A new sufficient condition for non-convex sparse recovery via weighted $\ell_r - \ell_1$ minimization

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Abstract. In this letter, we discuss the reconstruction of sparse signals from undersampled data, which belongs to the core content of compressed sensing. A new sufficient condition in terms of the restricted isometry constant (RIC) and restricted orthogonality constant (ROC) is first established for the performance guarantee of recently proposed non-convex weighted $\ell_r - \ell_1$ minimization in recovering (approximately) sparse signals that may be polluted by noise. To be specific, it is shown that if the RIC δ_s and ROC $\theta_{s,s}$ of measurement matrix obey

$$\delta_s + \nu(s)\theta_{s,s} < 1,$$

where $\nu(s)$ depends on s for given quantities, then any s-sparse signals in noiseless setting are guaranteed to be recovered accurately via solving the constrained weighted $\ell_r - \ell_1$ minimization optimization problem and any (approximately) s-sparse signals can be estimated robustly in the noisy case. In addition, we provide several pivotal remarks which indicate the recovery guarantee is much less restricted than the existing one. The results obtained contribute to proving the fidelity of the excellent weighted $\ell_r - \ell_1$ minimization method.

Key words. Compressed sensing; Constrained weighted $\ell_r - \ell_1$ minimization; Restricted isometry property; Nonconvex sparse recovery

1 Introduction

Recovering a high-dimensional sparse signal in the awareness of significantly fewer observations, probably perturbed by noise, is an essential topic in signal processing. This as well as other associating topics in compressed sensing [1–3] have received plenty of recent attention in a diverse set of areas, incorporating photography, holography, facial recognition, magnetic resonance imaging, etc.

In compressed sensing, one discusses the model below

$$y = Ax + e, \tag{1.1}$$

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where $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ $(m \ll n)$ is a given measurement matrix, $x \in \mathbb{R}^n$ is an original sparse signal, and $e \in \mathbb{R}^m$ is an additive noise. The objective is to estimate the sparse signal x in the knowledge of the measurement matrix A and the observation vector y. The signal x could be recovered by the well-known ℓ_1 minimization method suggested through Candès and Tao [1]

$$\min_{z\in\mathbb{R}^n} \|z\|_1 \text{ subject to } \|Ax - y\|_2 \leq \epsilon, \tag{1.2}$$

where $\epsilon > 0$ denotes the noise level. Especially, one takes $\epsilon = 0$ in the noise-free situation. One favourably employs this approach as an efficiency method for recovering a sparse signal in a wide range of scenarios, see, e.g. [5, 10, 20].

Recently, Zhou and Yu [4] introduced to substitute (1.2) for the recovery of sparse signal x is to think over the constrained weighted $\ell_r - \ell_1$ minimization model below

$$\hat{x} = \arg\min_{z\in\mathbb{R}^n} \{\|z\|_r^r - \alpha\|z\|_1^r \text{ subject to } \|Az - y\|_2 \leq \epsilon\}, \tag{1.3}$$

where $\|z\|_r^r = \sum_{i=1}^n |z_i|^r$, $0 < r \leq 1$, $0 \leq \alpha \leq 1$. Suppose $\alpha \neq 1$ in the case of $r = 1$ henceforth. It is easy to see that (1.3) goes to the conventional ℓ_r minimization model as $\alpha = 0$. They demonstrated that the (approximately) sparse signals could be accurately/robustly reconstructed with the model (1.3) in the noiseless/noisy situation under the r-restricted isometry property condition. Furthermore, with a number of numerical simulations, they indicated that the weighted $\ell_r - \ell_1$ minimization performs better than the other present classic methods, such as ADMM-Lasso [21], CoSaMP [22], IHT [23] and ℓ_{1-2} minimization [24]. In [4], the researchers analyzed in detail some other virtues from the constrained weighted $\ell_r - \ell_1$ minimization.

One of the highly extensively utilized criterion for the recovery of sparse signal is the restricted isometry property (RIP) proposed by Candès and Tao [1]. A vector u is s-sparse in the case that $\#\text{supp}(u) \leq s$, in which $\text{supp}(u) = \{j : u_j \neq 0\}$ is the support of $u = (u_1, u_2, \cdots, u_n)^\top \in \mathbb{R}^n$.

Definition 1.1. *For an $m \times n$ matrix A and an integer s with $1 \leq s \leq n$, the restricted isometry constant (RIC) of order s is the smallest constant such that for every s-sparse vector u, the following inequalities hold*

$$(1 - \delta_s)\|u\|_2^2 \leq \|Au\|_2^2 \leq (1 + \delta_s)\|u\|_2^2. \tag{1.4}$$

Let t be an integer with $1 \leq t \leq n$. If $s + t \leq n$, the restricted orthogonality constant (ROC) of order (s, t) is the smallest number that obeys

$$|\langle Au, Av\rangle| \leq \theta_{s,t}\|u\|_2\|v\|_2 \tag{1.5}$$

for every s-sparse vector u and t-sparse vector v such that the supports of u and v are disjoint.

In the references, researchers have introduced a lot of sufficient conditions regarding the RIP conditions for accurate/robust reconstruction of sparse signals exploiting the models (1.2) and (1.3), see, e.g. [5–17]. Especially, Tony Cai and Anru Zhang [7] showed that the condition $\delta_s + \theta_{s,s} < 1$ can ensure the accurate construction of every s-sparse signal in the noise-free situation using the ℓ_1 minimization. Simultaneously, for any $\varepsilon > 0$, the condition $\delta_s+\theta_{s,s}<1+\varepsilon$ is not sufficient to estimate accurately every s-sparse signal via any estimate approach. As for the weighted $\ell_r-\ell_1$ minimization model (1.3), in the literature, researchers provided some based-RIP conditions for reconstructing sparse signals. For example, Zhou and Yu [4] established the condition $\delta_{as}+b\delta_{(a+1)s}<b-1$ with $b=((as)^{1-r/2}-\alpha(as)^{r/2})/(s^{1-r/2}+\alpha s^{r/2})>1$ and $a>0$ being appropriately chosen such that as is an integer; Cai [15] extended Zhou and Yu's result to the context of block sparse recovery and low-rank matrix reconstruction; Zhang and Zhang [16] presented by inserting prior support information into the block version of weighted $\ell_r-\ell_1$ minimization model, a sufficient condition concerning block p-RIP; Zhou [17] firstly obtained the condition $\delta_{2s}<\tau/\sqrt{\tau^2+\gamma}$ with $\tau=\left(\frac{s-\alpha s^r}{s+\alpha s^r}\right)^{1/r}$ and $\gamma=\frac{2^{2/r-2}}{s}[(1+\alpha-\alpha 2^r)^{-2/r}(s+1)+1]$ for $s\geq 2$, and the condition based on high-order RIP $\delta_{ts}<\tau/\sqrt{\tau^2+\gamma_t}$ with $\tau=\left(\frac{s-\alpha s^r}{s+\alpha s^r}\right)^{1/r}$ and $\gamma_t=\frac{2^{2/r-2}}{s(t-1)^2}[(1+\alpha-\alpha 2^r)^{-2/r}(s(t-1)+1)+1]$ for $t\geq 3,\ s\geq 2$. In the present paper, a new sufficient condition with respect to RIC and ROC will be established to reconstruct accurately all s-sparse signals when there is no noise and estimate robustly nearly s-sparse signals when the observed signal is contaminated by noise utilizing the model (1.3). The gained results are the extension of advanced results in [7] and [12]. To our best of knowledge, this is the first sufficient condition based on RIP and ROP of stable recovery of sparse signals by employing the model (1.3).

The construct of this paper is as follows. In Sec. 2, we give some key lemmas which are necessary in the proof of the theorem. In Sec. 3, we establish the main result and several remarks, and the associating proof is deferred to the Sec. 4. The summarization is provided in the Sec. 5.

2 Preliminaries

We begin with the introducing of basic notations. For a vector $u\in\mathbb{R}^n$, u_T stands for a vector that equals to x on the index set $T\subseteq\{1,2,\cdots,n\}$ and zero otherwise. T^c represents the complement set of T in $\{1,2,\cdots,n\}$, i.e. $T^c=\{1,2,\cdots,n\}\backslash T$. $u_{\max(s)}$ indicates the vector u with all but the biggest s coordinates in absolute values put to zeros.

We first of all recall a central technical inequality utilized in the proof of the main result, which is developed in [7]. It gives a strategy for assessing the inner product $\langle u,v\rangle$ with the ROC in the case that merely one portion is sparse.

Lemma 2.1. *Let $s,t\leq n$ and $\rho\geq 0$. Assume that the supports of $u,v\in\mathbb{R}^n$ are disjoint and $\#supp(u)\leq s$. If $\|v\|_1\leq\rho t$ and $\|v\|_\infty\leq\rho$, then*

$$\langle Au,Av\rangle\leq\theta_{s,t}\|u\|_2\cdot\rho\sqrt{t}. \tag{2.6}$$

We then need the result below given by [7]. It presents an inequality respecting the sums of two arrays of the qth power of numbers greater than or equal to 0 in the knowledge of the inequality of their sums.

Lemma 2.2. *Assume that $\mu\geq 0$, $a_1\geq a_2\geq\cdots\geq a_n\geq 0$, and $\sum_{i=1}^s a_i+\mu\geq\sum_{i=s+1}^n a_i$, then for every $q\geq 1$,*

$$\sum_{i=s+1}^n a_i^q\leq s\left(\sqrt[q]{\frac{\sum_{i=1}^s a_i^q}{s}}+\frac{\mu}{s}\right)^q. \tag{2.7}$$

Lemma 2.3. *Let $\hat{x}$ be the solution of (1.3), recovery error $h = \hat{x} - x$, $T = supp(x_{\max(s)})$ and $S = supp(h_{\max(s)})$. Then,*

$$\|h_{S^c}\|_r^r \leq \|h_S\|_r^r + 2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r. \tag{2.8}$$

Proof. Since $\hat{x}$ is a minimizer of (1.3), we get

$$\|x+h\|_r^r - \alpha\|x+h\|_1^r \leq \|x\|_r^r - \alpha\|x\|_1^r.$$

That is,

$$\|x_T + h_T\|_r^r + \|x_{T^c} + h_{T^c}\|_r^r \leq \|x_T\|_r^r + \|x_{T^c}\|_r^r + \alpha\|x+h\|_1^r - \alpha\|x\|_1^r. \tag{2.9}$$

Observing that $\|h_T\|_r \leq \|h_S\|_r$ and $\|h_{S^c}\|_r \leq \|h_{T^c}\|_r$, combining with r-inverse triangular inequality, the left side (LS) of (2.9) yields

$$\begin{aligned} LS \geq &\|x_T\|_r^r - \|h_T\|_r^r + \|h_{T^c}\|_r^r - \|x_{T^c}\|_r^r \\ \geq &\|x_T\|_r^r - \|h_S\|_r^r + \|h_{S^c}\|_r^r - \|x_{T^c}\|_r^r. \end{aligned} \tag{2.10}$$

As for the right side (RS) of (2.9), by using the fact that $(a+b)^r \leq a^r + b^r$ for $a, b \geq 0$, one gets

$$RS \leq \|x_T\|_r^r + \|x_{T^c}\|_r^r + \alpha\|h\|_1^r. \tag{2.11}$$

Hence, together with (2.9)-(2.11), it implies

$$\|h_{S^c}\|_r^r \leq \|h_S\|_r^r + 2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r.$$

□

3 Main results

On the basis of the aforementioned preparations, we now give the main results in this section.

Theorem 3.1. *Set $\nu(s) = \left(\frac{2^{\frac{2}{r}-2}\alpha^{\frac{1}{r}}(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}\sqrt{n}}{s^{\frac{1}{r}-\frac{1}{2}}-2^{\frac{3}{r}-3}\alpha^{\frac{1}{r}}\sqrt{n}} + 2^{\frac{1}{r}-1}\right)$. Think out the signal reconstruction model (1.1) with $\|e\|_2 \leq \eta$ for some $\eta \geq \epsilon$. Assume that $\hat{x}$ is the solution of (1.3) with $\|e\|_2 \leq \epsilon$. If*

$$\delta_s + \nu(s)\theta_{s,s} < 1, \tag{3.12}$$

then

$$\|\hat{x} - x\|_2 \leq C_1\|x_{T^c}\|_r + C_2(\epsilon + \eta), \tag{3.13}$$

where

$$C_1 = \lambda^{-1}(s)\left\{\frac{2^{\frac{3}{r}-2}\theta_{s,s}(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}}{s^{\frac{1}{r}-\frac{1}{2}}(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})} + \frac{2^{\frac{4}{r}-3}}{s^{\frac{1}{r}-\frac{1}{2}}}\right\}$$

and

$$C_2 = \frac{(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}\sqrt{1+\delta_s}}{\lambda(s)(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})}$$

with

$$\lambda(s) = \left\{1-\sqrt{n}\left[\frac{2^{\frac{2}{r}-2}\alpha^{\frac{1}{r}}\theta_{s,s}(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}}{s^{\frac{1}{r}-\frac{1}{2}}(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})}+\frac{2^{\frac{3}{r}-3}\alpha^{\frac{1}{r}}}{s^{\frac{1}{r}-\frac{1}{2}}}\right]\right\}.$$

Remark 3.2. *When $\alpha = 0$ and $r = 1$, Theorem 3.1 reduces to Theorem 2.2 [7]. Since the condition (3.12) blends RIC δ_s and ROC $\theta_{s,s}$, it appears slightly complex. Surprisingly, by applying Lemma 3.1 [7], we can derive a brief condition as to (1.3), which is as follows*

$$\delta_s \leq \begin{cases} \frac{1}{1+2\nu}, & s \text{ is even}, s \geq 2; \\ \frac{\sqrt{s^2-1}}{\sqrt{s^2-1}+2s\nu}, & s \text{ is odd}, s \geq 3. \end{cases} \tag{3.14}$$

It should be emphasized that if we display (3.12) by (3.14), then Theorem 3.1 as before holds.

Remark 3.3. *Similar to what was done in Remark 3.2, by using the mutual coherence, we can also gain a concise condition for (1.3). The definition of mutual coherence (see, e.g. [14, 18]) is*

$$\mu = \max_{i\neq j}|A_i^\top A_j|,$$

where $A_i (i = 1, \cdots, n)$ is the ith column of A, and $\|A_i\|_2 = 1$, $i = 1, \cdots, n$. In addition, it is known from [18] that $\delta_s \leq (s-1)\mu$ and $\theta_{s,t} \leq \sqrt{st}\mu$. From (3.12), we can obtain

$$\mu < \frac{1}{s-1+s\nu(s)}. \tag{3.15}$$

Theorem 3.1 also holds under the condition (3.15). Newly, the reference [19] also presented a sufficient condition based on mutual coherence with (1.3), whose expression is

$$\mu < \frac{1}{s-1+2^{\frac{1}{r}-1}s^{\frac{1}{r}}}. \tag{3.16}$$

To compare the condition we established with that of [19], their upper bounds are given in Fig. 3.1. In the experiment, we take $\alpha = 0.2$ and $n = 100$. Observing Fig. 3.1, we can see that our conditions are better than theirs except for a few ranges of values of s and r.

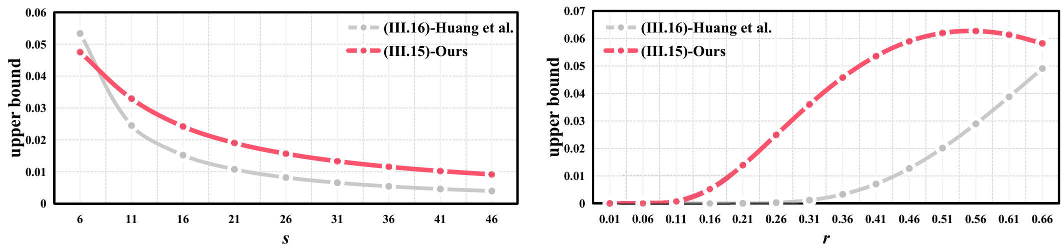
Remark 3.4. *Theorem 3.1 still holds (only replace n in $\nu(s)$ and $\lambda(s)$ by 1) provided that we substitute the model (1.3) with the following model*

$$\hat{x} = \arg\min_{z\in\mathbb{R}^n}\{\|z\|_r^r - \alpha\|z\|_2^r \text{ subject to } \|Az-y\|_2 \leq \epsilon\}. \tag{3.17}$$

Theorem 3.1 now returns to Theorem 1 [12] in the case of $\alpha = 1$ and $r = 1$. The resulting condition for (3.17) is

$$\delta_s + \frac{\sqrt{s}+\sqrt{2}-1}{\sqrt{s}-1}\theta_{s,s} < 1. \tag{3.18}$$

A comparison of condition (3.18) with that of [14] has been done in [12] and they showed that condition (3.18) is weaker than that of [12].



(a) (b)

Fig. 3.1: Comparison conditions (3.15) and (3.16) for (a) $r = 3/4$, (b) $s = 5$.

4 Proof

Proof of Theorem 3.1. Let $h = \hat{x} - x$. By Lemma 2.3, we get

$$\|h_{S^c}\|_r^r \leq \|h_S\|_r^r + 2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r. \tag{4.19}$$

In accordance with the feasibility of $\hat{x}$,

$$\|Ah\|_2 \leq \|Ax - y\|_2 + \|A\hat{x} - y\|_2 \leq \epsilon + \eta. \tag{4.20}$$

By employing Hölder inequality, we get

$$\|h_{S^c}\|_\infty^r \leq \frac{\|h_S\|_r^r}{s} \leq \frac{(s^{\frac{1}{r}-\frac{1}{2}}\|h_S\|_2)^r}{s} = \frac{\|h_S\|_2^r}{s^{\frac{r}{2}}},$$

which yields

$$\|h_{S^c}\|_\infty \leq \frac{\|h_S\|_2}{s^{\frac{1}{2}}} \leq \frac{2^{\frac{1}{r}-1}\|h_S\|_2}{s^{\frac{1}{2}}} + \frac{2^{\frac{2}{r}-2}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}}}. \tag{4.21}$$

Applying Lemma 2.2 to (4.19) with $q = 1/r$ and $\mu = 2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r$ deduces

$$\begin{aligned}
\|h_{S^c}\|_1 \leq & s\left((\frac{\|h_S\|_1}{s})^{\frac{1}{r}} + \frac{2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r}{s} \right)^{\frac{1}{r}} \\
\overset{(a)}{\leq} & s2^{\frac{1}{r}-1}\left(\frac{\|h_S\|_1}{s} + \frac{(2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r)^{\frac{1}{r}}}{s^{\frac{1}{r}}} \right) \\
\overset{(b)}{\leq} & s2^{\frac{1}{r}-1}\left(\frac{\|h_S\|_2}{s^{\frac{1}{2}}} + \frac{(2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r)^{\frac{1}{r}}}{s^{\frac{1}{r}}} \right) \\
\leq & s\left(\frac{2^{\frac{1}{r}-1}\|h_S\|_2}{s^{\frac{1}{2}}} + \frac{2^{\frac{2}{r}-2}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}}} \right),
\end{aligned} \tag{4.22}$$

where (a) is from the fact $(u_1^r + u_2^r)^{1/r} \leq 2^{\frac{1}{r}-1}(u_1 + u_2)$ for any $u_1, u_2 \geq 0$, and (b) is due to Cauchy-Schwarz inequality. It then follows from Lemma 2.1 that

$$\frac{\langle Ah_S, Ah_{S^c}\rangle}{\theta_{s,s}\sqrt{s}} \leq \|h_S\|_2 \left(\frac{2^{\frac{1}{r}-1}\|h_S\|_2}{s^{\frac{1}{2}}} + \frac{2^{\frac{2}{r}-2}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}}} \right). \tag{4.23}$$

As a result, we have

$$
\begin{aligned}
|\langle Ah, Ah_S\rangle| =& |\langle Ah_S, Ah_S\rangle + \langle Ah_S, Ah_{S^c}\rangle| \\
\geq& (1-\delta_s)\|h_S\|_2^2 - |\langle Ah_S, Ah_{S^c}\rangle| \\
\geq& (1-\delta_s)\|h_S\|_2^2 - \theta_{s,s}\|h_S\|_2 \left(2^{\frac{1}{r}-1}\|h_S\|_2 + \frac{2^{\frac{2}{r}-2}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}}\right).
\end{aligned}
\tag{4.24}
$$

On the other hand, by (4.20) and the definition of RIC, we get

$$
|\langle Ah_S, Ah\rangle| \leq \|Ah\|_2\|Ah_S\|_2 \leq (\epsilon+\eta)\sqrt{1+\delta_s}\|h_S\|_2. \tag{4.25}
$$

Combining with (4.24) and (4.25), it leads to

$$
(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})\|h_S\|_2 - \theta_{s,s}\frac{2^{\frac{2}{r}-2}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}} \leq (\epsilon+\eta)\sqrt{1+\delta_s}. \tag{4.26}
$$

Under the condition (3.12), it results in $1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s} > 0$, and thereby yields

$$
\|h_S\|_2 \leq \frac{1}{1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s}}\left(\frac{2^{\frac{2}{r}-2}\theta_{s,s}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}} + (\epsilon+\eta)\sqrt{1+\delta_s}\right). \tag{4.27}
$$

Exploiting again Lemma 2.2 to (4.19) with $q = 2/r$, it brings

$$
\begin{aligned}
\|h_{S^c}\|_2 \leq& s^{\frac{1}{2}}\left((\frac{\|h_S\|_2^2}{s})^{\frac{r}{2}} + \frac{2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r}{s}\right)^{\frac{1}{r}} \\
\leq& s^{\frac{1}{2}}2^{\frac{1}{r}-1}\left(\frac{\|h_S\|_2}{s^{\frac{1}{2}}} + \frac{(2\|x_{T^c}\|_r^r + \alpha\|h\|_1^r)^{\frac{1}{r}}}{s^{\frac{1}{r}}}\right) \\
\leq& 2^{\frac{1}{r}-1}\|h_S\|_2 + \frac{2^{\frac{2}{r}-2}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}}.
\end{aligned}
\tag{4.28}
$$

Combining with (4.27) and (4.28), we have

$$
\begin{aligned}
\|h\|_2 =& \sqrt{\|h_S\|_2^2 + \|h_{S^c}\|_2^2} \\
\leq& \left\{\|h_S\|_2^2 + \left[2^{\frac{1}{r}-1}\|h_S\|_2 + \frac{2^{\frac{2}{r}-2}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}}\right]^2\right\}^{\frac{1}{2}} \\
\leq& (1+2^{\frac{2}{r}-2})^{\frac{1}{2}}\|h_S\|_2 + \frac{2^{\frac{3}{r}-3}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}} \\
\leq& \frac{(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}}{1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s}}\left(\frac{2^{\frac{2}{r}-2}\theta_{s,s}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}} + (\epsilon+\eta)\sqrt{1+\delta_s}\right) \\
&+ \frac{2^{\frac{3}{r}-3}(2^{\frac{1}{r}}\|x_{T^c}\|_r + \alpha^{\frac{1}{r}}\|h\|_1)}{s^{\frac{1}{r}-\frac{1}{2}}},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\lambda(s)\|h\|_2 =: \left\{1-\sqrt{n}\left[\frac{2^{\frac{2}{r}-2}\alpha^{\frac{1}{r}}\theta_{s,s}(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}}{s^{\frac{1}{r}-\frac{1}{2}}(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})} + \frac{2^{\frac{3}{r}-3}\alpha^{\frac{1}{r}}}{s^{\frac{1}{r}-\frac{1}{2}}}\right]\right\}\|h\|_2 \\
\leq& \left\{\frac{2^{\frac{3}{r}-2}\theta_{s,s}(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}}{s^{\frac{1}{r}-\frac{1}{2}}(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})} + \frac{2^{\frac{4}{r}-3}}{s^{\frac{1}{r}-\frac{1}{2}}}\right\}\|x_{T^c}\|_r \\
&+ \frac{(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}\sqrt{1+\delta_s}}{1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s}}(\epsilon+\eta).
\end{aligned}
\tag{4.29}
$$

Making use of the condition (3.12), we obtain the upper bound estimation of recovery error

$$\begin{aligned}\|h\|_2 \leq &\lambda^{-1}(s)\left\{\frac{2^{\frac{3}{r}-2}\theta_{s,s}(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}}{s^{\frac{1}{r}-\frac{1}{2}}(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})}+\frac{2^{\frac{4}{r}-3}}{s^{\frac{1}{r}-\frac{1}{2}}}\right\}\|x_{T^c}\|_r \\ &+\frac{(1+2^{\frac{2}{r}-2})^{\frac{1}{2}}\sqrt{1+\delta_s}}{\lambda(s)(1-\delta_s-2^{\frac{1}{r}-1}\theta_{s,s})}(\epsilon+\eta). \end{aligned} \tag{4.30}$$

□

5 Conclusion

In this paper, we establish the proof for the performance guarantee of weighted $\ell_r - \ell_1$ minimization in recovering sparse signals which are possibly disturbed by noise. This work partially fills up the void of combining RIP and ROP, two powerful theoretical tools, to construct sufficient conditions for weighted $\ell_r - \ell_1$ minimization method to accurately/robustly recover sparse signals. Note that our current work only obtains a loose recovery condition and a recovery error bound. One future direction is to provide tighter results even the sharp ones.

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