Exact solutions to a carsharing pricing and relocation problem under uncertainty

Giovanni Pantuso
Department of Mathematical Sciences, University of Copenhagen,
Universitetsparken 5, 2100, Copenhagen, Denmark
gp@math.ku.dk.

Abstract

In this article we study the problem of jointly deciding carsharing prices and vehicle relocations. We consider carsharing services operating in the context of multi-modal urban transportation systems. Pricing decisions take into account the availability of alternative transport modes, and customer preferences with respect to these. In order to account for the inherent uncertainty in customer preferences, the problem is formulated as a mixed-integer two-stage stochastic program with integer decision variables at both stages. We propose an exact solution method for the problem based on the integer L-Shaped method which exploits an efficient exact algorithm for the solution of the subproblems. Tests on artificial instances based on the city of Milan illustrate that the method can solve, or find good solutions to, moderately sized instances for which a commercial solver fails. Furthermore, our results suggest that, by adjusting prices between different zones of the city, the operator can attract significantly more demand than with a fixed pricing scheme and that such a pricing scheme, coupled with a sufficiently large fleet, significantly reduces the relevance of staff-based relocations. A number of issues, that remain to be addressed in future research, are pointed out in our conclusions.

1 Introduction

High dependency on private vehicles and low occupancy rates increase car usage and congestion in many cities of the world, contributing to pollution and poor urban air quality [10]. Improvements of public transport [37] and road pricing measures [9, 13] have, to a large extent, failed to provide sustainable solutions [10, 1, 30]. In this context, shared mobility, and particularly carsharing, has emerged as a viable alternative, linked to, e.g., a decrease in congestion [11], pollution [29], land used [39] and transport costs [14, 32].

In reaction to the high flexibility demanded by users, modern carsharing services are commonly designed for *on-demand*, *short-term*, *one-way* usage [19]. That is, users are allowed to rent a car without reservation and return it as soon as, and wherever (within the operating area), their journey is completed. The drop-off location/station is thus typically different from the pick-up location/station. Such configuration poses new planning challenges to *carsharing operators* (CSOs). On-demand rentals make the CSO unaware of when, where and for how long new rentals will occur. One-way rentals create frequent imbalances in the distribution of vehicles, that is an accumulation of vehicles in low-demand zones, and vehicle shortage in high-demand zones [3, 6] with levels of service dropping accordingly. A central task for a CSO is to provide a distribution of vehicles in the business area compatible with demand tides and oscillations [47, 48].

As a prime form of response to these challenges, CSOs initiate staff-based vehicle relocations between stations/zones of the city before shortages occur and customer satisfaction levels drop [21, 19]. That is, CSO's staff reach designated cars and drives them to different places. This gives rise to the so called *Vehicle Relocation Problem* (VReP), which consists of determining the relocations to perform in order to prepare for future demand. The research literature covers several problem settings, different levels of detail and granularity of decisions, as well as different mathematical approaches, see e.g., [7, 48, 6, 8, 22, 24, 23, 33, 3, 2, 15, 17]. These studies are thoroughly reviewed and classified in a recent survey, see [19].

The survey also reports that staff-based rebalancing could be complemented by manipulating demand through dynamic pricing. In fact, users of urban mobility services typically choose among different transport modes (e.g., metro, carsharing, bikesharing) that vary in a number of key attributes including price, see e.g., [52, 16, 36]. However, the authors of the survey comment that, at present times, this issue is solely "an important part of future research" [19].

Compared with relocation decisions, carsharing pricing strategies have received limited attention in the research literature though the number of available studies is growing. We can distinguish two main categories of pricing strategies, which we refer to as *individual* and *collective*, according to their end recipient. Individual pricing strategies are targeted to individual users. They require an interaction between the CSO and the individual user by means of which the final trip price, pick-up or drop-off location are agreed upon. As an example, the operator sends the individual user offers in the form of discounts or bonuses in exchange to a trip configuration which the operator deems beneficial for the entire system. Collective pricing strategies are, instead, targeted to the entire user base. They have the scope of influencing the cumulative rental demand by, e.g., decreasing the price of rentals to/from selected zones, but do not require an interaction with the individual user, nor their reply as to whether the price is accepted or not. The approach proposed in this article belongs to the latter category.

Several pricing strategies can be classified as *individual*. In [43] a method is developed that identifies vehicles placed in low demand zones using idle time as a proxy. The method then offers the user a drop-off location with low expected idle time in exchange for a discount (e.g., free minutes of usage). The method is evaluated in a simulation framework based on the city of Vancouver. The authors report that the average vehicle idle time is decreased by up to 16 percent. In [46] it is assumed that each customer is interested in a trip between a specific origin and destination and is sensitive to price. The operator then offers a price for the given trip in order to, ideally, incentivize/prevent favorable/unfavorable car movements. They model the carsharing system as a continuous-time Markov chain where a pricing policy is input to the model. In [12] a user-based relocation method is presented in which the users are offered to leave the car in a location different from the one planned in exchange for a fare discount. The authors formulate the decision problem as mixed-integer nonlinear programming problem, and model customers preferences with respect to the offer of alternative drop-off locations expressing the corresponding utility as a functions of the distance between the desired and offered drop-off locations. In [41] a predictive, user-based, relocation strategy is introduced for station-based carsharing services. They assume that the CSO can offer each returning customers an incentive to relocate. That is, upon the arrival of a customer, the CSO determines whether to offer an incentive, what the incentive should be and where the vehicle should be relocated. Therefore, an optimization problem is solved upon the arrival of every customer. Estimated customer preferences are used to model their reaction to incentive offers. Station-based services are considered also in [27] where users send real-time trip requests to the CSO, specifying their origin and destination. Upon receiving requests, the CSO assigns vehicles to users, plans staff-based relocations and determines incentives for customers, to whom a car has already been assigned, in exchange for a change of destination. The focus on [49] is instead on free-floating services. The system they consider is organized as follows. Each user sends a request for a vehicle either on-demand or as a reservation for a future rental. With the request they are required to specify intended pick-up and drop-off locations and departure time. The system elaborates the available requests and responds to users with proposed service options. Each service option includes a pick-up and drop-off location, pick-up time, and price. Finally, in [45] a pricing scheme to induce user-based relocations is introduced, with focus on station-based one-way services. The pricing scheme consists of adding or deducting a fixed expense to the original expense to adjust the users' preferential pick-up and drop-off location. To study the user behavior a questionnaire is carried out. The pricing problem is finally formulated as an optimization problem.

A number of articles have also focused on *collective* pricing strategies. A mixed-integer nonlinear programming model is provided in [51] for the joint problem of deciding fleet size, trip pricing and staff-based relocations for an electric station-based carsharing service. The authors consider demand elasticity with respect to prices using a logit-based function. The authors test the method on a case study based in Singapore.

In [44] a pricing scheme is developed with the scope of influencing user demand and keeping the distribution of vehicle at a given balance level. After modeling the relationship between carsharing price and demand, the authors develop a nonlinear optimization model to define pricing schemes which minimize the deviation from inventory upper and lower bounds at each charging station.

A station-based electric one-way service is considered in [50]. The CSO is to decide charging schedules and service prices. The authors assume that rental demand is influenced by prices and adopt a linear elasticity function to express demand as a function of price. Electric carsharing services are considered also in [38] where a dynamic pricing scheme is proposed with the scope of solving imbalances in the distribution of vehicles, as well as facilitating vehicle-grid-integration. For each origin and destination station the operator can influence demand using two price adjustment levels. Rental demand is connected to prices via a price elasticity to a reference demand for a default price. Pricing decisions are made using a mixed-integer nonlinear program. Finally, in [28] a bilevel nonlinear mathematical programming model is proposed to determine carsharing prices and staff-based relocations. In the upper level, the carsharing operator determines vehicles relocations and prices. In the lower level, travelers choose travel modes from a cost-minimization perspective and demand is computed using a logit model. The authors assume a one-way station-based carsharing system in competition with private cars.

The method presented in the present paper can be classified as a collective pricing strategy and extends the available literature in a number of ways. First, the available methods do not take into account the impact of alternative transport modes on transport choices. Carsharing services live in multi-modal transport systems and failure to model this heterogeneity may result into myopic models of customers behavior. As an example, compared to [28], in the present paper customers choices are not limited to private or shared cars. Rather, we assume that customers may choose among any number of available transport services (e.g., bicycle and bus). This, in turn, can help the CSO set prices from/to a given zone also as a function of the alternatives available in the zone. Second, the articles that include demand elasticity, typically limit their attention to elasticity with respect to prices, see e.g., [51, 28] and [41]. In this article we allow the operator to model customers preferences with respect to any number of both exogenous and endogenous characteristics of the service such as, but not limited to, travel time and waiting time. Such elasticity is modeled using utility functions which yield a linear optimization problem as long as the function is linear in the endogenous characteristics of the service e.g., price. Finally, extending all available methods, we explicitly account for uncertainty with respect to customer preferences. That is, we consider that a portion of the preferences of each customer is unknown to the operator and is, as such, handled by means of a stochastic program.

The contributions of this article can be stated as follows.

1. We propose a two-stage integer stochastic programming model for the joint pricing and relocation problem. The central idea is to influence demand by acting on prices, and performing preventive relocations accordingly, in order to maximize expected profits. To model the interplay between pricing decisions and customers choices we follow the recipe first provided by [5] for integrating demand models within mixed integer linear optimization models. This framework consists of modeling user preferences, and the uncertainty therein, by means of utility functions. A discretization of the unknown portion of the utility, and the adoption of utility functions which are linear in the decision variables of the model, ensure that the resulting optimization model is linear. This framework has also been used by e.g., [34] in the context of parking services and [16] in the context of carsharing, and a more general description is provided in [35].

2. To solve the resulting stochastic program with integer variables at both stages, we propose an exact L-Shaped method that exploits a compact reformulation and efficient exact algorithm for the integer subproblems.

3. We provide empirical evidences on the performance of the algorithm and on the solutions obtainable, based on artificial instances built on data from the city of Milan. An instance generator is made available online.

The remainder of this article is organized as follows. In Section 2 we define the problem and clarify modeling assumptions. In Section 3 we provide an extensive formulation of the problem which has the scope of explicitly defining the relationship between pricing decisions and customer choices. We introduce the model, particularly the second-stage, in an extensive and discursive manner in order to make the interplay between pricing decisions and customers demand explicit. In Section 4 we describe an integer L-Shaped method to find exact solutions to the problem. This is enabled by a compact and more tractable reformulation of the second-stage problem where customers choices are pre-processed, and by an exact greedy algorithm for solving the second-stage problem, both described in Section 4. In Section 5 we present a set of artificial instances based on the carsharing services offered in the Italian city of Milan. The same instances as well as an instance generator are made available online. In Section 6 we present the results of a computational study. We shed lights on the efficiency of the algorithm and comment on the solutions obtainable by means of the model introduced. In Section 7 we draw final conclusions, point out existing limitations of this work and discuss possible avenues of future research.

2 Problem definition and assumptions

A CSO offers one-way, reservation-free, carsharing services and is faced with the problem of jointly deciding the prices to charge and relocations to perform in order to comply with demand. The characteristics of the service, and the perimeter of the corresponding decision problem, are clarified by following assumptions.

A0-Target periods The operating hours are partitioned into a number of distinct *target periods*, that is, portions of the operating hours in which the CSO may, in general, apply different prices and distributions of the fleet, see Figure 1. Before each target period, and the CSO must decide i) the prices to apply during the target period and ii) the relocations to perform in sight of the uncertain rental demand during the target period. The CSO plans for each target period independently based on updated system information (e.g., fleet distribution

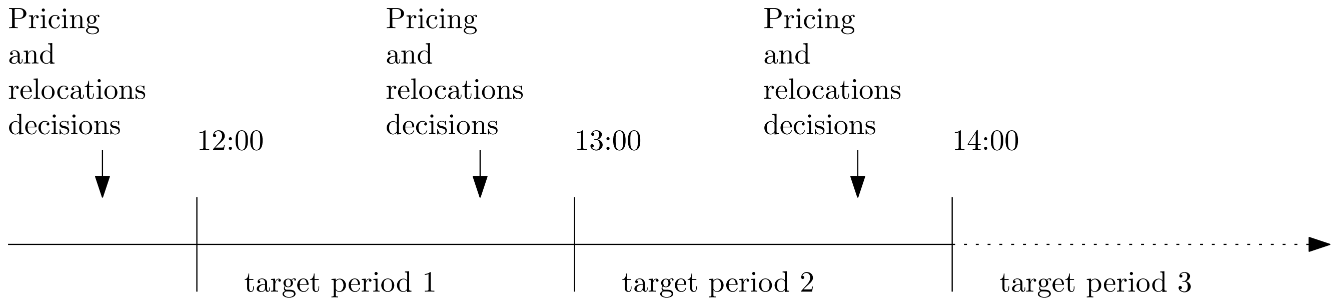


Figure 1: Target periods and decision timing. In this example target periods have a length of one hour and pricing and relocation decisions are made and implemented before the beginning of each the target period.

and individual vehicles' status) and demand outlook within the target period. In the example in Figure 1, before 12.00 the CSO must decide the prices to apply in target period 1 (12.00 – 13.00) and the relocations to perform before that. Thus, in this case the target period lasts for one hour. Similarly, one may consider longer target periods, e.g., morning hours and afternoon hours, as well as shorter ones, e.g., 30 minutes target periods, depending on how often it is sensible to adjust prices in the specific context.

A1-Business area The operating area is made of a finite set of locations, henceforth *zones*, see Figure 2. If the carsharing service is station-based a zone naturally represents a station. If the service is free-floating we assume that the business area is suitably partitioned into a number of zones, and each zone is represented by a suitable geographical location.

A2-Pricing scheme The price is made of a *per-minute fee* and a *drop-off fee*. The per-minute fee is valid throughout the day (i.e., in all target periods) and is independent of the origin and destination of the trip. Instead, the drop-off fee can be different in each target period and for each origin and destination. Figure 2 provides an example where the per-minute fee is Euro 0.2, independently of the origin and destination, while for each pair of zones a different drop-off fee is set. In the example, a drop-off fee of Euro 1.5 is charged if the car is picked up in zone z_1 and returned in zone z_3, while a drop-off fee of Euro -1 is charged if the car is picked up in zone z_3 and returned in zone z_2. Thus, we assume that the drop-off fee may also be negative to encourage desired movements of cars and increase demand. This setup generalizes the pricing schemes adopted in a number of carsharing services which typically charge a positive drop-off fee only if the customer returns the car in specific, unfavorable, zones of the city, or provide incentives, such as free driving minutes, to pick up a car in specific, unfavorable, zones. In order to keep the pricing scheme easy to communicate to customers, we assume the CSO must choose among only a finite set of possible drop-off fees. In the example of Figure 2 this set is Euro $\{-1, 0, 1, 1.5, 2\}$.

A3-Alternative transport services The business area offers a number of alternative transport services (e.g., public transport and bicycles) outside the control of the CSO. The alternative services may be different for each pair of zones. Each alternative service has unlimited capacity (i.e., each customer can choose any alternative service without decreasing their availability). In the example in Figure 2, carsharing is offered on all origin-destination pairs, while busses are not an available alternative for moving from zone z_3 to zone z_1, and riding a bicycle is not an option between z_2 and z_3 due to e.g., the absence of suitable bicycle lanes.

A4-Customers are informed The CSO is able to inform customers about the current price from their location to every other zone, prior to rentals. In the example of Figure 2, the CSO is able to inform a user in z_1 (e.g., on the mobile application used to locate the car) that, if the car is returned in z_2, there will be a drop-off fee of Euro 1.5, in addition to the per-minute fee. Customers are also aware about the availability of alternative transport services. Considering the example in Figure 2, a customer moving between z_1 and z_2 knows that they may use bus, bicycle, and carsharing. For all possible transport modes (including carsharing) the user knows the respective prices and characteristics (e.g., waiting time and travel time).

A5-Closed market A customer chooses exactly one transport service among the available ones. This corresponds to saying that a customer does not give up their trip. In the example of Figure 2, a customer moving from z_1 to z_2 will eventually choose to travel either by bicycle, carsharing or bus, and complete its journey.

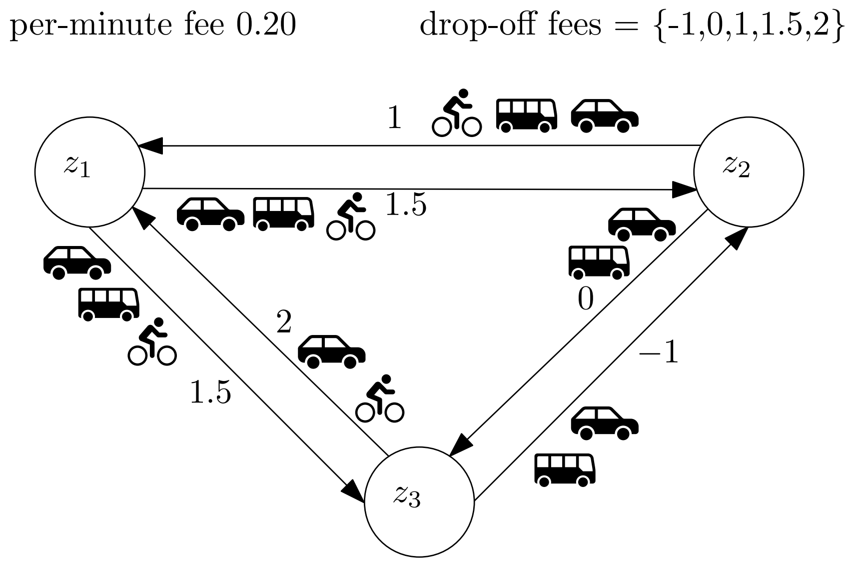


Figure 2: Zones, fees and alternative transport services. This example counts three zones, and three alternative transport services. Not all transport services are available between each pair of zones. Prices are expressed in Euro.

A6-Customers preferences The CSO is able to describe a portion of customers travel preferences as a function of different observable characteristics of the available transport services (e.g., travel time, price and waiting time). Nevertheless, the choice of each customer depends also on a number of additional elements not observable by the CSO. Therefore, customers preferences are partially unknown to the CSO. The unknown part of customers preferences is fully described by a probability distribution.

A7-Direct rentals Customers traveling with shared cars drive directly from their origin to their destination zone. This assumption is made for simplicity and is without loss of generality. Different travel patterns can be included simply by modeling customer-specific travel times in (4).

A8-Homogeneous fleet All shared vehicles are identical. This assumption is made for the sake of simplicity in the exposition of the reformulation of the second-stage problem and is without loss of generality. Throughout the text we will comment on the necessary modifications in case of a heterogeneous fleet.

A9-Profit maximization The CSO maximizes profits. While other objectives may be considered, such as maximizing demand served, or minimizing zonal deficit of cars, profits are the central objective of private carsharing operators.

A10-One-way trips For the sake of simplicity, we assume one-way trips. That is, customers move from their origin zone to a different zone. The model presented in Section 3 can however accommodate also round trips, provided a suitable specification of the parameters of the trip (e.g., duration).

Based on these assumptions, the problem can be briefly stated as follows. Given (a) a target period, (b) the cumulative mobility demand between each pair of zones in the target period, (c) usage and relocation costs, (d) the current distribution of cars, (e) a model of customers preferences including a probability distribution describing customer preferences unknown to the CSO, the CSO is to decide i) the drop-off fees to apply during the target period and ii) the relocations to perform in sight of the uncertain rental demand during the target period in order to maximize expected profits.

3 Mathematical model

Consider a urban area represented by a finite set $\mathcal{I}$ of zones (e.g., charging stations or a suitable partition of the business area) and a CSO offering a finite set of shared vehicles $\mathcal{V}$. Before the beginning of the target period, the CSO is to decide the drop-off fee between each pair of zones and the relocations to perform to better serve demand in the target period. At the time of planning, the fleet is geographically dispersed in the urban area as the result of previous rentals. Let decision variable z_{vi} be equal to 1 if vehicle v is made available for rental in (possibly relocated to) zone i in the target period, 0 otherwise. Let C^R_{vi} be the relocation cost born by the CS company to make vehicle v available in zone i. This cost is zero if the vehicle is initially in zone i and positive otherwise. Let $\mathcal{L}$ be a finite set of drop-off fees the CSO may apply. Let decision variable λ_{ijl} be equal to 1 if fee l is applied between zone i and and zone j, 0 otherwise. Finally, let $z := (z_{vi})_{i\in\mathcal{I},v\in\mathcal{V}}$ and $\lambda := (\lambda_{ijl})_{i,j\in\mathcal{I},l\in\mathcal{L}}$. The carsharing pricing and relocation problem is thus

$$\max -\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{I}} C^R_{vi} z_{vi} + Q(z,\lambda) \tag{1a}$$

$$\sum_{i\in\mathcal{I}} z_{vi} = 1 \qquad v\in\mathcal{V} \tag{1b}$$

$$\sum_{l\in\mathcal{L}} \lambda_{ijl} = 1 \qquad i,j\in\mathcal{I} \tag{1c}$$

$$z_{vi}\in\{0,1\} \qquad i\in\mathcal{I}, v\in\mathcal{V} \tag{1d}$$

$$\lambda_{ijl}\in\{0,1\} \qquad i,j\in\mathcal{I}, l\in\mathcal{L}. \tag{1e}$$

Constraints (1b) ensure that each vehicle is made available in exactly one zone. Constraints (1c) state that exactly one drop-off fee can be selected between each origin i and destination j. The objective function (1a) represents the expected profit obtained in the target period. The first term consists of the total relocation cost while the second term $Q(z,\lambda)$ represents the expected revenue from rentals as a result of pricing and relocation activities. The meaning of $Q(x,\lambda)$ will be made explicit by the end of this section as a result of the definition of the second-stage problem which we are now introducing.

Once relocation (z) and pricing (λ) decisions have been made, the CSO observes the consequent customers rentals. The business area offers a set $\mathcal{A}$ of alternative transport services, outside the control of the CSO, such as metro, busses and private bicycles. Each service has, in general, a different price and different characteristics. Let decision variable p_{vij} be the price of service $v\in\mathcal{V}\cup\mathcal{A}$ between zones i and j. The price of a carsharing ride between zones i and j is

$$p_{vij} = P^V T^{CS}_{ij} + \sum_{l\in\mathcal{L}} L_l \lambda_{ijl} \qquad \forall v\in\mathcal{V}, i,j\in\mathcal{I} \tag{2}$$

where parameter P^V is the carsharing per-minute fee, T^{CS}_{ij} the driving time between zones i and j and L_l the value of drop-off fee at level $l\in\mathcal{L}$ in some currency. Note that, in case of a heterogeneous fleet, it is simply necessary to make the per-minute fee and the driving time vehicle-dependent. Instead, the price of alternative services is entirely exogenous, that is

$$p_{vij} = P_{vij} \qquad \forall v\in\mathcal{A}, i,j\in\mathcal{I} \tag{3}$$

where parameter P_{vij} is the price of alternative service $v\in\mathcal{A}$ between i and $j\in\mathcal{I}$.

Let $\mathcal{K}$ be the set of customers, with $\mathcal{K}_i\subseteq\mathcal{K}$ being the set of customers traveling from zone $i\in\mathcal{I}$ and $\mathcal{K}_{ij}\subseteq\mathcal{K}_i$ the set of customers traveling from $i\in\mathcal{I}$ to $j\in\mathcal{I}$ in the target period.

Consider an individual customer k. The customer is faced with a choice among a finite number of alternative transport services that can bring them to their destination. Using the fairly standard assumption that customers maximize their utility, we can state that each service will provide the customer a different utility, and that the customer will choose the transport service that provides them the highest utility. This utility is known to the customer but not to the CSO.

Consider now the CSO. As we said, the CSO is not aware of the utility provided by the different services to each customer. Rather, the CSO is aware of a number of characteristics of the different services, primarily the price, p_{vij} and a some additional characteristics, say $\pi^1_{vij},\ldots,\pi^N_{vij}$ for service v between i and j (e.g., travel time and waiting time), as well as possibly some characteristics of the decision maker. Based on this, the CSO can specify a function that relates these, known, characteristics to the utility obtained by the customer. We denote this function as

$$F_k(p_{vij},\pi^1_{vij},\ldots,\pi^N_{vij})$$

However, there are additional elements that influence the utility that the CSO does not or cannot observe. For this reason, the utility is better represented by

$$F_k(p_{vij}, \pi^1_{vij}, \dots, \pi^N_{vij}) + \tilde{\xi}_{kv}$$

where $\tilde{\xi}_{kv}$ is a random variable that captures the difference between the utility that the CSO is able to model and the true utility observed by the customer. Different distributions for $\tilde{\xi}_{kv}$ will lead to different choice models. As an example, the popular Logit model is obtained when each $\tilde{\xi}_{kv}$ follows, independently, an identical extreme value distribution (Gumbel type I), and the Probit model is obtained when it follows a multivariate normal distribution. These, and additional choice models, as well as a discussion of their fundamental assumptions and limitations are discussed in, e.g., [42]. See also [4] for an exposition related to transport choices.

Let now u_{ijkv} be a decision variable which captures the utility obtained by customer $k \in \mathcal{K}$ when moving from i to $j \in \mathcal{I}$ using service $v \in \mathcal{V} \cup \mathcal{A}$. Given a realization ξ_{kv} of the random term $\tilde{\xi}_{kv}$ the utility is determined by

$$u_{ijkv} = F_k(p_{vij}, \pi^1_{vij}, \dots, \pi^N_{vij}) + \xi_{kv} \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, v \in \mathcal{V} \cup \mathcal{A} \tag{4}$$

Note that, since the observed characteristics of the different transport services, $\pi^1_{vij}, \dots, \pi^N_{vij}$, are given, constraints (4) are linear if $F_k(\cdot)$ is linear in p_{vij}, which is instead a decision variable.

Based on the utility provided by the different transport services, customers will make their choices. Let decision variable w_{ijkv} be equal to 1 if customer $k \in \mathcal{K}_{ij}$ chooses service $v \in \mathcal{V} \cup \mathcal{A}$, 0 otherwise. A customer will choose exactly one service (see Assumption A5 in Section 2)

$$\sum_{v \in \mathcal{V} \cup \mathcal{A}} w_{ijkv} = 1 \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij} \tag{5}$$

In order for a customer to choose a service, the service must be available. Let thus binary variable y_{ikv} be equal to 1 if service $v \in \mathcal{V} \cup \mathcal{A}$ is offered to customer $k \in \mathcal{K}_i$, 0 otherwise. Alternative services $v \in \mathcal{A}$ are always offered to customers whenever they are available at all, that is

$$y_{ikv} = Y_{vi} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}_i, v \in \mathcal{A} \tag{6}$$

where parameter Y_{vi} is equal to 1 if alternative service v is available in zone i, 0 otherwise. Conversely, a shared car $v \in \mathcal{V}$ may be offered to customers in zone i only if it is physically available at i, that is

$$y_{ikv} \leq z_{iv} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}_i, v \in \mathcal{V} \tag{7}$$

In addition, each car $v \in \mathcal{V}$ can be rented by only one customer. If more than one customers wish to use car v, the car is taken by the first customer arriving at the car. We assume that customers are indexed according to their arrival time at the car, i.e., customer k arrives before q if $k < q$. We impose that a vehicle is offered to a customer only if it is offered also to the customer arriving before them (who perhaps did not take it), that is:

$$y_{ikv} \leq y_{i(k-1)v} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}_i, v \in \mathcal{V} \tag{8}$$

A vehicle becomes unavailable for a customer if any customer has arrived before them and rented the car, that is:

$$z_{iv} - y_{ikv} = \sum_{j \in \mathcal{I}} \sum_{q \in \mathcal{K}_{ij} : q < k} w_{ijqv} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}_i, v \in \mathcal{V} \tag{9}$$

that is, if car v is in zone i ($z_{iv} = 1$), but it is not offered to customer k ($y_{ikv} = 0$) we obtain

$$1 = \sum_{j \in \mathcal{I}} \sum_{q \in \mathcal{K}_{ij} : q < k} w_{ijqv}$$

meaning that one customer has arrived before k and rented the car. On the other hand, if the car is offered to customer k, ($y_{ikv} = 1$), then it must be in zone i ($z_{iv} = 1$ – see (7)), and we obtain

$$0 = \sum_{j \in \mathcal{I}} \sum_{q \in \mathcal{K}_{ij} : q < k} w_{ijqv}$$

meaning that no customer arriving before k has taken the car. The same equality holds if the vehicle is not available at all ($z_{iv} = 0$ and $y_{ijkv} = 0$). Now that we have clarified how the availability of rental cars is regulated, we can state that a service can be chosen only if it is offered to the customer

$$w_{ijkv} \leq y_{ikv} \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, v \in \mathcal{V} \cup \mathcal{A} \tag{10}$$

Among the available services, the customer will chose the one yielding the highest utility. Therefore, for a given zone $i \in \mathcal{I}$, let decision variable ν_{ivwk} be equal to 1 if both services v and w in $\mathcal{V} \cup \mathcal{A}$ are available to customer $k \in \mathcal{K}_i$, 0 otherwise, and decision variable μ_{ijvwk} be equal to one if service $v \in \mathcal{V} \cup \mathcal{A}$ yields a greater utility than service $w \in \mathcal{V} \cup \mathcal{A}$ to customer $k \in \mathcal{K}_{ij}$ moving from i to j, 0 otherwise. The following constraints state that ν_{ivwk} is equal to one when both services v and w are available

$$y_{ikv} + y_{ikw} \leq 1 + \nu_{ivwk} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}_i, v, w \in \mathcal{V} \cup \mathcal{A}, \tag{11}$$

$$\nu_{ivwk} \leq y_{ikv} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}_i, v, w \in \mathcal{V} \cup \mathcal{A}, \tag{12}$$

$$\nu_{ivwk} \leq y_{ikw} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}_i, v, w \in \mathcal{V} \cup \mathcal{A}. \tag{13}$$

A service is chosen only if it yields the highest utility

$$w_{ijkv} \leq \mu_{ijvwk} \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, v, w \in \mathcal{V} \cup \mathcal{A} \tag{14}$$

that is, as soon as μ_{ijvwk} is set to 0 for some index w, w_{ijkv} is forced to take value 0 and service v is not chosen by customer k on i-j. The following constraints ensure that decision variable μ_{ijvwk} takes the correct value according to the utility

$$M_{ijk}\nu_{ivwk} - 2M_{ijk} \leq u_{ijkv} - u_{ijkw} - M_{ijk}\mu_{ijvwk} \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, v, w \in \mathcal{V} \cup \mathcal{A} \tag{15}$$

and

$$u_{ijkv} - u_{ijkw} - M_{ijk}\mu_{ijvwk} \leq (1 - \nu_{ivwk})M_{ijk} \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, v, w \in \mathcal{V} \cup \mathcal{A} \tag{16}$$

where constant M_{ijk} represents the greatest difference in utility between two services on $i - j$ for customer $k \in \mathcal{K}_{ij}$, that is $M_{ijk} \geq |u_{ijkv} - u_{ijkw}|, \forall v, w \in \mathcal{V} \cup \mathcal{A}$. Constraints (15)-(16) work as follows. When both services v and w are available ($\nu_{ivwk} = 1$) and $u_{ijkv} > u_{ijkw}$, (16) forces μ_{ijvwk} to take value 1, while (15) reduces to $0 \leq u_{ijkv} - u_{ijkw}$. When both service v and w are available and $u_{ijkv} < u_{ijkw}$, (15) forces μ_{ijvwk} to take value 0, while (16) reduces to $0 \geq u_{ijkv} - u_{ijkw}$. When one of the two services is not available ($\nu_{ivwk} = 0$), constraints (15)-(16) are satisfied irrespective of the value of μ_{ijvwk}. In case of ties ($u_{ijkv} = u_{ijkw}$) we impose

$$\mu_{ijvwk} + \mu_{ijwvk} \leq 1 \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, v, w \in \mathcal{V} \cup \mathcal{A} \tag{17}$$

A service can be preferred only if offered

$$\mu_{ijvwk} \leq y_{ikv} \qquad \forall i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, v, w \in \mathcal{V} \cup \mathcal{A} \tag{18}$$

Let decision variable α_{ijkvl} be equal to 1 if fare l is applied between i and j and customer k chooses shared car $v \in \mathcal{V}$, 0 otherwise. The following constraints ensure the relationship between λ_{ijl} and w_{ijkv} and α_{ijkvl}

$$\lambda_{ijl} + w_{ijkv} \leq 1 + \alpha_{ijkvl} \qquad \forall v \in \mathcal{V}, i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, l \in \mathcal{L} \tag{19}$$

$$\alpha_{ijkvl} \leq \lambda_{ijl} \qquad \forall v \in \mathcal{V}, i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, l \in \mathcal{L} \tag{20}$$

$$\alpha_{ijkvl} \leq w_{ijkv} \qquad \forall v \in \mathcal{V}, i, j \in \mathcal{I}, k \in \mathcal{K}_{ij}, l \in \mathcal{L} \tag{21}$$

That is, α_{ijkvl} is forced to take value 1 as soon as both λ_{ijl} and w_{ijkv} take value one, and value 0 as soon as either λ_{ijl} or w_{ijkv} take value 0.

Finally, for a given realization $\xi := (\xi_{kv})_{k \in \mathcal{K}, v \in \mathcal{V} \cup \mathcal{A}}$ of the random utility term $\tilde{\xi} := (\tilde{\xi}_{kv})_{k \in \mathcal{K}, v \in \mathcal{V} \cup \mathcal{A}}$ the second-stage profit can be formally expressed as

$$Q(z, \lambda, \xi) = \max \sum_{v \in \mathcal{V}} \sum_{(i,j) \in \mathcal{I} \times \mathcal{I}} \left(P^V T_{ij}^{CS} - C_{ij}^U\right) \sum_{k \in \mathcal{K}_{ij}} w_{ijkv} \tag{22a}$$

$$+\sum_{v\in\mathcal{V}}\sum_{(i,j)\in\mathcal{I}\times\mathcal{I}}\sum_{k\in\mathcal{K}_{ij}}\sum_{l\in\mathcal{L}} L_{ijl}\alpha_{ijkvl} \tag{22b}$$

$$\text{s.t.}\quad (2),(3),(4),(5),(6),(7),(8),(9), (10),(11),(12),(13),(14),(15),(16),(17),(18),(19),(20),(21) \tag{22c}$$

where C^U_{ij} is the cost born by the CSO when a vehicle is rented between i and j, (22a) represents the net revenue generated by the per-minute fee, and (22b) represents the income generated by the drop-off fee.

Thus, we can formally express the expected profit (i.e., the recourse function) as

$$\mathcal{Q}(z,\lambda) := \mathbb{E}_{\tilde{\xi}}\Big[Q(z,\lambda,\xi)\Big]$$

Problem (1) is a two-stage mixed-integer stochastic program with integer decision variables at both stages.

4 L-Shaped Method

We propose a multi-cut Integer L-Shaped method to find exact solutions to problem (1). The original, single-cut, version of the method was introduced by [26]. Assuming a set $\mathcal{S} = \{1,\ldots,S\}$ of scenarios (e.g., an iid sample) of $\tilde{\xi}$, each with probability π_s, the Master Problem (MP) can be formulated as

$$\max -\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{I}} C^R_{vi}z_{vi} + \sum_{s\in\mathcal{S}}\pi_s\phi_s \tag{23a}$$

$$\sum_{i\in\mathcal{I}} z_{vis} = 1 \qquad v\in\mathcal{V} \tag{23b}$$

$$\sum_{l\in\mathcal{L}} \lambda_{ijl} = 1 \qquad i\in\mathcal{I}, j\in\mathcal{J} \tag{23c}$$

$$z_{vi}\in\{0,1\} \qquad i\in\mathcal{I}, v\in\mathcal{V} \tag{23d}$$

$$\lambda_{ijl}\in\{0,1\} \qquad i\in\mathcal{I}, j\in\mathcal{I}, l\in\mathcal{L} \tag{23e}$$

$$\phi_s \text{ free} \qquad s\in\mathcal{S}. \tag{23f}$$

Let $\phi := (\phi_s)_{s\in\mathcal{S}}$. For each $s\in\mathcal{S}$, the second-stage problem $Q(z,\lambda,\xi_s)$ is solved as a subproblem. The L-Shaped method consists of solving MP in a Branch& Cut framework where optimality cuts are added at (integer) nodes of the tree. Observe that problem (1) has relatively complete recourse, that is, the second-stage problem $Q(z,\lambda,\xi_s)$ is feasible for every solution that satisfies the first-stage constraints. Consequently, the method requires only the definition of optimality cuts.

The practical viability of the method is enabled by a compact reformulation of $Q(z,\lambda,\xi_s)$, provided in Section 4.1, and an efficient exact algorithm for its solution, introduced in Section 4.2. We then provide the expression of optimality cuts and relaxation cuts in Section 4.3 and Section 4.4, respectively. The former are necessary to cut off solutions for which $\phi_s < Q(z,\lambda,\xi_s)$, the latter provide non-trivial lower bounds to $Q(z,\lambda,\xi_s)$ and are crucial for the efficiency of the algorithm. Final efficiency measures are provided in Section 4.5.

4.1 Compact formulation of the second-stage problem

For a given realization ξ of $\tilde{\xi}$, and first-stage decision (z,λ), the second-stage problem can be reformulated by preprocessing customer choices. The preprocessing phase ensures that the reformulation is linear regardless of whether the utility function adopted is linear in the price. Thus, any choice model can be used, without any restriction to utility function linear in the price.

We introduce the concept of a request. A request represents a customer who wishes to use carsharing for moving from its origin to its destination. For a given realization ξ, let the set $\mathcal{R}(\xi)$ be the set of requests. The set $\mathcal{R}(\xi)$ contains a request for each customer $k\in\mathcal{K}$ for which there exists at least one drop-off level $l\in\mathcal{L}$ such that the customer would prefer carsharing to alternative transport services, that is, for which $u_{ijkv} > u_{ijkw}$ with $v\in\mathcal{V}$ and $w\in\mathcal{A}$ for some choice of $l\in\mathcal{L}$ (note that all shared cars yield the same utility). Let $i(r)$, $j(r)$ and $k(r)$ be the origin, destination and customer of request r, respectively, and $l(r)$ the highest drop-off fee at which customer $k(r)$ would prefer carsharing to other services. Note that customer $k(r)$ would still prefer carsharing at any drop-off fee lower than $l(r)$ (under the reasonable assumption that the customer is sensitive to price). For each realization ξ of $\tilde{\xi}$ the set of requests can be populated in $\mathcal{O}\big(|\mathcal{K}| \times |\mathcal{L}| \times |\mathcal{A}|\big)$ operations as described in Algorithm 1.

Algorithm 1 Computation of $\mathcal{R}(\xi)$.

1: Input: ξ
2: $\mathcal{R}(\xi) \leftarrow \emptyset$
3: **for** customer $k \in \mathcal{K}$ **do**
4: $\quad i \leftarrow i(k)$, $j \leftarrow j(k)$ {$i(k)$ and $j(k)$ are the origin and destination of customer k.}
5: $\quad l^{MAX} \leftarrow -\infty$ {The highest drop-off fee at which customer k will choose carsharing.}
6: $\quad L_{l^{MAX}} \leftarrow -\infty$
7: $\quad$ **for** drop-off level $l \in \mathcal{L}$ **do**
8: $\quad\quad p^{CS} \leftarrow P^V T^{CS}_{ij} + L_l$ {Calculate the price of a carsharing ride.}
9: $\quad\quad U^{CS} \leftarrow F_k(P^{CS}, \pi^1_{vij}, \ldots, \pi^N_{vij}) + \xi_{kv}$ for some $v \in \mathcal{V}${Calculate the utility of carsharing.}
10: $\quad\quad$ **for** service $v \in \mathcal{A} : Y_{vi} = 1$ **do**
11: $\quad\quad\quad p^A_v \leftarrow P_{vij}$ {Calculate the price of a ride with alternative v.}
12: $\quad\quad\quad U^A_v \leftarrow F_k(P^A_v, \pi^1_{vij}, \ldots, \pi^N_{vij}) + \xi_{kv}$ {Calculate the utility of alternative v.}
13: $\quad\quad$ **end for**
14: $\quad\quad$ **if** $U^{CS} > \max_{v \in \mathcal{A}}\{U^A_v\}$ and $L_l > L_{l^{MAX}}$ **then**
15: $\quad\quad\quad l^{MAX} \leftarrow l$
16: $\quad\quad\quad L_{l^{MAX}} \leftarrow L_l$
17: $\quad\quad$ **end if**
18: $\quad$ **end for**
19: $\quad$ **if** $l^{MAX} > -\infty$ **then**
20: $\quad\quad r \leftarrow |\mathcal{R}(\xi)| + 1$ {In this case there exists a drop-off fee at which k prefers carsharing. Thus we create a request.}
21: $\quad\quad \mathcal{R}(\xi) \leftarrow \mathcal{R}(\xi) \cup \{r\}$
22: $\quad\quad i(r) \leftarrow i$
23: $\quad\quad j(r) \leftarrow j$
24: $\quad\quad k(r) \leftarrow k$
25: $\quad\quad l(r) \leftarrow l^{MAX}$
26: $\quad$ **end if**
27: **end for**
28: **return** $\mathcal{R}(\xi)$

Observe that, in case of a heterogeneous fleet, Algorithm 1 should be edited at lines $8-9$ and 14 to account for the fact that each vehicle gives, in general, a different utility.

Let then $R_{rl} = P^V T^{CS}_{i(r),j(r)} - C^U_{i(r),j(r)} + L_l$, for $l \leq l(r)$, be the net revenue generated if request r is satisfied at drop-off fee level l. Let $\mathcal{R}_r(\xi) = \{\rho \in \mathcal{R}(\xi) : i(\rho) = i(r), k(\rho) < k(r)\}$ be the set of requests which have a precedence over r. Let $\mathcal{R}_{ij}(\xi) = \{r \in \mathcal{R}(\xi) : i(r) = i, j(r) = j\}$. Let $\mathcal{L}_r(\xi) = \{l \in \mathcal{L} : L_l \leq L_{l(r)}\}$. Finally, let decision variable y_{vrl} be equal to 1 if request r is satisfied by vehicle v at level l, 0 otherwise. We can now reformulate the second-stage problem as follows.

$$Q(z, \lambda, \xi) = \max \sum_{r \in \mathcal{R}(\xi)} \sum_{v \in \mathcal{V}} \sum_{l \in \mathcal{L}_r(\xi)} R_{vrl} y_{vrl} \tag{24a}$$

$$\sum_{v \in \mathcal{V}} \sum_{l \in \mathcal{L}_r(\xi)} y_{vrl} \leq 1 \qquad r \in \mathcal{R}(\xi) \tag{24b}$$

$$\sum_{r \in \mathcal{R}(\xi)} \sum_{l \in \mathcal{L}_r(\xi)} y_{vrl} \leq 1 \qquad v \in \mathcal{V} \tag{24c}$$

$$\sum_{l \in \mathcal{L}_{r_1}(\xi)} y_{v,r_1,l} + \sum_{r_2 \in \mathcal{R}_{r_1}(\xi)} \sum_{l \in \mathcal{L}_{r_2}(\xi)} y_{v,r_2,l} \leq z_{v,i(r_1)} \qquad r_1 \in \mathcal{R}(\xi), v \in \mathcal{V} \tag{24d}$$

$$y_{v,r_1,l_1} + \sum_{r_2 \in \mathcal{R}_{r_1}(\xi)} \sum_{l_2 \in \mathcal{L}_{r_2}(\xi)} y_{v,r_2,l_2} + \sum_{v_1 \in \mathcal{V}: v_1 \neq v} y_{v_1,r_1,l_1}$$
$$\geq \lambda_{i(r_1),j(r_j),l_1} + z_{v,i(r_1)} - 1 \qquad r_1 \in \mathcal{R}(\xi), v \in \mathcal{V}, l_1 \in \mathcal{L}_{r_1}(\xi) \tag{24e}$$

$$\sum_{v \in \mathcal{V}} y_{vrl} \leq \lambda_{i(r),j(r),l} \qquad r \in \mathcal{R}(\xi), l \in \mathcal{L}_r(\xi) \tag{24f}$$

$$y_{vrl} \in \{0,1\} \qquad r \in \mathcal{R}(\xi), v \in \mathcal{V}, l \in \mathcal{L}_r(\xi) \tag{24g}$$

The objective function (24a) represents the net revenue obtained by the satisfaction customer requests. Constraints (24b) ensure that each request is satisfied at most once. Constraints (24c) ensure that each vehicle satisfies at most one request. Constraints (24d) state that a request can be satisfied by vehicle v only if the vehicle is in zone $i(r_1)$ and the vehicle has not been assigned to a customer with a lower index (that is, arriving at the vehicle before $k(r_1)$). Constraints (24e) state that a request r_1 at a certain level l_1 must be satisfied by vehicle v if level l_1 has been chosen ($\lambda_{i(r_1),j(r_1),l_1} = 1$) and the vehicle is available at $i(r_1)$ ($z_{v,i(r_1)} = 1$), unless the car has been used to satisfy the request of a customer with a higher priority (second term on the left-hand-side), or r_1 has been satisfied by another vehicle (third term on the left-hand-side). Constraints (24f) state that a request can be satisfied at level l only if level l is applied to all customers traveling between i and j. Note that z and λ are input data in problem (24).

4.2 Solution of the second-stage problem

Given a solution (z, λ) to MP and a scenario ξ_s, the optimal second stage profit $Q(z, \lambda, \xi_s)$ and solution can be computed by the greedy procedure sketched in Algorithm 2.

Algorithm 2 Greedy algorithm for computing $Q(z, \lambda, \xi_s)$ and its optimal solution.

INPUT: z, λ, $\mathcal{R}(\xi_s)$.
$\mathcal{V}_s^A \leftarrow \mathcal{V}$ {$\mathcal{V}_s^A$ is the set of available vehicles.}
$Y_{vrl} \leftarrow 0 \; \forall v \in \mathcal{V}, r \in \mathcal{R}(\xi_s), l \in \mathcal{L}_r(\xi_s)$.
$Q(z, \lambda, \xi_s) \leftarrow 0$
Sort requests $\mathcal{R}(\xi_s)$ in non-decreasing order of the customer index $k(r)$ {Remember that customers with a lower index have the precedence over customers with a higher index, see (9).}
for request $r \in \mathcal{R}(\xi_s)$ **do**
    $L_{i(r),j(r)} = \sum_{l \in \mathcal{L}} l \lambda_{i(r),j(r),l}$ {Identify the fee applied between $i(r)$ and $j(r)$.}
    if $L_{i(r),j(r)} \leq l(r)$ **then**
        for $v \in \mathcal{V}_s^A$ **do**
            if $z_{v,i(r)} = 1$ **then**
                $Y_{v,r,L_{i(r),j(r)}} \leftarrow 1$
                $\mathcal{V}_s^A \leftarrow \mathcal{V}_s^A \setminus \{v\}$ {Vehicle v becomes unavailable}
                $Q(z, \lambda, \xi_s) \leftarrow Q(z, \lambda, \xi_s) + R_{r,L_{i(r),j(r)}}$
            end if
        end for
    end if
end for
return $Q(z, \lambda, \xi_s)$ and $Y_{vrl} \forall v \in \mathcal{V}, r \in \mathcal{R}(\xi_s), l \in \mathcal{L}(\xi_s)$.

Algorithm 2 proceeds as follows. Given a solution (z, λ) to MP and the requests available in scenario ξ_s, the algorithm first initializes the solution Y_{vrl}, the objective value $Q(z, \lambda, \xi_s)$ and the set of available vehicles. Then it sorts the requests in non-decreasing order of the customer index $k(r)$. This is necessary to enforce that customers with a lower index have their request satisfied before customers with a higher index. The algorithm then iterates over the ordered requests. For each request it first checks whether the fee applied on its origin-destination pair, $L_{i(r),j(r)}$, is lower than then highest drop-off fee acceptable to the customer, $l(r)$. If this is the case, the algorithm looks for vehicles available at the origin of request r, $i(r)$. If one such vehicle v is found, the request is assigned to the vehicle at the current drop-off fee, i.e, $Y_{v,r,L_{i(r),j(r)}}$ is set to 1, vehicle v is made unavailable, and the revenue is increased by the revenue of request r, that is $R_{r,L_{i(r),j(r)}}$. The algorithm performs $\mathcal{O}\big(|\mathcal{R}(\xi_s)| \times |\mathcal{V}|\big)$ operations.

4.3 Optimality cuts

We are now concerned with finding a *valid set of optimality cuts*, that is a finite number of optimality cuts which enforce $\phi_s \leq Q(z, \lambda, \xi_s)$ for all $s \in \mathcal{S}$.

Assume an upper bound U_s on $\max_{z,\lambda} Q(z, \lambda, \xi_s)$ exists for all $s \in \mathcal{S}$. In the case of a homogeneous fleet a valid upper bound U_s on $\max_{z,\lambda} Q(z, \lambda, \xi_s)$ is

$$U_s = \sum_{r \in \mathcal{R}(\xi_s)} \max\{R_{r,l(r)}, 0\}$$

That is, the upper bound assumes that all, and only, the requests which generate a positive revenue are satisfied, and that these are satisfied at the highest drop-off fee $l(r)$. If the fleet is not homogeneous, we would have a vehicle-specific revenue R_{vrl} for satisfying request r at level l. In this case, a valid upper bound can be obtained by assuming all requests are satisfied by the vehicle which yields the highest non-negative revenue, i.e.,

$$U_s = \sum_{r \in \mathcal{R}(\xi_s)} \max\left\{ \max_{v \in \mathcal{V}} \{R_{v,r,l(r)}\}, 0 \right\}$$

Given a solution (z^t, λ^t) to MP (e.g., at a given node t of the Branch & Cut procedure), let $\mathcal{Z}_t^+ \subseteq \mathcal{V} \times \mathcal{I}$ and $\mathcal{Z}_t^- \subseteq \mathcal{V} \times \mathcal{I}$ be the set of tuples (v, i) for which $z_{vi}^t = 1$ and $z_{vi}^t = 0$, respectively. Similarly, let $\Lambda_t^+ \subseteq \mathcal{I} \times \mathcal{I} \times \mathcal{L}$ and $\Lambda_t^- \subseteq \mathcal{I} \times \mathcal{I} \times \mathcal{L}$ be the set of tuples (i, j, l) for which $\lambda_{ijl} = 1$ and $\lambda_{ijl} = 0$, respectively. Proposition 4.1 defines a valid set of optimality cuts.

Proposition 4.1. *Let (z^t, λ^t) be the t-th feasible solution to MP, and $Q(z, \lambda, \xi_s)$ its second-stage value for scenario s. The set of cuts*

$$\begin{aligned}\phi_s \leq &\Big(Q(z, \lambda, \xi_s) - U_s\Big)\Big(\sum_{(v,i) \in \mathcal{Z}_t^+} z_{vi} - \sum_{(v,i) \in \mathcal{Z}_t^-} z_{vi} + \sum_{(i,j,l) \in \Lambda_t^+} \lambda_{ijl} - \sum_{(i,j,l) \in \Lambda_t^-} \lambda_{ijl}\Big) \\ &+ U_s - \Big(Q(z, \lambda, \xi_s) - U_s\Big)\Big(|\mathcal{Z}_t^+| + |\Lambda_t^+| - 1\Big)\end{aligned} \tag{25}$$

defined for all (z^t, λ^t) feasible to MP is a valid set of optimality cuts.

Proof. It is sufficient to observe that, for $(z, \lambda) = (z^t, \lambda^t)$, we have

$$\Big(\sum_{(v,i) \in \mathcal{Z}_t^+} z_{vi} - \sum_{(v,i) \in \mathcal{Z}_t^-} z_{vi} + \sum_{(i,j,l) \in \Lambda_t^+} \lambda_{ijl} - \sum_{(i,j,l) \in \Lambda_t^-} \lambda_{ijl}\Big) = |\mathcal{Z}_t^+| + |\Lambda_t^+|$$

and optimality cut (25) reduces to

$$\phi_s \leq Q(z, \lambda, \xi_s), \qquad \forall s \in \mathcal{S}$$

On the other hand, if $(z, \lambda) \neq (z^t, \lambda^t)$ we get

$$\Big(\sum_{(v,i) \in \mathcal{Z}_t^+} z_{vi} - \sum_{(v,i) \in \mathcal{Z}_t^-} z_{vi} + \sum_{(i,j,l) \in \Lambda_t^+} \lambda_{ijl} - \sum_{(i,j,l) \in \Lambda_t^-} \lambda_{ijl}\Big) \leq |\mathcal{Z}_t^+| + |\Lambda_t^+| - 1$$

and, observing that $Q(z, \lambda, \xi_s) - U_s \leq 0$, the right-hand-side of the cut becomes greater than or equal to U_s.

Thus, since MP is a maximization problem, the set of cuts enforces $\phi_s \leq Q(z, \lambda, \xi_s)$ when $(z, \lambda) = (z^t, \lambda^t)$ and yields a valid upper bound on the remaining solutions. ■

4.4 Relaxation cuts

Ordinary Benders decomposition cuts can be derived by solving the LP relaxation of the subproblems $Q(z, \lambda, \xi_s)$. Relaxation cuts are not to be confused with optimality cuts as they are, in general, not tight at the point (z^t, λ^t) at which they are generated. However, relaxation cuts provide a, possibly, non-trivial upper bound on $Q(z, \lambda, \xi_s)$, that is an upper bound which might be lower than U_s for several (z, λ) solutions. A relaxation cut is obtained as follows

$$\phi_s \leq \sum_{r \in \mathcal{R}(\xi_s)} \pi_r^A + \sum_{v \in \mathcal{V}} \pi_v^B + \sum_{r \in \mathcal{R}(\xi_s)} \sum_{v \in \mathcal{V}} \pi_{rv}^C z_{v,i(r)} \tag{26a}$$

$$+ \sum_{r \in \mathcal{R}(\xi_s)} \sum_{v \in \mathcal{V}} \sum_{l \in \mathcal{L}_r(\xi_s)} \pi_{rvl}^D \Big(\lambda_{i(r),j(r),l} + z_{v,i(r)} - 1\Big) + \sum_{r \in \mathcal{R}(\xi_s)} \sum_{l \in \mathcal{L}_r(\xi_s)} \pi_{rl}^E \lambda_{i(r),j(r),l} \tag{26b}$$

where π_r^A, π_v^B, π_{rv}^C, π_{rvl}^D, π_{rl}^E are the values of the dual solution to $Q(z, \lambda, \xi_s)$ corresponding to constraints (24b), (24c), (24d), (24e) and (24f), respectively.

Figure 3: Municipality of Milan. Pins identify the ten locations of the city of Milan in the instances of [16]. The star identifies the Dome of Milan used as a reference point of the center of the city.

4.5 Other efficiency measures

MP exhibits symmetric solutions. If no vehicle is made available in a given zone i, all configurations of the drop-off fees between zone i and the remaining zones j are equivalent. In fact, no customer will be served on those (i, j) pairs. This problem can be solved by mean of the following constraints

$$\sum_{v \in \mathcal{V}} z_{vi} + \lambda_{ij1} \geq 1 \qquad \forall i, j \in \mathcal{I} \tag{27}$$

Constraints (27) enforce that, when no vehicle is available in zone i (i.e., $\sum_{v \in \mathcal{V}} z_{vi} = 0$) we arbitrarily chose drop-off fee number 1 (i.e., $\lambda_{ij1} = 1$). On the other hand, if $\sum_{v \in \mathcal{V}} z_{vi} \geq 1$ constraints (27) are satisfied regardless of the choice of a drop-off fee.

5 Test instances

In this section we present the test instances we used to run a computational study whose results are presented in Section 6. The test instances mimic carsharing services in the Italian city of Milan. The city hosts a number of carsharing companies and, according to the municipality of Milan [40], in 2018 there were a total of 3108 free-floating shared vehicles, with an average of 16851 daily rentals, and 149 station-based shared vehicles with an average of 108 daily rentals. We start by describing how the instances were constructed and finally we clarify the specific choices of control parameters for our tests. For the sake of replicability, an instance generator is made publicly available at the address https://github.com/GioPan/instancesPricingAndRepositioningProblem.

5.1 Zones and alternative transport services

We build upon, and expand, the instances used by [16]. The authors consider ten key locations in the business area of the city of Milan which we use as representatives of as many zones, thus setting $\mathcal{I} = \{1, \dots, 10\}$, see Figure 3. The authors consider as alternative transport services *public transport* (PT – consisting of a combination of busses, metro and superficial trains) and *bicycles* (B). Therefore we set $\mathcal{A} = \{PT, B\}$. For each pair of zones, the authors provide all the information necessary to calculate the utility as further explained in Section 5.2.

Table 1: Parameters and coefficients of the utility function.

Parameter	Meaning
T^{CS}_{vij}	Time spent riding a shared car between i and j when using service v. This quantity is strictly positive only when v is a carsharing service, otherwise it is 0.
T^{PT}_{vij}	Time spent in public transportation between i and j when using service v. This quantity is strictly positive only when v is PT, otherwise it is 0.
T^{B}_{vij}	Time spent riding a bicycle between i and j when using service v. This quantity is strictly positive only when v is B, otherwise it is 0.
T^{Walk}_{vij}	Walking time necessary when moving with transportation service v between i and j. This includes the walking time to the nearest service (e.g., shared car or bus stop), between connecting means (e.g., when switching between bus and metro to reach the final destination), and from to the final destination.
T^{Wait}_{vij}	Waiting time when using service v between i and j, and includes the waiting time for the service (e.g., bus or metro) as well as for connection.
$beta^P_k$	Price sensitivity of customer k.
β^{CS}_k	Time sensitivity of customer k when using a shared vehicle.
β^{PT}_k	Time sensitivity of customer k when using public transport.
β^{B}_k	Time sensitivity of customer k when riding a bicycle.
β^{Walk}_k	Time sensitivity of customer k when walking.
β^{Wait}_k	Time sensitivity of customer k when waiting.

5.2 Customers and utility functions

According to (4) each customer k is characterized by a known utility function and a random variable $\tilde{\xi}_{kv}$ which represents the portion of the preferences of the customer with respect to service v that the CSO cannot explain. We start by introducing the portion of the utility estimated by the CSO.

We adopt the utility function described in [31]. This function provides an estimate of the utility of each transport service as a function of price, travel time, walking time (e.g., to reach the service and from the service to destination) and waiting time. These represent the characteristics observable by the CSO.

For each customer $k \in \mathcal{K}$ traveling between i and j with transportation service v the utility function is

$$F_k(p_{vij}, T^{CS}_{vij}, T^{PT}_{vij}, T^{B}_{vij}, T^{Walk}_{vkij}, T^{Wait}_{vij}) = \beta^P_k p_{vij} + \beta^{CS}_k T^{CS}_{vij} + \beta^{PT}_k T^{PT}_{vij} + \tau(T^{B}_{vij})\beta^B_k T^B_{vij} + \tau(T^{Walk}_{vij})\beta^{Walk}_k T^{Walk}_{vij} + \beta^{Wait}_k T^{Wait}_{vij} \quad (28)$$

The meaning of each parameter and coefficient of function (28) is clarified in Table 1 and function $\tau : \mathbb{R} \to \mathbb{R}$, defined as $\tau(t) = \lceil \frac{t}{10} \rceil$, allows us to model the utility of cycling and walking as a piece-wise linear function: the utility of walking and cycling decreases faster as the walking and cycling time increases, see [31].

We use the β coefficients of the original utility function provided by [31] and marginally adapted to the carsharing context by [16] (e.g., price sensitivity has been adapted from Italian Lira to Euro). The values of the coefficients are the following: $\beta^{CS} = -1$, $\beta^{PT} = -2$, $\beta^{B} = -2.5$, $\beta^{Walk} = -3$ and $\beta^{Wait} = -6$. For the price sensitivity β^P the authors create two customer segments. They assign $\beta^P = -188.33$ if a customer belongs to the *lower-middle class* or $\beta^P = -70.63$ if a customer belongs to the *upper-middle class*. We randomly assigned customers (with equal probability) to the either upper-middle class ($\beta^P_k = -70.63$) or lower-middle class ($\beta^P_k = -188.33$). In more general cases, the parameters of utility functions can be estimated, provided the availability of data records on actual customers choices. The estimation procedure itself depends on several elements and underlying assumptions. As an example, a classical procedure to estimate the parameters of a Logit model is to maximize the log-likelihood function. Alternative methods include maximum simulated likelihood, simulated moments as well as Bayesian estimation. This topic is treated in detail in, e.g., [42].

For the time parameters (T-parameters) in Table 1 we use the values estimated by [16] on the actual transport services in the city of Milan in 2017. These values can also be found in the files accompanying the instance generator we make available online at https://github.com/GioPan/instancesPricingAndRepositioningProblem. It should be noted that, in more general applications, the T-parameters for the different services might change significantly during the day as a result of issues such as different traffic patterns, road congestion or time-varying public transport schedules. Therefore, the T-parameters should be understood as specific for the target period under consideration.

The price parameters are set as follows. The price of a bicycle ride is set to $P_{Bij} = 0$ for all (i, j) pairs, the price for public transport services is $P_{PT,ij} = 2$ (in Euro) for all (i, j) corresponding to the current price of an ordinary ticket valid for 90 minutes between each origin and destination within the municipality of Milan (price valid on November 2020). The per-minute fee of carsharing is set to $P_v^V = 0.265$ Euros per minute (the average of current per-minute fees offered by the CSOs in the city of Milan). The drop-off fees considered are $L_1 = -2$, $L_2 = -1$, $L_3 = 0$, $L_4 = 1$, $L_5 = 2$ Euro in the base case ($\mathcal{L} = \{1, 2, 3, 4, 5\}$). Further analysis on the drop-off fee will be described in Section 6.

5.3 Individual customers profiles

The utility function in Section 5.2 entails that all customers within a given class (upper- or lower-middle class) are characterized by identical preferences with respect to travel time, price, waiting time and walking time. Customers are told apart by their preferences with respect to unobserved features of the services, captured by $\tilde{\xi}_{kv}$.

However, the availability of large amounts of customer data may allow the CSO to profile customers at the individual level, i.e., to assign each customer an individual utility function. We are not aware of publicly available utility functions which are able distinguish between individual customers. Therefore, in order to assess the effect of individual customer profiles, at least on the the performance of the algorithm, we use an additional configuration in which an individual utility function for each customer is obtained by applying a random perturbation to the coefficients provided by [16]. Particularly, for each customer k, β_k^P will be uniformly drawn in $[-188.33, -70.63]$, where -188.33 is the β_k^P coefficient for lower-middle class customers and -70.63 is the β_k^P coefficient of upper-middle class customers in the general case, see Section 5.2. This allows us to obtain customers which can be anywhere between the upper- and lower-middle class.

The remaining β coefficients will be uniformly drawn in $[0.8\beta, 1.2\beta]$, where β is the value provided by [16]. As an example, for each k we will draw β_k^{PT} in $[-1.6, -2.4]$. The lower β_k^{PT} the less utility the customer will obtain for each minute spent in public transportation.

5.4 Uncertainty

The random term of the utility $\tilde{\xi}_{kv}$ is modeled as a Gumbel (Extreme Value type I) distribution with mean 0 and standard deviation σ. This corresponds to using a Logit choice model (see [42, 4]). The value of σ is set as the empirical standard deviation of $U_{ijkv} = F_k(p_{vij}, T_{vij}^{CS}, T_{vij}^{PT}, T_{vij}^{B}, T_{vij}^{Walk}, T_{vij}^{Wait})$ for all $i, j \in \mathcal{I}, v \in \mathcal{V} \cup \mathcal{A}, k \in \mathcal{K}_{ij}$. This entails that the expectation term in the objective function of (1) (i.e., $Q(z, \lambda)$) is a multidimensional integral that makes the solution of the problem prohibitive. For this reason we approximate $\tilde{\xi}_{kv}$ by iid samples drawn from its the underlying Gumbel distributions. The resulting discrete stochastic program goes under the name of *Sample Average Approximation* (SAA), see [25]. Its optimal objective value provides an unbiased estimator of the true objective value. The full model of the SAA is provided in Appendix A.

5.5 Position of customers and vehicles

We partition customers into sets $\mathcal{K}_i$ and then further into sets $\mathcal{K}_{ij}$ is such a way to test different configurations of demand, e.g., center to outskirt and vice-versa. Each one of the ten zones in our instances is characterized by a degree of "centrality". We use the walking distance from the *Dome of Milan* as a proxy of centrality, see Figure 3. Let d_i be the walking distance from zone $i \in \mathcal{I}$ to the Dome. Customers $\mathcal{K}$ are first randomly partitioned into disjoint subsets $\mathcal{K}_i$ with a probability π_i which depends on the centrality of the zone as follows

$$\pi_i = \frac{\gamma_i d_i}{\sum_{i \in \mathcal{I}} \gamma_i d_i} \tag{29}$$

where $\gamma_i = e^{-\alpha^{FROM} \Delta_i}$ with $\alpha^{FROM} \in [0, 1]$ and $\Delta_i = d_i - \sum_{i \in \mathcal{I}} d_i / |\mathcal{I}|$ is the deviation from the mean distance. In words, as α^{FROM} increases, the zones closer to the center (negative Δ_i) will receive a higher probability and the zones far from the center a lower probability, resulting in a higher concentration of customers in the central zones. Further, all customers assigned to a given zone i will be randomly assigned a destination zone j, and thus inserted into subset $\mathcal{K}_{ij}$, with a probability (29). This time $\gamma_j = e^{-\alpha^{TO} \Delta_i}$ with $\alpha^{TO} \in [0, 1]$. Again, as α^{TO} increases more customers will be directed to central zones. As an example, setting a low value of α^{FROM} and a high value of α^{TO} will create instances with higher demand from the outskirt to the center. The partitioning of customers is sketched in Algorithm 3.

Algorithm 3 Algorithm for the partition of customers into subsets $\mathcal{K}_i$ and $\mathcal{K}_{ij}$, $(i,j) \in \mathcal{I} \times \mathcal{I}$.

Input: $\mathcal{K}$, $\mathcal{I}$, d_i for $i \in \mathcal{I}$, $\alpha^{FROM} \in [0,1]$, $\alpha^{TO} \in [0,1]$
$\mathcal{K}_i = \mathcal{K}_{ij} \leftarrow \emptyset$ for $(i,j) \in \mathcal{I} \times \mathcal{I}$
for zone $i \in \mathcal{I}$ **do**
    Calculate $\Delta_i = d_i - \sum_{i \in \mathcal{I}} d_i / |\mathcal{I}|$
    Calculate $\gamma_i^{FROM} = e^{-\alpha^{FROM} \Delta_i}$ and $\gamma_i^{TO} = e^{-\alpha^{TO} \Delta_i}$
end for
for zone $i \in \mathcal{I}$ **do**
    Calculate $\pi_i^{FROM} = \frac{\gamma_i^{FROM} d_i}{\sum_{i \in \mathcal{I}} \gamma_i^{FROM} d_i}$
    Calculate $\pi_i^{TO} = \frac{\gamma_i^{TO} d_i}{\sum_{i \in \mathcal{I}} \gamma_i^{TO} d_i}$
end for
for Customer $k \in \mathcal{K}$ **do**
    Draw an origin zone i from $\mathcal{I}$ according to the probability distribution $(\pi_i)_{i \in \mathcal{I}}^{FROM}$
    $\mathcal{K}_i \leftarrow \mathcal{K}_i \cup \{k\}$
    Draw a destination zone j from $\mathcal{I}$ according to the probability distribution $(\pi_i)_{i \in \mathcal{I}}^{TO}$
    $\mathcal{K}_{ij} \leftarrow \mathcal{K}_{ij} \cup \{k\}$
end for
return $\mathcal{K}_i$ and $\mathcal{K}_{ij}$ for $(i,j) \in \mathcal{I} \times \mathcal{I}$

We assume the decision maker is a CSO with a fleet of $|\mathcal{V}|$ homogeneous vehicles. Each vehicle v is randomly assigned to an initial zone i according to probability (29), where $\gamma_i = e^{-\alpha^V \Delta_i}$ with $\alpha^V \in [0,1]$. Also in this case, as α^V increases more cars will be initially located in central zones.

5.6 Costs

We assume a fleet of Fiat 500 cars with classical combustion engine. The relocation cost C_{ij}^R, equal for all vehicles, is set as the cost of the fuel necessary for a ride between i and j, plus the per-minute salary of the driver multiplied by the driving time. The per-minute salary of the driver is set to 0.20 Euro/minute. It is calculated from the Italian national collective contract for logistics services valid at October 1st 2019 (available at https://www.lavoro-economia.it/ccnl/ccnl.aspx?c=328) as follows: the average per minute salary of the five lowest salary levels is increased by 30% to account for e.g., night shifts and holidays, yielding approximately 0.20 Euro/minute. Finally, the cost C_{ij}^U is set equal to the fuel necessary for a ride between i and j. The fuel consumption is calculated based on the specifics of a Fiat 500 petrol car and assuming an average speed of 50km/h and a fuel price of 1.60 Euro/liter. All the data data necessary to generate the instances described, as well as an instance generator implemented in Java, are made available at https://github.com/GioPan/instancesPricingAndRepositioningProblem.

5.7 Control parameters

The control parameters for the instances used in the tests are summarized in Table 2. For each control parameter we report the different values used in the tests. The control parameters were chosen in order to test different vehicles-to-customers ratios, ranging from 1/2 to 1/12, and different absolute values for the number of customers and vehicles. In addition, the different configurations of the parameters α^{FROM}, α^{TO} and α^V yield different configurations of demand (e.g., center to outskirt and outskirt to center) and of the carsharing system (e.g., vehicles located in the center and in the outskirt). The number of scenarios $|\mathcal{S}|$ is arbitrarily set equal to 10 on all instances. In Section 6.2 we provide insights on how the L-Shaped method scales with the number of scenarios. As explained in Section 5.4 scenarios represent i.i.d. samples of the underlying Gumbel distribution.

6 Results

This section is divided into three parts. First, we provide implementation details and setup of the experiments in Section 6.1. Following, in Section 6.2 we report on the performance of L-Shaped method especially in comparison with a commercial solver. Finally, in Section 6.3 we provide an analysis of the solutions and comment on their managerial implications.

Table 2: Control parameters used to generate the test instances.

Parameter	Meaning	Values
$\lvert\mathcal{V}\rvert$	the number of vehicles	50, 100, 200
$\lvert\mathcal{K}\rvert$	the number of customers	200,400,600
α^{FROM}	initial location of the customers	0.2, 0.8
α^{TO}	destination of the customers	0.2, 0.8
α^{V}	initial location of the vehicles	0.2, 0.8
Individual profiles	Whether each customer is profiled individually	Yes, No

6.1 Experiments setup

The L-Shaped method and the extensive form the SAAs (see Appendix A) were implemented in Java using the CPLEX 12.10 libraries. Particularly, the L-Shaped method was implemented by solving the master problem in a Branch & Cut framework and adding optimality cuts as lazy constraints at integer nodes. Unless otherwise specified, we used CPLEX's default parameters both when solving the extensive SAA and when using the L-Shaped method. This entails, e.g., a target relative optimality gap of 0.01%. The only exception, unless otherwise specified, is a time limit of 1800 seconds. In the L-Shaped method, relaxation cuts and optimality cuts were applied only at integer nodes throughout the entire tree. Particularly, relaxation cuts were crucial to the implementation. The performance of the algorithm without relaxation cuts was extremely poor. Tests were run on machines with 2×2.4 GHz `AMD Opteron 2431` 6 core CPU and 24Gb RAM. We remind the reader that, unless otherwise specified, the SAAs are solved with ten iid samples (scenarios) of random variable $\tilde{\xi}$, see Section 5.4. We stress that the number of scenarios does not represent the number of instances, as in a scenario-analysis procedure. Rather, by definition, the stochastic program takes into account all scenarios simultaneously. The impact of the number of scenarios on the computational complexity of the problems solved is assessed in Section 6.2.

6.2 Analysis of the L-Shaped Method

In the first part of the tests we compared the performance of the L-Shaped method to that of CPLEX for solving the SAA on all configurations of the control parameters in Table 2 for which $|\mathcal{K}| > |\mathcal{V}|$. The scope of our experiments is thus to obtain empirical evidences as to whether, and to what extent, the L-Shaped method scales better than using CPLEX without any decomposition strategy. The tables in this section report the optimality gap (`gap`) and elapsed time (`t` in seconds) for both CPLEX and the L-Shaped method. For the L-Shaped method they also report the optimality gap at the root node (`gapR`) and after 50% of the time limit – i.e., 50% of 1800 seconds – (`gap50`). All gaps are expressed as percentages and are calculated as $100 * |\texttt{bestbound} - \texttt{bestinteger}| / |\texttt{bestinteger}|$. The size of the SAA problem without decomposition is reported in Appendix B.

Table 3: Comparison of CPLEX and L-Shaped method on the instances with and $\alpha^{V} = 0.2$.

				CPLEX		L-Shaped			
$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{K}\rvert$	α^{FROM}	α^{TO}	`gap`	`t`	`gap`	`gapR`	`gap50`	`t`
50	200	0.2	0.2	0.0088	452.47	0.6967	18.6752	0.7466	1801.00
50	400	0.2	0.2	4.7820	1801.68	14.4808	74.9725	17.2825	1800.66
50	600	0.2	0.2	-	-	34.9493	89.2833	39.1895	1802.00
100	200	0.2	0.2	0.0084	553.92	0.0975	9.9177	0.0975	1801.38
100	400	0.2	0.2	-	-	2.5186	23.4470	5.3168	1801.07
100	600	0.2	0.2	-	-	14.1692	35.2426	14.8637	1821.75
200	400	0.2	0.2	-	-	0.0000	6.3509	-	156.78
200	600	0.2	0.2	-	-	0.0785	10.6491	0.7863	1806.39
50	200	0.2	0.8	0.0000	134.00	0.3381	17.4624	0.3381	1800.03
50	400	0.2	0.8	1.8164	1803.60	11.2732	64.1181	13.2796	1800.05
50	600	0.2	0.8	137.5269	1801.60	16.2842	91.0324	25.1233	1800.16
100	200	0.2	0.8	0.0000	346.80	0.2815	7.2972	0.2815	1800.55
100	400	0.2	0.8	-	-	0.6280	19.5696	0.8692	1800.42
100	600	0.2	0.8	-	-	11.1184	34.2304	12.2836	1804.76

200	400	0.2	0.8	-	-	0.0000	41.7344	-	119.53
200	600	0.2	0.8	-	-	0.3384	11.2541	0.3625	1820.99
50	200	0.8	0.2	0.1916	1800.45	1.9646	23.9553	2.2150	1800.28
50	400	0.8	0.2	10.3621	1806.04	19.1742	91.0136	25.5109	1800.03
50	600	0.8	0.2	-	-	50.0477	109.7915	52.8586	1800.02
100	200	0.8	0.2	0.0863	1803.15	0.5169	15.6382	0.5336	1801.16
100	400	0.8	0.2	-	-	6.5731	27.8935	9.8736	1801.15
100	600	0.8	0.2	-	-	24.8446	62.7364	24.9827	1812.75
200	400	0.8	0.2	-	-	0.2654	13.1386	0.4498	1810.83
200	600	0.8	0.2	-	-	5.1871	20.1720	5.7967	1821.70
50	200	0.8	0.8	0.1403	1800.78	0.7991	22.5586	0.8374	1800.33
50	400	0.8	0.8	3.1983	1803.68	15.2703	57.2480	20.6710	1802.52
50	600	0.8	0.8	-	-	46.6289	106.1663	48.8839	1804.99
100	200	0.8	0.8	0.2490	1801.28	0.7255	12.8107	0.7930	1801.43
100	400	0.8	0.8	-	-	3.4139	27.6540	5.4211	1803.38
100	600	0.8	0.8	-	-	24.8277	52.2813	26.1444	1800.02
200	400	0.8	0.8	-	-	0.0963	7.2607	0.0963	1801.18
200	600	0.8	0.8	-	-	6.9456	17.7706	7.3125	1846.31
				12.1823	885.47	9.8292	38.2289	12.1067	1701.43

We start by reporting the results on the default setup, that is, in which customers are not profiled at the individual level (i.e., customers have identical sensitivities to prices, driving time, waiting and walking time, see Section 5.2). Table 3 and Table 4 report the results on the instances with $\alpha^V = 0.2$ (more vehicles initially located in zones far from the center) and $\alpha^V = 0.8$ (more vehicles initially located in central zones), respectively. Each table reports on a total of 32 instances, one for each configuration of the control parameters. The results in Table 3 and Table 4 are rather similar. We observe that CPLEX is a viable alternative only for the smallest instances. As the number of vehicles grows CPLEX fails to deliver a feasible solution, and runs into memory problems. On the other hand, the L-Shaped method is able to provide a solution to all instances tested, and in many cases it provides a high quality solution, with a rather small optimality gap. We can also observe that, while the optimality gap at the root node is on average much higher than the final optimality gap, the gap after 50% of the allowed time is only a few percentage points higher. This illustrates, that the L-Shaped method may also deliver good solutions in a relatively short time (15 minutes).

Table 4: Comparison of CPLEX and L-Shaped method on the instances with $\alpha^V = 0.8$.

				CPLEX		L-Shaped			
$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{K}\rvert$	α^{FROM}	α^{TO}	`gap`	`t`	`gap`	`gapR`	`gap50`	`t`
50	200	0.2	0.2	0.0059	479.49	0.5028	24.4373	0.5411	1800.83
50	400	0.2	0.2	16.7040	1801.95	14.0036	48.9993	19.7977	1800.03
50	600	0.2	0.2	-	-	33.2558	81.8976	37.6588	1800.91
100	200	0.2	0.2	0.0091	1250.03	0.0904	15.8445	0.2180	1800.29
100	400	0.2	0.2	-	-	2.0635	22.9479	6.5261	1802.54
100	600	0.2	0.2	-	-	16.2161	37.5899	17.4882	1808.28
200	400	0.2	0.2	-	-	0.7891	12.0835	2.2191	1809.83
200	600	0.2	0.2	-	-	6.3879	20.9339	30.6798	1853.58
50	200	0.2	0.8	0.0099	246.36	0.6165	25.7909	0.9025	1800.87
50	400	0.2	0.8	2.0974	1804.45	11.7004	45.4173	15.7910	1802.50
50	600	0.2	0.8	-	-	31.6544	95.5335	36.7324	1801.59
100	200	0.2	0.8	0.0000	469.42	0.0291	13.9523	0.0291	1800.33
100	400	0.2	0.8	-	-	1.9427	25.7279	4.6547	1805.75
100	600	0.2	0.8	-	-	16.5597	34.3635	17.5988	1801.26
200	400	0.2	0.8	-	-	0.3674	11.6829	1.8109	1809.27
200	600	0.2	0.8	-	-	4.5425	19.4970	6.6028	1807.60
50	200	0.8	0.2	0.0000	76.22	0.0389	12.3461	0.0389	1800.51

50	400	0.8	0.2	4.1437	1804.33	15.1026	67.4814	21.7608	1803.37
50	600	0.8	0.2	-	-	44.8552	87.1632	48.1871	1803.69
100	200	0.8	0.2	0.0000	208.51	0.0000	9.4161	-	26.28
100	400	0.8	0.2	-	-	0.2335	12.5821	0.2726	1800.85
100	600	0.8	0.2	-	-	7.3760	25.0311	8.8577	1817.52
200	400	0.8	0.2	-	-	0.0020	8.6414	-	206.88
200	600	0.8	0.2	-	-	0.0021	11.9967	-	790.72
50	200	0.8	0.8	0.0083	78.21	0.0083	16.2539	-	34.46
50	400	0.8	0.8	34.2159	1802.63	9.6329	47.5271	12.1571	1802.80
50	600	0.8	0.8	-	-	37.0417	90.4674	40.3969	1800.35
100	200	0.8	0.8	0.0000	222.09	0.0000	9.2810	-	31.41
100	400	0.8	0.8	-	-	0.2077	19.7303	0.3026	1805.44
100	600	0.8	0.8	-	-	6.8777	27.1833	6.9310	1811.58
200	400	0.8	0.8	-	-	0.0000	42.1315	-	125.03
200	600	0.8	0.8	-	-	0.0294	8.4615	0.0306	1801.72
				4.7662	539.14	8.1916	32.2623	13.0072	1505.25

A sensible reduction of the optimality gap can be achieved by applying valid inequality (27). Such valid inequality has the effect of removing some symmetric solutions, as explained in Section 4.5. We observed that the addition of (27) decreased the average optimality gap from 9.82% to 9.24% on the instances with $\alpha^V = 0.2$ and from 8.19% to 7.10% on the instances with $\alpha^V = 0.8$. The addition of (27) resulted particularly beneficial on the instances which yielded the largest optimality gaps reported in Table 3 and Table 4. All the results on the performance of the L-Shaped method with the addition of (27) are reported in Appendix C.

We turn now our attention on the performance of the algorithm when each customer is profiled individually (see Section 5.3). The results in Table 5 and Table 6 are obtained with $\alpha^V = 0.2$ and 0.8, respectively. Valid inequality (27) is always added to the models.

Table 5: Comparison of CPLEX and L-Shaped method on the instances with $\alpha^V = 0.2$ and individual customer profiles.

				CPLEX		L-Shaped			
$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	gap	t	gap	gapR	gap50	t
50	200	0.2	0.2	0.0014	297.99	0.4747	21.5553	0.4747	1801.00
50	400	0.2	0.2	0.4151	1804.09	6.5981	59.3308	8.9622	1800.13
50	600	0.2	0.2	-	-	13.6751	52.5981	18.9363	1800.03
100	200	0.2	0.2	0.0000	221.91	0.0000	8.9488	-	16.47
100	400	0.2	0.2	-	-	0.4829	26.1941	0.8372	1801.67
100	600	0.2	0.2	-	-	9.2186	40.2920	9.2848	1805.22
200	400	0.2	0.2	-	-	0.0000	9.9094	-	229.96
200	600	0.2	0.2	-	-	0.0571	14.7626	0.0579	1801.39
50	200	0.2	0.8	0.0067	381.59	0.5159	20.9177	0.5159	1800.11
50	400	0.2	0.8	0.3913	1800.30	3.5957	43.2969	4.3436	1800.03
50	600	0.2	0.8	-	-	14.1098	63.4863	16.2944	1802.22
100	200	0.2	0.8	0.0000	199.66	0.1800	10.6250	0.1800	1800.13
100	400	0.2	0.8	-	-	0.1798	22.5452	0.3266	1800.02
100	600	0.2	0.8	-	-	9.1325	36.2728	9.8433	1808.65
200	400	0.2	0.8	-	-	0.0000	59.2478	-	165.06
200	600	0.2	0.8	-	-	0.1327	15.6153	0.4830	1800.02
50	200	0.8	0.2	0.0000	145.84	0.0632	22.1339	0.0632	1800.60
50	400	0.8	0.2	0.4584	1801.37	6.0223	57.8394	9.1647	1800.69
50	600	0.8	0.2	2.9868	1804.64	20.5953	74.1159	23.3852	1800.03
100	200	0.8	0.2	0.0000	374.57	0.1444	14.1475	0.1444	1800.12
100	400	0.8	0.2	0.2382	1806.73	0.9336	12.6012	1.4826	1800.09
100	600	0.8	0.2	-	-	14.7882	35.0665	14.8822	1815.08

200	400	0.8	0.2	-	-	0.1239	11.8485	0.1239	1800.50
200	600	0.8	0.2	-	-	4.6691	23.0727	5.0362	1827.23
50	200	0.8	0.8	0.0000	137.42	0.2137	21.9851	0.2137	1800.11
50	400	0.8	0.8	0.2161	1801.02	4.0149	53.8589	5.2497	1801.68
50	600	0.8	0.8	-	-	14.4585	70.6343	20.4297	1802.88
100	200	0.8	0.8	0.0000	367.32	0.0327	14.5185	0.0327	1800.03
100	400	0.8	0.8	-	-	2.0523	30.1037	3.4881	1800.07
100	600	0.8	0.8	-	-	10.3710	34.8283	17.2050	1814.54
200	400	0.8	0.8	-	-	0.0001	11.5481	-	170.66
200	600	0.8	0.8	-	-	1.7683	15.4424	2.2644	1807.74
				0.3367	867.80	3.8517	32.0441	6.0126	1498.71

The results illustrated in Table 5 and Table 6 are similar to those observed earlier in Table 3 and Table 4. CPLEX remains a viable option only for the smallest instances, while the L-Shaped method is able to find a solution, in some cases a high quality one, to all instances and to solve a number of them. We observe also that the average optimality gap appears sensibly lower compared to the results in Tables 3 and 4. Also in this case high quality solutions can be obtained already after 15 minutes.

A pattern in the optimality gaps reported in Tables 3 to 6 can be observed. The optimality gap appears inversely correlated with the vehicles-to-customers ratio. That is, the instances which yield the largest optimality gaps (50 vehicles and 400 customers, 50 vehicles and 600 customers, 100 vehicles and 600 customers) are those with the smallest vehicles-to-customers ratios among the instances tested (1/8, 1/12, 1/6, respectively). Supposedly, when vehicles are scarce compared to the number of customers, it becomes more challenging for the algorithm to identify, within the 30 minutes provided, a relocation and pricing plan which is able to satisfy demand in such a way to yield the highest profit. On the contrary, as the ratio increases, the model has more freedom to satisfy customers demand, and especially those requests generating the highest revenue.

Table 6: Comparison of CPLEX and L-Shaped method on the instances with $\alpha^V = 0.8$ and individual customer profiles.

				CPLEX		L-Shaped			
$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	gap	t	gap	gapR	gap50	t
50	200	0.2	0.2	0.0044	311.01	3.8564	17.7774	4.2676	1800.18
50	400	0.2	0.2	1.5001	1802.42	13.5540	60.3835	14.4840	1800.75
50	600	0.2	0.2	-	-	25.1954	105.0348	27.2104	1801.63
100	200	0.2	0.2	0.0099	1246.59	1.1923	7.5390	1.3117	1801.87
100	400	0.2	0.2	-	-	7.0935	33.3830	9.9477	1802.14
100	600	0.2	0.2	-	-	14.8632	29.7073	18.9278	1803.56
200	400	0.2	0.2	-	-	1.7218	22.1296	4.9610	1805.73
200	600	0.2	0.2	-	-	8.6047	17.3350	8.6047	1833.83
50	200	0.2	0.8	0.0097	375.72	1.6477	20.1198	2.0232	1800.04
50	400	0.2	0.8	0.9894	1802.12	11.4544	53.4891	13.7308	1801.20
50	600	0.2	0.8	-	-	20.1773	89.9520	24.8459	1803.77
100	200	0.2	0.8	0.0091	671.58	0.1778	6.9976	0.2153	1801.05
100	400	0.2	0.8	-	-	2.9674	32.4136	5.1617	1800.03
100	600	0.2	0.8	-	-	15.1294	32.5442	16.0408	1805.42
200	400	0.2	0.8	-	-	4.7056	28.3808	4.7289	1801.26
200	600	0.2	0.8	-	-	10.8675	26.4194	10.8675	1800.02
50	200	0.8	0.2	0.0003	73.80	0.0175	17.0926	0.0175	1800.08
50	400	0.8	0.2	0.2689	1802.31	2.5979	32.0731	3.4004	1800.03
50	600	0.8	0.2	-	-	14.2154	75.7595	17.8579	1800.19
100	200	0.8	0.2	0.0000	167.21	0.0000	14.6898	-	22.02
100	400	0.8	0.2	-	-	0.2919	18.9859	0.5424	1803.09
100	600	0.8	0.2	-	-	10.3980	40.7039	10.4926	1810.93
200	400	0.8	0.2	-	-	0.0000	61.0074	-	150.43

200	600	0.8	0.2	-	-	0.0296	12.2654	0.0328	1804.32
50	200	0.8	0.8	0.0068	64.66	0.2138	4.6541	0.2138	1800.14
50	400	0.8	0.8	0.0302	1804.08	1.7691	42.8840	2.3329	1800.18
50	600	0.8	0.8	-	-	10.0485	78.3238	13.5323	1800.02
100	200	0.8	0.8	0.0000	165.74	0.0000	1.4263	-	23.85
100	400	0.8	0.8	-	-	0.1842	6.4020	0.1842	1800.06
100	600	0.8	0.8	-	-	7.8771	36.8295	9.4600	1800.01
200	400	0.8	0.8	-	-	0.0000	61.6141	-	129.12
200	600	0.8	0.8	-	-	0.0469	13.3682	0.0471	1802.95
				0.2357	791.33	5.7899	33.6217	8.1178	1594.61

Table 7 and Table 8 report, for the instances with $\alpha^V = 0.2$ and 0.8, respectively, the results obtained by letting the L-Shaped method run for up to 5 hours (18000 seconds) with a target 1% optimality gap. We observe that the optimality gap goes down from an average of 9.82% to an average of 6.48% for the case with $\alpha^V = 0.2$ and from an average of 8.19% to an average of 5.14% for the case with $\alpha^V = 0.8$. For the case with individual customer profiles we obtain an average optimality gap of 2.48% and 3.31% for the case with $\alpha^V = 0.2$ and 0.8, respectively. All results on the instances with individual customer profiles are reported in Appendix D. These results are possibly of little practical use since, in a business context, solutions are most likely required in much shorter time. Nevertheless, they show that the model can provide useful bounds that may serve a reference point for example in the development of faster heuristic methods.

Table 7: Results of the L-Shaped method with the addition of eq. (27) on the instances with $\alpha^V = 0.2$ with a time limit of 18000 seconds and 1% target optimality gap.

$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	gap	gapR	gap50	t
50	200	0.2	0.2	0.9890	18.6752	-	159.36
50	400	0.2	0.2	10.2092	66.5573	11.1192	18000.12
50	600	0.2	0.2	22.9573	87.2440	25.9743	18003.41
100	200	0.2	0.2	0.6386	9.5524	-	29.63
100	400	0.2	0.2	0.9219	21.2273	-	3653.12
100	600	0.2	0.2	7.4259	35.8381	8.8381	18000.29
200	400	0.2	0.2	0.0000	6.3509	-	157.42
200	600	0.2	0.2	0.8479	10.4516	-	717.90
50	200	0.2	0.8	0.8965	17.6529	-	21.23
50	400	0.2	0.8	6.6519	62.4842	7.6630	18000.62
50	600	0.2	0.8	13.2604	90.2582	13.8926	18000.10
100	200	0.2	0.8	0.3083	7.2972	-	23.84
100	400	0.2	0.8	0.9374	19.6630	-	1164.27
100	600	0.2	0.8	6.3696	34.0425	6.9128	18008.18
200	400	0.2	0.8	0.0000	41.7344	-	128.15
200	600	0.2	0.8	0.7697	12.6665	-	705.42
50	200	0.8	0.2	1.3911	23.9553	1.5307	18000.20
50	400	0.8	0.2	15.1496	85.1665	15.6460	18000.37
50	600	0.8	0.2	36.5566	108.3757	37.5433	18000.09
100	200	0.8	0.2	0.9165	15.4557	-	151.94
100	400	0.8	0.2	2.7552	22.7406	3.2210	18008.25
100	600	0.8	0.2	13.9898	42.5987	14.5641	18010.22
200	400	0.8	0.2	0.4081	12.4280	-	332.82
200	600	0.8	0.2	1.2695	15.3980	2.6620	18000.07
50	200	0.8	0.8	0.9616	22.5871	-	839.48
50	400	0.8	0.8	11.8177	55.7576	13.0508	18001.72
50	600	0.8	0.8	33.2157	106.9156	34.4212	18000.09
100	200	0.8	0.8	0.9309	13.3519	-	364.48
100	400	0.8	0.8	1.8436	17.1371	2.0311	18001.11

100	600	0.8	0.8	10.3464	46.7222	15.3117	18004.26
200	400	0.8	0.8	0.5470	7.2607	-	296.25
200	600	0.8	0.8	2.3070	16.7484	2.7073	18009.94
				6.4872	36.0717	12.7700	9837.32

Table 8: Results of the L-Shaped method with the addition of eq. (27) on the instances with $\alpha^V = 0.8$ with a time limit of 18000 seconds and 1% target optimality gap.

$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	gap	gapR	gap50	t
50	200	0.2	0.2	0.9052	37.8092	-	186.24
50	400	0.2	0.2	10.1669	42.1383	10.4452	18000.24
50	600	0.2	0.2	21.5968	77.8433	22.0765	18000.11
100	200	0.2	0.2	0.9726	6.1394	-	106.15
100	400	0.2	0.2	1.2153	22.6182	1.4967	18000.48
100	600	0.2	0.2	5.8809	35.6069	7.0281	18000.17
200	400	0.2	0.2	0.9991	5.3451	-	1793.44
200	600	0.2	0.2	0.9964	10.6059	1.3911	14025.40
50	200	0.2	0.8	0.9928	33.4059	-	904.69
50	400	0.2	0.8	8.4548	42.0692	8.9256	18000.29
50	600	0.2	0.8	20.4095	96.0829	21.1479	18000.08
100	200	0.2	0.8	0.7544	4.7786	-	37.44
100	400	0.2	0.8	0.9978	20.3964	1.2145	16084.62
100	600	0.2	0.8	6.5475	35.6683	6.8735	18000.45
200	400	0.2	0.8	0.5994	9.3896	-	396.00
200	600	0.2	0.8	0.8804	17.6274	-	8150.06
50	200	0.8	0.2	0.5667	12.5954	-	12.95
50	400	0.8	0.2	9.1978	65.9956	9.6357	18000.12
50	600	0.8	0.2	28.8099	82.1348	30.6454	18002.77
100	200	0.8	0.2	0.4533	9.4214	-	22.96
100	400	0.8	0.2	0.9253	4.0455	-	204.01
100	600	0.8	0.2	3.7135	27.6410	4.3096	18000.61
200	400	0.8	0.2	0.0020	8.6414	-	220.92
200	600	0.8	0.2	0.5837	11.5793	-	613.09
50	200	0.8	0.8	0.6258	14.9683	-	12.69
50	400	0.8	0.8	6.2098	44.5269	7.6010	18002.40
50	600	0.8	0.8	27.1409	99.1525	27.7092	18000.46
100	200	0.8	0.8	0.9110	1.3040	-	22.28
100	400	0.8	0.8	0.9539	17.2446	-	276.26
100	600	0.8	0.8	2.2269	13.2578	2.6462	18005.33
200	400	0.8	0.8	0.0000	42.1315	-	124.68
200	600	0.8	0.8	0.0754	8.4615	-	419.24
				5.1489	30.0196	10.8764	8675.83

Finally, we report on the performance of the L-Shaped method and CPLEX as the number of scenarios (sample size) increases. The results reported above in this section have been obtained by arbitrarily using ten scenarios to approximate the underlying continuous random variable. Figures 4 and 5 report the optimality gap obtained with the L-Shaped method and CPLEX as the sample size increases, for the smallest ($|\mathcal{V}| = 50$, $|\mathcal{K}| = 200$) and largest ($|\mathcal{V}| = 200$, $|\mathcal{K}| = 600$) instances, respectively, with identical customers. Particularly, Figure 4a and Figure 4b report the gap of the L-Shaped method after 30 minutes and 5 hours, respectively, while Figure 4c and Figure 4d report the gap of CPLEX after 30 minutes and 5 hours, respectively. Similarly, Figure 5 reports the gaps of the L-Shaped method after after 30 minutes and 5 hours on the largest instance. Tests on the largest instances were conducted only for the L-Shaped method as it is already evident from Tables 3 and 4 that, on those instances, CPLEX fails to deliver solutions already with a sample of ten scenarios.

As intuition suggests, when using the L-Shaped method, the optimality gap grows with the number of scenarios, both on the smallest (Figure 4) and the largest (Figure 5) instances. For the smallest instances, with a time limit of 30 minutes the growth appears mild and, with 100 scenarios, the optimality gap remains in the neighborhood of 4% in the worst case, with an average optimality gap in the neighborhood of 2% (see Figure 4a). After 5 hours, the L-Shaped method drops the optimality gap even further, with a worst case gap in the neighborhoods of 2%. This gives room for a more dense approximation of the uncertainty, i.e., a larger sample size, compared to the 10 scenarios used in our previous tests. The performance of CPLEX on the same instances is dramatically worse (see Figure 4c). Given a 30-minute time limit, on the smallest sample sizes, CPLEX outperforms the L-Shaped method, but as the sample size grows to 50 or higher the solver's optimality gaps are orders of magnitude higher than those of the L-Shaped method (compare Figure 4c and Figure 4a). The performance of CPLEX improves with a 5 hours time limit (see Figure 4d), though performing much worse than the L-Shaped method at least for the largest sample sizes (compare Figure 4d and Figure 4b). On the largest instances, with a 30-minute time limit, the gap growth for the L-Shaped method remains limited up to a sample size of 25, but grows dramatically with a larger sample size, see Figure 5a. Also the variance of the optimality gap grows with the sample size, limiting the reliability of the method for large numbers of scenarios. Nevertheless, with a longer time limit the L-Shaped method is able to reduce the optimality gap approximately ten times on the largest sample sizes, see Figure 5b.

Summarizing, the results in Figures 4 and 5 illustrate that, for small instances, a solution time of 30 minutes is sufficient to obtain high quality solutions also with a more extensive approximation of the uncertainty. However, especially on the largest instances, a 30-minute time limit might result too small to accommodate for a better description of the uncertainty. Nevertheless, allowing the L-Shaped method to run for a longer time (e.g., 5 hours) can yield substantial reductions of the optimaility gap. In any case, Figure 4 illustrates that the L-Shaped method scales significantly better than CPLEX as the sample size increases.

The observed performance of the L-Shaped method allows us a conclusive reflection on the envisaged usage of the method. In Tables 3 to 6 we have let both the algorithm and CPLEX run for 30 minutes and observed that our algorithm scales significantly better. Therefore, depending on the practical operating needs of the the CSO, the algorithm might already provide a practice-ready tool. That is, if the CSO is able to wait for a solution for 30 minutes, our tests provide empirical evidences that the algorithm delivers a solution and often a high-quality one, with the vehicles-to-customers ratio being a strong driver of the quality of the solution obtained. Tables 3 to 6 also illustrate that the algorithm was able to find good solutions after 15 minutes (50% of the solution time, see column `gap50`). Therefore, if the CSO has tighter time requirements, a potential strategy is to terminate the algorithm earlier, knowing that this implies giving up something in terms of quality of the solution. However, our tests show that the optimality gap after 15 minutes is typical not dramatically higher than the optimality gap obtained after 30 minutes. Nevertheless, there might arise situations in which waiting for a solution for 30 or even 15 minutes might be impractical, e.g., if the demand landscape changes more frequently and relocation and pricing plans are required more often. In these cases, the proposed algorithm might be proven inefficient and one might have to consider developing faster heuristic algorithms. If this is the case, the performance of the L-Shaped method, and the dual bounds it delivers, provide a reliable benchmark. An example is provided in Appendix E where we test a simple Iterated Local Search and assess its performances using the bounds provided by the L-Shaped method. For the smallest instances the heuristic is able to provide primal solutions of quality comparable or even better than the L-Shaped method. Nevertheless, the quality of the solutions delivered drops significantly as the size of the instance increase, indicating that further refinement is needed. Finally, regardless of the solution time, the algorithm proposed may be used by the CSO to obtain solutions that allow them to support managerial choices or analyze policy implications, e.g., subsidies, plans to expand the fleet or to hire additional staff for relocations activities. Some insights on the impact of a pricing scheme on profits and demand are provided in Section 6.3.

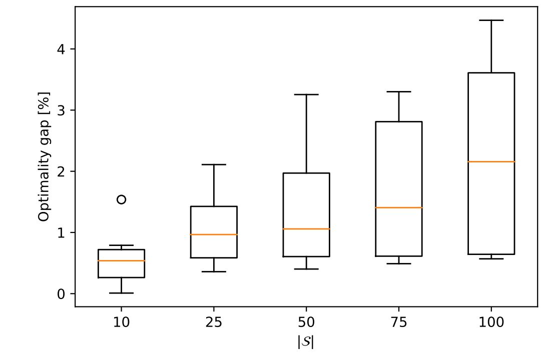
6.3 Analysis of the solutions

In this section we present some evidences based on the analysis of the solutions obtained by the proposed model. The analysis was performed using the default configuration, i.e., without individual customer profiles, as we believe it is a more realistic configuration to achieve by CSOs.

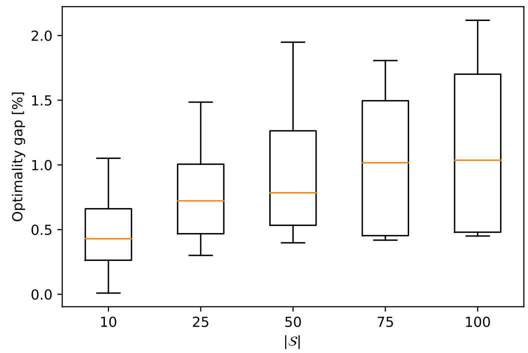
The analysis in this section is based on the results obtained on the instances with the largest number of customers (600) and with different distributions of vehicles and customers, namely

D1 Vehicles in the outskirt and demand from center to outskirt ($\alpha^V = 0.2$, $\alpha^{FROM} = 0.8$, $\alpha^{TO} = 0.2$)

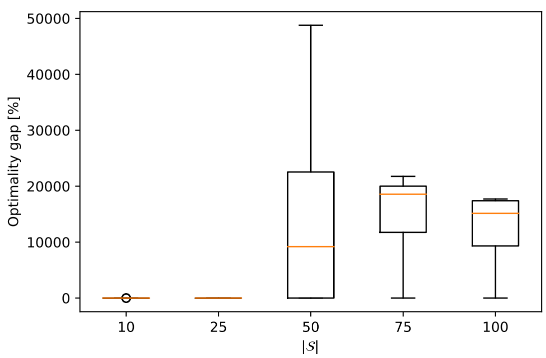
D2 Vehicles in the center and demand from outskirt to center ($\alpha^V = 0.8$, $\alpha^{FROM} = 0.2$, $\alpha^{TO} = 0.8$)



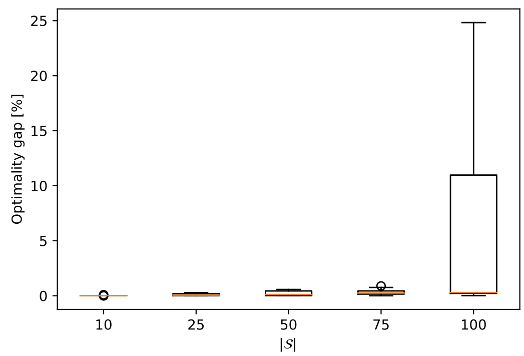
(a) L-Shaped method after 30 minutes.



(b) L-Shaped method after 5 hours.

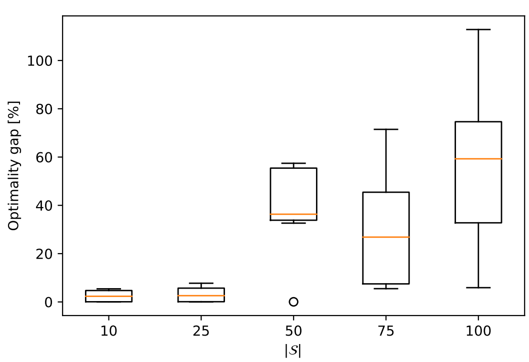


(c) CPLEX after 30 minutes.

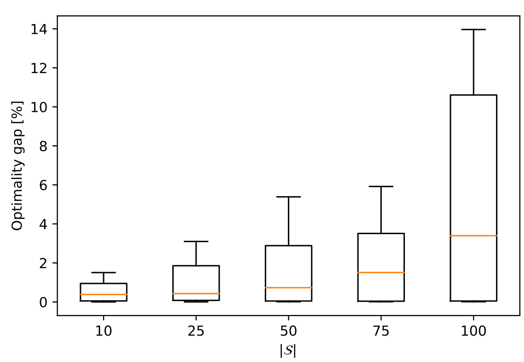


(d) CPLEX after 5 hours. Plot obtained removing an outlier with an optimality gap greater than 1000%.

Figure 4: Optimality gap obtained when using the L-Shaped method and CPLEX for different sample sizes $|\mathcal{S}|$ on the instances with $|\mathcal{V}| = 50$ and $|\mathcal{K}| = 200$. The optimality gap is computed as the average over all combinations of α^V, α^{FROM} and α^{TO} assuming identical customers and no additional valid indequality.



(a) L-Shaped method after 30 minutes.



(b) L-Shaped method after 5 hours.

Figure 5: Optimality gap obtained when using the L-Shaped method for different sample sizes $|\mathcal{S}|$ on the instances with $|\mathcal{V}| = 200$ and $|\mathcal{K}| = 600$. The optimality gap is computed as the average over all combinations of α^V, α^{FROM} and α^{TO} assuming identical customers and no additional valid indequality.

D3 Vehicles in the center and demand from center to outskirt ($\alpha^V = 0.8$, $\alpha^{FROM} = 0.8$, $\alpha^{TO} = 0.2$)

D4 Vehicles in the outskirt and demand from outskirt to center ($\alpha^V = 0.2$, $\alpha^{FROM} = 0.2$, $\alpha^{TO} = 0.8$)

The number of vehicles is set either to 50 or to 200, corresponding to vehicles-to-customers ratios of 1/12 and 1/3, respectively. It should be noted that, as pointed out by e.g., [18, 20], the number of available cars in a zone influences customer demand. In our approach, in which customers are considered at the individual level, the connection between the number of available cars and demand is handled jointly by the utility function and, especially, by the optimization model. That is, there is a potential demand, made of the users which, according to their utility function, would choose carsharing, if available, at the drop-off fee level set in the first stage, and there is a realized demand (i.e., actual rentals), which takes into account that not all potential customers may find an available car. This is done through constraints (9) that state that an available car is taken by the first customer arriving at the car. It should be further clarified that, in our experiments, we assume that customers do not wait for more cars to become available, i.e., the waiting time is always set to zero in our instances, corresponding to saying that, if no car is available, the customer will immediately choose another transport service. Indeed, in some real-life carsharing services, some waiting time could be taken into account. That is, when there are no cars available in the zone, some user might decide to wait until a car is returned. However, we believe the assumption that customers do not wait is the most appropriate especially in a free-floating one-way service, where both customers and the CSO have limited information on whether and when a new car will be returned close to the user.

We start by presenting the effect of pricing strategies on profits and relocations. For each distribution of customers and vehicles, we solved two configurations of the model. In the first configuration the prices were optimally set by model (1). In the second configuration the drop-off fee was set to 0 (the average of the drop-off fees considered) everywhere, corresponding to a situation in which the CSO applies only a per-minute fee and does not adjust prices with respect to the origin and destination and to the time of the day.

Table 9: Comparison of the solutions with and without dynamic pricing on the instances with 50 vehicles and 600 customers.

Distribution	Metric	With dynamic pricing	Without dynamic pricing
D1	Expected Profit [%]	100	81.78
	% of vehicles Relocated	26.0	10.0
	Min $\lvert\mathcal{R}(\xi)\rvert$	167	80
	Max $\lvert\mathcal{R}(\xi)\rvert$	195	107
	Expected % Requests satisfied	24	42
D2	Expected Profit [%]	100	66.06
	% of vehicles Relocated	22.0	2.0
	Min $\lvert\mathcal{R}(\xi)\rvert$	168	81
	Max $\lvert\mathcal{R}(\xi)\rvert$	187	105
	Expected % Requests satisfied	26	49
D3	Expected Profit [%]	100	65.05
	% of vehicles Relocated	18.0	6.0
	Min $\lvert\mathcal{R}(\xi)\rvert$	167	80
	Max $\lvert\mathcal{R}(\xi)\rvert$	195	107
	Expected % Requests satisfied	26	49
D4	Expected Profit [%]	100	66.36
	% of vehicles Relocated	10.0	0.0
	Min $\lvert\mathcal{R}(\xi)\rvert$	168	81
	Max $\lvert\mathcal{R}(\xi)\rvert$	187	105
	Expected % Requests satisfied	26	48

Tables 9 and 10 report a number of solution statistics for the case with 50 and 200 customers, respectively. Expected profits for the case without dynamic pricing are reported as a percentage of the expected profits with dynamic pricing. In both the case with 50 and 200 vehicles we observe that the expected profit without pricing is approximately 65 to 80% of the expected profit obtained by adjusting prices. The main driver of the higher expected profit generated by a pricing strategy is the higher number of requests generated, approximately double both in the case with 50 and in the case with 200 vehicles. That is, by adjusting prices the CSO is able to attract significantly more demand and increase competition.

Table 10: Comparison of the solutions with and without dynamic pricing on the instances with 200 vehicles and 600 customers.

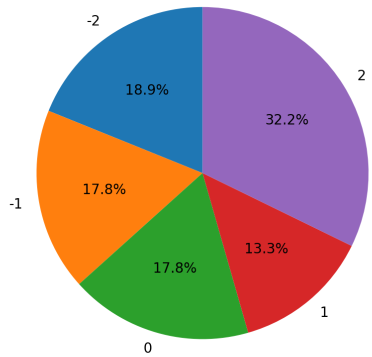
Distribution	Metric	With dynamic pricing [%]	Without dynamic pricing [%]
D1	Expected Profit [%]	100	70.12
	% of vehicles Relocated	0.5	1.5
	Min $\|\mathcal{R}(\xi)\|$	167	80
	Max $\|\mathcal{R}(\xi)\|$	195	107
	Expected % Requests satisfied	53	90
D2	Expected Profit [%]	100	70.87
	% of vehicles Relocated	1.5	0.5
	Min $\|\mathcal{R}(\xi)\|$	168	82
	Max $\|\mathcal{R}(\xi)\|$	187	99
	Expected % Requests satisfied	51	91
D3	Expected Profit [%]	100	73.94
	% of vehicles Relocated	0	0
	Min $\|\mathcal{R}(\xi)\|$	167	80
	Max $\|\mathcal{R}(\xi)\|$	195	107
	Expected % Requests satisfied	56	100
D4	Expected Profit [%]	100	71.18
	% of vehicles Relocated	0	0
	Min $\|\mathcal{R}(\xi)\|$	168	82
	Max $\|\mathcal{R}(\xi)\|$	187	99
	Expected % Requests satisfied	56	94

In the case with 200 vehicles and without dynamic pricing, the CSO is able to satisfy the great majority of the requests (more than 90% – see Table 10) performing very few, if any, relocations – we will return to this point later. However, in this case the CSO is able to attract less than (approximately) 1/6 of the customers. With dynamic pricing, the CSO is able to attract close to 1/3 of the customers and serve slightly more than 50% of the requests. In total, the number of rentals are approximately the same, with and without dynamic pricing. However, by adjusting prices the CSO is able to increase the revenue. In fact, as shown in Figure 6, the most used drop-off fee is the highest (2 Euro), illustrating that the CSO is able to exploit the higher willingness to pay of some customers.

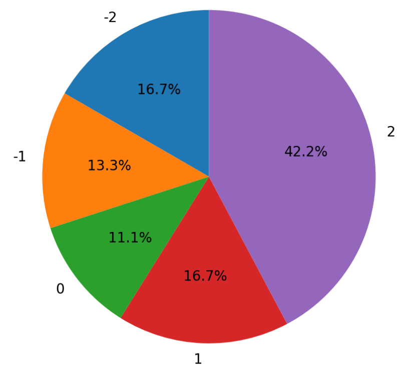
In the case with 50 vehicles (thus one vehicle every twelve potential customers), without dynamic pricing the CSO is able to satisfy approximately half of the total requests due to the reduced number of vehicles, see Table 9. Also in this case the number of requests is lower (approximately half) then the number of requests obtained by adjusting prices. With dynamic pricing, the CSO is able to attract close to 1/3 of the customers and to satisfy only approximately 25% of them. Also in this case, the CSO benefits from the higher competition. In more than 50% of the origin-destination pairs the CSO is able to apply the highest drop-off fee (2 Euro) see Figure 7 and to reposition the fleet in such a way to satisfy the requests with the highest revenue. Thus, a trend we observe in Table 9 is that a pricing strategy allows the CSO to attract more demand but to satisfy only part of it. While this allows the CSO to exploit competition, many customers do not see their wish to use carsharing satisfied. This negative user experience might have an impact in the long run. This is however beyond the scope of this study.

Interestingly, in the case with 200 vehicles (one every three customers – see Table 10) the need for relocations is almost null, regardless of how prices are set. The fleet is large enough to cover sufficiently well the entire business area and serve almost all requests. On the other hand, with a fleet 50 vehicles (one every twelve customers – see Table 9) the need for relocations is more evident. The fleet is now insufficient to cover the entire demand. In the case without dynamic pricing, fewer relocations are needed compared to the case with dynamic pricing. This is due to the lower demand attracted (approximately 80 to 107 requests with a fleet of 50 vehicles). Many more relocations are performed when dynamic pricing is applied as a consequence of the higher demand generated (at least 167 requests for 50 vehicles). Thus the CSO finds it beneficial to move vehicles where they can generate more revenue.

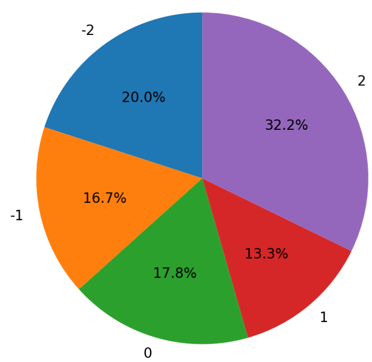
A natural follow up question is the impact of relocations on profits. Therefore, we focused on those instances



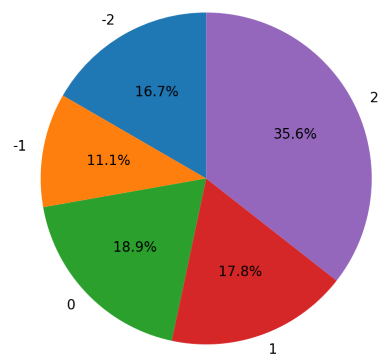
(a) Distribution $D1$



(b) Distribution $D2$

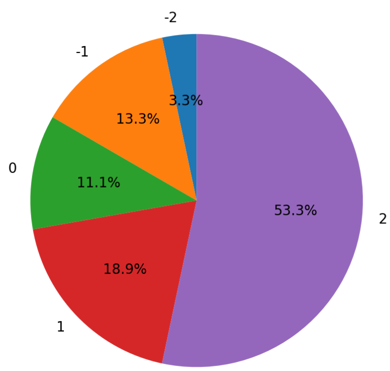


(c) Distribution $D3$

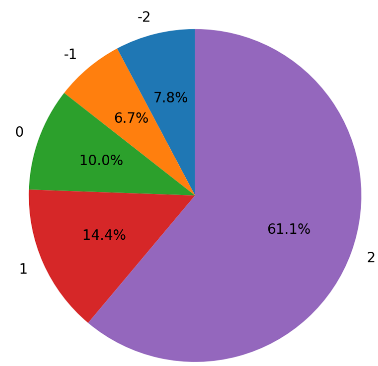


(d) Distribution $D4$

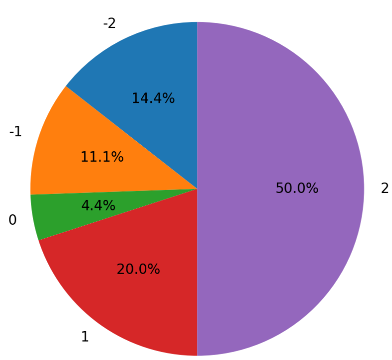
Figure 6: Drop-off fees adopted in the instances with 200 vehicles and 600 customers. Drop-off fees are expressed in Euro.



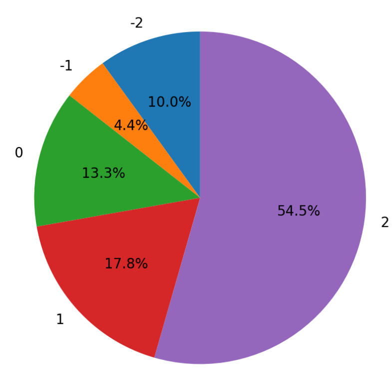
(a) Distribution $D1$



(b) Distribution $D2$



(c) Distribution $D3$



(d) Distribution $D4$

Figure 7: Drop-off fees adopted in the instances with 50 vehicles and 600 customers. Drop-off fees are expressed in Euro.

where more relocations were suggested, i.e., with 50 vehicles and dynamic pricing, see Table 9. We solved the same instances, but this time preventing the model from making any relocations. That is, vehicles were forced to remain in their initial positions. The results indicate that the expected profit without relocations is only marginally lower. Particularly, 98.5% for distribution $D1$, 98.3% for $D2$, 97.4% for $D3$ and 98.8% for $D4$. This is due to the fact that relocations are expensive and can yield only a very minor increase in revenue. That is, with 50 vehicles and always more than 167 requests, vehicles would always be rented, even if not relocated. By relocating a vehicle the CSO is able to charge a higher drop-off fee, but bears the relocation cost. This results in a marginal profit increase.

Relocations are however likely to generate a higher impact on profits when the vehicles-to-customers ratio is even smaller. We performed the same test with a vehicles-to-customers ratio equal to 1/100 (i.e., 10 vehicles 1000 customers). The results show that the profit without relocations was 70.07% with distribution $D1$, 75.26% with $D2$, 76.22% with distribution $D3$ and 78.07% with distribution $D4$ (these percentages are calculated on the best upper bounds since a near optimal solution was available for all instances). In all cases the percentage of vehicles relocated ranged between 10 and 20%, similar to the case with 50 vehicles and 600 vehicles. This means that the percentage of relocations remained approximately the same with a smaller vehicles-to-customers ratio, but had a much higher impact on profits. Thus, it appears that a dynamic pricing strategy, coupled with a sufficiently large fleet (say more than one vehicle every twelve customers in our case), decreases significantly the need for staff-based relocations. Otherwise, relocations remain an important tool even with a dynamic pricing strategy.

7 Conclusions, limitations and future work

We presented a novel optimization model for jointly deciding carsharing prices and relocations. The problem is modeled as a two-stage integer stochastic program in order to account for uncertainty in customers preferences. An exact solution algorithm based on the integer L-Shaped method has been proposed. Extensive tests have been performed on instances based on the municipality of Milan, in order to assess both the performance of the solution algorithm and the type of solutions obtained. The instances have been made available online for the sake of future research.

Results illustrate that, within times compatible with business practice, the method solves or finds a high quality solution to most instances. In addition, it finds a feasible solution to all instances considered. In contrast, CPLEX delivers a solution to only a few, small, instances.

The analysis of the solutions illustrates that a pricing strategy helps the CSO to significantly increase expected profits. This is due to the increased demand generated and the resulting competition. In our instances the demand was approximately doubled compared to a situation without dynamic pricing. This in turn generates higher expected profits by exploiting customer's higher willingness to pay. The results also show that, by adopting a zone-based pricing strategy and employing a large enough fleet, the impact of staff-based relocations on profits becomes marginal. On the other hand, the impact of relocations becomes more evident as the size of the fleet decreases.

A number of limitations remain to be addressed in future research, as we comment in what follows. A pricing strategy which varies with each origin and destination, or frequently throughout the day, may not be applicable in all contexts, or raise concerns related to the potential complexity for users who would rather prefer a simpler pricing strategy. The scope of this article was that of introducing a general model framework which could then be adapted to specific contexts and improved. For example, the model proposed can be easily adapted to different time and space resolutions, i.e., it is possible to define the length of the target period and the discretization of the business area based on the specific needs. In addition, the model may be easily modified to enforce that e.g., drop-off fees vary only according to the pick-up place or only according to the drop-off place. Future research may provide furter modifications and improvements.

The size of the instances used in this study is comparable with the size of the station-based carsharing in Milan which, according to [40], in 2018 counted 149 shared vehicles and, on average 108 daily rentals (see Table 10 for a comparison). Other examples are the station-based carsharing offered by *Letsgo* (https://letsgo.dk/), which currently operates a fleet of around 200 vehicles in Copenhagen, and *Vy* (https://www.vy.no/en/travelling-with-us/other-modes-of-transport/city-car) that operates a fleet of 250 vehicles in Oslo. Nevertheless, bigger fleets and a higher number of customers are likely to limit the practical efficiency of our exact method and call for faster, e.g., heuristic, methods.

The performance of the algorithm with respect to a higher number of zones remains to be assessed. Our instances, generated on the basis of [16], contained ten zones. Supposedly, a more granular discretization of the business area is likely to have a negative impact on the practical applicability of the method. However, the benefits of a finer partition of the business area into pricing zones is to be addressed by further research, particularly in the case of free-floating services. Effective discretization strategies and methods are, to our knowledge, still an open research question.

As reported by [52], the choice of carsharing users is also influenced by elements such as the type of vehicle and its proximity to the user. In addition, comfort, weather conditions, and purpose of the trip are all factors which might influence customers decisions. While proximity is considered in the form of walking time in the utility function we used in our experiments, the remaining elements are not captured explicitly, but are rather included in the portion of customer preferences that the CSO cannot explain. Future research might be set up to extend the model and utility function used in the tests in order to better capture customers behavior.

Several other sources of uncertainty affect the problem, that have not been considered in this study. These include, e.g., the total number of customers appearing in each zone, and their destination. Our model might account for this uncertainty by setting a sufficiently large number of customers. Travel times, both with carsharing and with alternative transport services are also, to a certain degree, uncertain in practice. The impact of this uncertainty on solutions and profits remains to be understood.

The analysis of the solutions indicates that, by dynamically adjusting prices, the CSO is able to attract significantly more demand. However, with a vehicles-to-customers ratio of 1 to 12 (see Table 9), the portion of the demand satisfied was, approximately, only 25%. That is, the majority of the customers who would have used carsharing did not have the chance to do so. As a consequence, users may perceive a low availability of the service. The effect of this in the long term remains to be clarified.

Finally, our model is currently unable to use pricing as a preventive measure to encourage a profitable distribution of the fleet. Consider two subsequent target periods, say t_1 and t_2, and two zones, say A and B. Assume that the CSO expects high demand in zone A in period t_2. They may, consequently, set a lower price or an incentive in t_1, for renting cars in zone B and delivering them in zone A, and/or disincentivize movements in the opposite direction. In order to be able to optimize these decisions the proposed model should be extended to account for a multistage decision process.

A Sample Average Approximation

Let $\xi_1, \dots, \xi_S$ be an S-dimensional iid sample of $\tilde{\xi}$ and let $\mathcal{S} = \{1, \dots, S\}$. Let decision variable y_{vrls} be equal to 1 if request $r \in \mathcal{R}(\xi_s)$ is satisfied by vehicle v at level l under realization s, 0 otherwise. The SAA of problem (1) can be stated as follows.

$$\max - \sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{I}} C^R_{vi} z_{vi} + \frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}\sum_{r\in\mathcal{R}(\xi_s)}\sum_{v\in\mathcal{V}}\sum_{l\in\mathcal{L}_r(\xi_s)} R_{rl} y_{vrls} \tag{30a}$$

$$\sum_{i\in\mathcal{I}} z_{vi} = 1 \qquad v \in \mathcal{V} \tag{30b}$$

$$\sum_{l\in\mathcal{L}} \lambda_{ijl} = 1 \qquad i \in \mathcal{I}, j \in \mathcal{J} \tag{30c}$$

$$\sum_{v\in\mathcal{V}}\sum_{l\in\mathcal{L}_r(\xi_s)} y_{vrls} \le 1 \qquad r \in \mathcal{R}(\xi_s), s \in \mathcal{S} \tag{30d}$$

$$\sum_{r\in\mathcal{R}(\xi_s)}\sum_{l\in\mathcal{L}_r(\xi_s)} y_{vrls} \le 1 \qquad v \in \mathcal{V}, s \in \mathcal{S} \tag{30e}$$

$$\sum_{l\in\mathcal{L}_{r_1}(\xi_s)} y_{v,r_1,l,s} - z_{v,i(r_1)} + \sum_{r_2\in\mathcal{R}_{r_1}(\xi_s)}\sum_{l\in\mathcal{L}_{r_2}(\xi_s)} y_{v,r_2,l,s} \le 0 \qquad r_1 \in \mathcal{R}(\xi_s), v \in \mathcal{V}.s \in \mathcal{S} \tag{30f}$$

$$y_{v,r_1,l_1,s} + \sum_{r_2\in\mathcal{R}_{r_1}(\xi_s)}\sum_{l_2\in\mathcal{L}_{r_2}(\xi_s)} y_{v,r_2,l_2,s} + \sum_{v_1\in\mathcal{V}:v_1\neq v} y_{v_1,r_1,l_1,s}$$
$$\ge \lambda_{i(r_1),j(r_j),l_1} + z_{v,i(r_1)} - 1 \qquad r_1 \in \mathcal{R}(\xi_s), v \in \mathcal{V}, l_1 \in \mathcal{L}_{r_1}(\xi_s), s \in \mathcal{S} \tag{30g}$$

$$\sum_{v\in\mathcal{V}} y_{vrls} \le \lambda_{i(r),j(r),l} \qquad r \in \mathcal{R}(\xi_s), l \in \mathcal{L}_r(\xi_s), s \in \mathcal{S} \tag{30h}$$

$$y_{vrls} \in \{0,1\} \qquad r \in \mathcal{R}(\xi_s), v \in \mathcal{V}, l \in \mathcal{L}_r(\xi_s), s \in \mathcal{S} \tag{30i}$$

$$z_{vi} \in \{0,1\} \qquad i \in \mathcal{I}, v \in \mathcal{V} \tag{30j}$$

$$\lambda_{ijl} \in \{0,1\} \qquad i \in \mathcal{I}, j \in \mathcal{I}, l \in \mathcal{L}. \tag{30k}$$

B Size of the instances

This section reports the size of the instances for the base case in Table 11 and for the case with customers profiled individually in Table 12.

Table 11: Size of the SAA model without decomposition for all instances tested in the base case.

$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	# Variables	# Constraints
50	200	0.2	0.2	95350	127528
50	400	0.2	0.2	199800	269257
50	600	0.2	0.2	296000	399868
100	200	0.2	0.2	190200	252478
100	400	0.2	0.2	399500	533561
100	600	0.2	0.2	591600	791919
200	400	0.2	0.2	798300	1061560
200	600	0.2	0.2	1182900	1576120
50	200	0.2	0.8	89450	119419
50	400	0.2	0.8	189750	254569
50	600	0.2	0.8	279350	375643
100	200	0.2	0.8	178500	236520
100	400	0.2	0.8	379200	504170
100	600	0.2	0.8	558300	743843
200	400	0.2	0.8	758100	1003471
200	600	0.2	0.8	1116500	1480645
50	200	0.8	0.2	94300	125998
50	400	0.8	0.2	195900	263086
50	600	0.8	0.2	287050	387577
100	200	0.8	0.2	188100	249448
100	400	0.8	0.2	391800	521542
100	600	0.8	0.2	573900	767780
200	400	0.8	0.2	783300	1038043
200	600	0.8	0.2	1147300	1527880
50	200	0.8	0.8	90300	120337
50	400	0.8	0.8	185750	249469
50	600	0.8	0.8	279500	375949
100	200	0.8	0.8	180100	238237
100	400	0.8	0.8	371300	494272
100	600	0.8	0.8	558800	744752
200	400	0.8	0.8	742300	983773
200	600	0.8	0.8	1117700	1482856
				452819	603174

Table 12: Size of the SAA model without decomposition for all instances tested with individual customers profiles.

$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	# Variables	# Constraints
50	200	0.2	0.2	84500	116461
50	400	0.2	0.2	171000	237127
50	600	0.2	0.2	250800	347389
100	200	0.2	0.2	168600	230662
100	400	0.2	0.2	341600	469628
100	600	0.2	0.2	501200	687990
200	400	0.2	0.2	682900	934729
200	600	0.2	0.2	1002100	1369291

50	200	0.2	0.8	77200	107281
50	400	0.2	0.8	163300	226723
50	600	0.2	0.8	239650	333007
100	200	0.2	0.8	153900	212381
100	400	0.2	0.8	326100	448923
100	600	0.2	0.8	478900	659508
200	400	0.2	0.8	651700	893323
200	600	0.2	0.8	957500	1312609
50	200	0.8	0.2	83100	114676
50	400	0.8	0.2	168100	232792
50	600	0.8	0.2	243900	338821
100	200	0.8	0.2	165900	227329
100	400	0.8	0.2	335700	460942
100	600	0.8	0.2	487300	670921
200	400	0.8	0.2	670900	917242
200	600	0.8	0.2	974100	1335121
50	200	0.8	0.8	79300	110035
50	400	0.8	0.8	161150	224428
50	600	0.8	0.8	243800	340351
100	200	0.8	0.8	158100	217835
100	400	0.8	0.8	321800	444378
100	600	0.8	0.8	487300	674153
200	400	0.8	0.8	643300	884479
200	600	0.8	0.8	974100	1341553
				389025	535065

C Effect of valid inequality

Table 13: Results of the L-Shaped method with the addition of eq. (27) on the instances with $\alpha^V = 0.2$.

$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	gap	gapR	gap50	t
50	200	0.2	0.2	0.6967	18.6752	0.6967	1800.20
50	400	0.2	0.2	17.2850	66.5573	20.6265	1800.52
50	600	0.2	0.2	30.5015	87.2440	39.2311	1809.94
100	200	0.2	0.2	0.0975	9.5524	0.0975	1800.98
100	400	0.2	0.2	1.9480	21.2273	3.9377	1805.67
100	600	0.2	0.2	11.6208	35.8381	14.7910	1801.95
200	400	0.2	0.2	0.0000	6.3509	-	139.41
200	600	0.2	0.2	0.0777	10.4516	0.3662	1800.78
50	200	0.2	0.8	0.3381	17.6529	0.3381	1800.63
50	400	0.2	0.8	9.6397	62.4842	11.6808	1801.21
50	600	0.2	0.8	19.5455	90.2582	20.6217	1800.03
100	200	0.2	0.8	0.2815	7.2972	0.2815	1800.13
100	400	0.2	0.8	0.7148	19.6630	1.0807	1801.47
100	600	0.2	0.8	9.8896	34.0425	11.4341	1812.93
200	400	0.2	0.8	0.0000	41.7344	-	115.74
200	600	0.2	0.8	0.4710	12.6665	0.6677	1814.99
50	200	0.8	0.2	1.9850	23.9553	2.2390	1800.13
50	400	0.8	0.2	21.8829	85.1665	25.2703	1800.02
50	600	0.8	0.2	44.9080	108.3757	62.3967	1803.83
100	200	0.8	0.2	0.5238	15.4557	0.5320	1801.09
100	400	0.8	0.2	6.1743	22.7406	9.0017	1806.48

100	600	0.8	0.2	23.6427	42.5987	23.6588	1816.20
200	400	0.8	0.2	0.3072	12.4280	0.3209	1804.22
200	600	0.8	0.2	6.7474	15.3980	17.5717	1865.22
50	200	0.8	0.8	0.8509	22.5871	0.9463	1800.15
50	400	0.8	0.8	15.4867	55.7576	19.9941	1801.46
50	600	0.8	0.8	40.1722	106.9156	41.2931	1807.37
100	200	0.8	0.8	0.7932	13.3519	0.8354	1802.32
100	400	0.8	0.8	4.4619	17.1371	7.3475	1803.98
100	600	0.8	0.8	19.1824	46.7222	23.6637	1823.21
200	400	0.8	0.8	0.0963	7.2607	0.0963	1800.78
200	600	0.8	0.8	5.6194	16.7484	6.8357	1823.23
				9.2482	36.0717	12.2618	1702.07

Table 14: Results of the L-Shaped method with the addition of eq. (27) on the instances with $\alpha^V = 0.8$.

$\|\mathcal{V}\|$	$\|\mathcal{K}\|$	α^{FROM}	α^{TO}	gap	gapR	gap50	t
50	200	0.2	0.2	0.4613	37.8092	0.5067	1800.37
50	400	0.2	0.2	11.6098	42.1383	13.2894	1802.87
50	600	0.2	0.2	30.3469	77.8433	39.0602	1807.97
100	200	0.2	0.2	0.1113	6.1394	0.2584	1800.27
100	400	0.2	0.2	4.0199	22.6182	6.3574	1800.15
100	600	0.2	0.2	14.6664	35.6069	15.9543	1805.19
200	400	0.2	0.2	0.9991	5.3451	1.0499	1810.37
200	600	0.2	0.2	4.5424	10.6059	14.3498	1889.13
50	200	0.2	0.8	0.8298	33.4059	1.0115	1800.11
50	400	0.2	0.8	11.8754	42.0692	16.4411	1801.42
50	600	0.2	0.8	24.6840	96.0829	32.7535	1800.96
100	200	0.2	0.8	0.0291	4.7786	0.0291	1800.47
100	400	0.2	0.8	2.1136	20.3964	3.4560	1805.67
100	600	0.2	0.8	12.3031	35.6683	14.0160	1817.99
200	400	0.2	0.8	0.2123	9.3896	0.4837	1800.23
200	600	0.2	0.8	3.7770	17.6274	4.4652	1833.46
50	200	0.8	0.2	0.0389	12.5954	0.0389	1800.08
50	400	0.8	0.2	13.3947	65.9956	15.1977	1800.04
50	600	0.8	0.2	34.8704	82.1348	38.2156	1803.88
100	200	0.8	0.2	0.0000	9.4214	-	26.88
100	400	0.8	0.2	0.2335	4.0455	0.2348	1801.63
100	600	0.8	0.2	9.1181	27.6410	11.0863	1801.40
200	400	0.8	0.2	0.0020	8.6414	-	205.30
200	600	0.8	0.2	0.0034	11.5793	-	786.63
50	200	0.8	0.8	0.0083	14.9683	-	24.39
50	400	0.8	0.8	9.8202	44.5269	12.5750	1801.95
50	600	0.8	0.8	30.9029	99.1525	36.1072	1808.71
100	200	0.8	0.8	0.0000	0.4394	-	21.84
100	400	0.8	0.8	0.1667	17.2446	0.2030	1801.88
100	600	0.8	0.8	6.1475	13.2578	6.5530	1805.95
200	400	0.8	0.8	0.0000	42.1315	-	122.34
200	600	0.8	0.8	0.0294	8.4615	0.0306	1802.96
				7.1037	29.9925	10.9125	1506.01

D Results on the instances with individual customer profiles after 5 hours

Table 15: Results of the L-Shaped method with the addition of eq. (27) on the instances with $\alpha^V = 0.2$ and individual customer profiles with a time limit of 18000 seconds and a 1% target optimality gap.

$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{K}\rvert$	α^{FROM}	α^{TO}	gap	gapR	gap50	t
50	200	0.2	0.2	0.9337	21.5553	-	120.82
50	400	0.2	0.2	3.0295	59.3308	4.1428	18002.76
50	600	0.2	0.2	10.1187	52.5981	11.3538	18000.13
100	200	0.2	0.2	0.1841	8.9488	-	16.69
100	400	0.2	0.2	0.8770	26.1941	-	841.63
100	600	0.2	0.2	2.6675	40.2920	3.4833	18000.07
200	400	0.2	0.2	0.1093	9.9094	-	224.15
200	600	0.2	0.2	0.3517	14.7626	-	479.14
50	200	0.2	0.8	0.8707	20.9177	-	11.71
50	400	0.2	0.8	1.8429	43.2969	2.5193	18000.92
50	600	0.2	0.8	9.9981	63.4863	10.4941	18000.11
100	200	0.2	0.8	0.2865	10.6250	-	19.35
100	400	0.2	0.8	0.9395	22.5452	-	535.48
100	600	0.2	0.8	2.8913	36.2728	4.4123	18008.31
200	400	0.2	0.8	0.0000	59.2478	-	181.93
200	600	0.2	0.8	0.5768	15.6153	-	726.62
50	200	0.8	0.2	0.7559	22.1339	-	44.38
50	400	0.8	0.2	1.5553	58.5075	1.7269	18000.08
50	600	0.8	0.2	14.8996	72.8427	17.8340	18000.08
100	200	0.8	0.2	0.8890	13.9807	-	121.73
100	400	0.8	0.2	0.9990	11.8809	-	1379.79
100	600	0.8	0.2	4.8890	35.7724	5.7060	18006.26
200	400	0.8	0.2	0.1760	11.8485	-	149.53
200	600	0.8	0.2	0.9180	23.4393	-	5103.61
50	200	0.8	0.8	0.9921	21.9851	-	82.89
50	400	0.8	0.8	1.7121	53.8589	2.1396	18002.06
50	600	0.8	0.8	9.9533	70.6343	10.2331	18000.25
100	200	0.8	0.8	0.9995	14.5185	-	101.45
100	400	0.8	0.8	0.9953	30.1037	-	4394.49
100	600	0.8	0.8	3.0919	34.8283	5.3525	18000.65
200	400	0.8	0.8	0.0001	11.5481	-	164.89
200	600	0.8	0.8	0.9972	15.4424	-	4699.97
				2.4844	31.5289	6.6165	7356.94

Table 16: Results of the L-Shaped method with the addition of eq. (27) on the instances with $\alpha^V = 0.8$ and individual customer profiles with a time limit of 18000 seconds and a 1% target optimality gap.

$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{K}\rvert$	α^{FROM}	α^{TO}	gap	gapR	gap50	t
50	200	0.2	0.2	1.5133	17.7774	1.7378	18000.07
50	400	0.2	0.2	8.8698	60.3835	11.4205	18000.10
50	600	0.2	0.2	16.0592	105.0348	19.0821	18000.14
100	200	0.2	0.2	0.9991	7.5390	-	2345.67
100	400	0.2	0.2	1.9678	33.3830	2.2880	18000.46
100	600	0.2	0.2	6.7921	29.7073	7.3807	18000.10
200	400	0.2	0.2	0.9852	22.1296	-	3051.54

200	600	0.2	0.2	1.6326	17.3350	3.1395	18000.54
50	200	0.2	0.8	1.1847	20.1198	1.3004	18000.58
50	400	0.2	0.8	8.7233	53.4891	9.5341	18000.09
50	600	0.2	0.8	14.6785	89.9520	16.2307	18000.09
100	200	0.2	0.8	0.5693	6.9976	-	43.73
100	400	0.2	0.8	1.7097	32.4136	1.9677	18000.31
100	600	0.2	0.8	6.3776	32.5442	7.1209	18000.09
200	400	0.2	0.8	0.9601	28.3808	-	4120.56
200	600	0.2	0.8	4.6562	26.4194	5.5152	18000.11
50	200	0.8	0.2	0.7444	17.0926	-	15.28
50	400	0.8	0.2	1.5675	32.0731	1.7184	18000.13
50	600	0.8	0.2	8.4393	75.7595	9.0644	18000.30
100	200	0.8	0.2	0.0000	14.6898	-	25.29
100	400	0.8	0.2	0.9903	18.9859	-	579.88
100	600	0.8	0.2	3.8100	40.7039	5.1358	18000.36
200	400	0.8	0.2	0.0000	61.0074	-	164.47
200	600	0.8	0.2	0.3060	12.2654	-	441.39
50	200	0.8	0.8	0.9938	4.6541	-	14.80
50	400	0.8	0.8	0.9997	42.8840	1.0556	12864.17
50	600	0.8	0.8	6.3284	78.3238	6.5753	18000.09
100	200	0.8	0.8	0.2054	1.4263	-	23.31
100	400	0.8	0.8	0.9188	6.4020	-	104.09
100	600	0.8	0.8	2.6157	36.8295	3.3959	18000.60
200	400	0.8	0.8	0.0000	61.6141	-	127.80
200	600	0.8	0.8	0.3572	13.3682	-	445.02
				3.3111	34.4277	6.3146	10324.10

E Example Heuristic

In this section we present a simple Iterated Local Search (ILS) to find primal solutions to the problem. In a nutshell, an ILS works as follows: given an initial (current at a generic iteration) solution it performs a local search. To escape local optima, the solution returned by the local search is randomly perturbed and the local search restarted. This procedure is repeated until a stopping criteria is met. In what follows we explain how this procedure is adapted to our problem.

We encode solutions ζ to the original problem using

- a vector $\Pi(\zeta) \in \mathbb{N}^{|\mathcal{V}|}$ defining the position of the vehicles. The v-th element of the vector is an integer $i \in \mathcal{I}$ identifying the zone where vehicle v becomes available (possibly after relocation).
- a matrix $\Lambda(\zeta) \in \mathbb{N}^{|\mathcal{I}|\times|\mathcal{I}|}$ defining the drop-off-fees. The $i-j$-th element of the matrix is an integer $l \in \mathcal{L}$ identifying the drop-off-bee between zone i and j.

Let $H(a,b)$ be the Hamming distance between two vectors or, in the case of matrices, concatenations of rows. We define two types of neighborhoods.

- $\mathcal{N}^{\Pi}(\zeta) = \left\{\zeta' | H(\Pi(\zeta), \Pi(\zeta')) = 1, \Lambda(\zeta) = \Lambda(\zeta')\right\}$. In words, it defines all solutions which can be obtained from ζ by changing solely one vehicle position.
- $\mathcal{N}^{\Lambda}(\zeta) = \left\{\zeta' | \Pi(\zeta = \Pi(\zeta'), H(\Lambda(\zeta)), \Lambda(\zeta')) = 1\right\}$. In words, it defines all solutions which can be obtained by changing solely one drop-off fee.

We define two *first improvement operators*:

- $f^{\Pi}(\zeta) : \mathbb{N}^{|\mathcal{V}|} \times \mathbb{N}^{|\mathcal{I}|\times|\mathcal{I}|} \rightarrow \mathcal{N}^{\Pi}(\zeta)$ scans the neighborhood $\mathcal{N}^{\Pi}(\zeta)$ and returns the first improving solution (i.e., with a higher fitness value) if it exists, ζ otherwise.
- $f^{\Lambda}(\zeta) : \mathbb{N}^{|\mathcal{V}|} \times \mathbb{N}^{|\mathcal{I}|\times|\mathcal{I}|} \rightarrow \mathcal{N}^{\Lambda}(\zeta)$ scans the neighborhood $\mathcal{N}^{\Lambda}(\zeta)$ and returns the first improving solution (i.e., with a higher fitness value) if it exists, ζ otherwise.

Let $z(\zeta)$ be the fitness function defining how each solution is evaluated. It is defined as the objective function of the original problem (1a). For each solution ζ considered, the second-stage revenue is computed as illustrated in Section 4.2. Given a solution ζ, we define two types of local search

- $LS^{\Pi}(\zeta) : \mathbb{N}^{|\mathcal{V}|} \times \mathbb{N}^{|\mathcal{I}| \times |\mathcal{I}|} \rightarrow \mathcal{N}^{\Pi}(\zeta)$ which performs a local search on the $\mathcal{N}^{\Pi}(\zeta)$ neighborhood using the $f^{\Pi}(\zeta)$ first improvement operator
- $LS^{\Lambda}(\zeta) : \mathbb{N}^{|\mathcal{V}|} \times \mathbb{N}^{|\mathcal{I}| \times |\mathcal{I}|} \rightarrow \mathcal{N}^{\Lambda}(\zeta)$ which performs a local search on the $\mathcal{N}^{\Lambda}(\zeta)$ neighborhood using the $f^{\Lambda}(\zeta)$ first improvement operator.

Algorithm 4 sketches the local search procedures.

Algorithm 4 Local Search

INPUT: ζ, Operator $f(\zeta)$ to use (i.e., $f^{\Pi}(\zeta)$ or $f^{\Lambda}(\zeta)$).
INPUT: `TIMELIMIT`
`FOUND=TRUE`
$\zeta^{CURRENT} = \zeta$
while $FOUND$ AND `ELAPSEDTIME≤TIMELIMIT` **do**
  $\zeta^{N} \leftarrow f(\zeta^{CURRENT})$
  if $z(\zeta^{N}) > z(\zeta^{CURRENT})$ **then**
    $\zeta^{CURRENT} \leftarrow \zeta^{N}$
  else
    `FOUND=FALSE`
  end if
end while
return $\zeta^{CURRENT}$

Finally, we define a function $P(\zeta, R) : \mathbb{N}^{|\mathcal{V}|} \times \mathbb{N}^{|\mathcal{I}| \times |\mathcal{I}|} \rightarrow \mathbb{N}^{|\mathcal{V}|} \times \mathbb{N}^{|\mathcal{I}| \times |\mathcal{I}|}$ that randomly re-assigns $R\%$ of the positions and $R\%$ of the fees. The entire Iterated Local Search is sketched in Algorithm 5.

Algorithm 5 Iterated Local Search

INPUT: `MAXRESTARTSWITHOUTIMPROVEWMENT`, `TIMELIMIT`, R
`#RESTARTSWITHOUTIMPROVEWMENT`$\leftarrow 0$
$\zeta^{CURRENT} \leftarrow$ random solution
$\zeta^{BEST} \leftarrow \zeta^{CURRENT}$
while `#RESTARTSWITHOUTIMPROVEWMENT≤MAXRESTARTSWITHOUTIMPROVEWMENT` AND `ELAPSEDTIME≤TIMELIMIT` **do**
  $\zeta^{N} \leftarrow LS^{\Pi}(LS^{\Lambda}(\zeta^{CURRENT}))$
  if $z(\zeta^{N}) < z(\zeta^{CURRENT})$ **then**
    `#RESTARTSWITHOUTIMPROVEWMENT`$\leftarrow$ `#RESTARTSWITHOUTIMPROVEWMENT`$+1$
  else
    `#RESTARTSWITHOUTIMPROVEWMENT`$\leftarrow 0$
  end if
  if $z(\zeta^{N}) > z(\zeta^{BEST})$ **then**
    $\zeta^{BEST} \leftarrow \zeta^{N}$
  end if
  $\zeta^{CURRENT} \leftarrow P(\zeta^{N}, R)$
end while
return ζ^{BEST}

In Algorithm 5 we set `MAXRESTARTSWITHOUTIMPROVEWMENT` to 3, `TIMELIMIT` to 1800 seconds, and R to 30%.

Tables 17 and 18 report the optimality gap and solution time of the ILS compared to that of the L-Shaped method on all instances with identical customer profiles. The optimality gap of the ILS is calculated using the bound delivered by the L-Shaped method as

$$\texttt{gap} = \frac{|\texttt{ILSOBJECTIVE} - \texttt{LSBESTBOUND}|}{|ILSOBJECTIVE| + 10^{-10}}$$

The tables show that, for the smallest instances, the performance of the ILS is comparable to that of the L-Shaped method. On a number of instances (e.g., with $|\mathcal{V}| = 50$) the ILS even delivers better primal solutions than the L-Shaped method. Nevertheless, as the size of the instances increases the performance of the ILS drops.

Table 17: Comparison of optimality gap and solution time obtained with ILS and the LS method on the instances with identical customer profiles and with $\alpha^V = 0.2$.

$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{K}\rvert$	α^{FROM}	α^{TO}	gapLS	gapILS	tLS	tILS
50	200	0.2	0.2	0.6967	1.2615	1800.20	1801.00
50	400	0.2	0.2	17.2850	14.4768	1800.52	1800.13
50	600	0.2	0.2	30.5015	21.5516	1809.94	1800.05
100	200	0.2	0.2	0.0975	0.3102	1800.98	1804.39
100	400	0.2	0.2	1.9480	3.9251	1805.67	1802.78
100	600	0.2	0.2	11.6208	44.7249	1801.95	1811.11
200	400	0.2	0.2	0.0000	276.9414	139.41	1810.90
200	600	0.2	0.2	0.0777	456.4824	1800.78	1812.87
50	200	0.2	0.8	0.3381	0.7718	1800.63	1800.57
50	400	0.2	0.8	9.6397	8.5835	1801.21	1801.20
50	600	0.2	0.8	19.5455	15.2189	1800.03	1801.78
100	200	0.2	0.8	0.2815	0.2815	1800.13	1801.84
100	400	0.2	0.8	0.7148	3.6945	1801.47	1801.23
100	600	0.2	0.8	9.8896	38.4997	1812.93	1807.95
200	400	0.2	0.8	0.0000	258.6824	115.74	1803.26
200	600	0.2	0.8	0.4710	421.6527	1814.99	1810.50
50	200	0.8	0.2	1.9850	3.9133	1800.13	1800.40
50	400	0.8	0.2	21.8829	17.4800	1800.02	1801.26
50	600	0.8	0.2	44.9080	33.0970	1803.83	1801.01
100	200	0.8	0.2	0.5238	1.3703	1801.09	1801.37
100	400	0.8	0.2	6.1743	8.1636	1806.48	1800.58
100	600	0.8	0.2	23.6427	52.5532	1816.20	1806.71
200	400	0.8	0.2	0.3072	260.9089	1804.22	1817.48
200	600	0.8	0.2	6.7474	386.4842	1865.22	1801.24
50	200	0.8	0.8	0.8509	0.9891	1800.15	1801.00
50	400	0.8	0.8	15.4867	12.6303	1801.46	1801.19
50	600	0.8	0.8	40.1722	28.5881	1807.37	1800.41
100	200	0.8	0.8	0.7932	1.4929	1802.32	1800.77
100	400	0.8	0.8	4.4619	5.6145	1803.98	1807.19
100	600	0.8	0.8	19.1824	45.0015	1823.21	1806.86
200	400	0.8	0.8	0.0963	242.2407	1800.78	1806.16
200	600	0.8	0.8	5.6194	307.5339	1823.23	1811.13

Table 18: Comparison of optimality gap and solution time obtained with ILS and the LS method on the instances with identical customer profiles and with $\alpha^V = 0.8$.

$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{K}\rvert$	α^{FROM}	α^{TO}	gapLS	gapILS	tLS	tILS
50	200	0.2	0.2	0.4613	1.2477	1800.37	1800.88
50	400	0.2	0.2	11.6098	10.5114	1802.87	1800.99
50	600	0.2	0.2	30.3469	23.8963	1807.97	1800.32
100	200	0.2	0.2	0.1113	0.4159	1800.27	1801.25
100	400	0.2	0.2	4.0199	6.9075	1800.15	1806.85
100	600	0.2	0.2	14.6664	42.8427	1805.19	1813.64
200	400	0.2	0.2	0.9991	235.4502	1810.37	1814.95
200	600	0.2	0.2	4.5424	347.5242	1889.13	1817.40

50	200	0.2	0.8	0.8298	1.2781	1800.11	1800.27
50	400	0.2	0.8	11.8754	8.8758	1801.42	1800.32
50	600	0.2	0.8	24.6840	21.8355	1800.96	1801.07
100	200	0.2	0.8	0.0291	1.0756	1800.47	1800.33
100	400	0.2	0.8	2.1136	5.9566	1805.67	1800.81
100	600	0.2	0.8	12.3031	36.9526	1817.99	1803.40
200	400	0.2	0.8	0.2123	219.8087	1800.23	1811.94
200	600	0.2	0.8	3.7770	332.4861	1833.46	1805.89
50	200	0.8	0.2	0.0389	0.3443	1800.08	1800.10
50	400	0.8	0.2	13.3947	12.2721	1800.04	1801.02
50	600	0.8	0.2	34.8704	26.9711	1803.88	1802.90
100	200	0.8	0.2	0.0000	0.0000	26.88	1800.97
100	400	0.8	0.2	0.2335	1.6940	1801.63	1802.90
100	600	0.8	0.2	9.1181	55.6383	1801.40	1810.57
200	400	0.8	0.2	0.0020	232.7449	205.30	1806.80
200	600	0.8	0.2	0.0034	331.4251	786.63	1816.44
50	200	0.8	0.8	0.0083	0.2765	24.39	1800.52
50	400	0.8	0.8	9.8202	8.4977	1801.95	1802.57
50	600	0.8	0.8	30.9029	21.7163	1808.71	1800.03
100	200	0.8	0.8	0.0000	0.0000	21.84	1800.08
100	400	0.8	0.8	0.1667	1.3339	1801.88	1805.07
100	600	0.8	0.8	6.1475	48.0809	1805.95	1809.60
200	400	0.8	0.8	0.0000	224.0099	122.34	1800.48
200	600	0.8	0.8	0.0294	285.5657	1802.96	1815.21

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