{d\in\mathbb{R}^n}{\text{minimize}} \quad m_k(d) \overset{\text{def}}{=} f_k + g_k^T d + \frac{1}{2} d^T H_k d + \frac{1}{3}\sigma_k \|d\|^3 \tag{2a}$$

$$\text{subject to} \quad A_k d + c_k = 0, \tag{2b}$$

where $f_k \overset{\text{def}}{=} f(x_k)$, $g_k \overset{\text{def}}{=} \nabla f(x_k)$, $A_k \overset{\text{def}}{=} A(x_k)$, $c_k \overset{\text{def}}{=} c(x_k)$, H_k denotes the Hessian $\nabla_{xx}\mathcal{L}(x_k, \lambda_k)$ or its approximation, $\sigma_k \in \mathbb{R}^+$ is an adaptive parameter in ARC.

Instead of solving subproblem (2) directly, we decompose the overall step via composite methods as follows.

$$d_k = n_k + t_k,$$

where n_k is called as a normal step which is used to satisfy feasibility condition, and t_k is called as a tangential step for ensuring sufficient decrease of the objective function's model, i.e.

$$A_k n_k + c_k = 0. \tag{3}$$

First, we can compute n_k by solving the following problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} n^T n && \text{(4a)} \\ \text{subject to} \quad & A_k n + c_k = 0. && \text{(4b)} \end{aligned}$$

Moreover, to ensure sufficient reduction in model function, we also require that the following condition

$$\|n_k\| \le \beta_1 \min \left\{ 1, \frac{\beta_2}{\sqrt{\sigma_k}^{\beta_3}} \right\} \frac{1}{\sqrt{\sigma_k}}, \tag{5}$$

where $\beta_1, \beta_2 > 0$ and $\beta_3 \in (0, 1)$.

However, problem (4) may not have a solution. So we distinguish two cases. Problem (2) is called compatible if n_k can be solved and satisfies (5). Otherwise, the algorithm goes to a feasibility restoration procedure, which is discussed in a while. First, we consider the case where problem (2) is compatible.

The normal step n_k has been obtained, next, we turn to the tangential step. The tangential step t_k is approximately expressed as

$$t_k = N_k u_k, \tag{6}$$

where u_k is the solution(or its approximation) of the following problem

$$\text{minimize} \quad f(x_k) + u^T g_k^N + \frac{1}{2} u^T H_k^N u + \frac{1}{3} \sigma_k \|u\|^3, \tag{7}$$

where

$$g_k^N \stackrel{\text{def}}{=} g^N(x_k) = N_k^T (g_k + H_k n_k) \quad \text{and} \quad H_k^N \stackrel{\text{def}}{=} H^N(x_k) = N_k^T H_k N_k \tag{8}$$

and the columns of N_k is an orthonormal basis for the null-space of A_k. The above problem is constructed by using reduced Hessian methods.

After computing u_k, from (6), we can rewrite

$$m_k^t(t_k) = f(x_k) + (g_k + H_k n_k)^T t_k + \frac{1}{2} (t_k)^T H_k t_k + \frac{1}{3} \sigma_k \|t_k\|^3. \tag{9}$$

Next, we discuss the issue of the acceptance of the trial point $x_k + d_k$.

Let the constraint violation

$$h(x) \stackrel{\text{def}}{=} \|c(x)\|.$$

In the following part, we use the same definition of a filter in [18], i.e. it is a set of pairs of the form (h_j, f_j), which satisfy $h_i < h_j$ or $f_i < f_j$. At the beginning, we can set $\mathcal{F}_0 = \varnothing$. More and more $(h(x), f(x))$ pairs would be added in the filter as the algorithm progresses.

We call that a trial point x is accepted by the filter $\mathcal{F}_k$ if and only if

$$h(x) \le (1-\gamma_h)h_j \quad \text{or} \quad f(x) \le f_j - \gamma_f h_j \tag{10}$$

holds for some $\gamma_h \in (0,1)$ and for all $(h_j, f_j) \in \mathcal{F}_k$, where $h_j \stackrel{\text{def}}{=} h(x_j)$.

The trial point $x_k + d_k$ is acceptable as x_{k+1} only if it is acceptable for $\mathcal{F}_k \bigcup (h_k, f_k)$.

Note that if an iterate x_k is accepted by $\mathcal{F}_k$, we add the pair (h_k, f_k) to the filter and remove from it every other pair $(h_j, f_j) \in \mathcal{F}_k$ such that both

$$h_j \ge h_k \quad \text{and} \quad f_j - \gamma_f h_j \ge f_k - \gamma_f h_k. \tag{11}$$

This measure means that a sequence $\{x_k\}$ is forced towards feasibility provided that $\{f(x_k)\}$ is monotonically decreasing and bounded below. However, this type of sequence $\{x_k\}$ could still be accepted even if it converges to a nonoptimal point. To avoid this from happening, we require that the model decrease satisfy the following condition

$$(m_k(0) - m_k(d_k))^\tau \left(\sqrt{\sigma_k}\right)^{\tau-1} > \kappa_h (h(x_k))^\phi, \tag{12}$$

where $\kappa_h > 0,\ \tau \ge 1$ and $\phi > 1$ are all fixed constants. If the condition (12) does not hold, x_k is added to the filter $\mathcal{F}_k$.

Next we consider the case where problem (2) is incompatible. In this case, we use the same strategy as in [18]. That is, the algorithm relies on the feasibility restoration procedure, whose purpose is to generate a new iterate $x_{k+1} = x_k + r_k$ which is acceptable for $\mathcal{F}_k$ and satisfies (5). We can do this by solving the following problem

$$\min_{x \in \mathbb{R}^n} h^2(x). \tag{13}$$

Many effective methods can be used to solve this problem and obtain r_k.

For convenience, we denote the set

$$\mathcal{A} = \{k \mid x_k \text{ is added to the filter}\}.$$

One finds that $\mathcal{F}_k \subsetneqq \mathcal{F}_{k+1} \iff k \in \mathcal{A}$. Let $\mathcal{A}_{\text{inc}}$ be the set of all indices of those iterations in which the feasibility of restoration procedure is invoked when problem (2) is not compatible.

A summary of the adaptive regularisation with cubic for equality constrained optimization algorithm framework is as follows.

Algorithm 1: Filter sequential ARC algorithm (FsARC).

Step 0. Initialization.

(i) Given starting point x_0, an initial $\sigma_0 > 0$ such that $\sigma_{\min} \leq \sigma_0$, an initial symmetric matrix H_0.

(ii) Set constants $0 < \eta_1 < \eta_2 < 1$, $1 < \gamma_1 \leq \gamma_2$, $\beta_1 \in (0, 1]$, $\beta_2 > 0$, β_3, β, κ_h, γ_h, $\gamma_f \in (0, 1)$, $\phi > 1$, $\tau \geq 1$.

(iii) Set $\mathcal{F}_0 = \varnothing$ and the iteration counter $k = 0$.

Step 1. Compute $f_k, g_k, h(x_k), A_k, N_k$.

Step 2. Stop if x_k is a stationary point of problem (1).

Step 3. If problem (2) is compatiable, ompute n_k by (4), compute t_k by (7) and set $d_k = n_k + t_k$. Compute ρ_k by

$$\rho_k = \frac{f(x_k) - f(x_k + d_k)}{m_k(0) - m_k(d_k)}. \tag{14}$$

Otherwise, add x_k to $\mathcal{F}_k$ and go to Step 9.

Step 4. If $x_k + d_k$ is not accepted by $\mathcal{F}_k$ and x_k, choose $\sigma_{k+1} \in [\gamma_1\sigma_k, \gamma_2\sigma_k]$, set $k = k + 1$, and go to Step 3.

Step 5. If $\rho_k < \eta_1$ and (12) hold, again set $x_{k+1} = x_k$, choose $\sigma_{k+1} \in [\gamma_1\sigma_k, \gamma_2\sigma_k]$, set $k = k + 1$, and go to Step 3.

Step 6. If (12) fails, add x_k to $\mathcal{F}_k$.

Step 7. Set $x_{k+1} = x_k + d_k$ and choose σ_{k+1} such that

$$\sigma_{k+1} \in \begin{cases} (\sigma_k, \gamma_1\sigma_k] & \text{if } \rho_k \in [\eta_1,\ \eta_2) \\ (0, \sigma_k] & \text{if } \rho_k \geq \eta_2. \end{cases}$$

Step 8. Compute H_{k+1}. Set $k = k + 1$ and go to Step 1.

Step 9. Feasibility restoration procedure.

9.1 Compute a new iterate point x_{k+1} by decreasing $h(x)$ such that x_{k+1} satisfies (10) and $(h(x_{k+1}), f(x_{k+1})) \notin \mathcal{F}_k$.

9.2 Determine σ_{k+1} and go to Step 8.

3 Global convergence

Assumptions G. Let $\{x_k\}$ be the sequence produced by Algorithm 1, where restoration iteration terminates successfully and the algorithm does not stop at a KKT point.

(G1) The iterations $\{x_k\} \subset X$, where X a closed, bounded domain $X \subset \mathbb{R}^n$.

(G2) $f(x)$ and $c(x)$ are differentiable on X, and $\nabla f(x)$ and $\nabla c(x)$ are Lipschitz-continuous over X.

(G3) There exists some constant $M_H > 0$, $\|H_k\| \leq M_H$ for all k.

(G4) H_k is semipositive definite on the null space of the Jocabian A_k^T for each k.

(G5) There exist constants $\delta_h, \kappa_n > 0$ satisfy that if $h(x_k) \leq \delta_h$,

$$k \notin \mathcal{A}_{\text{inc}} \quad \text{and} \quad \|n_k\| \leq \kappa_n h(x_k). \tag{15}$$

3.1 Preliminary results

The next lemma provides the reduction in f predicted from the subproblem (7).

Lemma 1 *Suppose that $k \notin \mathcal{A}_{\rm inc}$, and the step t_k^c is the Cauchy step for* (9). *Then*

$$f(x_k) - m_k^t(t_k) \ge \frac{\|g_k^N\|}{6\sqrt{2}} \min\left\{\frac{\|g_k^N\|}{1+\|H_k\|}, \frac{1}{2}\sqrt{\frac{\|g_k^N\|}{\sigma_k}}\right\} \tag{16}$$

for all $k \ge 0$.

Proof The Cauchy step t_k^c for (9) is

$$t_k^c = -\beta_k^c(g_k + H_k n_k) \ \text{ and } \ \beta_k^c = \arg\min_{\beta\in\mathbb{R}^+} m_k^t(-\beta(g_k + H_k n_k)). \tag{17}$$

Since $\beta \ge 0$, it follows from the Cauchy-Schwarz inequality and (17) that

$$\begin{aligned} & f(x_k) - m_k^t(t_k) \\ \ge\ & f(x_k) - m_k^t(-\beta(g_k + H_k n_k)) \\ =\ & \beta\|g_k + H_k n_k\|^2 - \frac{1}{2}\beta^2(g_k + H_k n_k)^T H_k (g_k + H_k n_k) - \frac{1}{3}\beta^3\sigma_k\|g_k + H_k n_k\|^3 \\ \overset{(8)}{\ge}\ & \beta\|g_k^N\|^2(1 - \frac{1}{2}\beta\|H_k\| - \frac{1}{3}\beta^2\sigma_k\|g_k^N\|). \end{aligned} \tag{18}$$

Denote

$$\begin{aligned} \hat\beta_k &= \frac{3}{2\sigma_k\|g_k^N\|}\left(-\frac{1}{2}\|H_k\| + \sqrt{\frac{1}{4}\|H_k\|^2 + \frac{4}{3}\sigma_k\|g_k^N\|}\right) \\ &= 2\left(\frac{1}{2}\|H_k\| + \sqrt{\frac{1}{4}\|H_k\|^2 + \frac{4}{3}\sigma_k\|g_k^N\|}\right)^{-1}. \end{aligned}$$

Then for $\beta \in [0, \hat\beta_k]$, when $1 - \frac{1}{2}\beta\|H_k\| - \frac{1}{3}\beta^2\sigma_k\|g_k^N\| \ge 0$ and $\beta \ge 0$, it follows that $f(x_k) \ge m_k^t(t_k)$.
Let

$$\theta_k \overset{\text{def}}{=} \frac{1}{\sqrt{2}\ \max\left\{1 + \|H_k\|, 2\sqrt{\sigma_k\|g_k^N\|}\right\}}. \tag{19}$$

By employing the inequalities

$$\begin{aligned} & \sqrt{\frac{1}{4}\|H_k\|^2 + \frac{4}{3}\sigma_k\|g_k^N\|} \\ \le\ & \frac{1}{2}\|H_k\| + \frac{2}{\sqrt{3}}\sqrt{\sigma_k\|g_k^N\|} \\ \le\ & 2\ \max\left\{\frac{1}{2}\|H_k\|, \frac{2}{\sqrt{3}}\sqrt{\sigma_k\|g_k^N\|}\right\} \\ \le\ & \sqrt{2}\ \max\left\{1 + \|H_k\|, 2\sqrt{\sigma_k\|g_k^N\|}\right\} \end{aligned}$$

and

$$\frac{1}{2}\|H_k\| \le \sqrt{2}\ \max\left\{1+\|H_k\|, 2\sqrt{\sigma_k\|g_k^N\|}\right\},$$

we can conclude that $0 < \theta_k \le \hat{\beta}_k$.

Then replace β in (18) with θ_k, we obtain that

$$\begin{aligned} & f(x_k) - m_k^t(t_k) \\ \ge & \frac{\|g_k^N\|^2(1-\frac{1}{2}\theta_k\|H_k\| - \frac{1}{3}\theta_k^2\sigma_k\|g_k^N\|)}{\sqrt{2}\ \max\left\{1+\|H_k\|, 2\sqrt{\sigma_k\|g_k^N\|}\right\}}. \end{aligned} \tag{20}$$

Combining the definition of θ_k in (19), it follows that $\theta_k\|H_k\| \le 1$ and $\theta_k^2\sigma_k\|g_k^N\| \le 1$. Hence, the numerator of (20) is bounded below by $\frac{1}{6}\|g_k^N\|^2$, which together with (20), implies that (16) holds.

The following result gives a critical bound on the tangential step.

Lemma 2 *Suppose that (G4) holds. Then the tangential step satisfies*

$$\|t_k\| \le \sqrt{3}\sqrt{\frac{\|g_k^N\|}{\sigma_k}}, \quad k \ge 0. \tag{21}$$

Proof Suppose that

$$\|t_k\| > \sqrt{3}\sqrt{\frac{\|g_k^N\|}{\sigma_k}} \tag{22}$$

for $k \ge 0$. Therefore, from (G4), we have

$$\begin{aligned} m_k^t(t_k) - f(x_k) &= (g_k + H_k n_k)^T t_k + \frac{1}{2}t_k^T H_k t_k + \frac{1}{3}\sigma_k\|t_k\|^3 \\ &\overset{(8)}{\ge} -\|t_k\|\|g_k^N\| + \frac{1}{3}\sigma_k\|t_k\|^3. \end{aligned}$$

Due to (22), $\frac{1}{3}\sigma_k\|t_k\|^3 - \|t_k\|\|g_k^N\| > 0$. Then $m_k^t(t_k) - f(x_k) > 0$, which contradicts (16). Hence the desired conclusion follows.

The following lemma estimates the model reduction, provided that the parameter σ_k is sufficiently large.

Lemma 3 *Suppose that (G1)-(G3) hold. Assume that* (5), (15) *and* (21) *hold, that* $k \notin \mathcal{A}_{\rm inc}$, *that*

$$\|g_k^N\| \ge \epsilon \tag{23}$$

for some $\epsilon > 0$*, and that*

$$\sigma_k \geq \max\left\{\left(\frac{\epsilon\sqrt{\epsilon}}{24\beta_1^2\beta_2^2\sqrt{6M}}\right)^{-\frac{1}{\beta_3}}, \left(\frac{\epsilon\sqrt{\epsilon}}{72\sqrt{2}\beta_1\beta_2 M}\right)^{\frac{2}{\beta_3}}, \left(\frac{2M_g}{\beta_1\beta_2 M_H}\right)^{-\frac{2}{1+\beta_3}}, \left(\frac{\epsilon\sqrt{\epsilon}}{144\sqrt{2}M_g\beta_1\beta_2}\right)^{-\frac{2}{\beta_3}}, \left(\frac{\epsilon\sqrt{\epsilon}}{24\sqrt{2}\beta_1^3\beta_2^3}\right)^{-\frac{2}{3\beta_3}}, \frac{4(1+M_H)^2}{9\epsilon}\right\} \stackrel{\text{def}}{=} \delta_1, \tag{24}$$

Then

$$m_k(0) - m_k(d_k) \geq \frac{\epsilon\sqrt{\epsilon}}{36\sqrt{2}}\sigma_k^{-\frac{1}{2}}.$$

Proof From the assumptions (G1)-(G3), (5) and (15), there exists $M > 0$ such that

$$\|g_k^N\| \leq M$$

for all k. Then, along with (21), it follows that

$$\begin{aligned} \|d_k\| &\leq \|n_k\| + \|t_k\| \\ &\leq \min\left\{\sqrt{3M} + \beta_1, \sqrt{3M} + \beta_1\beta_2\sigma_k^{-\frac{\beta_3}{2}}\right\}\sigma_k^{-\frac{1}{2}}. \end{aligned} \tag{25}$$

Therefore, combining the definition of m_k, (16) and (25), one finds that

$$\begin{aligned} &m_k(n_k) - m_k(d_k) \\ &= -(g_k + H_k n_k)^T t_k - \frac{1}{2}t_k^T H_k t_k + \frac{1}{3}\sigma_k(\|n_k\|^3 - \|d_k\|^3) \\ &\geq -(g_k + H_k n_k)^T t_k - \frac{1}{2}t_k^T H_k t_k - \frac{1}{3}\sigma_k\|t_k\|^3 - \frac{1}{3}\sigma_k(\|n_k\|^2\|t_k\| + \|n_k\|\|t_k\|^2) \\ &\stackrel{(16)}{\geq} \frac{1}{6\sqrt{2}}\|g_k^N\|\min\left\{\frac{\|g_k^N\|}{1+\|H_k\|}, \frac{1}{2}\sqrt{\frac{\|g_k^N\|}{\sigma_k}}\right\} - \frac{1}{3}\sigma_k(\|n_k\|^2\|t_k\| + \|n_k\|\|t_k\|^2) \\ &\stackrel{(24)}{\geq} \frac{1}{6\sqrt{2}}\epsilon\min\left\{\frac{\epsilon}{1+M_H}, \frac{\sqrt{\epsilon}}{2}\sigma_k^{-\frac{1}{2}}\right\} - \frac{\epsilon\sqrt{\epsilon}}{36\sqrt{2}}\sigma_k^{-\frac{1}{2}} \\ &\geq \frac{1}{6\sqrt{2}}\epsilon\min\left\{\frac{\epsilon}{2(1+M_H)}, \frac{\sqrt{\epsilon}}{3}\sigma_k^{-\frac{1}{2}}\right\}. \end{aligned} \tag{26}$$

Moreover, from (24), we can obtain that

$$\begin{aligned}
&|m_k(0) - m_k(n_k)| \\
&= |-g_k^T n_k - \frac{1}{2} n_k^T H_k n_k - \frac{1}{3}\sigma_k \|n_k\|^3| \\
&\leq M_g \|n_k\| + \frac{1}{2} M_H \|n_k\|^2 + \frac{1}{3}\sigma_k \|n_k\|^3 \\
&\leq 2M_g \beta_1 \beta_2 \sigma_k^{-\frac{1+\beta_3}{2}} + \frac{1}{3}\beta_1^3 \beta_2^3 \sigma_k^{-\frac{1+3\beta_3}{2}} \\
&\overset{(24)}{\leq} \frac{\epsilon\sqrt{\epsilon}}{36\sqrt{2}} \sigma_k^{-\frac{1}{2}}.
\end{aligned}$$

Combining this last inequality and (26), the claim is true.

Lemma 4 *Suppose that (G1) and (G5) hold and assume that*

$$h_k \leq \delta_h.$$

Then there exists some constant $\delta_c > 0$ *such that*

$$\delta_c h_k \leq \|n_k\|. \tag{27}$$

Proof Due to Cauchy-Schwarz inequality, calling upon (3) and (10), one finds that

$$h_k \leq \|A(x_k)\| \|n_k\| \tag{28}$$

for all k. Further, (G1) ensures that there exists a constant $\delta_c > 0$ such that

$$\max_{x \in X} \|A(x)\| \overset{\text{def}}{=} \frac{1}{\delta_c},$$

which together with (28) indicates that (27) holds. Therefore, the claim is true.

3.2 Feasibility

For the sake of brevity, the subset of successful steps is denoted as

$$\mathcal{S} \overset{\text{def}}{=} \{k \mid x_{k+1} = x_k + d_k\}.$$

Lemma 5 *Suppose that (G1)-(G3) and (G5) hold. Suppose also that* $|\mathcal{A}| < \infty$ *and that* (26) *holds for* $k \notin \mathcal{A}_{\text{inc}}$. *Then*

$$\lim_{k \to \infty} h_k = 0 \tag{29}$$

and n_k *satisfies* (27) *for all* $k \geq k_0$ *sufficiently large.*

Proof One finds that $|\mathcal{A}| < \infty$ means that there exists an integer $k_0 \in \mathbb{N}$ so that x_k is not acceptable for the filter for all $k \geq k_0$. Therefore, it is easy to get that $\rho_k \geq \eta_1$ holds from the mechanism of the algorithm. Consequently, it follows that

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq \eta_1(m_k(0) - m_k(d_k)) \\ &\geq \eta_1 \kappa_h^{\frac{1}{\tau}} (h(x_k))^{\frac{\phi}{\tau}} \sigma_k^{-\frac{\tau-1}{2\tau}} \\ &\geq 0. \end{aligned} \tag{30}$$

As a result, $\{f(x_k)\}$ is monotonically decreasing for $k \geq k_0$, It follows from (G1) and (G2) that $f_{\min} \leq f(x_k)$ and $0 \leq h(x_k) \leq h_{\max}$ for some constants $f_{\min}$ and $h_{\max} > 0$. Hence, we have that

$$\lim_{\substack{k \in \mathcal{S} \\ k \to \infty}} f(x_k) - f(x_{k+1}) = 0. \tag{31}$$

It follows from (30) that (29) holds. Moreover, $h_j = h_k$ holds if there exist unsuccessful iterations j which instantly follow the k, which also implies that (29) follows. Combining (15) and Lemma 4, the last desired conclusion follows.

The next lemma proves that there exists a subsequence such that $\{h_{k_i}\} \to 0$ if $|\mathcal{A}| = \infty$.

Lemma 6 *Suppose that (G1) and (G2) hold. Suppose also that $|\mathcal{A}| = \infty$. Then there exists a subsequence $\{k_i\} \subseteq \mathcal{A}$ such that*

$$\lim_{i \to \infty} h_{k_i} = 0 \tag{32}$$

The proof of this lemma follows in the same manner as the proof of Lemma 3.3 in [17].

Based on Lemma 5 and Lemma 6, we can show that the sequence $h(x_k) \to 0$ using the idea of Theorem 1 in [34].

Theorem 1 *Suppose that Assumptions G hold. Then*

$$\lim_{k \to \infty} h(x_k) = 0. \tag{33}$$

3.3 Optimality

Lemma 7 *Suppose that (G1) and (G2) hold. Suppose also that* (25) *holds, that $k \notin \mathcal{A}_{\text{inc}}$, and that n_k satisfies* (27). *Then*

$$h_k \leq \delta_{\bar{h}} \sigma_k^{-\frac{1+\beta_3}{2}} \tag{34}$$

and

$$h(x_k + d_k) \leq \delta_{\bar{h}} \sigma_k^{-1} \tag{35}$$

for some constant $\delta_{\bar{h}} \geq 0$.

Proof Due to $k \notin \mathcal{A}_{\mathrm{inc}}$, (5) together (27) give that

$$\delta_c h(x_k) \leq \|n_k\| \leq \beta_1\beta_2\sigma_k^{-\frac{1+\beta_3}{2}}, \tag{36}$$

which ensures that (34) holds. Then, from (G2) and the mean-value theorem, the constraint function $c(x)$ at $x_k + d_k$ reveals that

$$c(x_k + d_k) = c(x_k) + A_k d_k + \frac{1}{2} d_k^T \nabla_{xx} c(\xi_k) d_k,$$

where $\xi_k \in [x_k, x_k + d_k]$. Then, it follows from (25) that

$$\|d_k\| \ \leq (\beta_1 + \sqrt{3M})\sigma_k^{-\frac{1}{2}}. \tag{37}$$

Hence, it is clear from (G1) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \|c(x_k + d_k)\| \ &\leq \frac{1}{2} \max_{x \in X} \|\nabla_{xx} c(x)\| \|d_k\|^2 \\ &\leq \delta_{\bar{c}} (\beta_1 + \sqrt{3M})^2 \sigma_k^{-1}, \end{aligned}$$

where

$$\delta_{\bar{c}} \stackrel{\text{def}}{=} \frac{1}{2} \max_{x \in X} \|\nabla_{xx} c(x)\|$$

is a constant. Therefore, the conclusion follows with

$$\delta_{\bar{h}} = \max\left\{\delta_{\bar{c}}(\beta_1 + \sqrt{3M})^2, \beta_1\beta_2/\delta_c\right\}.$$

We continue our analysis by examining that iterations must be very successful when the parameter σ_k is sufficiently large.

Lemma 8 *Suppose that (G1)-(G3) hold. Suppose also that* (23) *and* (25) *hold, that $k \notin \mathcal{A}_{\mathrm{inc}}$, and that*

$$\sigma_k \geq \max\left\{\delta_1, \left(\frac{(1-\eta_2)\epsilon\sqrt{\epsilon}}{36\sqrt{2}C_f(\sqrt{3M} + \beta_1)^2}\right)^{-2}\right\} \stackrel{\text{def}}{=} \delta_2, \tag{38}$$

where $C_f = \frac{1}{2}\kappa_f + \frac{1}{2}M_H$ is a constant. Then

$$\rho_k \geq \eta_2.$$

Proof Combining (G1) and the mean value theorem on f, one finds that

$$f(x_k + d_k) = f(x_k) + g_k^T d_k + \frac{1}{2} d_k^T \nabla^2 f(\xi_k) d_k$$

for some $\xi_k \in [x_k, x_k + d_k]$.

From (G1) and (G2), we have that there exists a constant κ_f so that $\|\nabla^2 f(\xi_k)\| \le \kappa_f$, which together with (G3), yields that

$$\begin{aligned} f(x_k+d_k) - m_k(d_k) &= \frac{1}{2} d_k^T \nabla^2 f(\xi_k) d_k - \frac{1}{2} d_k^T H_k d_k - \frac{1}{3}\sigma_k \|d_k\|^3 \\ &\le \frac{1}{2} d_k^T \nabla^2 f(\xi_k) d_k + \frac{1}{2} d_k^T H_k d_k \\ &\le \frac{1}{2}\kappa_f \|d_k\|^2 + \frac{1}{2} M_H \|d_k\|^2 \\ &\le C_f \|d_k\|^2, \end{aligned} \tag{39}$$

where $C_f \stackrel{\text{def}}{=} \frac{1}{2}\kappa_f + \frac{1}{2}M_H$. Using the definition of ρ_k in (14), (38), (39) and Lemma 3, we deduce that

$$\begin{aligned} 1-\rho_k &= \frac{f(x_k+d_k) - m_k(d_k)}{m_k(0) - m_k(d_k)} \\ &\le \frac{36\sqrt{2} C_f (\sqrt{3M} + \beta_1)^2}{\epsilon\sqrt{\epsilon}} \sigma_k^{-\frac{1}{2}} \\ &\le 1 - \eta_2. \end{aligned}$$

Thus, the desired conclusion immediately follows.

Now, we prove that (12) will always holds when parameter σ_k is large enough.

Lemma 9 *Suppose that (G1)-(G3) hold. Suppose also that $k \notin \mathcal{A}_{\text{inc}}$,* (5)*,* (23) *and* (26) *hold, that n_k satisfies* (27)*, and that*

$$\sigma_k \ge \max\left\{ \delta_1, \left(\frac{(\epsilon\sqrt{\epsilon})^\tau}{(36\sqrt{2})^\tau \kappa_h \delta_h^\phi} \right)^{-\frac{2}{\phi(1+\beta_3)-1}} \right\} \stackrel{\text{def}}{=} \delta_3. \tag{40}$$

Then

$$(m_k(0) - m_k(d_k))^\tau (\sqrt{\sigma_k})^{\tau-1} \ge \kappa_h (h(x_k))^\phi.$$

Proof Using (40), Lemma 3 and Lemma 7, one finds that

$$\begin{aligned} (m_k(0) - m_k(d_k))^\tau (\sqrt{\sigma_k})^{\tau-1} &\ge \left(\frac{\epsilon\sqrt{\epsilon}}{36\sqrt{2}}\right)^\tau \sigma_k^{-\frac{1}{2}} \\ &\ge \kappa_h \delta_h^\phi \sigma_k^{-\frac{\phi(1+\beta_3)}{2}} \\ &\ge \kappa_h (h(x_k))^\phi. \end{aligned}$$

Therefore, the required result directly follows.

The following lemma ensures a sufficient reduction in f provided that the constraint violation is very small.

Lemma 10 *Suppose that (G1)-(G3) hold. Suppose also that* $k \notin \mathcal{A}_{\text{inc}}$*,* (23)*,* (26)*, and* (38) *hold, that* n_k *satisfies* (27)*, and that*

$$h_k \le \delta_{\bar{h}}^{-\frac{1}{\beta_3}} \left(\frac{\eta_2 \epsilon \sqrt{\epsilon}}{36\sqrt{2}\gamma_f} \right)^{\frac{1+\beta_3}{\beta_3}} \stackrel{\text{def}}{=} \delta_4. \tag{41}$$

Then

$$f(x_k) - f(x_k + d_k) \ge \gamma_f h(x_k).$$

Proof It follows from $k \notin \mathcal{A}_{\text{inc}}$, (23) and (38) that Lemmas 7-8 are applicable. Combining this result, (27) and (41), we deduce that

$$\begin{aligned} f(x_k) - f(x_k + d_k) &\ge \eta_2 (m_k(0) - m_k(d_k)) \\ &\ge \eta_2 \frac{\epsilon\sqrt{\epsilon}}{36\sqrt{2}} \sigma_k^{-\frac{1}{2}} \\ &\ge \eta_2 \frac{\epsilon\sqrt{\epsilon}}{36\sqrt{2}} \left(\frac{h(x_k)}{\delta_{\bar{h}}} \right)^{\frac{1}{1+\beta_3}} \\ &\ge \gamma_f h(x_k), \end{aligned}$$

which gives the desired result.

Next, we verify that if the σ_k is large and the constraint violation is small at x_k which is a nonoptimal point, problem (2) is always compatible.

Lemma 11 *Suppose that (G1)-(G3), and (G5) hold. Suppose also that* (23) *holds, that* (26) *holds for* $k \notin \mathcal{A}_{\text{inc}}$*, and that*

$$\sigma_k \ge \max \left\{ \gamma_2 \delta_2, \left(\frac{1}{\beta_2} \right)^{-\frac{2}{\beta_3}}, \left(\frac{(1-\gamma_h)\beta_1\beta_2}{\gamma_2 \kappa_n \delta_{\bar{h}}} \right)^{-\frac{2}{1-\beta_3}} \right\} \stackrel{\text{def}}{=} \delta_5. \tag{42}$$

Suppose furthermore that

$$h_k \le \min\{\delta_h, \delta_4\} \tag{43}$$

for all $k > 0$*. Then* $k \notin \mathcal{A}_{\text{inc}}$*.*

Proof Since $h_k \le \delta_h$, we deduce from (G5) and Lemma 4 that n_k satisfies (27). Moreover, it follows from $h_k \le \delta_h$ that (41) holds. Suppose for contradiction that $k \in \mathcal{A}_{\text{inc}}$. Using (5) and (42), we can conclude that

$$\|n_k\| > \beta_1 \beta_2 \sigma_k^{-\frac{1+\beta_3}{2}}. \tag{44}$$

Hence, the mechanism of the algorithm guarantees that $k - 1 \notin \mathcal{A}_{\text{inc}}$. To establish a contradiction, suppose that iteration $k-1$ is not successful. From (42) and $k-1 \notin \mathcal{A}_{\text{inc}}$, we can see that Lemmas 8 and 10 are applicable, which together with $h_k = h_{k-1}$, (15), and (41) yields

$$\rho_k \ge \eta_2 \quad \textit{and} \quad f(x_{k-1} + d_{k-1}) \le f(x_{k-1}) - \gamma_h h(x_{k-1}). \tag{45}$$

Meanwhile, note that x_{k-1} is accepted by the filter. If this iteration is unsuccessful, we can deduce that $x_{k-1} + d_{k-1}$ cannot be accepted by the filter and x_{k-1}. Combining this result and (45), we know

$$h(x_{k-1} + d_{k-1}) > (1 - \gamma_h)h(x_{k-1}) = (1 - \gamma_h)h(x_k).$$

However, Lemma 7 and the mechanism of the algorithm indicate that

$$(1 - \gamma_h)h(x_k) \le h(x_{k-1} + d_{k-1}) \le \delta_{\bar{h}}\sigma_{k-1}^{-1} \le \delta_{\bar{h}}\gamma_2\sigma_k^{-1}.$$

It follows from this last bound, (15) and (44) that

$$\beta_1\beta_2\sigma_k^{-\frac{1+\beta_3}{2}} < \|n_k\| \le \kappa_n h(x_k) \le \frac{\kappa_n\delta_{\bar{h}}\gamma_2}{1 - \gamma_h}\sigma_k^{-1}$$

and hence that

$$\sigma_k^{-\frac{1-\beta_3}{2}} > \frac{\beta_1\beta_2(1 - \gamma_h)}{\kappa_n\delta_{\bar{h}}\gamma_2}.$$

The last inequality is in contradiction with (42), which implies that the iteration $k-1$ is unsuccessful must fail. Consequently, we can see that the iteration $k-1$ is successful, which means that $h_k = h(x_{k-1} + d_{k-1})$. Using this result, (44), (15), and (35), one finds that

$$\beta_1\beta_2\sigma_k^{-\frac{1+\beta_3}{2}} < \|n_k\| \le \kappa_n h(x_k) \le \kappa_n\delta_{\bar{h}}\sigma_{k-1}^{-1} \le \kappa_n\delta_{\bar{h}}\gamma_2\sigma_k^{-1},$$

which contradicts (42) because $1 - \gamma_h < 1$. Thus, our assumption (44) does not hold and the lemma is proved.

Lemma 12 *Suppose that Assumptions G hold. Assume further that* $|\mathcal{A}| = \infty$ *and* (26) *holds for* $k \notin \mathcal{A}_{\text{inc}}$. *Then there exists a subsequence* $\{k_i\} \subseteq \mathcal{A}$ *so that*

$$\lim_{i\to\infty} \|g_{k_i}^N\| = 0. \tag{46}$$

Proof Let $\{k_i\}$ be any infinite subsequence of $\mathcal{A}$. To derive a contradiction, assume that there exists a subsequence $\{x_{k_i}\}$ such that

$$\|g_{k_i}^N\| \ge \epsilon_2 > 0 \tag{47}$$

for all i and some $\epsilon_2 > 0$. Suppose furthermore that

$$\sigma_{k_i}^{-\frac{1}{2}} \ge \epsilon_3 \tag{48}$$

for all $i \ge i_0$ and some $\epsilon_3 > 0$.

Using (15) and (32), we can get that

$$\lim_{i\to\infty} \|n_{k_i}\| = 0. \tag{49}$$

Hence, for sufficiently large i, (48) ensures that (5) follows, which shows that $k_i \notin \mathcal{A}_{\rm inc}$ for such i. Then, the proof of Lemma 3 gives that

$$|m_{k_i}(0) - m_{k_i}(n_{k_i})| \leq M_g\|n_{k_i}\| + \frac{1}{2}M_H\|n_{k_i}\|^2 + \frac{1}{3}\sigma_{k_i}\|n_{k_i}\|^3,$$

which together with (49) yields that

$$\lim_{i\to\infty}(m_{k_i}(0) - m_{k_i}(n_{k_i})) = 0. \tag{50}$$

It follows from (G3) and (26) that

$$m_{k_i}(0) - m_{k_i}(d_{k_i}) \geq \frac{\epsilon_2}{6\sqrt{2}}\min\left\{\frac{\epsilon_2}{2(1+M_H)}, \frac{1}{3}\sqrt{\epsilon_2}\epsilon_3\right\} \stackrel{\rm def}{=} \delta_6 > 0. \tag{51}$$

Then, we decompose the model decrease as

$$m_{k_i}(0) - m_{k_i}(d_{k_i}) = m_{k_i}(0) - m_{k_i}(n_{k_i}) + m_{k_i}(n_{k_i}) - m_{k_i}(d_{k_i}).$$

Substituting (50) and (51) into the above result, we have that

$$\liminf_{i\to\infty}(m_{k_i}(0) - m_{k_i}(d_{k_i})) \geq \delta_6 > 0. \tag{52}$$

Because x_{k_i} is acceptable for the filter, following the mechanism of the algorithm, we can conclude that either iteration $k_i \in \mathcal{A}_{\rm inc}$ or (12) does not hold. Due to $k_i \notin \mathcal{A}_{\rm inc}$, (12) dose not hold for i sufficiently large, namely,

$$(m_{k_i}(0) - m_{k_i}(d_{k_i}))^\tau(\sqrt{\sigma_k})^{\tau-1} < \kappa_h(h(x_k))^\phi. \tag{53}$$

Using this bound and (52), we immediately obtain that h_k is bounded away from zero for $i \to \infty$. But this is not possible because of (32). Hence (48) must fail and there exists a subsequence $\{k_l\} \subseteq \{k_i\}$ satisfies that

$$\lim_{l\to\infty}\frac{1}{\sqrt{\sigma_{k_l}}} = 0.$$

Next, we consider the set of indices $k_l > 0$ that are sufficiently large to guarantee that (40), (41) and (42) hold, and this is valid in view of (32). For these indices, Lemma 11 is applicable and we know that $k_l \notin \mathcal{A}_{\rm inc}$ for l sufficiently large. Thus, (53) follows for l sufficiently large. Nevertheless, Lemma 9 is also applicable, which contradicts (53), and therefore (47) must fail, giving the desired result.

The following analysis focuses on the finite subsequence, which means that $|\mathcal{A}_{\rm inc}|$ must be finite. In the following analysis, we denote $k_0 \geq 0$ as the last iteration such that x_{k_0-1} is acceptable for the filter.

Lemma 13 *Suppose that Assumptions G hold. Assume furthermore that* $|\mathcal{A}| < \infty$*, that* (26) *holds for* $k \notin \mathcal{A}_{\rm inc}$*, and that* (23) *holds for all* $k \geq k_0$*. Then*

$$\sigma_k \leq \sigma_{\max}$$

for all k *and some constant* $\sigma_{\max} > 0$*.*

Proof Due to Lemma 5, it is easy to find sufficiently large iteration $k_1 \geq k_0$ to guarantee that (43) holds. In addition, one finds that n_k satisfies (15) for all $k \geq k_1$. To derive a contradiction, we assume that iteration j is the first iteration following iteration k_1 such that

$$\sigma_j \geq \gamma_2 \max\left\{\frac{1}{\delta_2^2}, \frac{\delta_{\bar{h}}}{(1-\gamma_f)h^A}, \sigma_{k_1}\right\} \stackrel{\text{def}}{=} \gamma_2\delta_7, \tag{54}$$

where

$$h^A \stackrel{\text{def}}{=} \min_{i\in\mathcal{A}} h_i. \tag{55}$$

It follows from (54) that $\sigma_j \geq \gamma_2\sigma_{k_1}$, which guarantees that $j \geq k_1 + 1$. As a consequence, one finds that $j-1 \geq k_1$, and we can see that $j-1 \notin \mathcal{A}_{\text{inc}}$. (54) together with the mechanism of the algorithm yields that

$$\sigma_{j-1} \geq \frac{1}{\gamma_2}\sigma_j \geq \delta_7. \tag{56}$$

Following (54) and (56), it follows that (38) holds with k replaced by $j-1$. Hence, one finds that Lemma 8 indicates that

$$\rho_{j-1} \geq \eta_2. \tag{57}$$

Because n_{j-1} satisfies (15), Lemma 4 ensures that Lemma 7 is applicable. Combining this result, (54) and (56), we obtain that

$$h(x_{j-1}+d_{j-1}) \leq \delta_{\bar{h}}\sigma_{j-1}^{-1} \leq (1-\gamma_f)h^A. \tag{58}$$

Moreover, Lemma 10 is applicable because (54) and (56) guarantee that (38) follows and because (41) also follows for $j-1 \geq k_1$. Consequently, one finds that

$$f(x_{j-1}+d_{j-1}) \leq f(x_{j-1}) - \gamma_f h_{j-1},$$

which along with (58) guarantees that $x_{j-1}+d_{j-1}$ must be accepted by the filter and x_{j-1}. Using the above result, (57) and the mechanism of the algorithm, one finds that $\sigma_j \leq \sigma_{j-1}$, which is impossible because the iteration j is the first iteration following k_1 such that (54) follows. This indicates that $\sigma_k \leq \gamma_2\delta_7$ for all $k > k_1$.

If we define

$$\sigma_{\max} = \max\{\sigma_0, ..., \sigma_{k_1}, \gamma_2\delta_7\},$$

the desired conclusion follows.

Lemma 14 *Suppose that Assumptions G hold. Suppose furthermore that* $|\mathcal{A}| < \infty$ *and* (26) *holds for* $k \notin \mathcal{A}_{\text{inc}}$. *Then*

$$\liminf_{k\to\infty} \|g_k^N\| = 0. \tag{59}$$

Proof It follows from Lemma 5 that (15) holds for k sufficiently large. Meanwhile, from Lemma 5, we can get that (30) and (31) follow for any $k \in \mathcal{S}$, $k \geq k_0$. To establish a contradiction, suppose that (23) holds, and we see that

$$m_k(0) - m_k(d_k) = m_k(0) - m_k(n_k) + m_k(n_k) - m_k(d_k). \tag{60}$$

Furthermore, it follows from Lemma 3 that

$$|m_k(0) - m_k(n_k)| \leq M_g\|n_k\| + \frac{1}{2}M_H\|n_k\|^2 + \frac{1}{3}\sigma_k\|n_k\|^3,$$

which together with Lemma 5 and the second conclusion of (15) implies that

$$\lim_{k\to\infty}(m_k(0) - m_k(n_k)) = 0.$$

Using this limit, (30), (31), and (60), we have that

$$\lim_{\substack{k\to\infty \\ k\in\mathcal{S}}} (m_k(n_k) - m_k(d_k)) = 0. \tag{61}$$

But (G3), (26), (23), and Lemma 13 together indicate that

$$m_k(n_k) - m_k(d_k) \geq \frac{\epsilon}{6\sqrt{2}}\min\left\{\frac{\epsilon}{2(1+M_H)}, \frac{\sqrt{\epsilon}}{3}\sigma_{\max}^{-\frac{1}{2}}\right\} \tag{62}$$

for all $k \geq k_0$, which contradicts (61). Hence (23) cannot hold and the claim follows.

The following theorem gives the main theorem.

Theorem 2 *Suppose that Assumptions G hold. Suppose furthermore that* (26) *holds for* $k \notin \mathcal{A}_{\text{inc}}$. *Let* $\{x_k\}$ *be the sequence of iterates produced by the algorithm. Then there is a subsequence* $\{k_j\}$ *for which*

$$\lim_{j\to\infty} x_{k_j} = x_*$$

and x_* *is a first-order critical point for problem* (1).

Proof It follows from (G1), Theorem 1, Lemma 12, and Lemma 14 that

$$\lim_{j\to\infty} h_{k_j} = \lim_{j\to\infty}\|g_{k_j}^N\| = 0 \tag{63}$$

for some subsequence $\{k_j\}$, for any j, $k_j \notin \mathcal{A}_{\text{inc}}$.

We denote x_* as a limit point of the sequence $\{x_{k_j}\}$. It follows from (G1) that x_* is existent, and we assume that $\{k_l\} \subset \{k_j\}$ such that $\{x_{k_l}\}$ converges to x_*. Since $k_l \notin \mathcal{A}_{\text{inc}}$, for sufficiently large l, n_{k_l} satisfies (15) and converges to zero. Therefore, we deduce that $\lim_{j\to\infty} n_{k_j} = 0$. Furthermore, combining the definition of g_k^N in (8) and $\lim_{i\to\infty}\|g_{k_i}^N\| = 0$, one finds that

$$\lim_{j\to\infty}\|N_{k_j}^T g_{k_j}\| = 0.$$

It follows that x_* is first-order critical for problem (1), as announced.

4 Numerical Results

In this section, we present the numerical results of Algorithm 1 which have been performed on a desktop with Intel(R) Core(TM) i5-6200U CPU @ 2.30GHz 2.40GHz. Numerical testing was implemented as a MATLAB code and run under MATLAB version 9.4.0.813654 (R2018a).

In our implementation, the parameters: $\epsilon = 10^{-6}$, $\beta_1 = 0.1$, $\beta_2 = 100$, $\beta_3 = 0.01$, $\gamma_h = 10^{-5}$, $\kappa_h = 10^{-4}$, $\tau_2 = 2.01$, $\tau_1 = 2$, $\eta_1 = 0.01$, $\eta_2 = 0.9$. The computation terminates when $\|c(x_k)\| \leq \epsilon$ and $\|N_k^T g_k\| \leq \epsilon$ are both satisfied. The approach of updating H_k is implemented in the same way as that of Sections 4.4 and 4.5 in [24].

The numerical results are presented in Table (1) where the test problems are numbered in the same manner as in [21] and in [30]. For instance, HS49 is the problem 49 in [21] and S335 is the problem 335 in [30]. In these tables, NF and NC are the numbers of computation of the objective function and the numbers of computation of the constraint function.

Table 1: Numerical results of the Algorithm 1

Problem	Dimension n	Dimension m	NF,NC	$\|c(x_k)\|$	$\|N_k^T g_k\|$
AIRCRFTA	8	5	2	1.5932e-08	0
ARGTRIG	1000	1000	3	5.9215e-12	0
ARTIF	1002	1002	11	1.1690e-07	0
BDVALUE	1002	1000	2	6.4590e-08	0
BOOTH	2	2	1	0	0
BROYDN3D	1000	1000	11	2.4647e-07	0
BT1	2	1	230	1.7242e-11	3.3938e-07
BT2	3	1	17	4.0856e-14	1.2285e-08
BT3	5	3	2	0	1.6875e-11
BT4	3	2	7	3.1121e-09	6.2539e-09
BT5	3	2	4	2.0228e-10	4.1343e-09
BT6	5	2	12	4.5530e-07	2.8059e-07
BT7	5	3	7	9.9301e-16	1.2971e-12
BT8	5	2	1	5.2714e-08	1.0011e-09
BT9	4	2	18	7.1250e-13	5.2808e-08
BT10	2	2	0	5.6699e-07	0
BT11	5	3	10	1.6212e-12	7.8698e-09
BT12	5	3	7	5.7074e-11	5.5413e-07
BYRDSPHR	3	2	10	1.9179e-11	1.7688e-07
CLUSTER	2	2	0	7.0900e-08	0
GENHS28	10	8	5	3.0907e-16	1.9451e-09
GOTTFR	2	3	0	6.6266e-07	0
HATFLDF	3	3	2	1.5257e-09	0
HIMMELBA	2	2	1	0	0

Table 1 continued

Problem	Dimension		NF,NC	$\|c(x_k)\|$	$\|N_k^T g_k\|$
	n	m			
HIMMELBC	2	2	1	2.9738e-12	0
HIMMELBE	3	3	2	0	0
HS06	2	1	12	4.0238e-09	7.4575e-07
HS07	2	1	7	2.7397e-08	1.9318e-11
HS08	2	2	1	1.3802e-10	0
HS09	2	1	10	0	1.5432e-07
HS26	3	1	21	8.5907e-07	5.6719e-08
HS27	3	1	10	8.4069e-07	6.3436e-09
HS28	3	1	7	2.2204e-16	9.6102e-07
HS39	4	2	7	8.2445e-08	2.2212e-07
HS40	4	3	5	1.4497e-10	7.1395e-07
HS42	4	2	3	5.6563e-09	8.6474e-09
HS46	5	2	18	7.0778e-07	2.1696e-08
HS47	5	3	16	3.8059e-07	6.9730e-07
HS48	5	2	10	0	8.8392e-07
HS49	5	2	16	0	4.8668e-07
HS50	5	3	24	8.8818e-16	6.4451e-09
HS51	5	3	5	0	2.7372e-09
HS52	5	3	2	5.7220e-17	7.8894e-08
HS56	7	4	25	2.2147e-08	8.5192e-07
HS61	3	2	6	0	8.8818e-16
HS77	5	2	11	8.0086e-12	3.2480e-07
HS78	5	3	7	6.3686e-07	3.2077e-13
HS79	5	3	4	4.4723e-09	5.8091e-10
HYPCIR	2	2	1	4.1407e-12	0
MARATOS	2	1	5	2.9927e-12	1.0981e-11
MWRIGHT	5	3	7	5.3368e-14	1.9160e-13
POWELLBS	2	2	12	7.8350e-07	0
RECIPE	3	3	0	2.1935e-07	0
S216	2	1	7	5.1514e-13	5.9864e-09
S219	4	2	21	9.9854e-11	2.4031e-07
S235	3	1	13	5.3046e-13	3.3825e-12
S252	3	1	26	2.2768e-11	1.1438e-07
S254	3	2	12	1.0349e-10	3.2608e-08
S269	5	3	5	1.1102e-16	7.1813e-07
S316	2	1	1	1.3968e-11	1.0658e-14
S317	2	1	4	2.0060e-08	9.3806e-08
S318	2	1	5	3.5303e-11	7.7718e-10
S319	2	1	6	1.5499e-12	2.8066e-11
S320	2	1	7	3.9226e-10	4.0366e-08
S321	2	1	22	2.6645e-15	2.1227e-07
S322	2	1	6	1.1102e-16	3.5527e-13
S336	3	2	24	7.4727e-12	1.5978e-07

Table 1 continued

Problem	Dimension		NF,NC	$\|c(x_k)\|$	$\|N_k^T g_k\|$
	n	m			
S338	3	2	4	6.1065e-08	2.2151e-09
S344	3	1	8	1.4799e-09	7.2800e-07
S345	3	1	11	3.6108e-11	2.9972e-07
S373	9	6	16	2.3334e-11	1.3887e-07
S378	10	3	23	6.4950e-09	8.5723e-08
S394	20	12	20	1.3383e-09	4.3762e-10
S395	50	1	17	1.2510e-09	3.5049e-09
ZANGWIL3	3	3	0	5.4371e-07	0

5 Conclusion

We have introduced the adaptive regularization with cubic method in association with filter technique to solve nonlinear equality constrained programming, where composite-step method are introduced and the global convergence analysis is reported under some suitable assumptions. At the same time, numerical experiments verify its practical performance. However, the convergence study is not complete since local convergence properties are not discussed. To avoid Maratos effect and achieve fast convergence, we can introduce second-order corrections or other techniques in the algorithm, which is our next goal.

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Conflict of interest

The authors declare that they have no conflict of interest.

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