Dynamic programming in convex stochastic optimization

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Abstract

This paper studies the dynamic programming principle for general convex stochastic optimization problems introduced by Rockafellar and Wets in [30]. We extend the applicability of the theory by relaxing compactness and boundedness assumptions. In the context of financial mathematics, the relaxed assumptions are satisfied under the well-known no-arbitrage condition and the reasonable asymptotic elasticity condition of the utility function. Besides financial mathematics, we obtain several new results in linear and nonlinear stochastic programming and stochastic optimal control.

Keywords. Dynamic programming, stochastic programming, convexity

1 Introduction

Given a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$ (an increasing sequence of sub-σ-algebras of $\mathcal{F}$), consider the convex stochastic optimization problem

$$\text{minimize} \quad Eh(x) \quad \text{over } x \in \mathcal{N}, \qquad (SP)$$

where $\mathcal{N}$ is a linear space of stochastic processes $x = (x_t)_{t=0}^T$ adapted to $(\mathcal{F}_t)_{t \geq 0}$ (i.e., x_t is $\mathcal{F}_t$-measurable). We assume that x_t takes values in a Euclidean space $\mathbb{R}^{n_t}$ so the process $x = (x_t)_{t=0}^T$ takes values in $\mathbb{R}^n$ where $n := n_0 + \cdots + n_T$. The objective is defined on the space $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ of $\mathcal{F}$-measurable $\mathbb{R}^n$-valued functions x by

$$Eh(x) := \int_\Omega h(x(\omega), \omega) dP(\omega),$$

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where h is a convex normal integrand, i.e. an extended real-valued function on $\mathbb{R}^n \times \Omega$ such that $\omega \mapsto \operatorname{epi} h(\cdot, \omega)$ is a closed convex-valued measurable mapping; see e.g. [33, Chapter 14]. By [33, Proposition 14.28], $\omega \mapsto h(x(\omega), \omega)$ is measurable for all $x \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$. Here and in what follows, we define the integral of an extended real-valued random variable as $+\infty$ unless its positive part is integrable. The integral functional Eh is thus a well-defined extended real-valued convex function on $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$.

Problems of the form (SP) were first studied in [30, 13] where it was observed that many more specific stochastic optimization problems can be written in this unified format. These include more traditional formulations of stochastic programming, convex stochastic control and various problems in financial mathematics; see Section 5 below.

The articles [30, 13] gave formulations of the *dynamic programming* principle general enough to apply to the abstract stochastic optimization format (SP). They extend many earlier formulations of the stochastic dynamic programming principle such as those in [10, 1, 2, 11]. Like the present article, [30] studied the convex case where decisions are described by finite dimensional vectors. The article [13] extended the results to nonconvex problems where decisions are described by elements of Polish spaces. Both assumed that the objective is lower bounded and that the decisions are taken from a compact set, uniformly compact in [30]. The compactness assumptions were relaxed in [21] in the convex case and in [25] in the nonconvex case. This paper relaxes the lower boundedness assumption of the objective. This is interesting in many applications in financial mathematics as well as in mathematical programming such as linear stochastic optimization. In portfolio optimization problems, the relaxed conditions holds if the utility function satisfies the *reasonable asymptotic elasticity condition* that is extensively studied in financial mathematics; see e.g. [18, 28]. These problems are often set in a stochastic control format, which motivates the study of stochastic control beyond lower bounded objectives as well. We also provide new results on the theory of normal integrands concerning conditional independence. This allows us to derive many well-known results on Markov decision processes using the theory of normal integrands; see Section 5.

Like [30, 13], our approach builds on the notion of *conditional expectation of a normal integrand*, introduced in [7]. This allows for significant extensions to many better known formulations of stochastic dynamic programming while greatly simplifying the measurability questions that may come up in the dynamic programming recursion. More traditional formulations will be obtained as special cases in Section 5 below. A good illustration of potential measurability complications can be found in [5] which follows a different line of analysis not building on the theory of normal integrands; see [5, Section 1.2.II] and Section 5.3 below for a further comparison.

An extended real-valued random variable X is said to be *quasi-integrable* if either X^+ or X^- is integrable. Given a quasi-integrable X and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, there exists an extended real-valued $\mathcal{G}$-measurable random variable

$E^{\mathcal{G}}X$, almost surely unique, such that

$$E\left[\alpha(E^{\mathcal{G}}X)\right] = E\left[\alpha X\right] \quad \forall \alpha \in L^{\infty}_{+}(\Omega, \mathcal{G}, P).$$

The random variable $E^{\mathcal{G}}X$ is known as the $\mathcal{G}$-conditional expectation of X. The following extends the operation of conditional expectation to normal integrands. Sufficient conditions for its existence will be given in Section 2.

Definition 1. We say that a normal integrand $E^{\mathcal{G}}h$ is the $\mathcal{G}$-*conditional expectation* of a normal integrand h if it is the almost surely everywhere unique $\mathcal{G}$-measurable normal integrand such that

$$(E^{\mathcal{G}}h)(x) = E^{\mathcal{G}}[h(x)] \quad \text{a.s.}$$

for all $x \in L^0(\mathcal{G})$ for which $h(x)$ is quasi-integrable.

We will use the notations $x^t = (x_0, \ldots, x_t)$, $n^t = n_0 + \cdots + n_t$ and $E_t = E^{\mathcal{F}_t}$. We say that an adapted sequence $(h_t)_{t=0}^{T}$ of normal integrands $h_t : \mathbb{R}^{n^t} \times \Omega \to \overline{\mathbb{R}}$ solves the generalized *Bellman equations for h* if

$$\begin{aligned}
h_T &= E_T h, \\
h_t &= E_t \inf_{x_{t+1}} h_{t+1} \quad t = T - 1, \ldots, 0.
\end{aligned} \tag{BE}$$

More precisely, this means that there exists another sequence $(\tilde{h}_t)_{t=0}^{T-1}$ of normal integrands such that

$$\begin{aligned}
h_T &= E_T h, \\
\tilde{h}_t(x^t, \omega) &= \inf_{x_{t+1} \in \mathbb{R}^{n_{t+1}}} h_{t+1}(x^t, x_{t+1}, \omega), \\
h_t &= E_t \tilde{h}_t
\end{aligned} \tag{BE}$$

for $t = T - 1, \ldots, 0$. This paper gives sufficient conditions for the existence of the solutions of (BE); see Theorems 16 and 22 below. We also show that, when solutions exist, they provide useful characterizations of the optimum values and solutions of (SP); see Theorems 17 and 23 below. In the literature of stochastic control, results such as Theorems 17 and 23 relating the solutions of the Bellman equations to the solutions of the optimization problem are often called "verification theorems"; see e.g., [14]. Theorems 17 and 23 are illustrated by deriving several new results, e.g., in linear stochastic programming, stochastic control and financial mathematics.

2 Conditional expectations of normal integrands

The general theory of dynamic programming studied in this paper builds on conditional expectations of normal integrands. This section reviews the theory that will be used in the analysis in the subsequent sections. Most of the results below can be found in the literature but we include the simple proofs because that allows us to make some extensions to the existing theory.

2.1 Existence

Conditional expectations of convex normal integrands were introduced in [7]. The article [12] extended the definition to general $\mathcal{B} \otimes \mathcal{F}$-measurable lower bounded integrands. More general conditions for the existence of a conditional expectation of a normal integrand have been given in [35, 36, 9]. Our arguments and conditions for existence in Section 2.1 are largely from [7].

We say that a normal integrand h is *L-bounded* if there exist $\rho, m \in L^1$ with

$$h(x) \geq -\rho|x| - m \quad \forall x \in \mathbb{R}^n.$$

Condition 3 in the following lemma was used for the existence of a conditional expectation in [7, Theorem 2].

Lemma 1. *For a convex normal integrand h, the following are equivalent:*

1. *h is L-bounded,*

2. *there exist $v \in L^1(\mathbb{R}^n)$ and $m \in L^1$ such that*

$$h(x, \omega) \geq x \cdot v(\omega) - m(\omega),$$

3. *$\operatorname{dom} Eh^* \cap L^1 \neq \emptyset$.*

Proof. Let $v \in L^1$ such that $Eh^*(v) < \infty$ is finite. By Fenchel's inequality,

$$h(x, \omega) \geq x \cdot v - h^*(v, \omega) \geq -|v||x| - h^*(v, \omega),$$

so we may choose $\rho = |v|$ and $m = h^*(v)^+$. On the other hand, $h \geq -\rho|\cdot| - m$ can be written as $(h + \rho|\cdot|)^*(0) \leq m$. By [29, Theorem 16.4], this means that

$$\inf_{v \in \mathbb{R}^n} \{h^*(v) + \delta_{\mathbb{B}}(v/\rho)\} \leq m,$$

where the infimum is attained. By [33, Theorem 14.37], there is a $v \in L^0$ with $|v| \leq \rho$ and $h^*(v) \leq m$. $\qquad\square$

If a normal integrand h is L-bounded, then $h(x)$ is quasi-integrable and $E^{\mathcal{G}}[h(x)]$ is well-defined for every $x \in L^\infty(\mathcal{G})$. The following lemma shows that, for L-bounded normal integrands, it suffices to test with $x \in L^\infty(\mathcal{G})$ in the definition of conditional expectation.

Lemma 2. *Given an L-bounded normal integrand h, a $\mathcal{G}$-normal integrand $\bar{h}$ is the $\mathcal{G}$-conditional expectation of h if and only if*

$$\bar{h}(x) = E^{\mathcal{G}}[h(x)] \quad a.s.$$

for all $x \in L^\infty(\mathcal{G})$. In particular, if $h(x)$ is $\mathcal{G}$-measurable for every $x \in L^\infty(\mathcal{G})$, then $E^{\mathcal{G}}h = h$.

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Proof. It is clear that the condition holds if $\bar{h} = E^{\mathcal{G}}h$. On the other hand, if $\bar{h} \neq E^{\mathcal{G}}h$, then there exists an $x \in L^0(\mathcal{G})$ such that $h(x)$ is quasi-integrable and $\bar{h}(x) \neq E^{\mathcal{G}}[h(x)]$. For ν large enough,

$$\mathbb{1}_{\{|x| \leq \nu\}} \bar{h}(x) \neq \mathbb{1}_{\{|x| \leq \nu\}} E^{\mathcal{G}}[h(x)].$$

The left side equals $\mathbb{1}_{\{|x| \leq \nu\}} \bar{h}(\mathbb{1}_{\{|x| \leq \nu\}} x)$ while the right side equals

$$\begin{aligned}
\mathbb{1}_{\{|x| \leq \nu\}} E^{\mathcal{G}}[h(x)] &= \mathbb{1}_{\{|x| \leq \nu\}} E^{\mathcal{G}}[\mathbb{1}_{\{|x| \leq \nu\}} h(x)] \\
&= \mathbb{1}_{\{|x| \leq \nu\}} E^{\mathcal{G}}[\mathbb{1}_{\{|x| \leq \nu\}} h(\mathbb{1}_{\{|x| \leq \nu\}} x)] \\
&= \mathbb{1}_{\{|x| \leq \nu\}} E^{\mathcal{G}}[h(\mathbb{1}_{\{|x| \leq \nu\}} x)].
\end{aligned}$$

Thus, $\bar{h}(\mathbb{1}_{\{|x| \leq \nu\}} x) \neq E^{\mathcal{G}}[h(\mathbb{1}_{\{|x| \leq \nu\}} x)]$ so the condition fails. By [35, Lemma 6], the condition defines $\bar{h}$ uniquely almost surely everywhere. $\square$

Example 3. *Given $Q \in L^1(\mathbb{R}^{n \times n})$, $v \in L^1(\mathbb{R}^n)$, $m \in L^1$, the function*

$$h(x, \omega) := \frac{1}{2} x \cdot Q(\omega)x + x \cdot v(\omega) + m(\omega).$$

is a normal integrand, by [33, Example 14.29]. If Q is almost surely positive definite, then h is L-bounded and

$$(E^{\mathcal{G}}h)(x, \omega) = \frac{1}{2} x \cdot E^{\mathcal{G}}[Q](\omega)x + x \cdot (E^{\mathcal{G}}v)(\omega) + (E^{\mathcal{G}}m)(\omega).$$

Proof. Given $x \in L^\infty$, Lemma 59 gives

$$E^G[h(x)] = \frac{1}{2} x \cdot E^{\mathcal{G}}[Q](\omega)x + x \cdot (E^{\mathcal{G}}v)(\omega) + (E^{\mathcal{G}}m)(\omega),$$

so the claim follows from Lemma 2. $\square$

Theorem 4. *Let h be a real-valued normal integrand such that there exist $\bar{x} \in \mathbb{R}^n$ and $\rho \in L^1$ with $h(\bar{x}) \in L^1$ and*

$$|h(x) - h(x')| \leq \rho |x - x'| \quad \forall x, x' \in \mathbb{R}^n.$$

Then $E^{\mathcal{G}}h$ exists and it is characterized by

$$(E^{\mathcal{G}}h)(x) = E^{\mathcal{G}}[h(x)] \quad \forall x \in \mathbb{R}^n.$$

Moreover,

$$|(E^{\mathcal{G}}h)(x - x')| \leq (E^{\mathcal{G}}\rho)|x - x'| \quad \forall x, x' \in \mathbb{R}^n. \tag{1}$$

Proof. The assumptions imply that $h(x) \in L^1$ for all $x \in \mathbb{R}^n$ and that, by Jensen's inequality,

$$|E^{\mathcal{G}}[h(x)] - E^{\mathcal{G}}[h(x')]| \leq (E^{\mathcal{G}}\rho)|x - x'| \quad a.s.$$

for all $x, x' \in \mathbb{R}^n$. Let D be a countable dense set in $\mathbb{R}^n$ and define $\tilde{h}(x, \omega) := E^{\mathcal{G}}[h(x)](\omega)$ for each $x \in D$. By countability of D, there is a P-null set $N \in \mathcal{G}$ such that, for all $\omega \in \Omega \setminus N$,

$$|\tilde{h}(x, \omega) - \tilde{h}(x', \omega)| \leq (E^{\mathcal{G}}\rho)(\omega)|x - x'| \quad \forall x, x' \in D.$$

The function $\tilde{h}$ has a unique continuous extension to $\mathbb{R}^n \times (\Omega \setminus N)$. Finally, we extend the definition of $\tilde{h}$ to all of $\mathbb{R}^n \times \Omega$ by setting $\tilde{h}(\cdot, \omega) = 0$ for $\omega \in N$. The function $\tilde{h}$ thus constructed is a $\mathcal{G}$-measurable Caratheodory integrand and thus, normal. It is clear that it satisfies (1) as well.

If $x = \sum_{i=1}^{\nu} x^i \mathbb{1}_{A^i}$, where $x^i \in D$ and $A^i \in \mathcal{G}$ form a disjoint partition of Ω, we have

$$E^{\mathcal{G}}[h(x)] = E^{\mathcal{G}}[h(\sum_{i=1}^{\nu} x^i \mathbb{1}_{A^i})] = E^{\mathcal{G}}[\sum_{i=1}^{\nu} \mathbb{1}_{A^i} h(x^i)] = \sum_{i=1}^{\nu} \mathbb{1}_{A^i} \tilde{h}(x^i) = \tilde{h}(x).$$

Any $x \in L^\infty(\mathcal{G})$ is an almost sure limit of such simple random variables bounded by $\|x\|_{L^\infty}$, so dominated convergence for conditional expectations (see e.g. [34, Theorem II.7.2]) and scenariowise continuity of h and $\tilde{h}$ imply that $E^{\mathcal{G}}[h(x)] = \tilde{h}(x)$. Thus Lemma 2 implies the claim. $\square$

Counterexample 5. *The claim of Theorem 4 fails if ρ is not integrable. Indeed, let ξ be uniformly distributed on $(0, 1)$, $\eta > 1$ be nonintegrable independent of ξ and $\mathcal{G}$ be generated by ξ. Let $h(x, \omega) = \eta(\omega)|x - \xi(\omega)|$. Then $E^{\mathcal{G}}[h(x)] = +\infty$ for every constant $x \in \mathbb{R}$. On the other hand, choosing $x = \xi$, we have $E^{\mathcal{G}}[h(x)] = 0$, so $E^{\mathcal{G}}h$ is not characterized by constants even though h is Lipschitz and L-bounded convex normal integrand.*

Lemma 6. *Let h^1 and h^2 be L-bounded normal integrands with $h^1 \leq h^2$. Then $E^{\mathcal{G}}h^1 \leq E^{\mathcal{G}}h^2$ whenever the conditional expectations exist.*

Proof. For any $x \in L^\infty(\mathcal{G})$, $h^1(x)$ and $h^2(x)$ are quasi-integrable, so $h^1 \leq h^2$ implies

$$(E^{\mathcal{G}}h^1)(x) = E^{\mathcal{G}}[h_1(x)] \leq E^{\mathcal{G}}[h_2(x)] = (E^{\mathcal{G}}h^2)(x)$$

and the result follows from [35, Lemma 6]. $\square$

The following result is a monotone convergence theorem for conditional expectations of integrands.

Theorem 7. *Let $(h^\nu)_{\nu=1}^\infty$ be a nondecreasing sequence of L-bounded normal integrands and*

$$h = \sup_\nu h^\nu.$$

If each $E^{\mathcal{G}}h^\nu$ exists, then $E^{\mathcal{G}}h$ exists and

$$E^{\mathcal{G}}h = \sup_\nu E^{\mathcal{G}}h^\nu.$$

Proof. For any $x \in L^\infty(\mathcal{G})$ and $\alpha \in L^\infty(\mathcal{G}; \mathbb{R}_+)$, monotone convergence and Lemma 6 imply that

$$E[\alpha h(x)] = E[\alpha \sup_\nu h^\nu(x)] = \sup_\nu E[\alpha h^\nu(x)]$$
$$= \sup_\nu E[\alpha E^\mathcal{G}[h^\nu(x)]] = E[\alpha \sup_\nu E^\mathcal{G}[h^\nu(x)]].$$

Thus $\sup_\nu E^\mathcal{G} h^\nu = E^\mathcal{G} h$. $\square$

The following is our main result on the existence of conditional normal integrands.

Theorem 8. *An L-bounded (convex) normal integrand admits a conditional expectation and that is L-bounded (and convex) as well.*

Proof. Let h be an L-bounded normal integrand. Assume first that $h \leq m$ for some constant $m > 0$. By [33, Example 9.11],

$$h^\nu(x, \omega) := \inf_{x'} \{ h(x', \omega) + \nu \rho(\omega)|x - x'| \}$$

form a nondecreasing sequence of Caratheodory functions increasing pointwise to h. The assumed upper bound and the L-boundedness of h imply that h^ν satisfy the assumptions of Theorem 4. Thus, by Theorems 4 and 7, $E^\mathcal{G} h$ exists. To remove the assumption $h \leq m$, consider the nondecreasing sequence of functions $h^n(x) := \min\{h(x), m\}$ (which are normal integrands, by [33, Proposition 14.44]) and apply Theorem 7 again. The preservation of convexity follows from [36, Proposition 1.6.2]. $\square$

The truncation argument in the above proof is adapted from [35]. It gives the existence under slightly more general conditions than [7] who assumed the existence of a $\mathcal{G}$-measurable x such that $h(x)$ is integrable. More general existence results in the nonconvex case have been given in [36, 9].

2.2 Conditional expectations in operations

Most results in this section can be found in [7] [15] and [36] with the exception of Theorem 11 and parts 2 and 3 of Theorem 13 which seem new. Theorem 15 extends Corollary 1 of [7, Theorem 3] by slightly relaxing the assumptions on the domains of the integral functionals.

Recall that if ξ is a quasi-integrable random variable and $\mathcal{G}'$ is a sub-σ-algebra of $\mathcal{G}$, then
$$E^{\mathcal{G}'}[E^\mathcal{G} \xi] = E^{\mathcal{G}'} \xi;$$

see Lemma 60. This extends to normal integrands as follows.

Theorem 9 (Tower property)**.** *Assume that h is an L-bounded normal integrand, and $\mathcal{G}'$ is a sub-σ-algebra of $\mathcal{G}$. Then*

$$E^{\mathcal{G}'}(E^\mathcal{G} h) = E^{\mathcal{G}'} h.$$

Proof. By Theorem 8, $E^{\mathcal{G}}h$ exists and is L-bounded, so the result follows from the usual tower property (see Lemma 60) and Lemma 2. $\quad\square$

Given $\mathcal{H} \subset \mathcal{F}$, σ-algebras $\mathcal{G}$ and $\mathcal{G}'$ are *$\mathcal{H}$-conditionally independent* if

$$E^{\mathcal{H}}[1_{A'}1_A] = E^{\mathcal{H}}[1_{A'}]E^{\mathcal{H}}[1_A]$$

for every $A \in \mathcal{G}$ and $A' \in \mathcal{G}'$. A random variable w is *$\mathcal{H}$-conditionally independent* of $\mathcal{G}$ if $\sigma(w)$ and $\mathcal{G}$ are $\mathcal{H}$-conditionally independent. Likewise, we say that a normal integrand h is *$\mathcal{H}$-conditionally independent* of $\mathcal{G}$ if $\sigma(h)$ and $\mathcal{G}$ are $\mathcal{H}$-conditionally independent. Here $\sigma(h)$ is the smallest σ-algebra under which $\operatorname{epi} h$ is measurable. In other words, $\sigma(h)$ is generated by the family

$$\{(\operatorname{epi} h)^{-1}(O) \mid O \subset \mathbb{R}^{n+1} \text{ open}\}.$$

Example 10. *Let ξ be a random variable with values in a measurable space $(\Xi, \mathcal{A})$, H a $\mathcal{A}$-normal integrand on $\mathbb{R}^n$ and*

$$h(x,\omega) = H(x,\xi(\omega)).$$

If ξ is $\mathcal{H}$-conditionally independent of $\mathcal{G}$, then h is so too. Indeed, given an open $O \subset \mathbb{R}^{n+1}$, we have $(\operatorname{epi} h)^{-1}(O) = \xi^{-1}((\operatorname{epi} H)^{-1}(O))$, so $\sigma(h) \subseteq \sigma(\xi)$. To conclude, it suffices to note that sub-σ-algebras inherit conditional independence.

If an integrable random variable w is $\mathcal{H}$-conditionally independent of $\mathcal{G}$, then

$$E^{\mathcal{G} \vee \mathcal{H}}[w] = E^{\mathcal{H}}[w],$$

by Lemma 61. This extends to normal integrands as follows.

Theorem 11. *Let h be an L-bounded normal integrand $\mathcal{H}$-conditionally independent of $\mathcal{G}$. Then*

$$E^{\mathcal{G} \vee \mathcal{H}}h = E^{\mathcal{H}}h.$$

In particular, if h is independent of $\mathcal{G}$, then $E^{\mathcal{G}}h$ is deterministic.

Proof. Assume first that h satisfies the assumptions of Theorem 4. Then $E^{\mathcal{H}}h$ is characterized by

$$E^{\mathcal{H}}(h(x)) = (E^{\mathcal{H}}h)(x) \quad \forall x \in \mathbb{R}^n$$

and likewise for $E^{\mathcal{G} \vee \mathcal{H}}h$. Thus $E^{\mathcal{G} \vee \mathcal{H}}h = E^{\mathcal{H}}h$, by Lemma 61. The first claim now follows using Lipschitz regularizations as in the proof of Theorem 8. The second claim follows by taking $\mathcal{H}$ the trivial σ-algebra. $\quad\square$

Remark 12. *Conditional expectation is a linear operator on the linear space of normal integrands that satisfy the assumptions of Theorem 4.*

Proof. If h satisfies the assumptions of Theorem 4, then so does $-h$, and $h(x) \in L^1$ for every $x \in \mathbb{R}^n$. Thus

$$E^{\mathcal{G}}[-h](x) = E^{\mathcal{G}}[-h(x)] = -E^{\mathcal{G}}[h(x)] = -(E^{\mathcal{G}}h)(x) \quad \forall x \in \mathbb{R}^n,$$

so $E^{\mathcal{G}}[-h] = -E^{\mathcal{G}}h$, by Theorem 4. Additivity is proved similarly. $\quad\square$

The class of normal integrands is not a linear space, so one cannot hope for linearity of the conditional expectation, in general.

The product of two extended real numbers is defined as zero if one of them is zero while the nonnegative scalar multiple of a function h is defined by

$$(\alpha h)(x) := \begin{cases} \alpha h(x) & \text{if } \alpha > 0, \\ \delta_{\operatorname{cl dom} h}(x) & \text{if } \alpha = 0. \end{cases}$$

Equivalently,

$$(\alpha h)(x) := \alpha h(x) + \delta_{\operatorname{cl dom} h}(x) \quad \alpha \geq 0.$$

Theorem 13. *Let h, h^1 and h^2 be L-bounded normal integrands.*

1. *$h^1 + h^2$ is L-bounded and $E^{\mathcal{G}}(h^1 + h^2) = E^{\mathcal{G}} h^1 + E^{\mathcal{G}} h^2$.*

2. *If $\alpha \in L^1_+$, h is $\mathcal{G}$-measurable and αh is L-bounded, then $E^{\mathcal{G}}(\alpha h) = E^{\mathcal{G}}[\alpha] h$.*

3. *If $\alpha \in L^0_+(\mathcal{G})$ and αh is L-bounded, then $E^{\mathcal{G}}(\alpha h) = \alpha E^{\mathcal{G}} h$ if either α is strictly positive or $F^{\mathcal{G}}[\operatorname{cl dom} h] = \operatorname{cl dom} E^{\mathcal{G}} h$.*

Proof. Let $x \in L^{\infty}(\mathcal{G})$. Since $h^1(x)^-$ and $h^2(x)^-$ are integrable, 1 follows from Lemma 2 and the first part of Lemma 59. In 2,

$$\begin{aligned} E^{\mathcal{G}}[(\alpha h)(x)] &= E^{\mathcal{G}}[\alpha h(x) + \delta_{\operatorname{cl dom} h}(x)] \\ &= E^{\mathcal{G}}[\alpha] h(x) + \delta_{\operatorname{cl dom} h}(x), \\ &= (E^{\mathcal{G}}[\alpha] h)(x), \end{aligned}$$

by Lemma 59 so the claim follows from Lemma 2 again. In 3,

$$\begin{aligned} E^{\mathcal{G}}[(\alpha h)(x)] &= E^{\mathcal{G}}[\alpha h(x) + \delta_{\operatorname{cl dom} h}(x)] \\ &= \alpha E^{\mathcal{G}}[h(x)] + E^{\mathcal{G}}[\delta_{\operatorname{cl dom} h}(x)], \end{aligned}$$

by Lemma 59. If α is strictly positive or if $F^{\mathcal{G}}[\operatorname{cl dom} h] = \operatorname{cl dom} E^{\mathcal{G}} h$, we thus get

$$\begin{aligned} E^{\mathcal{G}}[(\alpha h)(x)] &= \alpha (E^{\mathcal{G}} h)(x) + \delta_{\operatorname{cl dom} E^{\mathcal{G}} h}(x) \\ &= (\alpha E^{\mathcal{G}} h)(x), \end{aligned}$$

which completes the proof. $\square$

The following shows that part 3 of Theorem 13 may fail without the extra assumptions.

Counterexample 14. *Let $\mathcal{G}$ be trivial, $\alpha = 0$ almost surely and $h(x, \omega) := \eta |x|$, where $\eta \notin L^1$ is nonnegative. Then $\alpha h = 0$ while $E^{\mathcal{G}} h = \delta_{\{0\}}$, so*

$$E^{\mathcal{G}}(\alpha h) \neq \alpha E^{\mathcal{G}} h.$$

Theorem 15. *Assume that h is a convex normal integrand such that there exists $x \in \operatorname{dom} Eh \cap L^0(\mathcal{G})$ and $v \in \operatorname{dom} Eh^* \cap L^1$ with $(x \cdot v)^- \in L^1$. Then*

$$E^{\mathcal{G}}(h^\infty) = (E^{\mathcal{G}}h)^\infty.$$

Proof. The difference quotients

$$h^\lambda(x', \omega) := \frac{h(x(\omega) + \lambda x', \omega) - h(x(\omega), \omega)}{\lambda}$$

define a sequence of normal integrands $(h^\lambda)_{\lambda=1}^\infty$ that, by convexity, increase pointwise to h^∞. By Fenchel's inequality,

$$h^\lambda(x', \omega) \geq x' \cdot v(\omega) + x \cdot v(\omega) - h^*(v(\omega), \omega)) - h(x(\omega), \omega),$$

so the claim follows from Theorems 7 and 13. $\qquad\square$

3 Dynamic programming for lower bounded objectives

The optimality conditions in Theorem 17 are essentially from [21] but formulated here more generally. The existence results for the generalized Bellman equations for lower bounded integrands in Theorem 16 are from [21]. They extend those of [30, 13] in the convex case by relaxing compactness assumption on the set of feasible strategies. Lemma 6 of [21] gives also the converse of Theorem 16 in the sense that Assumption 1 necessarily holds if the generalized Bellman equations admit a solution and the sets N_t are linear for all t.

This section studies dynamic programming in the case where h is *lower bounded* in the sense that there exists an $m \in L^1$ such that

$$h(x, \omega) \geq m(\omega) \quad \forall x \in \mathbb{R}^n$$

almost surely.

Assumption 1. Problem (SP) is feasible, h is lower bounded and

$$\mathcal{L} := \{x \in \mathcal{N} \mid h^\infty(x) \leq 0\}$$

is a linear space.

The following is from [21, Lemma 4].

Theorem 16. *Under Assumption 1, (BE) has a unique solution (h_t) of lower bounded normal integrands and*

$$N_t := \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x^t, \omega) \leq 0, \ x^{t-1} = 0\}$$

are linear-valued for all t. In this case, if $x \in \mathcal{L}$ is such that $x^{t-1} = 0$ then $x_t \in N_t$ almost surely.

Theorem 17 below, shows that, if the Bellman equations (BE) admit a solution $(h_t)_{t=0}^T$, then the optimal solutions $\bar{x} \in \mathcal{N}$ of (SP) are characterized by scenariowise minimization of h_t.

We will denote the projection of the set $\mathcal{N}$ of adapted strategies to its first t components by

$$\mathcal{N}^t := \{x^t \mid x \in \mathcal{N}\} = \left\{(x_{t'})_{t'=0}^t \mid x_{t'} \in L^0(\Omega, \mathcal{F}_{t'}, P; \mathbb{R}^{n_{t'}})\right\}.$$

The lower boundedness condition in the following result will be relaxed in Section 4 below.

Theorem 17. *Assume that h is lower bounded, (SP) is feasible and that the Bellman equations (BE) admit a solution $(h_t)_{t=0}^T$. Then each h_t is lower bounded,*

$$\inf (SP) = \inf_{x^t \in \mathcal{N}^t} Eh_t(x^t)$$

for all $t = 0, \ldots, T$ and, moreover, an $\bar{x} \in \mathcal{N}$ solves (SP) if and only if

$$\bar{x}_t \in \operatorname*{argmin}_{x_t \in \mathbb{R}^{n_t}} h_t(\bar{x}^{t-1}, x_t) \quad a.s. \tag{OP}$$

for all $t = 0, \ldots, T$. If

$$N_t(\omega) := \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x^t, \omega) \le 0, \ x^{t-1} = 0\}$$

is linear-valued for all $t = 0, \ldots, T$, then there exists an optimal $x \in \mathcal{N}$ with $x_t \perp N_t$ almost surely.

Proof. Let $x \in \mathcal{N}$. By [33, Theorem 14.60],

$$E\tilde{h}_{t-1}(x^{t-1}) = \inf_{x_t \in L^0(\mathcal{F}_t)} Eh_t(x^{t-1}, x_t).$$

Since $\tilde{h}_{t-1}$ is bounded from below, $E\tilde{h}_{t-1} = Eh_{t-1}$ on $\mathcal{N}^{t-1}$ and thus

$$\inf_{x^{t-1} \in \mathcal{N}^{t-1}} Eh_{t-1}(x^{t-1}) = \inf_{x^t \in \mathcal{N}^t} Eh_t(x^t).$$

By induction, $\inf (SP) = \inf_{x^t \in \mathcal{N}^t} Eh_t(x^t)$.

To prove the second claim, note first that

$$\bar{x}^t \in \operatorname*{argmin}_{x^t \in \mathcal{N}^t} Eh_t(x^t)$$

if and only if

$$\bar{x}^{t-1} \in \operatorname*{argmin}_{x^{t-1} \in \mathcal{N}^{t-1}} Eh_{t-1}(x^{t-1}) \quad \text{and} \quad \bar{x}_t \in \operatorname*{argmin}_{x_t \in L^0(\mathcal{F}_t)} Eh_t(\bar{x}^{t-1}, x_t),$$

where, by the second part of [33, Theorem 14.60], the second inclusion means that

$$\bar{x}_t \in \operatorname*{argmin}_{x_t \in \mathbb{R}^{n_t}} h_t(\bar{x}^{t-1}, x_t) \quad \text{a.s.}$$

An $\bar{x} \in \mathcal{N}$ solves (SP) if and only if $\bar{x}$ minimizes Eh_T. A backward recursion shows that optimal solutions satisfy (OP). The converse follows from a forward recursion.

Applying Theorem 57 recursively forward in time shows that (OP) has an $\mathcal{F}_t$-measurable solution $\bar{x}_t \perp N_t$ almost surely for all $t = 0, \ldots, T$. The last claim thus follows from the second one. □

Theorem 17 can be thought of as a discrete-time version of a "verification theorem" which, in the context of continuous-time stochastic control gives conditions under which solutions of the Hamilton-Jacobi-Bellman equation characterize the optimum values and solutions of optimal control problems.

4 Dynamic programming for L-bounded objectives

This section extends the results of the previous section by relaxing the lower boundedness assumption on h. The extensions are largely based on Lemma 19 which first appeared in [26] where it was used to extend the main result of [21] on the lower semicontinuity of the optimum value function of (SP). The extension is based on the interplay of the space $\mathcal{N}$ of adapted strategies with the set

$$\mathcal{N}^{\perp} := \{v \in L^1 \mid E[x \cdot v] = 0 \ \forall x \in \mathcal{N}^{\infty}\},$$

where $\mathcal{N}^{\infty} := \mathcal{N} \cap L^{\infty}$. The following gives an alternative expression for $\mathcal{N}^{\perp}$.

Lemma 18. $\mathcal{N}^{\perp} = \{v \in L^1 \mid E_t[v_t] = 0 \quad t = 0, \ldots, T\}$.

Proof. We have

$$E[x \cdot v] = \sum_{t=0}^{T} E[x_t \cdot v_t]$$

so $v \in \mathcal{N}^{\perp}$ if and only if $E[x_t \cdot v_t] = 0$ for all $x_t \in L^{\infty}(\mathcal{F}_t)$. The claim now follows from Lemma 59. □

More interestingly, we have the following.

Lemma 19. *Let $x \in \mathcal{N}$ and $v \in \mathcal{N}^{\perp}$. If $E[x \cdot v]^+ \in L^1$, then $E[x \cdot v] = 0$.*

Proof. Assume first that $T = 0$. Defining $x^{\nu} := \mathbb{1}_{\{|x| \leq \nu\}} x$, we have $x^{\nu} \in \mathcal{N}^{\infty}$, so $E[x^{\nu} \cdot v] = 0$ and thus, $E[x^{\nu} \cdot v]^- = E[x^{\nu} \cdot v]^+$. Since $[x^{\nu} \cdot v] \leq [x \cdot v]^+ \in L^1$, Fatou's lemma gives

$$E[x \cdot v]^- \leq \liminf_{\nu \to \infty} E[x^{\nu} \cdot v]^- = \liminf_{\nu \to \infty} E[x^{\nu} \cdot v]^+ \leq E[x \cdot v]^+$$

so $[x \cdot v]^- \in L^1$ as well. Since $|x^{\nu} \cdot v| \leq |x \cdot v|$, dominated convergence theorem gives $E[x \cdot v] = \lim E[x^{\nu} \cdot v] = 0$.

Assume now that the claim holds for every $(T-1)$-period model. Defining $x^\nu := \mathbb{1}_{\{|x_0| \leq \nu\}} x$, we have

$$[\sum_{t=1}^{T} x_t^\nu \cdot v_t]^+ = [x^\nu \cdot v - x_0^\nu \cdot v_0]^+ \leq [x^\nu \cdot v]^+ + [x_0^\nu \cdot v_0]^- \leq [x \cdot v]^+ + [x_0^\nu \cdot v_0]^-,$$

where the right side is integrable. Thus, $E[\sum_{t=1}^{T} x_t^\nu \cdot v_t] = 0$, by the induction hypothesis. Since $x_0^\nu \in L^\infty$, we also have $E[x_0^\nu \cdot v_0] = 0$ so $E[x^\nu \cdot v] = 0$. This implies $E[x \cdot v] = 0$ just like in the case $T = 0$. □

The following illustrates Lemma 19 with the stochastic integral of an adapted process with respect to a martingale; see [16].

Example 20. *Assume that $n_t = d$ for all t and let s be a d-dimensional martingale, i.e. an adapted integrable stochastic process such that $E_t[\Delta s_{t+1}] = 0$ for all t. If $x \in \mathcal{N}$ is such that*

$$E[\sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}]^+ < \infty,$$

then $E[\sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}] = 0$. This follows from Lemma 19 with $v \in \mathcal{N}^\perp$ defined by $v_t = \Delta s_{t+1}$.

The extensions of Theorem 16 and Theorem 17 below are based on the following lemma that allows us to reduce a more general problem to one with a lower bounded integrand.

Lemma 21. *Assume that there exists $p \in \mathcal{N}^\perp$ such that the normal integrand $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$ is lower bounded. Then* (BE) *has a solution for h if and only if* (BE) *has a solution for k. In this case, the solutions are unique and related by*

$$k_t(x^t, \omega) = h_t(x^t, \omega) - x^t \cdot E_t p_t(\omega). \tag{2}$$

Proof. If (BE) admits a solution for h, there exist sequences $(\tilde{h}_t)_{t=0}^{T-1}$ and $(h_t)_{t=0}^{T}$ of normal integrands that satisfy (BE). Since k is lower bounded, there exists $m \in L^1$ such that

$$h(x, \omega) \geq x \cdot p(\omega) - m(\omega).$$

We claim that,

$$h_t(x^t, \omega) \geq x^t \cdot [E_t p^t](\omega) - [E_t m](\omega) \tag{3}$$

for $t = T, \ldots, 0$ and

$$\tilde{h}_t(x^t, \omega) \geq x^t \cdot [E_{t+1} p^t](\omega) - [E_{t+1} m](\omega) \tag{4}$$

for $t = T - 1, \ldots, 0$. Indeed, if (3) holds, then, since $E_t p_t = 0$, we get (4) for $t - 1$. If (4) holds, then (3) follows from Lemma 6 and Example 3. Since $h_T = E_T h$, (3) holds for T so the claim follows by induction on t.

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The lower bounds in (4) and (3) imply that the sequences $(\tilde{k}_t)_{t=0}^{T-1}$ and $(k_t)_{t=0}^{T}$ defined by

$$\tilde{k}_t(x^t, \omega) := \tilde{h}_t(x^t, \omega) - x^t \cdot [E_{t+1}p^t](\omega),$$
$$k_t(x^t, \omega) := h_t(x^t, \omega) - x^t \cdot [E_t p^t](\omega)$$

satisfy (BE) for k. Indeed, by Theorem 13 and Example 3, $E_t \tilde{k}_t = k_t$ while, since $E_t p_t = 0$,

$$\tilde{k}_{t-1}(x^{t-1}, \omega) = \tilde{h}_{t-1}(x^{t-1}, \omega) - x^{t-1} \cdot [E_t p^{t-1}](\omega),$$
$$= \inf_{x_t \in \mathbb{R}^{n_t}} \{h_t(x^t, \omega) - x^t \cdot [E_t p^t](\omega)\}$$
$$= \inf_{x_t \in \mathbb{R}^{n_t}} k_t(x^{t-1}, x_t, \omega).$$

Thus, $(k_t)_{t=1}^{T}$ solves (BE) for k. Conversely, assume that $(k_t)_{t=0}^{T}$ and $(k_t)_{t=0}^{T-1}$ solve (BE) for k. By (16), k_t are lower bounded. Similar argument as above, then shows that the functions

$$\tilde{h}_t(x^t, \omega) := \tilde{k}_t(x^t, \omega) + x^t \cdot [E_{t+1}p^t](\omega),$$
$$h_t(x^t, \omega) := k_t(x^t, \omega) + x^t \cdot [E_t p^t](\omega)$$

satisfy (BE) for h. This completes the proof. $\qquad\square$

The following extends the existence result in Theorem 16 by relaxing the lower boundedness assumption on h.

Theorem 22. *Assume that there exists $p \in \mathcal{N}^\perp$ such that the normal integrand $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$ is lower bounded and*

$$\{x \in \mathcal{N} \mid k^\infty(x) \le 0 \ a.s.\}$$

is a linear space. Then (BE) has a unique solution $(h_t)_{t=0}^{T}$ of L-bounded normal integrands and

$$N_t(\omega) := \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x^t, \omega) \le 0, \ x^{t-1} = 0\}$$

are linear-valued for all t.

Proof. By Theorem 16, the Bellman equations associated with k have a unique solution $(k_t)_{t=0}^{T}$ of lower bounded normal integrands and the measurable mappings
$$\hat{N}_t(\omega) := \{x_t \in \mathbb{R}^{n_t} \mid k_t^\infty(x^t, \omega) \le 0, \ x^{t-1} = 0\}$$

are linear-valued for all t. Thus, by Lemma 21, (BE) associated with h has a unique solution of L-bounded normal integrands and, since $E_t p_t = 0$, we have $N_t = \hat{N}_t$ almost surely. $\qquad\square$

The following generalizes Theorem 17 by relaxing the lower boundedness assumption on h.

Theorem 23. *Assume that there exists $p \in \mathcal{N}^\perp$ such that the normal integrand $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$ is lower bounded and $Ek(x) = Eh(x)$ for all $x \in \mathcal{N}$. If (SP) is feasible and the Bellman equations (BE) admit a solution $(h_t)_{t=0}^T$, then*

$$\inf (SP) = \inf_{x^t \in \mathcal{N}^t} Eh_t(x^t)$$

for all $t = 0, \ldots, T$ and, moreover, an $\bar{x} \in \mathcal{N}$ solves (SP) if and only if

$$\bar{x}_t \in \underset{x_t \in \mathbb{R}^{n_t}}{\operatorname{argmin}} \, h_t(\bar{x}^{t-1}, x_t) \quad a.s. \tag{OP}$$

for all $t = 0, \ldots, T$. If

$$N_t(\omega) := \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x^t, \omega) \leq 0, \ x^{t-1} = 0\}$$

is linear-valued for all $t = 0, \ldots, T$, then there exists an optimal $x \in \mathcal{N}$ with $x_t \perp N_t$ almost surely.

Proof. By Lemma 21, (BE) has a solution for k. Since k is lower bounded and since $Eh = Ek$ on $\mathcal{N}$, Theorem 17 says that

$$\inf (SP) = \inf_{x^t \in \mathcal{N}^t} Ek_t(x^t) \tag{5}$$

for all $t = 0, \ldots, T$ and that an $\bar{x} \in \mathcal{N}$ solves (SP) if and only if

$$\bar{x}_t \in \underset{x_t \in \mathbb{R}^{n_t}}{\operatorname{argmin}} \, k_t(\bar{x}^{t-1}, x_t) \quad \text{a.s.}$$

for all $t = 0, \ldots, T$.

By definition of $(h_t)_{t=0}^T$, we always have $\inf (SP) \geq \inf_{x^t \in \mathcal{N}^t} Eh_t(x^t)$. By (2), there exist $m_t \in L^1$ such that

$$h_t(x^t) \geq x^t \cdot E_t p^t - m_t.$$

If $Eh_t(x^t) < \infty$, then $E[x^t \cdot (E_t p^t)] < \infty$, so Lemma 19 gives $Ek_t(x^t) = Eh_t(x^t)$. Thus,

$$\inf (SP) = \inf_{x^t \in \mathcal{N}^t} Eh_t(x^t),$$

which proves the first claim. Since $E_t p_t = 0$, we have

$$\underset{x_t}{\operatorname{argmin}} \, k_t(x^{t-1}(\omega), x_t, \omega) = \underset{x_t}{\operatorname{argmin}} \, h_t(x^{t-1}(\omega), x_t, \omega),$$

which proves the second claim. The last claim follows from that of Theorem 17 since $E_t p_t = 0$. $\square$

We combine the assumptions of Theorems 16 and 23 into the following.

Assumption 2. Problem (SP) is feasible and there exists $p \in \mathcal{N}^\perp$ such that

1. $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$ is lower bounded,

2. $Ek(x) = Eh(x)$ for all $x \in \mathcal{N}$,

3. $\{x \in \mathcal{N} \mid k^\infty(x) \leq 0 \text{ a.s.}\}$ is a linear space.

Note that, if h is lower bounded, one can take $p = 0$ so Assumption 2 reduces to Assumption 1. Sufficient conditions are given in Lemmas 25 and 26 below. Applications in Section 5 illustrate these conditions further. In particular, in financial mathematics, the assumption is related to the existence of a martingale measure of the price process.

The following combines Theorems 16 and 23.

Theorem 24. *Under Assumption 2, (BE) has a unique solution $(h_t)_{t=0}^T$,*

$$\inf (SP) = \inf_{x^t \in \mathcal{N}^t} Eh_t(x^t)$$

for all $t = 0, \ldots, T$, (SP) has a solution and the solutions $\bar{x} \in \mathcal{N}$ of (SP) are characterized by

$$\bar{x}_t \in \operatorname*{argmin}_{x_t \in \mathbb{R}^{n_t}} h_t(\bar{x}^{t-1}, x_t) \quad a.s. \quad t = 0, \ldots, T. \tag{OP}$$

The following gives sufficient conditions for Assumption 2.

Lemma 25. *Assumption 2 holds if (SP) is feasible and*

1. *there exists a $p \in \mathcal{N}^\perp$ and $m \in L^1$ and $\epsilon > 0$ with*

$$h(x, \omega) \geq \lambda x \cdot p(\omega) - m(\omega) \quad \forall \lambda \in [1 - \epsilon, 1 + \epsilon],$$

2. *$\{x \in \mathcal{N} \mid h^\infty(x) \leq 0 \text{ a.s.}\}$ is a linear space.*

In this case

$$\{x \in \mathcal{N} \mid h^\infty(x) \leq 0 \text{ a.s.}\} = \{x \in \mathcal{N} \mid k^\infty(x) \leq 0 \text{ a.s.}\}.$$

The lower bound in 1 can be written equivalently as

$$h(x, \omega) \geq x \cdot p(\omega) + \epsilon |x \cdot p(\omega)| - m(\omega)$$

or as

$$\lambda p \in \operatorname{dom} Eh^* \quad \forall \lambda \in [1 - \epsilon, 1 + \epsilon].$$

Proof. We have

$$h(x, \omega) \geq x \cdot p(\omega) - m(\omega),$$
$$h(x, \omega) - x \cdot p(\omega) \geq \epsilon x \cdot p(\omega) - m(\omega).$$

Let $x \in \mathcal{N}$. If either $Eh(x) < \infty$ or $E[h(x) - x \cdot p] < \infty$, the above inequalities and Lemma 19 give $E[x \cdot p] = 0$, so

$$Eh(x) = E[h(x) - x \cdot p].$$

The above inequalities also give

$$h^\infty(x, \omega) \geq x \cdot p(\omega),$$
$$h^\infty(x, \omega) - x \cdot p(\omega) \geq \epsilon x \cdot p(\omega).$$

If either $h^\infty(x, 0) \leq 0$ or $h^\infty(x, 0) - x \cdot p \leq 0$ almost surely, then $x \cdot p \leq 0$ almost surely. Lemma 19 then implies $x \cdot p = 0$ almost surely, so

$$\{x \in \mathcal{N} \mid h^\infty(x) \leq 0 \text{ a.s.}\} = \{x \in \mathcal{N} \mid h^\infty(x) - x \cdot p \leq 0 \text{ a.s.}\}.$$

The given linearity condition thus implies that in Assumption 2. $\square$

Lemma 26. *Assumption 2 holds if* (SP) *is feasible and there exists* $p \in \mathcal{N}^\perp$ *such that*

1. $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$ *is lower bounded,*

2. $\{x \in \mathcal{N} \mid x \in \text{dom}\, h \text{ a.s.}\} \subset \text{dom}\, Eh,$

3. $\{x \in \mathcal{N} \mid k^\infty(x) \leq 0 \text{ a.s.}\}$ *is a linear space.*

Proof. Let $x \in \mathcal{N}$ be such that either $Ek(x)$ or $Eh(x)$ is finite. Then $x \in \text{dom}\, h$ almost surely, so $x \in \text{dom}\, Eh$ by 2. By 1, there exists $m \in L^1$ such that $h(x) = k(x) + x \cdot p \geq x \cdot p - m$, so $E[x \cdot p] = 0$ by Lemma 19. Thus $Ek(x) = Eh(x)$. $\square$

Counterexample 27. *Without Assumption 2, it is possible that* (SP) *does not have a solution albeit* (BE) *has a unique solution and there is a unique* x *satisfying* (OP).

Indeed, let $n_t = 1$, $\alpha \in L^2(\mathcal{F}_0)$ *and* $p \in \mathcal{N}^\perp$ *be such that* $E[\alpha p_0] = \infty$ *and consider*
$$h(x, \omega) := \frac{1}{2}|x_0 - \alpha(\omega)|^2 + x_0 p_0(\omega).$$

Here $h_0(x, \omega) = \frac{1}{2}|x_0 - \alpha(\omega)|^2$, *so* $x \in \mathcal{N}$ *satisfies* (OP) *if and only if* $x_0 = \alpha$. *For such* x, $Eh(x) = \infty$.

Counterexample 28. *It is possible that* (BE) *has a solution,*

$$\mathcal{L} := \{x \in \mathcal{N} \mid h^\infty(x) \leq 0\}$$

is a linear space but N_t *in Theorem 23 is not linear-valued.*

Indeed, let $T = 0$, $n_0 = 2$, $\mathcal{F}_0$ *trivial*, $p \in \mathcal{N}^\perp$ *with* p *nonzero almost surely and consider*
$$h(x, \omega) := (x_0^2 - x_0^1)^+ + \delta_{\mathbb{R}_+}(x_0^2) + x_0^1 p_0^1(\omega).$$

Here $h_0(x, \omega) = (x_0^2 - x_0^1)^+ + \delta_{\mathbb{R}_+}(x_0^2)$, *so* $N_0 = \{x_0 \mid 0 \leq x_0^2 \leq x_0^1\}$ *while* $\mathcal{L} = \{x \mid 0 \leq x_0^2 \leq x_0^1, x_0^1 \cdot p_0^1 \leq 0 \text{ a.s.}\} = \{0\}$.

5   Applications

This section applies Theorems 17 and 23 to some well-known instances of (SP). Many of the existence results and optimality conditions below have previously been known only under more restrictive compactness and boundedness conditions. In the case of portfolio optimization in Section 5.5 below, we extend earlier results by allowing for portfolio constraints.

5.1   Mathematical programming

Consider the problem

$$
\begin{aligned}
&\text{minimize} && Ef_0(x) \quad \text{over } x \in \mathcal{N}, \\
&\text{subject to} && f_j(x) \leq 0 \quad j = 1, \ldots, l \ a.s., && (MP) \\
& && f_j(x) = 0 \quad j = l+1, \ldots, m \ a.s.,
\end{aligned}
$$

where f_j, $j = 0, \ldots, m$ are convex normal integrands with f_j affine for $j > l$. The problem fits the general framework with

$$
h(x, \omega) := \begin{cases} f_0(x, \omega) & \text{if } f_j(x, \omega) \leq 0, j = 1 \ldots, l, f_j(x, \omega) = 0, j = l+1, \ldots, m \\ +\infty & \text{otherwise.} \end{cases}
$$

Indeed, by [33, Example 1.32 and Proposition 14.33], h is a normal integrand. Problem (MP) is essentially from [31] where it was analyzed through convex duality. We have extended the formulation by the inclusion of affine equality constraints.

Assumption 3.

1. (MP) is feasible,

2. $\{x \in \mathcal{N} \mid f_j^\infty(x) \leq 0 \ j = 0, \ldots l, f_j^\infty(x) = 0 \ j = l+1, \ldots, m\}$ is a linear space,

3. there exists a $p \in \mathcal{N}^\perp$, $m \in L^1$ and an $\epsilon > 0$ such that

$$
f_0(x) \geq x \cdot p + \epsilon |x \cdot p| - m \quad \text{a.s.}
$$

   for all $x \in \mathbb{R}^n$ with $f_j(x) \leq 0$ for all $j = 1, \ldots, l$ and $f_j(x) = 0$ for all $j = l+1, \ldots, m$ almost surely.

Assumption 3 relaxes the conditions imposed in [30]. The last two conditions are sufficient for the conditions of Lemma 25 which imply Assumption 2. The last condition in Assumption 3 holds, in particular, if f_0 is bounded from below by an integrable function as one can then take $p = 0$. A direct application of Theorem 24 gives the following.

Theorem 29. *Under Assumption 3, (MP) has a solution.*

Example 30 (Linear programming). *In the special of Example 30, Assumption* 3 *means that*

1. (MP) *is feasible,*

2. $\{x \in \mathcal{N} \mid c \cdot x \leq 0, Ax \in K\}$ *is a linear space,*

3. *there exists a* $p \in \mathcal{N}^\perp$, $m \in L^1$ *and an* $\epsilon > 0$ *such that*

$$c \cdot x \geq x \cdot p + \epsilon|x \cdot p| - m \quad a.s.$$

for all $x \in \mathbb{R}^n$ *with* $Ax - b \in K$ *almost surely.*

The above are merely examples how the results of Section 4 can be used. In some applications, the conditions of Lemma 26 could be more convenient. The above only gives the existence of solutions. When the constraint matrix in Example 30 has a block-diagonal form, one can write the dynamic programming recursion in a more familiar form; see Example 48 below.

5.2 Optimal stopping

Let R be a real-valued adapted stochastic process and consider the *optimal stopping problem*

$$\text{maximize} \quad ER_\tau \quad \text{over } \tau \in \mathcal{T}, \tag{OS}$$

where $\mathcal{T}$ is the set of *stopping times*, i.e. measurable functions $\tau : \Omega \to \{0, \ldots, T+1\}$ such that $\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$ for each $t = 0, \ldots, T$. Choosing $\tau = T+1$ is interpreted as not stopping at all. Accordingly, we define $R_{T+1} := 0$. Consider also the problem

$$\underset{x \in \mathcal{N}_+}{\text{maximize}} \quad E\sum_{t=0}^{T} R_t x_t \quad \text{subject to} \quad x \geq 0, \ \sum_{t=0}^{T} x_t \leq 1 \ a.s. \tag{ROS}$$

This is a convex relaxation of (OS) which is obtained from (ROS) by adding the constraint that $x_t \in \{0,1\}$. Indeed, the feasible strategies x are then in one-to-one correspondence with stopping times via

$$x_t = \begin{cases} 1 & \text{if } t = \tau, \\ 0 & \text{if } t \neq \tau. \end{cases}$$

The following motivates the relaxation.

Lemma 31. *If* $R \in L^1$, *then* $\sup(OS) = \sup(ROS)$ *and* $\text{argmax}(ROS)$ *is the closed convex hull of strategies* x *that correspond to optimal solutions of* (OS).

Proof. As in the proof of [24, Lemma 2], it can be shown that the processes corresponding to stopping times are the extreme points of the feasible set of (ROS). By Banach-Alaoglu theorem, the feasible set of (ROS) is compact in the weak topology that L^∞ has as the dual of L^1. Thus, by Krein–Milman theorem,

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the feasible set of (ROS) is the closed convex hull of those x corresponding to stopping times.

If $R \in L^1$, the objective is weakly continuous so the relaxation does not affect the optimum value. It is easy to verify, by contradiction, that extreme points of the weakly compact $\mathrm{argmax}\,(ROS)$ are extreme points of the feasible set, which, by the Krein-Milman theorem again, proves the last claim. $\qquad\square$

Problem (ROS) fits the general framework with $n_t = 1$ for all t and

$$
h(x,\omega) = \begin{cases} -\sum_{t=0}^{T} R_t(\omega)x_t & \text{if } x \geq 0 \text{ and } \sum_{t=0}^{T} x_t \leq 1, \\ +\infty & \text{otherwise}, \end{cases}
$$

for an adapted real-valued process R and $x_{-1} := 0$.

Let S be the *Snell envelope* of R, i.e. the adapted stochastic process given by

$$
S_{T+1} := 0
$$
$$
S_t := \max\{R_t, E_t S_{t+1}\}.
$$

The Snell envelope is the smallest supermartingale that dominates the positive part R^+ of the reward process R. Indeed, let $\tilde{S}$ be another supermartingale that dominates R^+. Then $\tilde{S}_T \geq S_T$ and

$$
\tilde{S}_t \geq \max\{R_t, E_t \tilde{S}_{t+1}\}
$$

so $\tilde{S}_t \geq S_t$ for all t, by induction.

Theorem 32. *Assume that $R \in L^1$. The optimum value of (ROS) coincides for all $t = 0, \ldots, T$ with that of*

$$
\underset{x^t \in \mathcal{N}^t}{\text{maximize}} \quad E\left[\sum_{s=0}^{t} R_s x_s + E_t[S_{t+1}](1 - \sum_{s=1}^{t} x_s)\right]
$$

$$
\text{subject to} \quad x^t \geq 0, \; \sum_{s=0}^{t} x_s \leq 1 \quad a.s.
$$

In particular, the optimum value of (ROS) is ES_0. An $x \in \mathcal{N}$ is optimal if and only if

$$
x_t \in \underset{x_t \in \mathbb{R}}{\mathrm{argmax}} \left\{ (R_t - E_t[S_{t+1}])x_t \;\middle|\; x_t \in [0, 1 - \sum_{s=0}^{t-1} x_s] \right\} \quad a.s. \quad t = 0, \ldots, T.
$$

In particular, optimal values of (OS) and (ROS) coincide, (OS) admits optimal solutions $\tau \in \mathcal{T}$ and they are characterized by $R_\tau = S_\tau$ almost surely.

Proof. The domain of h is contained in the unit simplex almost surely, so $h^\infty = \delta_{\{0\}}$ and, since $R \in L^1$, h has an integrable lower bound. Thus, by Theorem 22,

BE has a solution $(h_t)_{t=0}^T$. To prove the claims concerning (ROS), it suffices, by Theorem 17, to show that

$$h_t(x^t, \omega) = \sum_{s=0}^t \left[-R_s(\omega)x_s + \delta_{\mathbb{R}_+}(x_s) \right] - E_t[S_{t+1}](\omega)(1 - \sum_{s=0}^t x_s) + \delta_{\mathbb{R}_+}(1 - \sum_{s=0}^t x_s).$$

Since R is adapted, h is $\mathcal{F}_T$-measurable so $h_T = h$ and the claim holds for $t = T$ since $S_{T+1} = 0$. Assume that the claim holds for t. We get

$\tilde{h}_{t-1}(x_{t-1}, \omega)$

$:= \inf_{x_t \in \mathbb{R}} h_t(x^t, \omega)$

$= \inf_{x_t \in \mathbb{R}} \left\{ \sum_{s=0}^t \left[-R_s(\omega)x_s + \delta_{\mathbb{R}_+}(x_s) \right] - E_t[S_{t+1}](\omega)(1 - \sum_{s=0}^t x_s) \ \middle| \ 0 \le x_t \le 1 - \sum_{s=0}^{t-1} x_s \right\}$

$= \sum_{s=0}^{t-1} \left[-R_s(\omega)x_s + \delta_{\mathbb{R}_+}(x_s) \right] - \sup_{x_t \in \mathbb{R}} \left\{ R_t(\omega)x_t + E_t[S_{t+1}](\omega)(1 - \sum_{s=0}^t x_s) \ \middle| \ 0 \le x_t \le 1 - \sum_{s=0}^{t-1} x_s \right\}$

$= \sum_{s=0}^{t-1} \left[-R_s(\omega)x_s + \delta_{\mathbb{R}_+}(x_s) \right] - \max\{ R_t(\omega), E_t[S_{t+1}](\omega) \}(1 - \sum_{s=0}^{t-1} x_s) + \delta_{\mathbb{R}_+}(1 - \sum_{s=0}^{t-1} x_s)$

$= \sum_{s=0}^{t-1} \left[-R_s(\omega)x_s + \delta_{\mathbb{R}_+}(x_s) \right] - S_t(\omega)(1 - \sum_{s=0}^{t-1} x_s) + \delta_{\mathbb{R}_+}(1 - \sum_{s=0}^{t-1} x_s).$

By Example 3, the $\mathcal{F}_{t-1}$-conditional expectation of the second last term is $E_{t-1}[S_t](\omega)(\sum_{s=0}^{t-1} x_s - 1)$, so the claim holds for $t - 1$. It is clear that the argmax over x_t always contains either 0 or $1 - \sum_{s=0}^{t-1} x_s$. Thus, if $\sum_{s=0}^{t-1} x_s$ takes values in $\{0, 1\}$, we can choose an optimal x_t such that $\sum_{s=0}^t x_s$ takes values in $\{0, 1\}$. Thus an induction gives an optimal strategy taking values in $\{0, 1\}$. $\square$

The above is classical in stochastic analysis (see e.g. [27]) but our proof via the convex relaxation (ROS) seems new. The process $(R_t)_{t=0}^T$ is *Markov* if, for all t, $(R_{t+1}, \ldots, R_T)$ is R_t-conditionally independent of $\mathcal{F}_t$.

Remark 33. *If $R \in L^1$ is a Markov process, then $S_t = \psi_t(R_t)$ for some measurable function ψ_t. In particular, the optimal stopping times $\tau \in \mathcal{T}$ are characterized by the condition $R_\tau = \psi_\tau(R_\tau)$.*

Proof. Clearly, $S_{T+1} := 0$ is of the required form. Assume that $S_{t+1} = \psi_{t+1}(R_{t+1})$ so that $S_t = \max\{R_t, E_t[\psi_{t+1}(R_{t+1})]\}$. By the conditional independence and Doob-Dynkin lemma [17, Lemma 1.13], $E_t[\psi_{t+1}(R_{t+1})] = E^{\sigma(R_t)}[\psi_{t+1}(R_{t+1})] = \hat{\psi}_t(R_t)$ for some measurable function $\hat{\psi}_t$. Defining $\psi_t(x) := \max\{x, \hat{\psi}_t(x)\}$, we get $S_t = \psi_t(R_t)$, so the claim follows by induction. $\square$

5.3 Optimal control

Consider the optimal control problem

$$
\begin{aligned}
&\text{minimize} \qquad E\left[\sum_{t=0}^{T} L_t(X_t, U_t)\right] \quad \text{over } (X, U) \in \mathcal{N}, \\
&\text{subject to} \qquad \Delta X_t = A_t X_{t-1} + B_t U_{t-1} + W_t \quad t = 1, \dots, T
\end{aligned}
\tag{OC}
$$

where the *state* X and the *control* U are processes with values in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, A_t and B_t are $\mathcal{F}_t$-measurable random matrices, W_t is an $\mathcal{F}_t$-measurable random vector and the functions L_t are convex normal integrands. The linear constrains in (OC) are called the *system equations*. Problem (OC) fits the general framework with $x = (X, U)$ and

$$
h(x, \omega) = \sum_{t=0}^{T} L_t(X_t, U_t, \omega) + \sum_{t=1}^{T} \delta_{\{0\}}(\Delta X_t - A_t(\omega)X_{t-1} - B_t(\omega)U_{t-1} - W_t(\omega)).
$$

The special structure in (OC) allows us to express the solution $(h_t)_{t=0}^{T}$ of the general Bellman equations (BE) in terms of normal integrands $J_t : \mathbb{R}^N \times \Omega \to \overline{\mathbb{R}}$ and $I_t : \mathbb{R}^N \times \mathbb{R}^M \times \Omega \to \overline{\mathbb{R}}$ that solve the following "dynamic programming" equations

$$
\begin{aligned}
J_{T+1} &= 0 \\
I_{t+1}(X_t, U_t) &= J_{t+1}(X_t + A_{t+1}X_t + B_{t+1}U_t + W_{t+1}) \\
J_t(X_t) &= \inf_{U_t \in \mathbb{R}^M} E_t(L_t + I_{t+1})(X_t, U_t).
\end{aligned}
\tag{6}
$$

Note that J_t is a function only of x_t and ω while the functions h_t in the general Bellman equations (BE) may depend on the whole path x^t of x up to time t. The functions J_t are often called the "value functions" or "cost-to-go functions". This terminology is well justified by Theorem 35 below. Informally, (6) can be written in the more familiar form

$$
\begin{aligned}
J_{T+1} &= 0 \\
J_t(X_t) &= \inf_{U_t \in \mathbb{R}^M} E_t[L_t(X_t, U_t) + J_{t+1}(X_t + A_{t+1}X_t + B_{t+1}U_t + W_{t+1})],
\end{aligned}
$$

which are the equations studied in e.g. [4, Section 2.2], [5, Section 1.2] and [8, Proposition 4.12] when applied to the convex case. We have introduced the functions I_t to clarify that the conditional expectations are taken here in the sense of normal integrands. This resolves many of the measurability problems that arise in earlier formulations. The optimality conditions and the existence results for the convex control problems in Theorems 35 and 38 below seem new in the presented generality.

Essentially, our formulation of (OC) in terms of normal integrands is what [5, Section 1.2.II] calls "semicontinuous models". In [5], these models were, however, interpreted quite narrowly with the exclusion of even the classical

linear quadratic model in Example 42. The discrepancy seems to have come from the belief that "in the usual stochastic programming model, the controls cannot influence the distribution of future states"; see page 12 of [5]. It is clear in (OC), however, that the controls do influence the distribution of future states. What is crucial for convexity is that the state equations are affine. In nonconvex dynamic programming recursions of [13, 25], even this assumption can be relaxed.

Compared to the general Bellman equations (BE), (6) provides a significant dimension-reduction with respect to time: the optimal control does not depend on the past states. This is often referred to as the "dynamic programming principle". This reduction is essentially due to the time-separable structure of (OC) where, conditionally on the current state, the future is "independent" of the previous states. We will see a further dimension-reduction with respect to scenarios when the random elements in the problem exhibit certain form of independence; see Example 41 below. Under appropriate conditions, the functions J_t in (6) turn out to be deterministic.

When applied to (OC), Theorem 17 gives the following.

Theorem 34. *Assume that* (OC) *is feasible,* L_t *are lower bounded and that* $(I_t, J_t)_{t=0}^T$ *is a solution of* (6). *Then the optimum value of the optimal control problem coincides with that of*

$$\text{minimize} \qquad E\left[\sum_{s=0}^{t-1}(E_t L_s)(X_s, U_s) + J_t(X_t)\right] \quad \text{over } (X^t, U^t) \in \mathcal{N}^t,$$

$$\text{subject to} \qquad \Delta X_s = A_s X_{s-1} + B_s U_{s-1} + W_s \quad s = 1, \dots, t$$

for all $t = 0, \dots, T$ *and, moreover, an* $(\bar{X}, \bar{U}) \in \mathcal{N}$ *solves* (OC) *if and only if it satisfies the system equations and*

$$\bar{U}_t \in \underset{U_t \in \mathbb{R}^M}{\text{argmin}} \, E_t(L_t + I_{t+1})(\bar{X}_t, \bar{U}_t).$$

for all $t = 0, \dots, T$. *If*

$$N_t(\omega) = \{(X_t, U_t) \in \mathbb{R}^{n_t} \mid X_t = 0, \ (E_t(L_t + I_{t+1}))^\infty(0, U_t, \omega) \leq 0\}$$

is linear-valued for all $t = 0, \dots, T$, *then there exists an optimal* $x \in \mathcal{N}$ *with* $x_t \perp N_t$ *almost surely.*

Rather than proving Theorem 34 directly, we will prove the following more general result the proof of which is based on Theorem 23. In addition to $p \in \mathcal{N}^\perp$, the assumptions involve $y_t \in L^1(\mathbb{R}^N)$. Throughout, we set $y_0 := y_{T+1} := 0$ and $X_{T+1} := 0$.

Theorem 35. *Assume that* (OC) *is feasible,* $(I_t, J_t)_{t=0}^T$ *is an* L-*bounded solution of* (6) *and that there exists* $p \in \mathcal{N}^\perp$, $y \in L^1$ *and* $m_t \in L^1$ *such that* $A_t^* y_t, B_t^* y_t$ *and* $W_t \cdot y_t$ *are integrable for all* t *and*

1. $E \sum_{t=0}^{T} L_t(X_t, U_t) = E \sum_{t=0}^{T} [L_t(X_t, U_t) - (X_t, U_t) \cdot p_t]$ *for all* $(X, U) \in \mathcal{N}$ *satisfying the system equations,*

2. $L_t(X_t, U_t) \geq (X_t, U_t) \cdot p_t - X_t \cdot \Delta y_{t+1} - (A_{t+1} X_t + B_{t+1} U_t + W_{t+1}) \cdot y_{t+1} - m_t$ *almost surely for all* t.

Then the optimum value of the optimal control problem coincides with that of

$$\text{minimize} \quad E\left[\sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s) + J_t(X_t)\right] \quad \text{over } (X^t, U^t) \in \mathcal{N}^t,$$

$$\text{subject to} \quad \Delta X_s = A_s X_{s-1} + B_s U_{s-1} + W_s \quad s = 1, \dots, t$$

for all $t = 0, \dots, T$ *and, moreover, an* $(\bar{X}, \bar{U}) \in \mathcal{N}$ *solves* (OC) *if and only if it satisfies the system equations and*

$$\bar{U}_t \in \underset{U_t \in \mathbb{R}^M}{\operatorname{argmin}} E_t(L_t + I_{t+1})(\bar{X}_t, \bar{U}_t).$$

for all $t = 0, \dots, T$. *If*

$$N_t(\omega) = \{(X_t, U_t) \in \mathbb{R}^{n_t} \mid X_t = 0, \ (E_t(L_t + I_{t+1}))^{\infty}(0, U_t, \omega) \leq 0\}$$

is linear-valued for all $t = 0, \dots, T$, *then there exists an optimal* $x \in \mathcal{N}$ *with* $x_t \perp N_t$ *almost surely.*

Proof. Let $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$. Condition 1 means that $Ek = Eh$ on $\mathcal{N}$. For any (X, U) satisfying the system equations, the lower bound in 2 can be written as

$$L_t(X_t, U_t) \geq (X_t, U_t) \cdot p_t - X_t \cdot \Delta y_{t+1} - \Delta X_{t+1} \cdot y_{t+1} - m_t.$$

Summing up, shows that k is lower bounded. All the claims follow from Theorem 23 once we show that $(h_t)_{t=0}^{T}$ given by

$$h_t(x^t, \omega) := \sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s, \omega) + E_t(L_t + I_{t+1})(X_t, U_t)$$

$$+ \sum_{s=1}^{t} \delta_{\{0\}}(-\Delta X_s + A_s X_{s-1} + B_s U_{s-1} + W_s) \tag{7}$$

is an L-bounded solution of (BE). Indeed, by Theorem 23, the optimum value of (OC) can then be expressed as

$$\inf_{x^t \in \mathcal{N}^t} E h_t(x^t) = \inf_{x^t \in \mathcal{N}^t} \{ \sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s, \omega) + E_t(L_t + I_{t+1})(X_t, U_t) \mid$$

$$\Delta x_s = A_s X_{s-1} + B_s U_{s-1} + W_s \ s = 1, \dots, t \}$$

$$= \inf_{x^t \in \mathcal{N}^t} \{ \sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s, \omega) + J_t(X_t) \mid$$

$$\Delta x_s = A_s X_{s-1} + B_s U_{s-1} + W_s \ s = 1, \dots, t \},$$

where the second inequality holds by [33, Theorem 14.60], Also, if (7) holds, then by Theorem 23, an $(\bar{X}, \bar{U}) \in \mathcal{N}$ solves (OC) if and only if $(\bar{X}_t, \bar{U}_t)$ minimizes (7) almost surely.

Assume that (7) solve (BE) from t onwards. Then

$$
\begin{aligned}
\tilde{h}_{t-1}(x^{t-1}) &= \inf_{x_t \in \mathbb{R}^{n_t}} h_t(x^{t-1}, x_t) \\
&= \sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s) \\
&\quad + \inf_{(X_t, U_t)} \{ E_t(L_t + I_{t+1})(X_t, U_t) \mid X_t = X_{t-1} + A_t X_{t-1} + B_t U_{t-1} + W_t \} \\
&\quad + \sum_{s=1}^{t-1} \delta_{\{0\}}(-\Delta X_s + A_s X_{s-1} + B_s U_{s-1} + W_s) \\
&= \sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s) + J_t(X_{t-1} + A_t X_{t-1} + B_t U_{t-1} + W_t) \\
&\quad + \sum_{s=1}^{t-1} \delta_{\{0\}}(-\Delta X_s + A_s X_{s-1} + B_s U_{s-1} + W_s) \\
&= \sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s) + I_t(X_{t-1}, U_{t-1}) \\
&\quad + \sum_{s=1}^{t-1} \delta_{\{0\}}(-\Delta X_s + A_s X_{s-1} + B_s U_{s-1} + W_s)
\end{aligned}
\tag{8}
$$

By Theorem 13 again,

$$
\begin{aligned}
h_{t-1}(x^{t-1}) &= E_{t-1} \tilde{h}_{t-1}(x^{t-1}) \\
&= \sum_{s=0}^{t-2} (E_{t-1} L_s)(X_s, U_s, \omega) + E_{t-1}(L_{t-1} + I_t)(X_{t-1}, U_{t-1}) \\
&\quad + \sum_{s=1}^{t-1} \delta_{\{0\}}(-\Delta X_s + A_s X_{s-1} + B_s U_{s-1} + W_s).
\end{aligned}
$$

We have $h_T = E_T h$, by Theorem 13. Thus, by induction, the normal integrands h_t given by (7) solve (BE). $\qquad\square$

Remark 36. *The lower bounds in Theorem 35 hold if and only if*

$$
E[L_t^*(p_t - (\Delta y_{t+1} + A_{t+1}^* y_{t+1}, B_{t+1}^* y_{t+1})) - W_{t+1} \cdot y_{t+1}] < \infty,
$$

for all t. Here $y_0 := y_{T+1} := 0$, $A_{T+1} := 0$ and $B_{T+1} := 0$. The assumption is closely connected with the feasibility of a problem dual to (OC); *see [23].*

Proof. By Fenchel's inequality, for feasible (X, U),

$$L_t(X_t, U_t) + L_t^*(p_t - (\Delta y_{t+1} + A_{t+1}^* y_{t+1}, B_{t+1}^* y_{t+1})) - W_{t+1} \cdot y_{t+1}$$
$$\geq (X_t, U_t) \cdot p_t - X_t \cdot (\Delta y_{t+1} + A_{t+1}^* y_{t+1}) - U_t \cdot (B_{t+1}^* y_{t+1}) - W_{t+1} \cdot y_{t+1}$$
$$= (X_t, U_t) \cdot p_t + X_t \cdot y_t - (X_t + A_{t+1} X_t + B_{t+1} U_t + W_{t+1}) \cdot y_{t+1}$$
$$= (X_t, U_t) \cdot p_t + X_t \cdot y_t - (X_{t+1}) \cdot y_{t+1}$$
$$= (X_t, U_t) \cdot p_t - X_t \cdot \Delta y_{t+1} - \Delta X_{t+1} \cdot y_{t+1},$$

which gives the equivalence. $\square$

Theorem 16 gives the following existence result for (6).

Theorem 37. *Assume that L_t are lower bounded and that the set*

$$\left\{ (X, U) \in \mathcal{N} \,\middle|\, \sum_{t=0}^{T} L_t^\infty(X_t, U_t) \leq 0, \ \Delta X_t = A_t X_{t-1} + B_t U_{t-1} \right\}$$

is linear. Then (6) has a unique solution $(J_t, I_t)_{t=0}^{T+1}$, each J_t and I_t is lower bounded and

$$N_t(\omega) := \{ (X_t, U_t) \in \mathbb{R}^{n_t} \mid X_t = 0, \ (E_t(L_t + I_{t+1}))^\infty(0, U_t, \omega) \leq 0 \}$$

is linear-valued for all t.

Rather than proving Theorem 37 directly, we will prove the following more general result the proof of which is based on Theorem 22.

Theorem 38. *Assume that there exists $p \in \mathcal{N}^\perp$, $y \in L^1$ and $m_t \in L^1$ such that*

1. *the set*

$$\left\{ (X, U) \in \mathcal{N} \,\middle|\, \sum_{t=0}^{T} (L_t^\infty(X_t, U_t) - p_t \cdot (X_t, U_t)) \leq 0, \ \Delta X_t = A_t X_{t-1} + B_t U_{t-1} \right\}$$

   is linear,

2. *$L_t(X_t, U_t) \geq (X_t, U_t) \cdot p_t - X_t \cdot \Delta y_{t+1} - (A_{t+1} X_t + B_{t+1} U_t + W_{t+1}) \cdot y_{t+1} - m_t$ almost surely for all t.*

3. *$A_t^* y_t$, $B_t^* y_t$ and $W_t \cdot y_t$ are integrable and $E_t[A_t^* y_t] = A_t^* E_t y_t$, $E_t[B_t^* E_t y_t] = B_t^* E_t y_t$ and $E_t[W_t \cdot y] = W_t \cdot E_t y_t$ for all t.*

Then (6) has a unique solution $(J_t, I_t)_{t=0}^{T+1}$, each J_t and I_t is L-bounded and

$$N_t(\omega) := \{ (X_t, U_t) \in \mathbb{R}^{n_t} \mid X_t = 0, \ (E_t(L_t + I_{t+1}))^\infty(0, U_t, \omega) \leq 0 \}$$

is linear-valued for all t.

Proof. Summing up the lower bounds in 2 shows that $k(x,\omega) := h(x,\omega) - x \cdot p(\omega)$ is lower bounded while 1 means that $\{x \in \mathcal{N} \mid k^\infty(x) \leq 0\}$ is a linear space. By Theorem 22, (BE) has a unique solution $(h_t)_{t=0}^T$ for h and

$$N_t(\omega) := \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x^t, \omega) \leq 0, \ x^{t-1} = 0\}$$

is linear-valued for all t.

Assume that $(I_{t'}, J_{t'})_{t'=t+1}^{T+1}$ are normal integrands satisfying (6) from time $t+1$ onwards, and that

$$\tilde{h}_t(x^t) = \sum_{s=0}^t (E_{t+1}L_s)(X_s, U_s) + I_{t+1}(X_t, U_t)$$

$$+ \sum_{s=1}^t \delta_{\{0\}}(\Delta X_s - A_s X_{s-1} + B_s U_{s-1} + W_s),$$

$$J_{t+1}(X_{t+1}) \geq X_{t+1} \cdot E_{t+1}y_{t+1} - \sum_{t'=t+1}^T E_{t+1}m_{t'}.$$

Here E_{T+1} is defined as the identity mapping so the above hold for $t = T$. The lower bound gives

$$I_{t+1}(X_t, U_t) = J_{t+1}(X_t + A_{t+1}X_t + B_{t+1}U_t + W_{t+1})$$

$$\geq (X_t + A_{t+1}X_t + B_{t+1}U_t + W_{t+1}) \cdot E_{t+1}y_{t+1} - \sum_{t'=t+1}^T E_{t+1}m_{t'}$$

$$= X_t \cdot E_{t+1}y_{t+1} + X_t \cdot A_{t+1}^* E_{t+1}y_{t+1} + U_t \cdot B_{t+1}^* E_{t+1}y_{t+1}$$

$$+ W_{t+1} \cdot E_{t+1}y_{t+1} - \sum_{t'=t+1}^T E_{t+1}m_{t'}. \tag{9}$$

Under conditions 2 and 3, all the terms in the expression of $\tilde{h}_t$ above are L-bounded, by Theorem 8. By Theorems 13 and 9,

$$h_t(x^t) = (E_t \tilde{h}_t)(x^t)$$

$$= \sum_{s=0}^{t-1} (E_t L_s)(X_s, U_s) + E_t(L_t + I_{t+1})(X_t, U_t)$$

$$+ \sum_{s=1}^t \delta_{\{0\}}(\Delta X_s - A_s X_{s-1} + B_s U_{s-1} + W_s).$$

In particular,

$$N_t(\omega) = \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x_t) \leq 0, \ x^{t-1} = 0\}$$
$$= \{(X_t, U_t) \in \mathbb{R}^{n_t} \mid X_t = 0, \ (E_t(L_t + I_{t+1}))^\infty(0, U_t) \leq 0\},$$

which is linear by Theorem 22. Thus, by Theorem 57, the functions

$$J_t(X_t) := \inf_{U_t} E_t(L_t + I_{t+1})(X_t, U_t)$$

and

$$I_t(X_{t-1}, U_{t-1}) := J_t(X_{t-1} + A_t X_{t-1} + B_t U_{t-1} + W_t)$$

are convex normal integrands. As in (8),

$$
\begin{aligned}
\tilde{h}_{t-1}(x^{t-1}) &= \sum_{s=0}^{t-1}(E_t L_s)(X_s, U_s) + J_t(X_{t-1} + A_t X_{t-1} + B_t U_{t-1} + W_t) \\
&\quad + \sum_{s=1}^{t-1}\delta_{\{0\}}(-\Delta X_s + A_s X_{s-1} + B_s U_{s-1} + W_s) \\
&= \sum_{s=0}^{t-1}(E_t L_s)(X_s, U_s) + I_t(X_{t-1}, U_{t-1}) \\
&\quad + \sum_{s=1}^{t-1}\delta_{\{0\}}(-\Delta X_s + A_s X_{s-1} + B_s U_{s-1} + W_s).
\end{aligned}
$$

By Lemma 6 and Example 3, conditions 2 and 3 give

$$
\begin{aligned}
(E_{t+1}L_t)(X_t, U_t) &\geq (X_t, U_t) \cdot E_{t+1} p_t - X_t \cdot E_{t+1}[\Delta y_{t+1}] \\
&\quad - X_t \cdot A_{t+1}^* E_{t+1} y_{t+1} - U_t \cdot B_{t+1}^* E_{t+1} y_{t+1}] \\
&\quad - W_{t+1} \cdot E_{t+1} y_{t+1} - E_{t+1} m_t.
\end{aligned}
$$

Combining this with (9) gives

$$(E_{t+1}L_t + I_{t+1})(X_t, U_t) \geq (X_t, U_t) \cdot E_{t+1} p_t + X_t \cdot E_{t+1} y_t - \sum_{t'=t}^{T} E_{t+1} m_{t'}.$$

By Lemma 6 and Example 3,

$$E_t(L_t + I_{t+1})(X_t, U_t) \geq X_t \cdot E_t y_t - \sum_{t'=t}^{T} E_t m_{t'}.$$

and thus,

$$J_t(X_t) \geq X_t \cdot E_t y_t - \sum_{t'=t}^{T} E_t m_{t'}.$$

The claim thus follows by induction on t. $\qquad\square$

Example 39. *Assumptions of Theorems* 35 *and* 38 *hold if* (OC) *is feasible,*

1. $\{(X,U) \in \mathcal{N} \mid \sum_{t=0}^{T} L_t^\infty(X_t, U_t) \leq 0, \ \Delta X_t = A_t X_{t-1} + B_t U_{t-1}\}$ *is linear.*

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2. *There exists $p \in \mathcal{N}^{\perp}$ and $\epsilon > 0$ such that for every $\lambda \in (1 + \epsilon, 1 - \epsilon)$ there exist $y \in L^1$ and $m \in L^1$ such that $A_t^* y_t, B_t^* y_t, y_t \cdot W_t$ are integrable and*

$$L_t(X_t, U_t) \geq \lambda(X_t, U_t) \cdot p_t - X_t \cdot \Delta y_{t+1} - \Delta X_{t+1} \cdot y_{t+1} - m_t$$

for all feasible (X, U) and for all t. Here $X_{t+1} := y_0 := y_{T+1} := 0$.

Proof. The second assumption clearly implies the lower bound in both theorems. Summing up the lower bounds in 2, we see that Lemma 25 is applicable, which implies the rest of the assumptions. $\qquad \square$

Remark 40. *Denoting $Q_t := E_t(L_t + I_{t+1})$, we can write (6) as*

$$Q_{T+1} := 0,$$
$$I_t(X_{t-1}, U_{t-1}) := \inf_{U_t \in \mathbb{R}^M} Q_t(X_{t-1} + A_t X_{t-1} + B_t U_{t-1} + W_t, U_t),$$
$$Q_{t-1} := E_{t-1}(L_{t-1} + I_t)$$

and the optimality condition as

$$U_t \in \operatorname*{argmin}_{U_t \in \mathbb{R}^M} Q_t(X_t, U_t).$$

If $(Q_t)_{t=0}^T$ satisfies the above, then the functions

$$J_t(X_t) := \inf_{U_t \in \mathbb{R}^M} Q_t(X_t, U_t)$$

satisfy (6). Formulation of the dynamic programming recursion in terms of the Q-functions is often used in the context of reinforcement learning e.g. in [3].

Example 41 (Conditional independence). *Assume that L_t are L-bounded, that $(J_t)_{t=0}^{\infty}$ and $(I_t)_{t=0}^{\infty}$ are an L-bounded solution of (6) and that there is a sequence of sigma-algebras $(\mathcal{H}_t)_{t=0}^T$ such that*

1. *L_t is $\mathcal{H}_t$-conditionally independent of $\mathcal{F}_t$ and $\sigma(A_{t+1}, B_{t+1}, W_{t+1})$ is $\mathcal{H}_{t+1} \vee \mathcal{H}_t$-conditionally independent of $\mathcal{F}_t$.*

2. *$\mathcal{H}_{t+1}$ is $\mathcal{H}_t$-conditionally independent of $\mathcal{F}_t$.*

Then J_t is $\mathcal{H}_t$-measurable and

$$E_t(L_t + I_{t+1}) = E^{\mathcal{H}_t}(L_t + I_{t+1}).$$

In particular, if $(\mathcal{H}_t)_{t=0}^{T+1}$ are mutually independent and A_t, B_t and W_t are $\mathcal{H}_t$-measurable , then each $E_t I_{t+1}$ is deterministic. If, in addition, L_t is independent of $\mathcal{F}_t$, then J_t is deterministic.

Proof. Assume that J_{t+1} is $\mathcal{H}_{t+1}$-measurable. Then

$$I_{t+1}(X_t, U_t) = J_{t+1}(X_t + A_{t+1} X_t + B_{t+1} U_t + W_{t+1})$$

is $\sigma(A_{t+1}, B_{t+1}, W_{t+1}) \vee \mathcal{H}_{t+1}$-measurable. By [17, Proposition 6.8], conditions 1 and 2 imply that I_{t+1} is $\mathcal{H}_t$-conditionally independent of $\mathcal{F}_t$. By Theorem 11,

$$E_t(L_t + I_{t+1}) = E^{\mathcal{H}_t}(L_t + I_{t+1}),$$

so J_t is $\mathcal{H}_t$ measurable, by Theorem 57. By Theorem 11, condition 1 implies that J_T is $\mathcal{H}_T$-measurable, so the claim follows from induction.

Under the additional assumptions, I_{t+1} is $\mathcal{H}_{t+1}$-measurable, so mutual independence of $\mathcal{H}_t$ gives, by Theorem 11, that

$$E_t I_{t+1} = E^{\mathcal{H}_t} I_{t+1}$$

is deterministic. When, in addition, L_t is independent of $\mathcal{F}_t$,

$$E_{t-1}(L_{t-1} + I_t)$$

is deterministic as well, and so too is J_t. □

Example 42 (Linear-quadratic control). *Consider* (OC) *in the case*

$$L_t(X_t, U_t) = \frac{1}{2} X_t \cdot Q_t X_t + \frac{1}{2} U_t \cdot R_t U_t,$$

where $Q_t \in L^1(\mathbb{R}^{N \times N})$ and $R_t \in L^1(\mathbb{R}^{M \times M})$ are random symmetric positive semi-definite matrices. Assume that the recursion

$$\begin{aligned}
K_T &:= E_T[Q_T], \\
K_{t-1} &:= E_{t-1}[Q_{t-1} + (I + A_t)^* K_t (I + A_t)] \\
&\quad - \frac{1}{2} E_{t-1}[(I + A_t)^* K_t B_t](E_{t-1}[R_{t-1} + B_t^* K_t B_t])^{-1} E_{t-1}[B_t^* K_t (I + A_t)]
\end{aligned}$$

is well-defined, $W_t \cdot K_t A_t$ and $W_t \cdot K_t A_t$ are integrable, W_t has zero mean and is independent of $\mathcal{F}_{t-1}$ and of $A_{t'}$, $B_{t'}$, $Q_{t'}$ and $R_{t'}$ for $t' \geq t$. Then

$$J_t(X_t) = \frac{1}{2} X_t \cdot K_t X_t + \frac{1}{2} \sum_{t'=t+1}^{T} E_t[W_{t'} \cdot Q_{t'} W_{t'}]$$

and the optimal control is given by $\bar{U}_t = -\Lambda_t X_t$, where

$$\Lambda_t = (E_t[R_t + B_{t+1}^* K_{t+1} B_{t+1}])^{-1} E_t[B_{t+1}^* K_{t+1}(I + A_{t+1})].$$

Note that if, in addition, Q_t, R_t, A_{t+1} and B_{t+1} are independent of $\mathcal{F}_t$, then the matrices K_t and Λ_t are deterministic with

$$\begin{aligned}
K_T &:= E[Q_T], \\
K_{t-1} &:= E[Q_{t-1} + (I + A_t)^* K_t (I + A_t)] \\
&\quad - \frac{1}{2} E[(I + A_t)^* K_t B_t](E[R_{t-1} + B_t^* K_t B_t])^{-1} E[B_t^* K_t (I + A_t)]
\end{aligned}$$

and

$$\Lambda_t = (E[R_t] + E[B_{t+1}]^* K_{t+1} E[B_{t+1}])^{-1} E[B_{t+1}]^* K_{t+1} E[I + A_{t+1}].$$

The backward recursion for $(K_t)_{t=0}^T$ are known as Riccati equations. The matrices K_t are well-defined e.g. when Q_t, R_t, A_t and B_t are all deterministic and R_t are positive definite.

Proof. By Example 3,

$$E_t L_t(X_t, U_t) = \frac{1}{2} X_t \cdot E_t[Q_t] X_t + \frac{1}{2} U_t \cdot E_t[R_t] U_t,$$

for all t, so

$$J_T(X_T) = \frac{1}{2} X_T \cdot E_T[Q_T] X_T.$$

Assume that the expression for $J_{t'}$ in the claim is valid for $t' = t, \ldots, T$. We get

$$
\begin{aligned}
I_t(X_{t-1}, U_{t-1}) &= J_t((I + A_t) X_{t-1} + B_t U_{t-1} + W_t) \\
&= \frac{1}{2} X_{t-1} \cdot (I + A_t)^* K_t (I + A_t) X_{t-1} + \frac{1}{2} U_{t-1} \cdot B_t^* K_t B_t U_{t-1} \\
&\quad + \frac{1}{2} W_t \cdot K_t W_t + X_{t-1} \cdot (I + A_t)^* K_t B_t U_{t-1} \\
&\quad + W_t \cdot K_t(X_{t-1} + A_t X_{t-1} + B_t U_{t-1}) + \frac{1}{2} \sum_{t'=t+1}^{T} E_t[W_{t'} \cdot Q_{t'} W_{t'}].
\end{aligned}
$$

When W_t has zero mean and is independent of $\mathcal{F}_{t-1}$, $Q_{t'}$, $R_{t'}$, $A_{t'}$ and $B_{t'}$ for $t' \geq t$ and W_t is independent of $\mathcal{F}_{t-1}$, Example 3 gives

$$
\begin{aligned}
E_{t-1}(L_{t-1} + I_t)(X_{t-1}, U_{t-1}) &= \frac{1}{2} X_{t-1} \cdot E_{t-1}[Q_{t-1} + (I + A_t)^* K_t(I + A_t)] X_{t-1} \\
&\quad + \frac{1}{2} U_{t-1} \cdot E_{t-1}[R_{t-1} + B_t^* K_t B_t] U_{t-1} \\
&\quad + X_{t-1} \cdot E_{t-1}[(I + A_t)^* K_t B_t] U_{t-1} + \frac{1}{2} \sum_{t'=t}^{T} E_{t-1}[W_{t'} \cdot Q_{t'} W_{t'}].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
J_{t-1}(X_{t-1}) &= \inf_{U_{t-1} \in \mathbb{R}^M} E_{t-1}(L_{t-1} + I_t)(X_{t-1}, U_{t-1}) \\
&= \frac{1}{2} X_{t-1} \cdot E_{t-1}[Q_{t-1} + (I + A_t)^* K_t(I + A_t)] X_{t-1} + \frac{1}{2} E_{t-1}[W_t \cdot K_t W_t] \\
&\quad + \inf_{U_{t-1} \in \mathbb{R}^M} \{ \frac{1}{2} U_{t-1} \cdot E_{t-1}[R_{t-1} + B_t^* K_t B_t] U_{t-1} + X_{t-1} \cdot E_{t-1}[(I + A_t)^* K_t B_t] U_{t-1} \} \\
&= \frac{1}{2} X_{t-1} \cdot K_{t-1} X_{t-1} + \frac{1}{2} \sum_{t'=t}^{T} E_t[W_{t'} \cdot Q_{t'} W_{t'}],
\end{aligned}
$$

where the infimum is attained by the $\bar{U}_t$ in the statement. The claim thus follows by induction on t. $\qquad\square$

5.4 Problems of Lagrange

Consider the problem

$$\text{minimize} \quad E\sum_{t=0}^{T} K_t(x_t, \Delta x_t) \quad \text{over } x \in \mathcal{N}, \tag{L}$$

where x is a process of fixed dimension d, K_t are convex normal integrands and $x_{-1} := 0$. This fits the general format with

$$h(x, \omega) = \sum_{t=0}^{T} K_t(x_t, \Delta x_t, \omega).$$

Problem (L) is a special case of a stochastic problem of Bolza studied in [32] whose objective contains an additional convex function of (x_0, x_T). This section analyzes the dynamic programming principle for (L). The results below seem to be new.

If in the linear stochastic programming model Example 30, the constraints can be grouped as

$$T_t \Delta x_t + W_t x_t - b_t \in C_t \quad t = 0, \dots, T \tag{10}$$

for given random matrices T_t and W_t and cones C_t, then the problem in Example 30 is an instance of (L) with

$$K_t(\Delta x_t, x_t, \omega) = \begin{cases} c_t(\omega) \cdot x_t & \text{if } T_t(\omega)\Delta x_t + W_t(\omega)x_t - b_t(\omega) \in C_t(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Note also that the problem of optimal control (OC) is a special case of (L) with $x_t = (X_t, U_t)$ and

$$K_t(x_t, \Delta x_t) = \begin{cases} L_t(X_t, U_t) & \text{if } \Delta X_t = A_t X_{t-1} + B_t U_{t-1} + W_t, \\ +\infty & \text{otherwise.} \end{cases}$$

Other examples can be found e.g. in financial mathematics; see [6, Example 3.6].

Much like in the optimal control problem (OC), the time-separable structure here allows us to express the solutions of the generalized Bellman equations (BE) in terms of normal integrands V_t and $\tilde{V}_t$ on $\mathbb{R}^d \times \Omega$ that solve the following dynamic programming equations

$$V_T = 0,$$
$$\tilde{V}_{t-1}(x_{t-1}, \omega) = \inf_{x_t \in \mathbb{R}^d} \{(E_t K_t)(x_t, \Delta x_t, \omega) + V_t(x_t, \omega)\}, \tag{11}$$
$$V_{t-1} = E_{t-1}\tilde{V}_{t-1}.$$

Note that V_t is a function only of x_t and ω while the functions h_t in the general Bellman equations (BE) may depend on the whole path of x up to time t.

Theorem 16 gives the following existence result for (L).

Theorem 43. *Assume that* (L) *is feasible,* K_t *are lower bounded and* $(V_t)_{t=0}^T$ *is a solution of* (11). *Then*

$$\inf(L) = \inf_{x^t \in \mathcal{N}^t} E\left[\sum_{s=0}^{t}(E_t K_s)(x_s, \Delta x_s) + V_t(x_t)\right] \quad t = 0, \ldots, T,$$

for all $t = 0, \ldots, T$ *and, moreover, an* $\bar{x} \in \mathcal{N}$ *solves* (L) *if and only if*

$$x_t \in \underset{x_t \in \mathbb{R}^d}{\operatorname{argmin}}\{(E_t K_t)(x_t, \Delta x_t) + V_t(x_t)\} \quad a.s.$$

for all $t = 0, \ldots, T$. *If*

$$N_t(\omega) = \{x_t \in \mathbb{R}^d \mid (E_t K_t)^\infty(x_t, x_t, \omega) + V_t^\infty(x_t, \omega) \le 0\}$$

is linear-valued for all $t = 0, \ldots, T$, *then there exists an optimal* $x \in \mathcal{N}$ *with* $x_t \perp N_t$ *almost surely.*

Rather than proving Theorem 43 directly, we will prove the following more general result the proof of which is based on Theorem 23. In addition to the "shadow price of information" $p \in \mathcal{N}^\perp$, the assumptions in this section involve other "dual variables" $y_t \in L^1(\mathbb{R}^d)$. Throughout, we set $y_{T+1} := 0$.

Theorem 44. *Assume that* (L) *is feasible,* $(V_t)_{t=0}^T$ *is an L-bounded solution of* (11) *and that there exists* $p \in \mathcal{N}^\perp$, $y \in L^1$ *and* $m_t \in L^1$ *such that*

1. $E \sum K_t(x_t, \Delta x_t) = E \sum [K_t(x_t, \Delta x_t) - x_t \cdot p_t]$ *for all* $x \in \mathcal{N}$,

2. $K_t(x_t, \Delta x_t) \ge x_t \cdot (p_t + \Delta y_{t+1}) + \Delta x_t \cdot y_t - m_t$ *almost surely for all* t.

Then

$$\inf(L) = \inf_{x^t \in \mathcal{N}^t} E\left[\sum_{s=0}^{t}(E_t K_s)(x_s, \Delta x_s) + V_t(x_t)\right] \quad t = 0, \ldots, T,$$

for all $t = 0, \ldots, T$ *and, moreover, an* $\bar{x} \in \mathcal{N}$ *solves* (L) *if and only if*

$$x_t \in \underset{x_t \in \mathbb{R}^d}{\operatorname{argmin}}\{(E_t K_t)(x_t, \Delta x_t) + V_t(x_t)\} \quad a.s.$$

for all $t = 0, \ldots, T$. *If*

$$N_t(\omega) = \{x_t \in \mathbb{R}^d \mid (E_t K_t)^\infty(x_t, x_t, \omega) + V_t^\infty(x_t, \omega) \le 0\}$$

is linear-valued for all $t = 0, \ldots, T$, *then there exists an optimal* $x \in \mathcal{N}$ *with* $x_t \perp N_t$ *almost surely.*

Proof. Summing up the lower bounds in 2 shows that $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$ is lower bounded while 1 means that $Ek = Eh$ on $\mathcal{N}$. Assume that $(V_t)_{t=0}^T$ is

an L-bounded solution of (11). All the claims follow from Theorem 23 once we show that $(h_t)_{t=0}^T$ given by

$$h_t(x^t, \omega) := \sum_{s=0}^{t} (E_t K_s)(x_s, \Delta x_s, \omega) + V_t(x_t, \omega)$$

is an L-bounded solution of (BE). Assume that $(h_{t'})_{t'=t}^T$ satisfies (BE) from time t onwards. We get

$$\tilde{h}_{t-1}(x^{t-1}, \omega) := \inf_{x_t \in \mathbb{R}^d} h_t(x^{t-1}, x_t, \omega)$$

$$= \sum_{s=0}^{t-1} (E_t K_s)(x_s, \Delta x_s, \omega) + \inf_{x_t \in \mathbb{R}^d} \{(E_t K_t)(x_t, \Delta x_t, \omega) + V_t(x_t, \omega)\}$$

$$= \sum_{s=0}^{t-1} (E_t K_s)(x_s, \Delta x_s, \omega) + \tilde{V}_{t-1}(x_{t-1}).$$

By Theorem 13 again,

$$h_{t-1}(x^t, \omega) = E_{t-1} \tilde{h}_{t-1}.$$

Since $V_T = 0$, we have $h_T = E_T h$, by Theorem 13. Thus, by induction, $(h_t)_{t=0}^T$ solves (BE). $\square$

The following gives sufficient conditions for the existence of solutions to the Bellman equations (11).

Theorem 45. *Assume that K_t are lower bounded and that*

$$\{x \in \mathcal{N} \mid \sum_{t=0}^{T} (K_t^\infty(x_t, \Delta x_t) - x_t \cdot p_t) \leq 0\}$$

is a linear space. Then (11) has a unique solution $(V_t)_{t=0}^T$, each V_t is lower bounded and

$$N_t(\omega) := \{x_t \in \mathbb{R}^d \mid (E_t K_t)^\infty(x_t, x_t, \omega) + V_t^\infty(x_t, \omega) \leq 0\} \qquad (12)$$

is linear-valued for all t.

Rather than proving Theorem 45 directly, we will prove the following more general result the proof of which is based on Theorem 22.

Theorem 46. *Assume that there exists $p \in \mathcal{N}^\perp$, $y \in L^1$ and $m_t \in L^1$ such that*

1. $\{x \in \mathcal{N} \mid \sum_{t=0}^{T} (K_t^\infty(x_t, \Delta x_t) - x_t \cdot p_t) \leq 0\}$ *is a linear space,*

2. $K_t(x_t, \Delta x_t) \geq x_t \cdot (p_t + \Delta y_{t+1}) + \Delta x_t \cdot y_t - m_t$ *almost surely for all t.*

Then (11) has a unique solution $(V_t)_{t=0}^T$, each V_t is L-bounded and

$$N_t(\omega) := \{x_t \in \mathbb{R}^d \mid (E_t K_t)^\infty(x_t, x_t, \omega) + V_t^\infty(x_t, \omega) \leq 0\} \qquad (13)$$

is linear-valued for all t.

Proof. Summing up the lower bounds in 2 shows that $k(x, \omega) := h(x, \omega) - x \cdot p(\omega)$ is lower bounded while 1 means that $\{x \in \mathcal{N} \mid k^\infty(x) \leq 0\}$ is a linear space. By Theorem 22, (BE) has a unique solution $(h_t)_{t=0}^T$ for h and

$$N_t(\omega) := \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x^t, \omega) \leq 0, \ x^{t-1} = 0\} \tag{14}$$

is linear-valued for all t.

Assume that $(V_{t'})_{t'=t}^T$ are normal integrands satisfying (11) from time t onwards, and that

$$h_t(x^t) = \sum_{s=0}^t (E_t K_s)(x_s, \Delta x_s) + V_t(x_t)$$

$$V_t(x_t) \geq -x_t \cdot (E_t y_{t+1}) - E_t \sum_{t'=t+1}^T m_{t'} \tag{15}$$

almost surely. We get

$$\tilde{h}_{t-1}(x^{t-1}) = \inf_{x_t \in \mathbb{R}^d} h_t(x^{t-1}, x_t)$$

$$= \sum_{s=0}^{t-1} (E_t K_s)(x_s, \Delta x_s) + \tilde{V}_{t-1}(x_{t-1}), \tag{16}$$

where

$$\tilde{V}_{t-1}(x_{t-1}) := \inf_{x_t \in \mathbb{R}^d} \{(E_t K_t)(x_t, \Delta x_t) + V_t(x_t)\}.$$

The equation in (15) and the linearity of (14) imply that (13) is linear. Thus, by Theorem 57, $\tilde{V}_{t-1}$ is a normal integrand.

By Lemma 6 and Example 3, condition 2 implies

$$(E_t K_t)(x_t, \Delta x_t) \geq x_t \cdot E_t \Delta y_{t+1} + \Delta x_t \cdot E_t y_t - E_t m_t$$

Combining with the inequality in (15) gives

$$\tilde{V}_{t-1}(x_{t-1}) \geq -x_{t-1} \cdot E_t y_t - E_t \sum_{t'=t}^T m_{t'}.$$

By Theorem 8, there is a unique normal integrand V_{t-1} such that $V_{t-1} = E_{t-1} \tilde{V}_{t-1}$. By Lemma 6 and Example 3, V_{t-1} satisfies the inequality in (15). Taking conditional expectations on both sides of (8) and using Theorems 13 and 9, we get

$$h_{t-1}(x^{t-1}) = \sum_{s=0}^{t-1} (E_{t-1} K_s)(x_s, \Delta x_s) + V_{t-1}(x_{t-1}).$$

For $t = T$, $V_T = 0$ so (15) holds by Theorem 13. The claim thus holds by induction on t. $\qquad\square$

The following analogue of Lemma 25 gives sufficient conditions for the assumptions of Theorem 44.

Example 47. *Assumptions of Theorems 44 and 46 hold if* (L) *is feasible,*

1. $\{x \in \mathcal{N} \mid \sum_{t=0}^{T} K_t^{\infty}(x_t, \Delta x_t) \leq 0\}$ *is a linear space,*

2. *there exists* $p \in \mathcal{N}^{\perp}$ *and* $\epsilon > 0$ *such that for every* $\lambda \in (1 - \epsilon, 1 + \epsilon)$, *there exist* $y \in L^1$ *and* $m_t \in L^1$ *with*

$$K_t(x_t, \Delta x_t) \geq x_t \cdot (\lambda p_t + \Delta y_{t+1}) + \Delta x_t \cdot y_t - m_t$$

for all t.

Proof. The second assumption clearly implies the lower bound in both theorems. Summing up the lower bounds in 2, we see that Lemma 25 is applicable, which implies the rest of the assumptions. $\square$

The linearity condition in Example 47 holds, in particular, if $K_t^{\infty} \geq 0$ and $K_t^{\infty}(x_t, x_t) > 0$ for every $t = 0, \ldots, T$ and $x_t \neq 0$. Indeed, in this case

$$\{x \in \mathbb{R}^n \mid \sum_{t=0}^{T} K_t^{\infty}(x_t, \Delta x_t) \leq 0\} = \{0\}$$

almost surely. The last condition in Example 47 will be discussed in [23].

Example 48 (Block-diagonal stochastic LP). *Consider the stochastic linear programming problem from Example 30 and assume that the constraint* $Ax - b \in K$ *can be written as*

$$T_t \Delta x_t + W_t x_t - b_t \in C_t \quad t = 0, \ldots, T.$$

This is an instance of (L) *with*

$$K_t(\Delta x_t, x_t, \omega) = \begin{cases} c_t(\omega) \cdot x_t & \text{if } T_t(\omega)\Delta x_t + W_t(\omega)x_t - b_t(\omega) \in C_t(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

If T_t, W_t, b_t *and* C_t *are* $\mathcal{F}_t$-measurable, then K_t *is an* $\mathcal{F}_t$-measurable convex normal integrand and the dynamic programming recursion (11) can be written as*

$$V_T = 0,$$
$$\tilde{V}_{t-1}(x_{t-1}, \omega) = \inf_{x_t \in \mathbb{R}^d} \{c_t(\omega) \cdot x_t + V_t(x_t, \omega) \mid T_t(\omega)\Delta x_t + W_t(\omega)x_t - b_t(\omega) \in C_t(\omega)\},$$
$$V_{t-1} = E_{t-1}\tilde{V}_{t-1}.$$

This is the classic formulation for linear stochastic programs. Theorem 46 gives sufficient conditions for this to be well-defined.

Existence of solutions to Bellman equations for linearly constrained problems have been established also in [20].

Example 49 (Conditional independence)**.** *Assume that K_t are L-bounded, that $(V_t)_{t=0}^T$ and $(\tilde{V}_t)_{t=0}^T$ are an L-bounded solution of (11) and that there is a sequence of sigma-algebras $(\mathcal{H}_t)_{t=0}^T$ such that $\mathcal{H}_t \subseteq \mathcal{F}_t$ and*

1. K_t is $\mathcal{H}_t$-conditionally independent of $\mathcal{F}_t$,

2. $\mathcal{H}_{t+1}$ is $\mathcal{H}_t$-conditionally independent of $\mathcal{F}_t$.

Then $\tilde{V}_t$ is $\mathcal{H}_{t+1}$-measurable and

$$V_t = E^{\mathcal{H}_t} \tilde{V}_t.$$

In particular, if $(\mathcal{H}_t)_{t=0}^{T+1}$ are mutually independent, then V_t are deterministic.

Proof. By definition, $V_T = 0$ is $\mathcal{H}_T$-measurable. Assume now that V_{t+1} is $\mathcal{H}_{t+1}$-measurable. By Theorem 11, condition 1 implies $E_{t+1}K_{t+1}$ is $\mathcal{H}_{t+1}$-measurable. Thus, $\tilde{V}_t$ is then $\mathcal{H}_{t+1}$-measurable as well. By Theorem 11 again, condition 2 implies $V_t = E^{\mathcal{H}_t} \tilde{V}_t$, so the first two claims follow by induction. The last claim follows from Theorem 11. $\square$

Remark 50. *Even under the assumptions of Theorems 44 and 46, $K_t(x_t, \Delta x_t)$ need not be integrable for an optimal solution x of (L). Indeed, defining*

$$K_t(x_t, \Delta x_t, \omega) := x_t \cdot \Delta y_{t+1}(\omega) + y_t(\omega) \cdot \Delta x_t,$$

for a process $y \in L^1$ with $y_{T+1} := 0$, the assumptions of Example 47 are satisfied with $p = 0$. Since $\sum_{t=0}^T K_t(x_t, \Delta x_t, \omega) = 0$, any $x \in \mathcal{N}$ is optimal, but $K_t(x_t, \Delta x_t)$ need not be integrable.

5.5 Financial mathematics

Let $s = (s_t)_{t=0}^T$ be an adapted $\mathbb{R}^J$-valued stochastic process describing the unit prices of traded assets in a perfectly liquid financial market. Consider the problem of finding a dynamic trading strategy $z = (z_t)_{t=0}^T$ that provides the "best hedge" against the financial liability of delivering a random amount $c \in L^0$ of cash at time T. If we measure our risk preferences over random cashflows with the "expected shortfall" associated with a nondecreasing nonconstant convex "loss function" $V : \mathbb{R} \to \overline{\mathbb{R}}$, the problem can be written as

$$
\begin{aligned}
\text{minimize} \quad & EV\left(c - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\right) \quad \text{over} \quad x \in \mathcal{N}, \\
\text{subject to} \quad & x_t \in D_t \quad t = 0, \dots, T \text{ a.s.,}
\end{aligned}
\tag{ALM}
$$

where D_t is a random $\mathcal{F}_t$-measurable set describing possible portfolio constraints. We will assume $D_T = \{0\}$, which means that all positions have to be closed at the terminal date. Note that nondecreasing convex loss functions V are in one-to-one correspondence with nondecreasing concave utility functions U via $V(c) = -U(-c)$.

Problem (ALM) is standard in financial mathematics although it is based on quite unrealistic assumptions on the financial market. In particular, it assumes that we can buy and sell arbitrary quantities of all assets at prices given by s. It also assumes that one can lend and borrow arbitrary amounts of cash at zero interest rate. Under these assumptions, the random variable c can be thought of as the difference of the claim to be hedged and the initial wealth and the sum in the objective can be interpreted as the proceeds from trading from time $t = 0$ to $t = T$.

Problem (ALM) fits the general framework with

$$h(x,\omega) = V\left(c(\omega) - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}(\omega), \omega\right) + \sum_{t=0}^{T-1} \delta_{D_t(\omega)}(x_t, \omega).$$

We start by giving sufficient conditions for the main existence results in Section 4. It turns out that, in the absence of portfolio constraints, the linearity condition in Lemma 25 becomes the classical *no-arbitrage* condition

$$x \in \mathcal{N}, \ \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \geq 0 \ a.s. \implies \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} = 0 \ a.s.; \quad \text{(NA)}$$

see Remark 52 below. The lower bound in Lemma 25 holds, in particular, if there exists a martingale measure $Q \ll P$ such that

$$cy, \ V^*(dQ/dP) \in L^1,$$

V is deterministic and either

$$AE_-(V) := \limsup_{u \to -\infty} \frac{uV'(u)}{V(u)} < 1 \quad \text{or} \quad AE_+(V) := \liminf_{u \to +\infty} \frac{uV'(u)}{V(u)} > 1;$$

see Remark 53. The above limits are known as "asymptotic elasticies" of V. The conditions above are satisfied by most familiar loss functions such as the exponential, logarithmic and power functions.

The problem (ALM) fits the general framework with

$$h(x,\omega) = V\left(c(\omega) - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}(\omega), \omega\right) + \sum_{t=0}^{T-1} \delta_{D_t(\omega)}(x_t, \omega).$$

As before, we assume that V is a nondecreasing, nonconstant convex normal integrand on $\Omega \times \mathbb{R}$. This implies, in particular that, the recession function V^∞ of V is nondecreasing and strictly positive on strictly positive reals.

Theorem 51 (Existence of solutions). *Assume that*

1. there exist $y \in L^0$ and $\epsilon > 0$ such that

$$cy, \ y\Delta s_t, \ \sigma_{D_t}(E_t[y\Delta s_{t+1}]), \ V^*(\lambda y) \in L^1 \quad \forall t, \ \lambda \in [1-\epsilon, 1+\epsilon]$$

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2. the set $\mathcal{L} = \{x \in \mathcal{N} \mid \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \geq 0, \ x_t \in D_t^\infty \ P\text{-a.s.}\}$ is linear.

Then (ALM) admits optimal solutions.

Proof. By Theorems 23 and 24, it suffices to show that the two conditions in Lemma 25 hold. By Fenchel's inequality, $V(u) \geq \lambda u y - V^*(\lambda y)$ and

$$\delta_{D_t}(x_t) \geq \lambda x_t \cdot (E_t[y \Delta s_{t+1}]) - \lambda \sigma_{D_t}(E_t[y \Delta s_{t+1}])$$

so

$$h(x) \geq \lambda cy - V^*(\lambda y) - \lambda \sum_{t=0}^{T-1} x_t \cdot [y \Delta s_{t+1} - E_t[y \Delta s_{t+1}] - \lambda \sum_{t=0}^{T-1} \sigma_{D_t}(E_t[y \Delta s_{t+1}]).$$

Choosing $p_t = E_t[y \Delta s_{t+1}] - y \Delta s_{t+1}$ and

$$m = \max_{\lambda \in [1-\epsilon, 1+\epsilon]} \{V^*(\lambda y) + \lambda \sum_{t=0}^{T-1} \sigma_{D_t}(E_t[y \Delta s_{t+1}]) - \lambda cy\}$$

gives the first condition of Lemma 25. Indeed, by convexity of V^*, the maximum in the expression of m is attained scenariowise at $\lambda = 1 - \epsilon$ or $\lambda = 1 + \epsilon$ so m is integrable as the pointwise maximum of two integrable functions. By [33, Theorem 9.3],

$$h^\infty(x, \omega) = V^\infty\left(-\sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}(\omega), \omega\right) + \sum_{t=0}^{T-1} \delta_{D_t^\infty(\omega)}(x_t, \omega)$$

so condition 2 in Lemma 25 means that

$$\mathcal{L} = \{x \in \mathcal{N} \mid V^\infty\left(-\sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\right) \leq 0, \ x_t \in D_t^\infty \ P\text{-a.s.}\}$$

is a linear space. Since V is not a constant function, we have $V^\infty(u) > 0$ for $u > 0$ and hence

$$V^\infty(-\sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}) \leq 0 \quad \Leftrightarrow \quad \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \geq 0.$$

Thus,

$$\mathcal{L} = \{x \in \mathcal{N} \mid \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \geq 0, \ x_t \in D_t^\infty \ P\text{-a.s.}\},$$

so condition 2 in Lemma 25 holds. $\qquad\square$

The above extends the existence results in Theorems 2.7 and 2.10 of [28] by allowing for portfolio constraints and more general utility functions.

Remark 52. *In the absence of portfolio constraints, condition 2 in Theorem* 51 *becomes* (NA).

Remark 53. *Condition 1 in Theorem* 51 *holds, in particular, if V is lower bounded since then, $EV^*(0) < \infty$ so one can simply take $y = 0$. More generally, the condition holds if there exists a $y \in L^1$ such that*

$$cy, \ \sigma_{D_t}(E_t[y\Delta s_{t+1}]), \ V^*(y) \in L^1 \quad \forall t$$

and one of the following conditions hold

A There exist $\lambda \in (0,1)$, $\bar{y} \in \mathrm{dom}\, EV^*$ and $C > 0$ such that

$$V^*(\lambda y, \omega) \le CV^*(y, \omega) \quad \forall y \in [0, \bar{y}(\omega)].$$

B There exist $\lambda > 1$, $\bar{y} \in \mathrm{dom}\, EV^*$ and $C > 0$ such that

$$V^*(\lambda y, \omega) \le CV^*(y, \omega) \quad \forall y \ge \bar{y}(\omega).$$

If V is deterministic, then condition A holds if $AE_-(V) < 1$ while B holds if $AE_+(V) > 1$.

Proof. The first claim is obvious. As to the second, let $y \in \mathrm{dom}\, EV^*$. Under A,

$$
\begin{aligned}
EV^*(\lambda y) &= E\mathbb{1}_{\{y \le \bar{y}\}}V^*(\lambda y) + E\mathbb{1}_{\{y > \bar{y}\}}V^*(\lambda y) \\
&\le E\mathbb{1}_{\{y \le \bar{y}\}}V^*(\lambda y) + E\mathbb{1}_{\{y > \bar{y}\}}\max\{V^*(\lambda \bar{y}), V^*(y)\} \\
&\le E\mathbb{1}_{\{y \le \bar{y}\}}CV^*(y) + E\mathbb{1}_{\{y > \bar{y}\}}\max\{CV^*(\bar{y}), V^*(y)\},
\end{aligned}
$$

where the first inequality comes from the convexity of V^*. Since $\bar{y}, y \in \mathrm{dom}\, EV^*$, the last expression is integrable. Under B,

$$
\begin{aligned}
V^*(\lambda y) &= \mathbb{1}_{\{y \le \bar{y}\}}V^*(\lambda y) + \mathbb{1}_{\{y > \bar{y}\}}V^*(\lambda y) \\
&\le \mathbb{1}_{\{y \le \bar{y}\}}\max\{V^*(y), V^*(\lambda \bar{y})\} + \mathbb{1}_{\{y > \bar{y}\}}V^*(\lambda y) \\
&\le \mathbb{1}_{\{y \le \bar{y}\}}\max\{V^*(y), CV^*(\bar{y})\} + \mathbb{1}_{\{y > \bar{y}\}}CV^*(y),
\end{aligned}
$$

where the first inequality comes from the convexity of V^*. Thus, under both A and B, $\lambda \, \mathrm{dom}\, EV^* \subseteq \mathrm{dom}\, EV^*$ so condition 1 in Theorem 51 holds.

We have $AE_-(V) < p$ if and only if there exists $\bar{u} < 0$ such that

$$V'(u) \le pV(u)/u \quad \forall u \le \bar{u},$$

while $AE_+(V) > p$ if and only if there exists $\bar{u} > 0$ such that

$$V'(u) \ge pV(u)/u \quad \forall u \ge \bar{u},$$

Thus, the last claim follows from [22, Lemmas 20 and 21]. $\qquad \square$

When applied directly to (ALM), the generalized Bellman equations (BE) do not provide much information about the solutions. We will thus reformulate (ALM) as an optimal control problem and find that the optimal investment strategy at time t depends on the past allocations only through the current level of wealth. If s is componentwise almost surely nonzero, we can write (ALM) as a stochastic control problem

$$
\begin{aligned}
\text{minimize} \quad & EV\,(c - X_T) \quad \text{over} \quad (X, U) \in \mathcal{N}, \\
\text{subject to} \quad & X_0 = 0 \\
& \Delta X_t = R_t \cdot U_{t-1} \quad \forall t = 1, \ldots, T \\
& U_t \in \tilde{D}_t \quad \forall t = 0, \ldots, T
\end{aligned}
$$

where X_t is the wealth generated by the trading strategy up to time t, $R_t^j := \Delta s_t^j / s_{t-1}^j$ is the rate of return on asset j, $U_t^j := s_t^j x_t^j$ is the amount of cash invested in asset j over the period $(t, t+1]$ and

$$
\tilde{D}_t(\omega) = \{ U \in \mathbb{R}^J \mid (U^j / s_t^j(\omega))_{j \in J} \in D_t(\omega) \}.
$$

This formulation can be extended as follows.

Example 54. *Consider the problem*

$$
\begin{aligned}
\text{minimize} \quad & EV\,(c - X_T) \quad \text{over} \quad (X, U) \in \mathcal{N}, \\
\text{subject to} \quad & X_0 = w \\
& \Delta X_t = R_t \cdot U_{t-1} \quad \forall t = 1, \ldots, T \\
& (X_t, U_t) \in \hat{D}_t \quad \forall t = 0, \ldots, T - 1,
\end{aligned}
$$

where $\hat{D}_t$ is an $\mathcal{F}_t$-measurable random set in $\mathbb{R} \times \mathbb{R}^J$ and $w \in L^0(\mathcal{F}_0)$ is a given initial wealth. The dependence of the portfolio constraint on the wealth X_t has practical significance. For instance,

$$
\hat{D} = \{ (X_t, U_t) \mid \mathbb{1} \cdot U_t \leq \alpha X_t \}
$$

describes a capital requirement where at most the proportion α of the current wealth can be invested in the risky assets while the constraint

$$
\hat{D} = \{ (X_t, U_t) \mid \mathbb{1} \cdot U_t = X_t \}
$$

would require the whole wealth to be invested in the risky assets. Various combinations of the above e.g., with short-selling constraints, can be incorporated into the constraints $\hat{D}$.

The above model fits the control format with $N = 1$, $M = |J|$, $A_t = 0$, $B_t = R_t$, $W_t = 0$ and

$$
\begin{aligned}
L_T(X_T, U_T) &= V(c - X_T), \\
L_t(X_t, U_t) &= \delta_{\hat{D}_t}(X_t, U_t), \\
L_0(X_0, U_0) &= \delta_{\{w\}}(X_0) + \delta_{\hat{D}_0}(X_t, U_t).
\end{aligned}
$$

Assuming that V and c are $\mathcal{F}_T$-measurable, the dynamic programming equations (6) can be written as

$$J_T(X_T) = V(c - X_T)$$
$$I_{t+1}(X_t, U_t) = J_{t+1}(X_t + R_{t+1} \cdot U_t) \quad t = T-1, \ldots, 0,$$
$$J_t(X_t) = \inf_{U_t \in \mathbb{R}^J} \{(E_t I_{t+1})(X_t, U_t) \mid (U_t, X_t) \in \hat{D}_t\} \quad t = T-1, \ldots, 1,$$
$$J_0(X_0) = \delta_{\{w\}}(X_0) + \inf_{U_0 \in \mathbb{R}^J} \{(E_0 I_1)(X_0, U_0) \mid (X_0, U_0) \in \hat{D}_0\}.$$

$$(17)$$

The optimality conditions (17) imply that the optimal portfolio U_t depends on the past decisions only via X_t.

Example 55 (Exponential utility). *Consider the model of Example 54 in the case where $V(u) = \exp(\rho u)/\rho$ and $\hat{D}_t(\omega) = \mathbb{R} \times \tilde{D}_t(\omega)$. A simple induction argument in (17) gives*

$$J_t(X_t) = \alpha_t V(-X_t),$$

where $\alpha_T = \exp(\rho c)$ and

$$\alpha_t = \inf_{U_t \in \tilde{D}_t} E_t[\alpha_{t+1} \exp(-\rho R_{t+1} \cdot U_t)].$$

An adapted portfolio process $\bar{U}$ satisfies the optimality condition if and only if $\bar{U}_t$ achieves the infimum above almost surely for all t. Note that the amounts of wealth U_t invested in the risky assets do not depend on the level of wealth X_t.

Assuming further that each R_t is independent of $\mathcal{F}_{t-1}$ and that $\tilde{D}_t$ and c are deterministic, we see that α_t are constants and the cost-to-go functions J_t are deterministic. If, in addition, if R_t is normally distributed with mean μ_t and covariance Σ_t, then, by the properties of moment generating functions,

$$E_t[\alpha_{t+1} \exp(-\rho R_{t+1} \cdot U_t)] = \alpha_{t+1} E[\exp(-\rho R_{t+1} \cdot U_t)]$$
$$= \alpha_{t+1} \exp(-\rho U_t \cdot \mu_t + \frac{\rho^2}{2} U_t \cdot \Sigma_t U_t),$$

so, when there are no portfolio constraints, it is optimal to take $U_t = \Sigma_t^{-1} \mu_t / \rho$. This is a classic finding in portfolio optimization going back to [19].

6 Appendix

The following is a reformulation of [29, Theorem 9.2].

Lemma 56. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ be convex and assume that*

$$N := \{x \in \mathbb{R}^n \mid f^\infty(x, 0) \le 0\}$$

is linear. Then

$$p(u) := \inf_{x \in \mathbb{R}^n} f(x, u, \omega)$$

is a convex lsc function and

$$p^\infty(u) = \inf_{x \in \mathbb{R}^n} f^\infty(x, u).$$

Moreover, $\operatorname{argmin} f(\cdot, u) \cap N^\perp \neq \emptyset$ *for all* $u \in \mathbb{R}^m$.

Proof. Since N is a convex cone, the linearity condition means that

$$f^\infty(x, u) \leq 0, \ f^\infty(-x, -u) > 0 \quad \Rightarrow \quad A(x, u) \neq 0,$$

where $A(x, u) = u$. Thus, by [29, Theorem 9.2], p is a lsc convex function, the formula for p^∞ is valid and the infimum in the definition of p is attained. That the infimum is attained by $x \in N^\perp$ follows from [29, Corollary 8.6.1] □

The following was used in the proof of Theorem 17 and some of its corollaries in Section 5.

Theorem 57. *Let f be a convex normal integrand on $\mathbb{R}^n \times \mathbb{R}^m \times \Omega$ and assume that the set-valued mapping*

$$N(\omega) = \{x \in \mathbb{R}^n \mid f^\infty(x, 0, \omega) \leq 0\}$$

is linear-valued. Then

$$p(u, \omega) := \inf_{x \in \mathbb{R}^n} f(x, u, \omega)$$

is a normal integrand on $\mathbb{R}^m \times \Omega$ and

$$p^\infty(u, \omega) = \inf_{x \in \mathbb{R}^n} f^\infty(x, u, \omega).$$

Moreover, given a $u \in \mathcal{L}^0(\mathcal{F})$, there is an $x \in \mathcal{L}^0(\mathcal{F})$ with $x(\omega) \perp N(\omega)$ and

$$p(u(\omega), \omega) = f(x(\omega), u(\omega), \omega).$$

Proof. By Lemma 56, the linearity condition implies for every $\omega \in \Omega$, that the set

$$S(\omega) := N(\omega)^\perp \cap \operatorname*{argmin}_x f(x, u, \omega)$$

is nonempty and that $p(\cdot, \omega)$ is a lower semicontinuous convex function with

$$p^\infty(u, \omega) = \inf_{x \in \mathbb{R}^n} f^\infty(x, u, \omega).$$

By [33, Proposition 14.47], the lower semicontinuity implies that p is a normal integrand. By [33, Exercise 14.54, Proposition 14.33, Proposition 14.11, Theorem 14.37], the set S is measurable so, by [33, Corollary 14.6], it admits a measurable selection x. □

The three lemmas below extend classical properties of expectations and conditional expectations to general, possibly nonintegrable random variables.

Lemma 58. *Given extended real-valued random variables ξ_1 and ξ_2, we have*

$$E[\xi_1 + \xi_2] = E[\xi_1] + E[\xi_2]$$

under any of the following:

1. *$\xi_1^+, \xi_2^+ \in L^1$ or $\xi_1^-, \xi_2^- \in L^1$.*

2. *$\xi_1 \in L^1$ or $\xi_2 \in L^1$,*

3. *ξ_1 or ξ_2 is $\{0, +\infty\}$-valued.*

Proof. If both ξ_1 and ξ_2 are positive, the equation holds by monotone convergence theorem. Assume now that $\xi_1, \xi_2 \in L^1$ and let $\xi = \xi_1 + \xi_2$. We have $\xi_+ - \xi_- = \xi_1^+ - \xi_1^- + \xi_2^+ - \xi_2^-$ so $\xi^+ + \xi_1^- + \xi_2^- = \xi^- + \xi_1^+ + \xi_2^+$ and

$$E\xi^+ + E\xi_1^- + E\xi_2^- = E\xi^- + E\xi_1^+ + E\xi_2^+.$$

Rearranging gives the equality. In general,

$$(\xi_1 + \xi_2)^+ \le \xi_1^+ + \xi_2^+, \quad (\xi_1 + \xi_2)^- \le \xi_1^- + \xi_2^-,$$
$$\xi_i^+ \le (\xi_i + \xi_j)^+ + \xi_j^-, \quad \xi_i^- \le (\xi_i + \xi_j)^- + \xi_j^+.$$

If $\xi_1^+, \xi_2^+ \in L^1$, the first inequality implies $(\xi_1 + \xi_2)^+ \in L^1$. If $\xi_i^- \notin L^1$, the last inequality gives $(\xi_i + \xi_j)^- \notin L^1$ so the equation holds. The sufficiency of $\xi_1^-, \xi_2^- \in L^1$ follows by an analogous argument. Assume now $\xi_1 \in L^1$. If $\xi_2^+ \in L^1$, the equation holds by 1 while if $\xi_2^+ \notin L^1$, the second inequality gives $(\xi_1 + \xi_2)^+ \notin L^1$, so the equation holds again. Claim 3 is clear. $\square$

Lemma 59. *Let ξ_1 and ξ_2 be extended real-valued random variables.*

1. *If ξ_1 and ξ_2 are quasi-integrable and satisfy any of the conditions in Lemma 58, then $\xi_1 + \xi_2$ is quasi-integrable and*

$$E^{\mathcal{G}}[\xi_1 + \xi_2] = E^{\mathcal{G}}[\xi_1] + E^{\mathcal{G}}[\xi_2].$$

2. *If ξ_2 and $(\xi_1 \xi_2)$ are quasi-integrable, and ξ_1 is $\mathcal{G}$-measurable, then*

$$E^{\mathcal{G}}[\xi_1 \xi_2] = \xi_1 E^{\mathcal{G}}[\xi_2].$$

Proof. Let $\alpha \in L_+^\infty(\mathcal{G})$. In 1, Lemma 58 gives

$$E[\alpha(\xi_1 + \xi_2)] = E[\alpha \xi_1] + E[\alpha \xi_2] = E[\alpha E^{\mathcal{G}} \xi_1] + E[\alpha E^{\mathcal{G}} \xi_2] = E[\alpha(E^{\mathcal{G}} \xi_1 + E^{\mathcal{G}} \xi_2)].$$

To prove 2, note first that the claim is clear if ξ_1 is bounded and either non-positive or nonnegative. Let ξ_1^ν be the projection ξ_1 to the $\mathbb{B}_\nu$. Since $\xi_1\xi_2$ is quasi-integrable, 1 gives

$$E^\mathcal{G}[\xi_1^\nu\xi_2] = E^\mathcal{G}[1_{\{\xi_1^\nu\geq 0\}}\xi_1^\nu\xi_2] + E^\mathcal{G}[1_{\{\xi_1^\nu<0\}}\xi_1^\nu\xi_2]$$
$$= 1_{\{\xi_1^\nu\geq 0\}}\xi_1^\nu E^\mathcal{G}[\xi_2] + 1_{\{\xi_1^\nu<0\}}\xi_1^\nu E^\mathcal{G}[\xi_2]$$
$$= \xi_1^\nu E^\mathcal{G}[\xi_2].$$

The last term converges almost surely to $\xi_1 E^\mathcal{G}[\xi_2]$. On the other hand, by 1,

$$E^\mathcal{G}[\xi_1^\nu\xi_2] = E^\mathcal{G}[(\xi_1^\nu\xi_2)^+ - (\xi_1^\nu\xi_2)^-] = E^\mathcal{G}[(\xi_1^\nu\xi_2)^+] + E^\mathcal{G}[-(\xi_1^\nu\xi_2)^-].$$

Since both terms on the right are monotone in ν and one of them is bounded,

$$\lim E^\mathcal{G}[\xi_1^\nu\xi_2] = \lim E^\mathcal{G}[(\xi_1^\nu\xi_2)^+] + \lim E^\mathcal{G}[-(\xi_1^\nu\xi_2)^-].$$

By conditional monotone convergence (see, e.g., [34, Theorem II.7.2]),

$$\lim E^\mathcal{G}[\xi_1\xi_2] = E^\mathcal{G}[(\xi_1\xi_2)^+] + E^\mathcal{G}[-(\xi_1\xi_2)^-],$$

where, by 1, the right side equals $E^\mathcal{G}[\xi_1\xi_2]$. $\qquad\square$

Lemma 60. *Assume that ξ is quasi-integrable and that $\mathcal{G}' \subseteq \mathcal{G} \subset \mathcal{F}$. Then*

$$E^{\mathcal{G}'}\xi = E^{\mathcal{G}'}[E^\mathcal{G}\xi].$$

and

$$E\xi = E[E^\mathcal{G}\xi].$$

Proof. The first claim follows directly from the definition. As to the second, apply Lemma 59 and Lemma 58 to $\xi = \xi^+ - \xi^-$. $\qquad\square$

The following is essentially [17, Proposition 6.6 and Corollary 6.7]

Lemma 61. *Given σ-algebras $\mathcal{G}'$, $\mathcal{G}$ and $\mathcal{H}$, the following are equivalent:*

1. *$\mathcal{G}'$ and $\mathcal{G}$ are $\mathcal{H}$-conditionally independent,*

2. *$E^\mathcal{H}[w'w] = E^\mathcal{H}[w']E^\mathcal{H}[w]$ for every $w' \in L^1(\mathcal{G}')$ and $w \in L^\infty(\mathcal{G})$,*

3. *$E^{\mathcal{G}\vee\mathcal{H}}[w'] = E^\mathcal{H}[w']$ for every $w' \in L^1(\mathcal{G}')$.*

In this case, if $\mathcal{G}$ and $\mathcal{H}$ are independent, so too are $\mathcal{G}$ and $\mathcal{G}'$.

Proof. The first implies the second by the monotone class theorem. When 2 holds, we have, for any $w' \in L^1(\mathcal{G}')$, $A \in \mathcal{G}$ and $B \in \mathcal{H}$,

$$E[E^\mathcal{H}[w']1_{A\cap B}] = E[E^\mathcal{H}[w'1_A]1_B] = E[w'1_A1_B] = E[E^{\mathcal{G}\vee\mathcal{H}}[w']1_{A\cap B}],$$

and, by the monotone class theorem, this extends from sets of the form $A \cap B$ to any set in $\mathcal{G} \vee \mathcal{H}$. Thus 2 implies 3. Assuming 3, we have, for $A' \in \mathcal{G}'$ and $A \in \mathcal{G}$,

$$E^{\mathcal{H}}[1_A 1_{A'}] = E^{\mathcal{H}}[E^{\mathcal{G} \vee \mathcal{H}}[1_A 1_{A'}]] = E^{\mathcal{H}}[1_A E^{\mathcal{G} \vee \mathcal{H}} 1_{A'}]$$
$$= E^{\mathcal{H}}[1_A E^{\mathcal{H}} 1_{A'}] = E^{\mathcal{H}}[1_A] E^{\mathcal{H}}[1_{A'}],$$

so 1 holds.

Assume now in addition that $\mathcal{G}$ and $\mathcal{H}$ are independent. Given any $A' \in \mathcal{G}'$ and $A \in \mathcal{G}$, we get

$$E[1_{A'} 1_A] = E[E^{\mathcal{H}}[1_{A'} 1_A]] = E[E^{\mathcal{H}}[1_{A'}] E^{\mathcal{H}}[1_A]]$$
$$= E[E^{\mathcal{H}}[1_{A'}] E[1_A]] = E[1_{A'}] E[1_A],$$

which implies the independence if $\mathcal{G}$ and $\mathcal{G}'$. $\qquad\square$

References

[1] E. M. L. Beale. On minimizing a convex function subject to linear inequalities. *J. Roy. Statist. Soc. Ser. B.*, 17:173–184; discussion, 194–203, 1955. (Symposium on linear programming.).

[2] R. Bellman. *Dynamic programming*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2010. Reprint of the 1957 edition, With a new introduction by Stuart Dreyfus.

[3] D. Bertsekas. *Reinforcement Learning and Optimal Control*. Athena Scientific optimization and computation series. Athena Scientific, 2019.

[4] D. P. Bertsekas. *Dynamic programming and stochastic control*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Mathematics in Science and Engineering, 125.

[5] D. P. Bertsekas and S. E. Shreve. *Stochastic optimal control*, volume 139 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978. The discrete time case.

[6] S. Biagini, T. Pennanen, and A.-P. Perkkiö. Duality and optimality conditions in stochastic optimization and mathematical finance. *Journal of Convex Analysis*, 25, 2018.

[7] J.-M. Bismut. Intégrales convexes et probabilités. *J. Math. Anal. Appl.*, 42:639–673, 1973.

[8] P. Carpentier, J.-P. Chancelier, G. Cohen, and M. De Lara. *Stochastic multi-stage optimization*, volume 75 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2015. At the crossroads between discrete time stochastic control and stochastic programming.

[9] C. Choirat, C. Hess, and R. Seri. A functional version of the birkhoff ergodic theorem for a normal integrand: A variational approach. *Annals of probability*, 31(1):63–92, 2003.

[10] G. B. Dantzig. Linear programming under uncertainty. *Management Sci.*, 1:197–206, 1955.

[11] E. B. Dynkin. Probabilistic concave dynamic programming. *Mat. Sb. (N.S.)*, 87(129):490–503, 1972.

[12] E. B. Dynkin and I. V. Evstigneev. Regular conditional expectation of correspondences. *Teor. Verojatnost. i Primenen.*, 21(2):334–347, 1976.

[13] I. V. Evstigneev. Measurable selection and dynamic programming. *Math. Oper. Res.*, 1(3):267–272, 1976.

[14] W. H. Fleming and R. W. Rishel. *Deterministic and stochastic optimal control*. Applications of Mathematics, No. 1. Springer-Verlag, Berlin-New York, 1975.

[15] F. Hiai and H. Umegaki. Integrals, conditional expectations, and martingales of multivalued functions. *J. Multivariate Anal.*, 7(1):149–182, 1977.

[16] J. Jacod and A. N. Shiryaev. Local martingales and the fundamental asset pricing theorems in the discrete-time case. *Finance Stoch.*, 2(3):259–273, 1998.

[17] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[18] D. Kramkov and W. Schachermayer. The condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999.

[19] R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics*, 3:373–413, 1969.

[20] P. Olsen. When is a multistage stochastic programming problem well-defined? *SIAM J. Control Optim.*, 14(3):518–527, 1976.

[21] T. Pennanen and A.-P. Perkkiö. Stochastic programs without duality gaps. *Mathematical Programming*, 136(1):91–110, 2012.

[22] T. Pennanen and A.-P. Perkkiö. Convex duality in optimal investment and contingent claim valuation in illiquid markets. *Finance and Stochastics*, 22(4):733–771, Oct 2018.

[23] T. Pennanen and A.-P. Perkkiö. Duality in convex stochastic optimization. *Manuscript*, 2022.

[24] T. Pennanen and A.-P. Perkkiö. Optimal stopping without snell envelopes. *Proceedings of the American Mathematical Society*, 2022.

[25] T. Pennanen, A.-P. Perkkiö, and M. Rásonyi. Existence of solutions in non-convex dynamic programming and optimal investment. *Mathematics and Financial Economics*, pages 1–16, 2016.

[26] A.-P. Perkkiö. Stochastic programs without duality gaps for objectives without a lower bound. *manuscript*, 2016.

[27] G. Peskir and A. Shiryaev. *Optimal stopping and free-boundary problems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2006.

[28] M. Rásonyi and L. Stettner. On utility maximization in discrete-time financial market models. *Ann. Appl. Probab.*, 15(2):1367–1395, 2005.

[29] R. T. Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[30] R. T. Rockafellar and R. J.-B. Wets. Nonanticipativity and L^1-martingales in stochastic optimization problems. *Math. Programming Stud.*, (6):170–187, 1976. Stochastic systems: modeling, identification and optimization, II (Proc. Sympos., Univ Kentucky, Lexington, Ky., 1975).

[31] R. T. Rockafellar and R. J.-B. Wets. The optimal recourse problem in discrete time: L^1-multipliers for inequality constraints. *SIAM J. Control Optimization*, 16(1):16–36, 1978.

[32] R. T. Rockafellar and R. J.-B. Wets. Deterministic and stochastic optimization problems of Bolza type in discrete time. *Stochastics*, 10(3-4):273–312, 1983.

[33] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1998.

[34] A. N. Shiryaev. *Probability*, volume 95 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1996. Translated from the first (1980) Russian edition by R. P. Boas.

[35] L. Thibault. Espérances conditionnelles d'intégrandes semi-continus. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 17(4):337–350, 1981.

[36] A. Truffert. Conditional expectation of integrands and random sets. *Ann. Oper. Res.*, 30(1-4):117–156, 1991. Stochastic programming, Part I (Ann Arbor, MI, 1989).