DUALITY ASSERTIONS IN VECTOR OPTIMIZATION W.R.T. RELATIVELY SOLID CONVEX CONES IN REAL LINEAR SPACES

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Abstract. We derive duality assertions for vector optimization problems in real linear spaces based on a scalarization using recent results concerning the concept of relative solidness for convex cones (i.e., convex cones with nonempty intrinsic cores). In our paper, we consider an abstract vector optimization problem with generalized inequality constraints and investigate Lagrangian type duality assertions for (weak, proper) minimality notions. Our interest is neither to impose a pointedness assumption nor a solidness assumption for the convex cones involved in the solution concept of the vector optimization problem. We are able to extend the well-known Lagrangian vector duality approach by Jahn (published in Math. Prog. 25, 1983) to such a setting.

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1. INTRODUCTION

Duality theory is a fundamental tool in mathematics. Particularly, the dual variational principles of Dirichlet and Thompson or the pair of dual programs in linear or nonlinear convex optimization (for instance in approximation theory) are very useful from the theoretical as well as computational point of view. It could be beneficial to consider a dual problem to a given optimization problem if the dual problem has (under additional conditions) the same optimal value as the given primal optimization problem. In particular, the solution of the dual problem could be done with other methods of variational analysis or numerical mathematics. The objective function values of feasible elements of one problem generate bounds for the objective function values of the other problem. Duality statements can be used in order to derive effective primal-dual algorithms for solving the given optimization problem. Moreover, there are important interpretations of the dual variables, instancing the Lagrangian method, saddle points, equilibrium points of two person games, shadow prices in economics, perturbation methods or dual variational principles.

Certainly, the advantages mentioned above require an appropriately constructed dual program. Anyway, these advantages are incentive enough, to look for dual problems in vector optimization with corresponding useful properties too. There are a lot of books and (survey)

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papers, which are dedicated to that aim (see for instance Adán and Novo [3], Breckner [7], Jahn [19], [20], [21], Boţ, Grad [4], Boţ, Grad and Wanka [5], Boţ, Wanka [6], Hernández, Löhne, Rodríguez-Marín, Tammer [17], Löhne [24], [25], [26], Luc [28], Sawaragi, Nakayama, Tanino [34] and references therein).

In scalar as well as in vector optimization, there are three important approaches to construct a dual problem:

- Conjugation,
- Lagrangian,
- Axiomatic Duality

(see Luc [28]).

Specifically in vector optimization, one has to take care how to handle the structure of the "inner" optimization problem in the Lagrangian and Conjugation approach. Nevertheless, there is no unified approach to dualization in vector optimization. Naturally, the solution of a vector optimization problem is not a singleton, but in general it is a set of efficient elements. This leads to difficulties since by solving the "inner" optimization problem in the Lagrangian or Conjugation approach, one obtains a set of solutions. Furthermore, the definition of infimum (or supremum) of a subset of a partially ordered space performs an important role in the progress of duality theory in vector optimization. An interesting discussion of these aspects is presented by Pallaschke, Rolewicz in [31] and by Nakayama in [29].

There are at least three essential ideas which are employed for overcoming the challenges that arise when generalizing well-known duality statements from scalar optimization theory to vector optimization in the literature (see Löhne, Tammer [27]):

- The first idea and a very popular approach is the employment of a scalarization in the formulation of the dual problem (see Schönfeld [35], Breckner [7], Jahn [19, 20, 21], Boţ, Grad [4], Boţ, Grad, Wanka [5, Chapter 4], Göpfert, Riahi, Tammer, Zălinescu [11], Gutierrez, Huerga, Novo, Tammer [15] and references therein). Separation results, scalarization techniques and well-known duality assertions from scalar optimization are applied in order to develop appropriate dual problems for the primal vector optimization problem, to show duality statements or in order to solve the dual problem. This procedure is applied in the proofs of duality statements in many papers (see e.g. [5], [6] and references therein). Nevertheless, this approach has the disadvantage that even in the case of linear vector optimization a duality gap may arise, while the usual assumptions for duality statements are satisfied as shown in [16]. In order to close the duality gap, one has to add assumptions that are not necessary in duality theory for scalar optimization problems.

  However, especially in the convex case, it is possible to derive useful duality assertions, corresponding optimality conditions and primal-dual algorithms based on a scalarization by linear functionals. The advantage of this approach is the beneficial structure of the dual problem and the formulation of the duality assertions for the natural solution concepts ((weak, proper) efficiency) of the primal and dual vector optimization problem.
- Taking into account the observation that a dual vector optimization problem is naturally set-valued, a second category of dual problems is derived in the literature (see Boţ, Grad and Wanka [5, Chapter 7], Corley [8], Luc [28], Nakayama [29], Tammer [37], Dolecki,

Malivert, [9], Pallaschke, Rolewicz [31], Song [36], Hamel, Heyde, Löhne, Tammer, Winkler [16], Sawaragi, Nakayama, Tanino [34], Tanino [38, 39], Tanino, Sawaragi [40]). In these papers, duality statements for vector optimization problems are derived without a scalarization "from the beginning". Rather, the dual problem is considered as set-valued optimization problem. By embedding the primal problem into a family of set-valued optimization problems depending from perturbation parameters and employing an extension of Fenchel's inequality, Tanino [39] has shown weak duality assertions taking into account the set-valued structure of the primal and dual vector optimization problem. Moreover, employing the relationship between a map and its biconjugate, Tanino has derived a strong duality statement. Furthermore, a set-valued approach in combination with Lagrangian techniques and perturbations of marginal relations is employed in order to prove duality statements for general vector optimization problems (where the solution concept is given by a transitive, translation-invariant relation) by Dolecki, Malivert in [9].

Duality assertions in this approach are (mostly) formulated for solutions of set-valued optimization problems that are more complicated to generate than solutions of vector optimization problems.

- Based on solution concepts with respect to the supremum and infimum in the sense of a vector lattice, a third type of dual problems in vector optimization is discussed in the literature. In order to prove duality assertions, Pallaschke and Rolewicz [31] supposed special conditions concerning the order in the image space. Employing corresponding notions of infimum and supremum in the sense of utopia minimum (maximum), a duality theory for objective functions with values in vector lattices is drived in [31]. Solution concepts based on infimal (supremal) sets are introduced in the paper by Nieuwenhuis [30]. For these solution concepts, Nieuwenhuis [30] and Taninio [38, 39] developed a duality theory. The solution concepts based on infimal sets are closely connected to the concept of weakly efficient elements. An extension of these concepts to infimal sets being tightly related to other kinds of efficiency is studied by Dolecki and Malivert in [9]. Löhne and Tammer [27] discussed an embedding of the image space of the vector optimization problem into a complete lattice without linear structure, namely a sublattice of the power set of the original image space. In [27], the primal and dual problems are set-valued optimization problems and hence interconnected with the approach of the second category.

  Also in the lattice approach to duality, the duality assertions are not formulated for the natural solution concepts ((weak, proper) efficiency) in vector optimization. Although there are relationships to the natural solution concepts in vector optimization (see Löhne [26, Section 3.5]), set-valued terms are involved in the formulation of the dual problem.

The duality statements in our paper are related to the first approach based on a scalarization. As already mentioned, the advantage of this approach is that the statements are formulated for the natural solution concepts ((weak, proper) efficiency) in vector optimization. Furthermore, the approach based on scalarization leads to useful and applicable optimality conditions that can be employed for deriving primal-dual algorithms, especially proximal point algorithms or geometric algorithms for solving vector-valued location problems (see [11, Chapter 4]). Predicated on a scalarization, we develop our duality assertions using a real-valued Lagrangian function.

Taking into account these advantages, we follow the approach to duality by Jahn [19, 20, 21] based on a scalarization for problems with linear topological image spaces. Because there are interesting applications for entropy problems, in mathematical finance and risk theory, we derive our results for vector optimization problems where the objective function takes its values in a linear space. Especially, in arbitrage theory (see the class of financial market models in [32, Section 2], and Riedel [33]), the investigations are done in the framework of general real linear spaces of financial positions instead of linear topological spaces to improve the applicability for practical purposes. Moreover, in the literature related to monetary measures of risk (see Föllmer, Schied [10]), the class of capital positions is adopted to be the linear space of bounded functions containing the constants.

We employ solution concepts formulated with respect to domination structures where generalized algebraic interior notions (the relative algebraic interior (or the intrinsic core) and the lineality space of a convex cone) are involved (recently developed by Günther, Khazayel and Tammer in [14], compare also Adán and Novo [2, 1], Grad and Pop [13]). Furthermore, we approve that new generalized convexity assumptions are fulfilled and suppose generalized regularity assumptions in the context of our algebraic interior notions in order to prove novel strong duality statements.

2. Preliminaries

Throughout the paper, let $E \neq \{0\}$ be a real linear space, and let

$$E' = \{x' : E \to \mathbb{R} \mid x' \text{ is linear}\}$$

be the algebraic dual space of E. It is well-known (see, e.g., Khan, Tammer and Zălinescu [22, Sec. 6.3]) that E can be endowed with the convex core topology τ_c (which is the strongest locally convex topology on E), that is generated by the family of all the semi-norms defined on E. According to [22, Prop. 6.3.1], the topological dual space of E, namely $(E, \tau_c)^*$, is exactly the algebraic dual space E'.

In what follows, $\mathbb{R}_+$ denotes the set of nonnegative real numbers, while $\mathbb{P} := \mathbb{R}_{++}$ denotes the set of positive real numbers. For any two points x and $\bar{x}$ in E, the closed, the open, the half-open line segments are given by

$$[x,\bar{x}] := \{(1-\lambda)x + \lambda\bar{x} \mid \lambda \in [0,1]\}, \qquad (x,\bar{x}) := \{(1-\lambda)x + \lambda\bar{x} \mid \lambda \in (0,1)\},$$
$$[x,\bar{x}) := \{(1-\lambda)x + \lambda\bar{x} \mid \lambda \in [0,1)\}, \qquad (x,\bar{x}] := \{(1-\lambda)x + \lambda\bar{x} \mid \lambda \in (0,1]\}.$$

Consider any set $\Omega \subseteq E$. Let the smallest affine subspace of E containing Ω be denoted by $\operatorname{aff}\Omega$. The following two (algebraic) interiority notions will be of special interest (c.f. Holmes [18, pp. 7-8]):

- the algebraic interior (or the core) of Ω, which is given as

$$\operatorname{cor}\Omega := \{x \in \Omega \mid \forall v \in E \; \exists \varepsilon > 0 : \; x + [0,\varepsilon] \cdot v \subseteq \Omega\},$$

- the relative algebraic interior (or the intrinsic core) of Ω, which is defined by

$$\operatorname{icor}\Omega := \{x \in \Omega \mid \forall v \in \operatorname{aff}(\Omega - \Omega) \; \exists \varepsilon > 0 : \; \mathrm{x} + [0,\varepsilon] \cdot \mathrm{v} \subseteq \Omega\}.$$

Recall that Ω is solid if $\operatorname{cor}\Omega \neq \emptyset$; relatively solid if $\operatorname{icor}\Omega \neq \emptyset$. It is well-known that

$$\operatorname{cor}\Omega = \begin{cases} \operatorname{icor}\Omega & \text{if } \operatorname{aff}\Omega = E, \\ \emptyset & \text{otherwise,} \end{cases}$$

and if Ω is relatively solid,

$$\operatorname{cor}\Omega \neq \emptyset \iff \operatorname{aff}\Omega = E.$$

The algebraic closure of Ω is defined using all linearly accessible points of Ω (c.f. Holmes [18, p. 9]) as

$$\operatorname{acl}\Omega := \{x \in E \mid \exists \bar{x} \in \Omega : \; [\bar{x}, x) \subseteq \Omega\}.$$

It is well-known that

$$\operatorname{int}_{\tau_c}\Omega \subseteq \operatorname{rint}_{\tau_c}\Omega \subseteq \operatorname{icor}\Omega \subseteq \Omega \subseteq \operatorname{acl}\Omega \subseteq \operatorname{cl}_{\tau_c}\Omega \subseteq \operatorname{aff}\Omega,$$

where $\operatorname{cl}_{\tau_c}\Omega$, $\operatorname{int}_{\tau_c}\Omega$ and $\operatorname{rint}_{\tau_c}\Omega$ denotes the closure, the interior and the relative interior of Ω with respect to the convex core topology τ_c, respectively. As usual, a set $\Omega \subseteq E$ is said to be convex if $(x, \bar{x}) \subseteq \Omega$ for all $x, \bar{x} \in \Omega$. Having a convex set $\Omega \subseteq E$, it is known that

$$\operatorname{acl}\Omega \subseteq \operatorname{cl}_{\tau_c}\Omega, \quad \operatorname{cor}\Omega = \operatorname{int}_{\tau_c}\Omega, \quad \operatorname{icor}\Omega = \operatorname{rint}_{\tau_c}\Omega,$$

where all stated sets are convex as well. If, in addition, Ω is relatively solid, then we have

$$\operatorname{acl}\Omega = \operatorname{cl}_{\tau_c}\Omega.$$

Recall that Ω is algebraically closed if $\operatorname{acl}\Omega = \Omega$; τ_c-closed if $\operatorname{cl}_{\tau_c}\Omega = \Omega$.

The following separation result will be used later for proving our main strong converse duality theorem (see Theorem 4.4 and Lemma 4.3).

Proposition 2.1. *Assume that $\Omega \subseteq E$ is convex and τ_c-closed set, and $x \in E$. Then, the following assertions are equivalent:*

1° $x \notin \Omega$.
2° $\exists x' \in E', \alpha \in \mathbb{R}, \forall \omega \in \Omega : \; x'(x) < \alpha \leq x'(\omega)$.

Proof. Applying a separation result by Jahn [21, Th. 3.18] for the space (E, τ_c), we easily get this proposition. □

Recall that a cone $K \subseteq E$ (i.e., $0 \in K = \mathbb{R}_+ \cdot K$) is convex if $K + K = K$; nontrivial if $\{0\} \neq K \neq E$; pointed if $\ell(K) := K \cap (-K) = \{0\}$. The set $\ell(K)$ is known as the lineality space of K. Notice that $\ell(K) \subseteq K \subseteq \operatorname{aff} K$, and K is a linear subspace of E if and only if $K = \ell(K)$.

In the sequel, we assume that

$$K \subseteq E \text{ is a convex cone with } K \neq \ell(K). \tag{2.1}$$

The (algebraic) dual cone of K is defined by

$$K^+ := \{x' \in E' \mid \forall k \in K : \; x'(k) \geq 0\}.$$

It is well-known that $\operatorname{acl} K^+ = K^+ = (\operatorname{acl} K)^+ = (\operatorname{cl}_{\tau_c} K)^+$. Moreover, if K is relatively solid, then

$$\begin{aligned} \emptyset \neq K^+ \setminus \ell(K^+) &= \{x' \in E' \mid \forall k \in \operatorname{icor} K : \; x'(k) > 0\} \\ &= \{x' \in K^+ \setminus \{0\} \mid \exists k \in \operatorname{icor} K : \; x'(k) > 0\} \end{aligned}$$

(see Khazayel et al. [23, Th. 4.8 and Rem. 5.8]). Moreover, we have

$$\mathbb{P}\cdot(K^+\setminus\ell(K^+)) = K^+\setminus\ell(K^+) = (K^+\setminus\ell(K^+)) + (K^+\setminus\ell(K^+)) = (K^+\setminus\ell(K^+)) + K^+,$$

hence $K^+\setminus\ell(K^+)$ is convex. If K (respectively, K^+) is solid, then K^+ (respectively, K) is pointed. Let us further define the following two subsets of K^+,

$$K^\# := \{y' \in E' \mid \forall k \in K\setminus\{0\} : \; y'(k) > 0\},$$
$$K^\& := \{y' \in E' \mid \forall k \in K\setminus\ell(K) : \; y'(k) > 0\} \subseteq K^+\setminus\ell(K^+).$$

It is easy to check that

$$\mathbb{P}\cdot K^\# = K^\# = K^\# + K^\# = K^\# + K^+ \quad \text{and} \quad \mathbb{P}\cdot K^\& = K^\& = K^\& + K^\& = K^\& + K^+,$$

hence $K^\#$ and $K^\&$ are convex sets. Obviously, we have

$$K^\# = \begin{cases} K^\& & \text{if } K \text{ is pointed,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 2.1 ([23, Th. 4.1, Cor. 4.5]). *Suppose that K is τ_c-closed and satisfies (2.1). Then, the following assertions hold:*

1° $\operatorname{icor} K^+ \subseteq K^\&$.
2° *If E has finite dimension, then* $\operatorname{icor} K^+ = K^\&$.
3° *If K^+ is relatively solid, then K is pointed* $\iff$ $K^\# \supseteq K^\&$ $\iff$ $K^\# \neq \emptyset$.

3. Vector optimization

Let us assume that a real linear space E is preordered by a cone K such that (2.1) is fulfilled. It is well-known that K induces on E a preorder relation $\leqq_K$ defined, for any two points $y, \bar{y} \in E$, by

$$y \leqq_K \bar{y} \quad :\Longleftrightarrow \quad y \in \bar{y} - K.$$

For notational convenience, we consider the binary relations $\leq_K$ and $<_K$ that are defined, for any two points $y, \bar{y} \in E$, by

$$y \leq_K \bar{y} \quad :\Longleftrightarrow \quad y \in \bar{y} - (K\setminus\ell(K)),$$
$$y <_K \bar{y} \quad :\Longleftrightarrow \quad y \in \bar{y} - \operatorname{icor} K.$$

3.1. **Minimality concepts.** Consider any nonempty set $Y \subseteq E$. We are going to recall definitions of some minimality concepts in vector optimization (see Jahn [21, Ch. 4] or Khan, Tammer and Zalinescu [22, Sec. 2.4] for an overview).

Definition 3.1 (Minimality). A point $\bar{y} \in Y$ is said to be a minimal element of Y w.r.t. K if for any $y \in Y$ the condition $y \leqq_K \bar{y}$ implies $\bar{y} \leqq_K y$. The set of all minimal elements of Y w.r.t. K is denoted by

$$\operatorname{MIN}(Y,K) := \{\bar{y} \in Y \mid \forall y \in Y : \; y \leqq_K \bar{y} \;\Rightarrow\; \bar{y} \leqq_K y\}.$$

Notice that

$$\operatorname{MIN}(Y,K) = \{\bar{y} \in Y \mid \nexists y \in Y : \; y \leq_K \bar{y}\}.$$

Definition 3.2 (Weak minimality)**.** A point $\bar{y} \in Y$ is said to be a weakly minimal element of Y w.r.t. K if there is no $y \in Y$ such that $y <_K \bar{y}$. The set of all minimal elements of Y w.r.t. K is denoted by

$$\mathrm{WMIN}(Y,K) := \{\bar{y} \in Y \mid \nexists y \in Y : \; y <_K \bar{y}\}.$$

As usual for Henig-type proper efficiency concepts (see Günther, Khazayel and Tammer [14]), (generalized) dilating cones for the cone K (which satisfies (2.1)) will play an important role. The considered proper efficiency concepts in this paper will be based on two specific families of convex cones

$$\mathscr{C}(K) := \{C \subseteq E \mid C \text{ is a convex cone with } K \setminus \ell(K) \subseteq \operatorname{icor} C \text{ and } C \neq \ell(C)\}$$

and

$$\mathscr{D}(K) := \{D \subseteq E \mid D \text{ is a nontrivial, convex cone with } K \setminus \ell(K) \subseteq \operatorname{cor} D\}.$$

It is easy to check that $\mathscr{D}(K) \subseteq \mathscr{C}(K)$, and $K \subseteq \operatorname{acl}(K \setminus \ell(K)) \subseteq \operatorname{acl}(\operatorname{icor} C) = \operatorname{acl} C$ for $C \in \mathscr{C}(K)$, as well as $K \subseteq \operatorname{acl} D$ for $D \in \mathscr{D}(K)$.

Definition 3.3 (Proper minimality in the sense of Henig)**.**
A point $\bar{y} \in Y$ is said to be a classical Henig properly minimal element of Y w.r.t. K if there is $D \in \mathscr{D}(K)$ such that $\bar{y} \in \mathrm{MIN}(Y,D)$. The set of all classical Henig properly minimal elements of Y w.r.t. K is denoted by $\mathrm{PMIN}_c(Y,K)$.

In particular, we like to consider the following Henig-type solution concept, which was recently studied by Günther, Khazayel and Tammer [14] in more detail.

Definition 3.4 (Extended proper minimality in the sense of Henig)**.**
A point $\bar{y} \in Y$ is said to be a Henig properly minimal element of Y w.r.t. K if there is $C \in \mathscr{C}(K)$ such that $\bar{y} \in \mathrm{MIN}(Y,C)$. The set of all Henig properly minimal elements of Y w.r.t. K is denoted by $\mathrm{PMIN}(Y,K)$.

Next, we present some important relationships between the considered solutions concepts.

Lemma 3.1. *Suppose that K satisfies (2.1). Then, the following assertions hold:*

1° $\mathrm{PMIN}_c(Y,K) \subseteq \mathrm{PMIN}(Y,K) \subseteq \mathrm{MIN}(Y,K) \subseteq \mathrm{WMIN}(Y,K)$.
2° *If $\mathscr{C}(K) = \mathscr{D}(K)$ (e.g., if K is solid), then* $\mathrm{PMIN}_c(Y,K) = \mathrm{PMIN}(Y,K)$.
3° *If $\mathscr{C}(K) = \emptyset$ ($\iff \mathscr{D}(K) = \emptyset \iff K^{\&} = \emptyset$), then* $\mathrm{PMIN}_c(Y,K) = \mathrm{PMIN}(Y,K) = \emptyset$.
4° $\mathrm{PMIN}_c(Y,K) = \bigcup_{D \in \mathscr{D}(K)} \mathrm{MIN}(Y,D) = \bigcup_{D \in \mathscr{D}(K)} \mathrm{WMIN}(Y,D)$.
5° $\mathrm{PMIN}(Y,K) = \bigcup_{C \in \mathscr{C}(K)} \mathrm{MIN}(Y,C) = \bigcup_{C \in \mathscr{C}(K)} \mathrm{WMIN}(Y,C)$.

Proof. This lemma is a direct consequence of Günther, Khazayel and Tammer [14, Lem. 4.3 and 4.4] by applying it for the identity mapping $f = \mathrm{id}_E$ on E and $\Omega := Y$. □

Lemma 3.2. *Suppose that K is relatively solid and satisfies (2.1). Then, the following assertions are satisfied:*

1° $\mathrm{WMIN}(Y,K) = \mathrm{WMIN}(Y+K,K) \cap Y$,
$\mathrm{MIN}(Y,K) = \mathrm{MIN}(Y+K,K) \cap Y$,
$\mathrm{PMIN}_c(Y,K) = \mathrm{PMIN}_c(Y+K,K) \cap Y$,
$\mathrm{PMIN}(Y,K) = \mathrm{PMIN}(Y+K,K) \cap Y$.

2° *If K is pointed, then*
$\mathrm{WMIN}(Y,C) = \mathrm{WMIN}(Y+K,C)$ *for any* $C \in \mathscr{C}(K)$,
$\mathrm{MIN}(Y,K) = \mathrm{MIN}(Y+K,K)$,
$\mathrm{PMIN}_c(Y,K) = \mathrm{PMIN}_c(Y+K,K)$,
$\mathrm{PMIN}(Y,K) = \mathrm{PMIN}(Y+K,K)$.

Proof. 1° It is easy to check that $\mathrm{WMIN}(Y,K) \subseteq \mathrm{WMIN}(Y+K,K)$, $\mathrm{MIN}(Y,K) \subseteq \mathrm{MIN}(Y+K,K)$, $\mathrm{PMIN}_c(Y,K) \subseteq \mathrm{PMIN}_c(Y+K,K)$ and $\mathrm{PMIN}(Y,K) \subseteq \mathrm{PMIN}(Y+K,K)$ taking into account some ideas used in the proofs of [14, Th. 5.4 and 5.5] (with $f := \mathrm{id}_E$ and $\Omega := Y$). Since $Y \subseteq Y+K$, we directly get the reverse inclusions in 1°.

2° Assume that K is pointed. First, we show that $\mathrm{MIN}(Y+K,K) \subseteq Y$. Consider $z \in \mathrm{MIN}(Y+K,K)$, i.e., $z = y+k$ for some $y \in Y$ and $k \in K$. If $k \in K \setminus \{0\}$, then $y \in y+k-K \setminus \{0\} = z-K \setminus \{0\}$, hence $z \notin \mathrm{MIN}(Y+K,K)$, a contradiction. Thus, $z = y \in Y$.

Let us show that $\mathrm{WMIN}(Y+K,C) = \mathrm{WMIN}(Y,C)$ for any $C \in \mathscr{C}(K)$. As in the proof of [14, Th. 5.2] (with $f := \mathrm{id}_E$ and $\Omega := Y$), we have $\mathrm{WMIN}(Y,C) \subseteq \mathrm{WMIN}(Y+K,C)$. Consider $z \in \mathrm{WMIN}(Y+K,C)$, i.e., $z = y+k$ for some $y \in Y$ and $k \in K$. If $k \in K \setminus \{0\}$, then $y \in y+k-K \setminus \{0\} \subseteq z - \mathrm{icor}\, C$, hence $z \notin \mathrm{WMIN}(Y+K,C)$, a contradiction. Thus, $z = y \in \mathrm{WMIN}(Y+K,C) \cap Y \subseteq \mathrm{WMIN}(Y,C)$.

Finally, using Lemma 3.1 (5°) we get

$$\mathrm{PMIN}(Y,K) = \bigcup_{C \in \mathscr{C}(K)} \mathrm{WMIN}(Y,C) = \bigcup_{C \in \mathscr{C}(K)} \mathrm{WMIN}(Y+K,C) = \mathrm{PMIN}(Y+K,K).$$

Similarly, since $\mathscr{D}(K) \subseteq \mathscr{C}(K)$ we get $\mathrm{PMIN}_c(Y,K) = \mathrm{PMIN}_c(Y+K,K)$ taking into account Lemma 3.1 (4°). □

In the next lemma, we state some localization results for weakly minimal elements:

Lemma 3.3. *Suppose that K is relatively solid and satisfies* (2.1). *Then, the following assertions are satisfied:*

1° *If* $K \subseteq \mathrm{aff}(Y-Y)$, *then* $\mathrm{WMIN}(Y,K) \subseteq Y \setminus \mathrm{icor}\, Y$.
2° *If Y is solid, then* $\mathrm{WMIN}(Y,K) \subseteq Y \setminus \mathrm{cor}\, Y$.
3° $\mathrm{WMIN}(Y,K) \subseteq Y \setminus \mathrm{icor}(Y+K)$.
4° *If K is solid, then* $\mathrm{WMIN}(Y,K) \subseteq Y \setminus \mathrm{cor}(Y+K) = Y \setminus (Y + \mathrm{cor}\, K)$.

Proof. All assertions follow from Günther, Khazayel and Tammer [14, Lem. 3.3] (with $f := \mathrm{id}_E$ and $\Omega := Y$). □

In our vector dual problems (see Section 4), we will be interested to find maximal elements of a dual image set $Y \subseteq E$ w.r.t. the convex cone K. Thus, it is convenient to define the following sets

$$\mathrm{MAX}(Y,K) := \mathrm{MIN}(Y,-K), \qquad \mathrm{WMAX}(Y,K) := \mathrm{WMIN}(Y,-K),$$
$$\mathrm{PMAX}_c(Y,K) := \mathrm{PMIN}_c(Y,-K), \qquad \mathrm{PMAX}(Y,K) := \mathrm{PMIN}(Y,-K).$$

3.2. **Linear scalarization.** Our vector duality approach will be based on the employment of a linear scalarization in the formulation of the dual problem. Scalarization concepts and well-known duality assertions from scalar optimization are applied in order to develop appropriate dual problems for the primal vector optimization problem, to show duality statements or in order to solve the dual problem. Therefore, in this section we recall known linear scalarization results.

Consider any nonempty set $Y \subseteq E$. The following five propositions are direct consequences of the results in [14].

Proposition 3.1 ([14, Th. 5.1]). *Suppose that K satisfies (2.1). Then:*

1° *For any $y' \in K^+ \setminus \ell(K^+)$, we have* $\operatorname{argmin}_{y\in Y} y'(y) \subseteq \mathrm{WMIN}(Y,K)$.

2° *For any $y' \in K^\&$, we have* $\operatorname{argmin}_{y\in Y} y'(y) \subseteq \mathrm{PMIN}_c(Y,K) \subseteq \mathrm{PMIN}(Y,K)$.

3° *For any $y' \in K^+$ with* $\operatorname{argmin}_{y\in Y} y'(y) = \{\bar{y}\}$ *for some $\bar{y} \in Y$, we have* $\bar{y} \in \mathrm{MIN}(Y,K)$.

4° *Assume that K is pointed. For any $y' \in K^\#$, we have* $\operatorname{argmin}_{y\in Y} y'(y) \subseteq \mathrm{PMIN}_c(Y,K) \subseteq \mathrm{PMIN}(Y,K)$.

Proposition 3.2 ([14, Th. 5.2]). *Suppose that K is relatively solid and satisfies (2.1). In addition, assume that $Y+K$ is relatively solid and convex. Then, the following assertions hold:*

1°

$$\mathrm{WMIN}(Y,K) \subseteq \bigcup_{y' \in K^+\setminus\{0\}} \operatorname{argmin}_{y\in Y} y'(y).$$

2° *If K^+ is pointed, then*

$$\mathrm{WMIN}(Y,K) = \bigcup_{y' \in K^+\setminus\{0\}} \operatorname{argmin}_{y\in Y} y'(y).$$

3° *If $\bar{y} \in \mathrm{WMIN}(Y,K)$ and $\bar{y} + \operatorname{icor} K \subseteq \operatorname{icor}(Y+K)$, then*

$$\bar{y} \in \bigcup_{y' \in K^+\setminus\ell(K^+)} \operatorname{argmin}_{y\in Y} y'(y).$$

4° *If $\mathrm{WMIN}(Y,K) + \operatorname{icor} K \subseteq \operatorname{icor}(Y+K)$, then*

$$\mathrm{WMIN}(Y,K) = \bigcup_{y' \in K^+\setminus\ell(K^+)} \operatorname{argmin}_{y\in Y} y'(y).$$

Proposition 3.3 ([14, Th. 5.3]). *Suppose that K satisfies (2.1), and $Y+K$ is convex. If either K and $Y+K$ are relatively solid (e.g, if E has finite dimension) or $Y+K$ is solid, then*

$$\mathrm{MIN}(Y,K) \subseteq \bigcup_{y' \in K^+\setminus\{0\}} \operatorname{argmin}_{y\in Y} y'(y).$$

Proposition 3.4 ([14, Th. 5.4]). *Suppose that K satisfies (2.1), and $Y+K$ is convex. Then:*

1°

$$\mathrm{PMIN}_c(Y,K) = \bigcup_{y' \in K^\&} \operatorname{argmin}_{y\in Y} y'(y).$$

2° *If K is τ_c-closed, and E has finite dimension, then*

$$\mathrm{PMIN}_c(Y,K) = \bigcup_{y' \in \operatorname{icor} K^+} \operatorname{argmin}_{y\in Y} y'(y).$$

Proposition 3.5 ([14, Th. 5.5]). *Suppose that K is relatively solid and satisfies (2.1). In addition, assume that $Y+K$ is convex. Then:*

1° *If* $\bar{y} \in \mathrm{PMIN}(Y,K)$ *and* $\bar{y} + \mathrm{icor}\, K \subseteq \mathrm{icor}(Y+K)$*, then*

$$\bar{y} \in \bigcup_{y' \in K^{\&}} \mathrm{argmin}_{y \in Y}\, y'(y).$$

2° *If* $\mathrm{PMIN}(Y,K) + \mathrm{icor}\, K \subseteq \mathrm{icor}(Y+K)$*, then*

$$\mathrm{PMIN}(Y,K) = \bigcup_{y' \in K^{\&}} \mathrm{argmin}_{y \in Y}\, y'(y).$$

3° *If* K *is* τ_c*-closed,* E *has finite dimension, and* $\mathrm{PMIN}(Y,K) + \mathrm{icor}\, K \subseteq \mathrm{icor}(Y+K)$*, then*

$$\mathrm{PMIN}(Y,K) = \bigcup_{y' \in \mathrm{icor}\, K^{+}} \mathrm{argmin}_{y \in Y}\, y'(y).$$

Remark 3.1. The previous five propositions can also be applied to the set $Y+K$ in the role of Y. Since K is a convex cone, we have $Y+K+K = Y+K$.

4. Duality for abstract vector optimization problems

4.1. **Abstract vector optimization problem.** Given two real linear spaces X and E, a nonempty feasible set $\Omega \subseteq X$, and a vector-valued objective function $f : X \to E$, we consider the following vector optimization problem:

$$\begin{cases} f(x) \to \min \text{ w.r.t. } K \\ x \in \Omega, \end{cases} \tag{P}$$

where the image space E is preordered by a cone K such that (2.1) is fulfilled. Naturally, one is searching for (weakly, properly) efficient solutions of the problem (P), i.e., one is interested to compute elements of

$$\mathrm{Eff}(\Omega \mid f,K) := f^{-1}[\mathrm{MIN}(f[\Omega],K)], \qquad \mathrm{WEff}(\Omega \mid f,K) := f^{-1}[\mathrm{WMIN}(f[\Omega],K)],$$
$$\mathrm{PEff}_c(\Omega \mid f,K) := f^{-1}[\mathrm{PMIN}_c(f[\Omega],K)], \qquad \mathrm{PEff}(\Omega \mid f,K) := f^{-1}[\mathrm{PMIN}(f[\Omega],K)].$$

In the following, we are primarily interested in computing elements of

$$\mathrm{MIN}(f[\Omega],K), \quad \mathrm{WMIN}(f[\Omega],K), \quad \mathrm{PMIN}_c(f[\Omega],K) \quad \text{and} \quad \mathrm{PMIN}(f[\Omega],K),$$

but also the elements of the sets

$$\mathrm{MIN}(f[\Omega]+K,K), \quad \mathrm{WMIN}(f[\Omega]+K,K), \quad \mathrm{PMIN}_c(f[\Omega]+K,K) \quad \text{and} \quad \mathrm{PMIN}(f[\Omega]+K,K)$$

are of interest, where $f[\Omega]+K$ is known as the upper image of (P).

We assume that the feasible set in (P) has the following structure

$$\Omega := \{x \in \widehat{\Omega} \mid g(x) \in -C\},$$

where $\widehat{\Omega} \subseteq X$ is a nonempty, convex set, V is a linear space endowed with the convex cone $C \subseteq V$, and $g : \widehat{\Omega} \to V$.

Remark 4.1. If C is a nontrivial, convex cone in V, then $g(x) \in -C$ in the formulation of Ω describes generalized inequality constraints. By putting $C := \{0\}$ (hence $C^+ = V$), one could model the case of equality constraints since $g(x) \in \{0\} \iff g(x) = 0$. In particular, the setting $X := \mathbb{R}^n$, $E := \mathbb{R}^m$, $K := \mathbb{R}^m_+$, $\widehat{\Omega} := \mathbb{R}^n_+$, $V := \mathbb{R}^q$, $C := \mathbb{R}^q_+$ (inequality constraints) or $C := \{0\}$ (equality constraints) has great importance in applications (e.g., in economics).

4.2. **Primal and dual optimization problems.** We consider now the following surrogate scalarized problems to the **primal vector problem** (P) employing different classes of linear functionals:

- $\mathscr{P}_1$: Compute elements of $\bigcup_{y' \in K^{\&}} \operatorname{argmin}_{y \in f[\Omega]+K} y'(y)$;
- $\mathscr{P}_2$: Compute elements of $\bigcup_{y' \in K^+ \setminus \ell(K^+)} \operatorname{argmin}_{y \in f[\Omega]+K} y'(y)$.

We consider the **(Lagrangian-type) dual vector problem** to (P) given by

$$\begin{cases} \mathrm{id}_E(y) \to \max \text{ w.r.t. } K \\ y \in \Psi, \end{cases} \tag{D}$$

where $\Psi \subseteq E$ is the dual feasible set and id_E represents the identity mapping on the linear space E. More precisely, we are interested in the following instances of (D):

- $\mathscr{D}_1$: Compute elements of $\mathrm{MAX}(\mathrm{id}_E[\Psi^1], K) = \mathrm{MAX}(\Psi^1, K)$, where
$$\Psi^1 := \{\bar{y} \in E \mid \exists k' \in K^{\&}, c' \in C^+ \, \forall x \in \widehat{\Omega} : \; (k' \circ f + c' \circ g)(x) \geq k'(\bar{y})\};$$
- $\mathscr{D}_2$: Compute elements of $\mathrm{WMAX}(\mathrm{id}_E[\Psi^2], K) = \mathrm{WMAX}(\Psi^2, K)$, where
$$\Psi^2 := \{\bar{y} \in E \mid \exists k' \in K^+ \setminus \ell(K^+), c' \in C^+ \, \forall x \in \widehat{\Omega} : \; (k' \circ f + c' \circ g)(x) \geq k'(\bar{y})\}.$$

Remark 4.2. Duality assertions for an abstract vector optimization problem of type (P) were initially studied by Jahn [19], [21, Sec. 8.2] in the setting that f and g are cone-convex maps, where both image spaces E and V are assumed to be topological linear spaces.

In our work, we concentrate on an algebraic setting in real linear spaces. Instead of assuming K-convexity of f we will deal in certain results with a K-convexlike function f (i.e., $f[\Omega]+K$ is a convex set). In particular, our interest is neither to impose a pointedness assumption (thus the case $K^{\#} = \emptyset \neq K^{\&}$ may happen) nor a solidness assumption for K and its dual cone K^+. In certain results related to the weak minimality (weak maximality) concept, we will impose a relative solidness assumption on K. Therefore, the duality approach presented in the next sections can be seen as an extension of the approach proposed by Jahn [19], [21, Sec. 8.2].

Remark 4.3. It is easy to check that the following hold:

1° For any $y' \in K^{\&}$, we have

$$\operatorname{argmin}_{y \in f[\Omega]+K} y'(y) = \operatorname{argmin}_{y \in f[\Omega]+\ell(K)} y'(y) \supseteq \operatorname{argmin}_{y \in f[\Omega]} y'(y) = f[\Omega] \cap \operatorname{argmin}_{y \in f[\Omega]+K} y'(y),$$

and if K is pointed, then

$$\operatorname{argmin}_{y \in f[\Omega]+K} y'(y) = \operatorname{argmin}_{y \in f[\Omega]} y'(y).$$

2° For any $y' \in K^+ \setminus \ell(K^+)$, we have

$$\operatorname{argmin}_{y \in f[\Omega]+K} y'(y) = \operatorname{argmin}_{y \in f[\Omega]+(K \setminus \operatorname{icor} K)} y'(y) \supseteq \operatorname{argmin}_{y \in f[\Omega]} y'(y) = f[\Omega] \cap \operatorname{argmin}_{y \in f[\Omega]+K} y'(y).$$

4.3. **Weak vector duality.** In the weak duality assertions, we show the relationships between the primal and dual image sets without any additional (convexity, regularity) assumptions.

Theorem 4.1. *The following assertions hold:*

1° $\Psi^1 \subseteq \{\bar{y} \in E \mid \exists k' \in K^{\&} \, \forall y \in f[\Omega] : \; k'(\bar{y}) \leq k'(y)\}$
$= \{\bar{y} \in E \mid \exists k' \in K^{\&} \, \forall y \in f[\Omega]+K : \; k'(\bar{y}) \leq k'(y)\}$.

2° $\Psi^2 \subseteq \{\bar{y} \in E \mid \exists k' \in K^+ \setminus \ell(K^+) \, \forall y \in f[\Omega] : \; k'(\bar{y}) \leq k'(y)\}$
$= \{\bar{y} \in E \mid \exists k' \in K^+ \setminus \ell(K^+) \, \forall y \in f[\Omega]+K : \; k'(\bar{y}) \leq k'(y)\}$.

Proof. 1° We fix some $\bar{y} \in \Psi^1$. Then, there are $k' \in K^{\&}$ and $c' \in C^+$ such that

$$(k' \circ f + c' \circ g)(x) \geq k'(\bar{y}) \quad \text{for all } x \in \widehat{\Omega}.$$

Observing that $(c' \circ g)(x) \leq 0$ for $c' \in C^+$ and $x \in \Omega$, we get

$$(k' \circ f)(x) \geq k'(\bar{y}) \quad \text{for all } x \in \Omega,$$

or equivalently,

$$k'(y) \geq k'(\bar{y}) \quad \text{for all } y \in f[\Omega]. \tag{4.1}$$

Since $k'(k) \geq 0$ for all $k \in K$, and $0 \in K$, we get that (4.1) is equivalent to

$$k'(y) \geq k'(\bar{y}) \quad \text{for all } y \in f[\Omega] + K.$$

The proof of assertion 1° is complete,

2° The proof is similar to the one for 1°. □

Theorem 4.2. *The following assertions hold:*

1° *If $p \in f[\Omega] + K$ and $d \in \Psi^1$, then $d - p \notin K \setminus \ell(K)$.*
2° *If $\bar{y} \in (f[\Omega] + K) \cap \Psi^1$, then $\bar{y}$ solves both $\mathscr{P}_1$ and $\mathscr{D}_1$.*
3° *If $p \in f[\Omega] + K$ and $d \in \Psi^2$, then $d - p \notin \operatorname{icor} K$.*
4° *If $\bar{y} \in (f[\Omega] + K) \cap \Psi^2$, then $\bar{y}$ solves both $\mathscr{P}_2$ and $\mathscr{D}_2$.*

Proof. 1° Take $p \in f[\Omega] + K$ and $d \in \Psi^1$. On the contrary, assume that $d - p \in K \setminus \ell(K)$. For any $k' \in K^{\&}$, we have $k'(d - p) > 0$, or equivalently, $k'(d) > k'(p)$. This contradicts Theorem 4.1 (1°).

2° Let $\bar{y} \in (f[\Omega] + K) \cap \Psi^1$. With 1°, we get $d \notin \bar{y} + (K \setminus \ell(K))$ for all $d \in \Psi^1$, hence $\bar{y} \in \mathrm{MAX}(\Psi^1, K)$, i.e., $\bar{y}$ solves $\mathscr{D}_1$. By Theorem 4.1 (1°), it follows that $\bar{y}$ solves $\mathscr{P}_1$.

3° Take $p \in f[\Omega] + K$ and $d \in \Psi^2$. On the contrary, assume that $d - p \in \operatorname{icor} K$. Then, for any $k' \in K^+ \setminus \ell(K^+)$ we have $k'(d - p) > 0$, or equivalently, $k'(d) > k'(p)$. This contradicts Theorem 4.1 (2°).

4° Let $\bar{y} \in (f[\Omega] + K) \cap \Psi^2$. Assertion 3° yields $d \notin \bar{y} + \operatorname{icor} K$ for all $d \in \Psi^2$, hence $\bar{y} \in \mathrm{WMAX}(\Psi^2, K)$, i.e., $\bar{y}$ solves $\mathscr{D}_2$. By Theorem 4.1 (2°), we conclude that $\bar{y}$ solves $\mathscr{P}_2$. □

4.4. **Strong vector duality.** In order to state strong duality assertions for our vector problems, we need some duality results and regularity conditions for scalarized problems.

Consider a scalar function $\varphi : \widehat{\Omega} \to \mathbb{R}$. The classical Lagrangian function $L_\varphi : \widehat{\Omega} \times C^+ \to \mathbb{R}$ is defined by

$$L_\varphi(x, c') := \varphi(x) + (c' \circ g)(x) \quad \text{for any } x \in \widehat{\Omega}.$$

According to Jahn [21, Def. 8.6], the scalar problem

$$\inf_{x \in \Omega} \varphi(x) \tag{4.2}$$

is called

- **normal** if
$$\inf_{x \in \Omega} \varphi(x) = \sup_{c' \in C^+} \inf_{x \in \widehat{\Omega}} L_\varphi(x, c');$$
- **stable** if it is normal and
$$\sup_{c' \in C^+} \inf_{x \in \widehat{\Omega}} L_\varphi(x, c')$$
has a solution.

Lemma 4.1. *Define*

$$\Omega(\bar{C}) := \{x \in \widehat{\Omega} \mid g(x) \in -\bar{C}\}$$

for any convex cone $\bar{C} \subseteq V$. Then, we have

$$\inf_{x\in\Omega(C)} \varphi(x) \geq \inf_{x\in\Omega(\mathrm{cl}_{\tau_c} C)} \varphi(x) = \inf_{x\in\widehat{\Omega}} \sup_{c'\in C^+} L_\varphi(x,c'). \tag{4.3}$$

Proof. For showing the equality in (4.3), assume for simplicity that C is τ_c-closed (i.e., $\mathrm{cl}_{\tau_c} C = C$). First, we show that

$$\sup_{c'\in C^+} L_\varphi(x,c') = \varphi(x) + \sup_{c'\in C^+} (c' \circ g)(x) = \begin{cases} \varphi(x) & \text{for } g(x) \in -C, \\ +\infty & \text{for } g(x) \notin -C. \end{cases} \tag{4.4}$$

Consider $x \in \widehat{\Omega}$. Clearly, if $g(x) \in -C$, then $(c' \circ g)(x) \leq 0$ for all $c' \in C^+$, hence $\sup_{c'\in C^+} (c' \circ g)(x) = 0$ and $\sup_{c'\in C^+} L_\varphi(x,c') = \varphi(x)$. Now, assume that $g(x) \notin -C$. By the separation result in Proposition 2.1, we get some $x' \in C^+$ and $\alpha \in \mathbb{R}$ such hat

$$x'(g(x)) > \alpha \geq x'(-c) \quad \text{for all } c \in C. \tag{4.5}$$

Assuming $x'(c) < 0$ for some $c \in C$, we get a contradiction in (4.5), since $tc \in C$ for all $t \geq 0$ and $x'(-tc) = -tx'(c) \to +\infty$ for $t \to +\infty$. Thus, we have $x' \in C^+$. From (4.5), we get for $c = 0$ the condition $x'(g(x)) > 0$. Due to $sx' \in C^+$ for all $s \geq 0$, and $(sx')(g(x)) \to +\infty$ for $s \to +\infty$, we have $\sup_{c'\in C^+} (c' \circ g)(x) = +\infty$, hence $\sup_{c'\in C^+} L_\varphi(x,c') = +\infty$. Therefore, (4.4) holds true.

Under the assumption that C is τ_c-closed, we conclude that

$$\inf_{x\in\Omega(\mathrm{cl}_{\tau_c} C)} \varphi(x) = \inf_{x\in\Omega(C)} \varphi(x) = \inf_{x\in\widehat{\Omega}} \sup_{c'\in C^+} L_\varphi(x,c').$$

If $C(\subseteq \mathrm{cl}_{\tau_c} C)$ is not necessarily τ_c-closed, then we get

$$\inf_{x\in\Omega(C)} \varphi(x) \geq \inf_{x\in\Omega(\mathrm{cl}_{\tau_c} C)} \varphi(x) = \inf_{x\in\widehat{\Omega}} \sup_{c'\in(\mathrm{cl}_{\tau_c} C)^+} L_\varphi(x,c') = \inf_{x\in\widehat{\Omega}} \sup_{c'\in C^+} L_\varphi(x,c'),$$

which shows the desired formula (4.3). □

According to the well-established Lagrangian duality theory, the problem (4.2) (respectively, problem (4.3) under τ_c-closedness of C) can be seen as the scalar **primal problem** while the corresponding scalar **dual problem** is

$$\sup_{c'\in C^+} \inf_{x\in\widehat{\Omega}} L_\varphi(x,c'). \tag{4.6}$$

Remark 4.4. From scalar generalized Lagrangian duality theory it is well-known that **weak duality** between the primal problem (4.2) and the dual problem (4.6) holds true, i.e.,

$$\sup_{c'\in C^+} \inf_{x\in\widehat{\Omega}} L_\varphi(x,c') \leq \inf_{x\in\widehat{\Omega}} \sup_{c'\in C^+} L_\varphi(x,c') \leq \inf_{x\in\Omega} \varphi(x). \tag{4.7}$$

Strong duality between the primal problem (4.2) and the dual problem (4.6) means that (4.6) has a solution and

$$\sup_{c'\in C^+} \inf_{x\in\widehat{\Omega}} L_\varphi(x,c') = \inf_{x\in\Omega} \varphi(x).$$

Notice, if C is τ_c-closed, then $\inf_{x\in\Omega} \varphi(x) = \inf_{x\in\widehat{\Omega}} \sup_{c'\in C^+} L_\varphi(x,c')$ by Lemma 4.1.

Lemma 4.2. (4.2) *is stable if and only if* $\inf_{x\in\Omega}\varphi(x)=\inf_{x\in\widehat{\Omega}}L_\varphi(x,\bar{c}')$ *for some* $\bar{c}'\in C^+$.

Proof. Clearly, if (4.2) is stable, then

$$\inf_{x\in\Omega}\varphi(x)=\sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_\varphi(x,c')=\inf_{x\in\widehat{\Omega}}L_\varphi(x,\bar{c}')$$

for some $\bar{c}'\in C^+$.

Assume that $\inf_{x\in\Omega}\varphi(x)=\inf_{x\in\widehat{\Omega}}L_\varphi(x,\bar{c}')$ for some $\bar{c}'\in C^+$. By the weak duality condition in (4.7), we get

$$\sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_\varphi(x,\bar{c}')\le\inf_{x\in\Omega}\varphi(x)=\inf_{x\in\widehat{\Omega}}L_\varphi(x,\bar{c}')\le\sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_\varphi(x,c'),$$

hence (4.2) is stable. □

Remark 4.5. According to Grad [12, Rem. 2.10] (applied for (E,τ_c) and (V,τ_c)), if φ is convex, g is C-convex, $\widehat{\Omega}$ is convex, and the **generalized Slater condition**

- (SC): $\exists\bar{x}\in\widehat{\Omega}:\ g(\bar{x})\in-\operatorname{cor}C$

is valid, then strong duality between the primal problem (4.2) and the dual problem (4.6) holds true. Several other regularity conditions for ensuring strong duality are given by Boţ, Grad and Wanka in [5, Sec. 3.2.3, 3.3.3] and by Grad in [12, Sec. 2.2.2].

We now consider the Lagrangian $L_{k'\circ f}$ associated to our primal vector optimization problem (P) scalarized by functionals $k'\in K^{\&}$ or $k'\in K^+\setminus\ell(K^+)$.

For deriving strong vector duality results, we need the following **regularity conditions**:

- (R1): For any $k'\in K^{\&}$ the scalar problem $\inf_{x\in\Omega}(k'\circ f)(x)$ is normal, i.e.,
$$\inf_{x\in\Omega}(k'\circ f)(x)=\sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c').$$
- (R2): For any $k'\in K^+\setminus\ell(K^+)$ the scalar problem $\inf_{x\in\Omega}(k'\circ f)(x)$ is normal, i.e.,
$$\inf_{x\in\Omega}(k'\circ f)(x)=\sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c').$$
- (R3): For any $k'\in K^{\&}$ the scalar problem $\inf_{x\in\Omega}(k'\circ f)(x)$ is stable, i.e., there is $c'\in C^+$ such that
$$\inf_{x\in\Omega}(k'\circ f)(x)=\inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c').$$
- (R4): For any $k'\in K^+\setminus\ell(K^+)$ the scalar problem $\inf_{x\in\Omega}(k'\circ f)(x)$ is stable, i.e., there is $c'\in C^+$ such that
$$\inf_{x\in\Omega}(k'\circ f)(x)=\inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c').$$

Since $K^{\&}\subseteq K^+\setminus\ell(K^+)\subseteq K^+$ we have $(R4)\Rightarrow(R3)\Rightarrow(R1)$ and $(R4)\Rightarrow(R2)\Rightarrow(R1)$.

Remark 4.6. Assume that f is a K-convex function on the convex set $\widehat{\Omega}$. Applying [21, Lem. 2.7 (b)], for any $k'\in K^+$ the composition $k'\circ f$ is a convex function on $\widehat{\Omega}$. Thus, if g is C-convex on $\widehat{\Omega}$ and the generalized Slater condition (SC) is valid, then (R1), (R2), (R3) and (R4) are satisfied.

In the following theorem, we derive strong direct duality statements.

Theorem 4.3. *The following assertions hold:*

1° *If $\bar{y}$ solves $\mathscr{P}_1$, and the regularity condition (R3) holds true, then $\bar{y}$ solves $\mathscr{D}_1$.*
2° *If $\bar{y}$ solves $\mathscr{P}_2$, and the regularity condition (R4) holds true, then $\bar{y}$ solves $\mathscr{D}_2$.*

Proof. 1° Assume that $\bar{y} \in \operatorname{argmin}_{y \in f[\Omega]+K} k'(y)$ for some $k' \in K^{\&}$ and $\bar{y} = f(\bar{x}) + k$ for some $\bar{x} \in \Omega$ and $k \in K$. Obviously, we have $k'(k) = 0$ and $f(\bar{x}) \in \operatorname{argmin}_{y \in f[\Omega]} k'(y)$. Then, using (R3) there is $c' \in C^+$ with

$$k'(\bar{y}) = k'(f(\bar{x})) = \inf_{x \in \Omega} (k' \circ f)(x) = \inf_{x \in \widehat{\Omega}} (k' \circ f + c' \circ g)(x),$$

hence

$$k'(\bar{y}) \leq (k' \circ f + c' \circ g)(x) \quad \text{ for all } x \in \widehat{\Omega}.$$

We conclude that $\bar{y} \in (f[\Omega]+K) \cap \Psi^1$, and by Theorem 4.2 (2°), we infer that $\bar{y}$ solves $\mathscr{D}_1$.

2° Assume that $\bar{y} \in \operatorname{argmin}_{y \in f[\Omega]+K} k'(y)$ for some $k' \in K^+ \setminus \ell(K^+)$ and $\bar{y} = f(\bar{x}) + k$ for some $\bar{x} \in \Omega$ and $k \in K$. Clearly, we have $k'(k) = 0$ and $f(\bar{x}) \in \operatorname{argmin}_{y \in f[\Omega]} k'(y)$. By the regularity condition (R4) there is $c' \in C^+$ with

$$k'(\bar{y}) = k'(f(\bar{x})) = \inf_{x \in \Omega} (k' \circ f)(x) = \inf_{x \in \widehat{\Omega}} (k' \circ f + c' \circ g)(x),$$

and so

$$k'(\bar{y}) \leq (k' \circ f + c' \circ g)(x) \quad \text{ for all } x \in \widehat{\Omega}.$$

Thus, $\bar{y} \in (f[\Omega]+K) \cap \Psi^2$, and by Theorem 4.2 (4°), we get that $\bar{y}$ solves $\mathscr{D}_2$. □

For proving our strong converse duality theorem, we need the following lemma.

Lemma 4.3. *Assume that $f[\Omega]+K$ is τ_c-closed and convex. Then, we have:*

1° *If the regularity condition (R1) holds true, and $\Psi^1 \neq \emptyset$, then $E \setminus (f[\Omega]+K) \subseteq \operatorname{cor} \Psi^1$.*
2° *If the regularity condition (R2) holds true, and $\Psi^2 \neq \emptyset$, then $E \setminus (f[\Omega]+K) \subseteq \operatorname{cor} \Psi^2$.*

Proof. 1° Let (R1) be satisfied. Take some $\bar{y} \in E \setminus (f[\Omega]+K)$. By the separation result in Proposition 2.1, there are $y' \in E'$ and $\alpha \in \mathbb{R}$ such that

$$y'(\bar{y}) < \alpha \leq y'(y) \quad \text{ for all } y \in f[\Omega]+K. \tag{4.8}$$

Obviously, we have $y' \in K^+ \setminus \{0\}$. Since $\Psi^1 \neq \emptyset$, there is $\tilde{y} \in \Psi^1$, and so by Theorem 4.1 (1°), there exists $k' \in K^{\&}$ such that

$$k'(\tilde{y}) \leq k'(y) \quad \text{ for all } y \in f[\Omega]+K. \tag{4.9}$$

Since $\mathbb{P} \cdot K^{\&} + \mathbb{R}_+ \cdot K^+ = K^{\&}$, we have

$$z'_\lambda := \lambda k' + (1-\lambda) y' \in K^{\&} \quad \text{ for all } \lambda \in (0,1].$$

By (4.8) and (4.9), for all $\lambda \in (0,1]$ and all $y \in f[\Omega]+K$, we have

$$z'_\lambda(y) = (\lambda k' + (1-\lambda)y')(y) \geq \lambda k'(\tilde{y}) + (1-\lambda)\alpha.$$

Then, for a sufficiently small λ it follows

$$z'_\lambda(y) \geq \lambda k'(\tilde{y}) + (1-\lambda)\alpha > \lambda k'(\bar{y}) + (1-\lambda)y'(\bar{y}) = z'_\lambda(\bar{y}) \quad \text{ for all } y \in f[\Omega]+K,$$

hence

$$z'_\lambda(\bar{y}) < \lambda k'(\tilde{y}) + (1-\lambda)\alpha \leq \inf_{y \in f[\Omega]+K} z'_\lambda(y) = \inf_{x \in \Omega} (z'_\lambda \circ f)(x).$$

Due to (R1), there is $c' \in C^+$ such that

$$z'_\lambda(\bar{y}) < \inf_{x \in \widehat{\Omega}} (z'_\lambda \circ f + c' \circ g)(x).$$

Finally, we conclude

$$\bar{y} \in \{y \in E \mid \exists k' \in K^{\&}, c' \in C^+ : \inf_{x \in \widehat{\Omega}} (k' \circ f + c' \circ g)(x) > k'(y)\} \subseteq \operatorname{cor} \Psi^1.$$

The proof of 1° is complete.

2° Let (R2) be satisfied. Take some $\bar{y} \in E \setminus (f[\Omega] + K)$. By the separation result in Proposition 2.1, there are $y' \in E'$ and $\alpha \in \mathbb{R}$ such that

$$y'(\bar{y}) < \alpha \le y'(y) \quad \text{for all } y \in f[\Omega] + K. \tag{4.10}$$

Obviously, we have $y' \in K^+ \setminus \{0\}$. Since $\Psi^2 \neq \emptyset$, there is $\tilde{y} \in \Psi^2$, and so by Theorem 4.1 (2°), there exists $k' \in K^+ \setminus \ell(K^+)$ such that

$$k'(\tilde{y}) \le k'(y) \quad \text{for all } y \in f[\Omega] + K. \tag{4.11}$$

Since $\mathbb{P} \cdot (K^+ \setminus \ell(K^+)) + \mathbb{R}_+ \cdot K^+ = K^+ \setminus \ell(K^+)$, we have

$$z'_\lambda := \lambda k' + (1-\lambda) y' \in K^+ \setminus \ell(K^+) \quad \text{for all } \lambda \in (0,1].$$

By (4.10) and (4.11), for all $\lambda \in (0,1]$ and all $y \in f[\Omega] + K$, we obtain

$$z'_\lambda(y) = (\lambda k' + (1-\lambda) y')(y) \ge \lambda k'(\tilde{y}) + (1-\lambda)\alpha.$$

Then, for a sufficiently small λ it follows

$$z'_\lambda(y) \ge \lambda k'(\tilde{y}) + (1-\lambda)\alpha > \lambda k'(\bar{y}) + (1-\lambda) y'(\bar{y}) = z'_\lambda(\bar{y}) \quad \text{for all } y \in f[\Omega] + K,$$

hence

$$z'_\lambda(\bar{y}) < \lambda k'(\tilde{y}) + (1-\lambda)\alpha \le \inf_{y \in f[\Omega]+K} z'_\lambda(y) = \inf_{x \in \Omega} (z'_\lambda \circ f)(x).$$

Due to (R2), there is $c' \in C^+$ such that

$$z'_\lambda(\bar{y}) < \inf_{x \in \widehat{\Omega}} (z'_\lambda \circ f + c' \circ g)(x).$$

Hence, we get

$$\bar{y} \in \{y \in E \mid \exists k' \in K^+ \setminus \ell(K^+), c' \in C^+ : \inf_{x \in \widehat{\Omega}} (k' \circ f + c' \circ g)(x) > k'(y)\} \subseteq \operatorname{cor} \Psi^2.$$

□

Remark 4.7. If $f[\Omega] + K$ is relatively solid and convex, then $f[\Omega] + K$ is τ_c-closed if and only if $f[\Omega] + K$ is algebraically closed.

Under a closedness and convexity assumption on the upper image $f[\Omega] + K$ we can state the following strong converse duality statements.

Theorem 4.4. *Assume that $f[\Omega] + K$ is τ_c-closed and convex. The following assertions hold:*

1° *If $\bar{y}$ solves $\mathscr{D}_1$, and the regularity condition (R1) holds true, then $\bar{y}$ solves $\mathscr{P}_1$.*

2° *If $\bar{y}$ solves $\mathscr{D}_2$, and the regularity condition (R2) holds true, then $\bar{y}$ solves $\mathscr{P}_2$.*

Proof. 1° Assume that $\bar{y}$ solves $\mathscr{D}_1$. Clearly, by Lemma 3.3 (2°) we have $\bar{y} \notin \operatorname{cor} \Psi^1$, and therefore $\bar{y} \in f[\Omega]+K$ by Lemma 4.3 (1°). By Theorem 4.2 (2°), since $\bar{y} \in (f[\Omega]+K) \cap \Psi^1$ we infer that $\bar{y}$ solves $\mathscr{P}_1$.

2° Assume that $\bar{y}$ solves $\mathscr{D}_2$. Lemma 3.3 (2°) yields $\bar{y} \notin \operatorname{cor} \Psi^2$, and so $\bar{y} \in f[\Omega]+K$ by Lemma 4.3 (2°). Since $\bar{y} \in (f[\Omega]+K) \cap \Psi^2$, by Theorem 4.2 (4°) we get that $\bar{y}$ solves $\mathscr{P}_2$. □

4.5. **Main vector duality theorems.** In this section, we will present our main vector duality theorems for the original primal vector problem (P) and the corresponding dual vector problem (D) as given in Sections 4.1 and 4.2. In problem (P), we will fix the weak minimality concept / proper minimality concept and in (D) the weak maximal concept / maximal concept. The proofs of our vector duality theorems are based on the previous shown results and in particular are using linear scalarization results derived in [14] (see Section 3.2).

Now, consider (P) with the weak minimality concept, and (D) with the weak maximal concept and $\Psi := \Psi^2$.

Theorem 4.5. *Suppose that K is relatively solid and satisfies (2.1). In addition, assume that $f[\Omega]+K$ is relatively solid and convex, and that the regularity condition (R4) holds true. Then, the following assertions hold:*

1° *If $\bar{y} \in f[\Omega] \cap \Psi^2$, then $\bar{y} \in \operatorname{WMIN}(f[\Omega],K)$ and $\bar{y} \in \operatorname{WMAX}(\Psi^2,K)$.*

2° *If $\bar{y} \in \operatorname{WMIN}(f[\Omega],K)$ and $\bar{y}+\operatorname{icor} K \subseteq \operatorname{icor}(f[\Omega]+K)$, then $\bar{y} \in \operatorname{WMAX}(\Psi^2,K)$.*

3° *If $\operatorname{WMIN}(f[\Omega],K)+\operatorname{icor} K \subseteq \operatorname{icor}(f[\Omega]+K)$, then*

$$\operatorname{WMIN}(f[\Omega],K) = \bigcup_{y' \in K^+ \setminus \ell(K^+)} \operatorname{argmin}_{y \in f[\Omega]} y'(y) = f[\Omega] \cap \operatorname{WMAX}(\Psi^2,K) = f[\Omega] \cap \Psi^2.$$

4° *If $\operatorname{WMIN}(f[\Omega]+K,K)+\operatorname{icor} K \subseteq \operatorname{icor}(f[\Omega]+K)$, and $f[\Omega]+K$ is algebraically closed, then*

$$\operatorname{WMIN}(f[\Omega]+K,K) = \bigcup_{y' \in K^+ \setminus \ell(K^+)} \operatorname{argmin}_{y \in f[\Omega]+K} y'(y) = \operatorname{WMAX}(\Psi^2,K).$$

Proof. 1° Follows by Proposition 3.1 (1°), Theorem 4.2 (4°) and Remark 4.3.

2° Follows by Proposition 3.2 (3°) and Theorem 4.3 (2°) and Remark 4.3.

3° Follows by 1°, 2° and Proposition 3.2 (4°).

4° Proposition 3.2 (4°) applied for $f[\Omega]+K$ in the role of Y yields

$$\operatorname{WMIN}(f[\Omega]+K,K) = \bigcup_{y' \in K^+ \setminus \ell(K^+)} \operatorname{argmin}_{y \in f[\Omega]+K} y'(y).$$

Moreover, by Theorem 4.4 (2°) we get $\operatorname{WMIN}(f[\Omega]+K,K) \supseteq \operatorname{WMAX}(\Psi^2,K)$. The converse inclusion $\operatorname{WMIN}(f[\Omega]+K,K) \subseteq \operatorname{WMAX}(\Psi^2,K)$ follows by Theorem 4.3 (2°). □

In the following, we consider (P) with the proper minimality concept (in the sense of Definition 3.3), and (D) with the weak maximal concept and $\Psi := \Psi^1$.

Theorem 4.6. *Suppose that K satisfies (2.1), and $f[\Omega]+K$ is convex. Moreover, suppose that the regularity condition (R3) holds true. Then:*

1° *If $\bar{y} \in f[\Omega] \cap \Psi^1$, then $\bar{y} \in \operatorname{PMIN}_c(f[\Omega],K)$ and $\bar{y} \in \operatorname{MAX}(\Psi^1,K)$.*

2° *If $\bar{y} \in \operatorname{PMIN}_c(f[\Omega],K)$, then $\bar{y} \in \operatorname{MAX}(\Psi^1,K)$.*

3°

$$\mathrm{PMIN}_c(f[\Omega],K)=\bigcup_{y'\in K^{\&}}\mathrm{argmin}_{y\in f[\Omega]}\,y'(y)=f[\Omega]\cap\mathrm{MAX}(\Psi^1,K)=f[\Omega]\cap\Psi^1.$$

4° *If $f[\Omega]+K$ is τ_c-closed, then*

$$\mathrm{PMIN}_c(f[\Omega]+K,K)=\bigcup_{y'\in K^{\&}}\mathrm{argmin}_{y\in f[\Omega]+K}\,y'(y)=\mathrm{MAX}(\Psi^1,K).$$

5° *If $f[\Omega]+K$ is τ_c-closed, and K is pointed, then*

$$\mathrm{PMIN}_c(f[\Omega]+K,K)=\mathrm{PMIN}_c(f[\Omega],K)=\bigcup_{y'\in K^{\#}}\mathrm{argmin}_{y\in f[\Omega]}\,y'(y)=\mathrm{MAX}(\Psi^1,K).$$

Proof. 1° Follows by Proposition 3.1 (2°), Theorem 4.2 (2°) and Remark 4.3.
2° Follows by Proposition 3.4 (1°), Theorem 4.3 (1°) and Remark 4.3.
3° Follows by 1°, 2° and Proposition 3.4 (1°).
4° Proposition 3.4 (1°) applied for $f[\Omega]+K$ in the role of Y yields

$$\mathrm{PMIN}_c(f[\Omega]+K,K)=\bigcup_{y'\in K^{\&}}\mathrm{argmin}_{y\in f[\Omega]+K}\,y'(y).$$

Moreover, by Theorem 4.4 (1°) we get $\mathrm{PMIN}_c(f[\Omega]+K,K)\supseteq\mathrm{MAX}(\Psi^2,K)$. The converse inclusion $\mathrm{PMIN}_c(f[\Omega]+K,K)\subseteq\mathrm{MAX}(\Psi^2,K)$ follows by Theorem 4.3 (1°).
5° If K is pointed, then $\ell(K)=\{0\}$ and $K^{\&}=K^{\#}$, hence the conclusion follows from assertion 4°, Remark 4.3 and Lemma 3.2 (2°).

□

In the next theorem, we consider (P) with the proper minimality concept (in the sense of Definition 3.4) and (D) with the maximal concept.

Theorem 4.7. *Suppose that K is relatively solid and satisfies (2.1). In addition, assume that $f[\Omega]+K$ is convex, and that the regularity condition (R3) holds true. Then:*

1° *If $\bar{y}\in f[\Omega]\cap\Psi^1$, then $\bar{y}\in\mathrm{PMIN}(f[\Omega],K)$ and $\bar{y}\in\mathrm{MAX}(\Psi^1,K)$.*
2° *If $\bar{y}\in\mathrm{PMIN}(f[\Omega],K)$ and $\bar{y}+\mathrm{icor}\,K\subseteq\mathrm{icor}(f[\Omega]+K)$, then $\bar{y}\in\mathrm{MAX}(\Psi^1,K)$.*
3° *If $\mathrm{PMIN}(f[\Omega],K)+\mathrm{icor}\,K\subseteq\mathrm{icor}(f[\Omega]+K)$, then*

$$\mathrm{PMIN}(f[\Omega],K)=\bigcup_{y'\in K^{\&}}\mathrm{argmin}_{y\in f[\Omega]}\,y'(y)=f[\Omega]\cap\mathrm{MAX}(\Psi^1,K)=f[\Omega]\cap\Psi^1.$$

4° *If $\mathrm{PMIN}(f[\Omega]+K,K)+\mathrm{icor}\,K\subseteq\mathrm{icor}(f[\Omega]+K)$, and $f[\Omega]+K$ is τ_c-closed, then*

$$\mathrm{PMIN}(f[\Omega]+K,K)=\bigcup_{y'\in K^{\&}}\mathrm{argmin}_{y\in f[\Omega]+K}\,y'(y)=\mathrm{MAX}(\Psi^1,K).$$

5° *If $\mathrm{PMIN}(f[\Omega]+K,K)+\mathrm{icor}\,K\subseteq\mathrm{icor}(f[\Omega]+K)$, the set $f[\Omega]+K$ is τ_c-closed, and K is pointed, then*

$$\mathrm{PMIN}(f[\Omega]+K,K)=\mathrm{PMIN}(f[\Omega],K)=\bigcup_{y'\in K^{\#}}\mathrm{argmin}_{y\in f[\Omega]}\,y'(y)=\mathrm{MAX}(\Psi^1,K).$$

Proof. 1° Follows by Proposition 3.1 (2°), Theorem 4.2 (2°) and Remark 4.3.

2° Follows by Proposition 3.5 (1°), Theorem 4.3 (1°) and Remark 4.3.

3° Follows by 1°, 2° and Proposition 3.5 (2°).

4° Proposition 3.5 (2°) applied for $f[\Omega]+K$ in the role of Y yields

$$\mathrm{PMIN}(f[\Omega]+K,K)=\bigcup_{y'\in K^{\&}}\operatorname{argmin}_{y\in f[\Omega]+K}y'(y).$$

Moreover, by Theorem 4.4 (1°) we get $\mathrm{PMIN}(f[\Omega]+K,K)\supseteq\mathrm{MAX}(\Psi^2,K)$. The converse inclusion $\mathrm{PMIN}(f[\Omega]+K,K)\subseteq\mathrm{MAX}(\Psi^2,K)$ follows by Theorem 4.3 (1°).

5° If K is pointed, then $\ell(K)=\{0\}$ and $K^{\&}=K^{\#}$, hence the conclusion follows from assertion 4°, Remark 4.3 and Lemma 3.2 (2°). □

4.6. **Dual vector problem.** In this section, we like to take a closer look on the structure of the following two instances of our vector dual problem (D):

- Compute elements of $\mathrm{MAX}(\Psi^1,K)$, where
$$\Psi^1=\{\bar{y}\in E\mid \exists k'\in K^{\&},c'\in C^+\,\forall x\in\widehat{\Omega}:\ (k'\circ f+c'\circ g)(x)\geq k'(\bar{y})\};$$
- Compute elements of $\mathrm{WMAX}(\Psi^2,K)$, where
$$\Psi^2=\{\bar{y}\in E\mid \exists k'\in K^+\setminus\ell(K^+),c'\in C^+\,\forall x\in\widehat{\Omega}:\ (k'\circ f+c'\circ g)(x)\geq k'(\bar{y})\}.$$

Remark 4.8. Obviously, the sets Ψ^1 and Ψ^2 have the following representations:

$$\Psi^1=\{\bar{y}\in E\mid \exists k'\in K^{\&},c'\in C^+:\ \inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c')\geq k'(\bar{y})\},$$
$$\Psi^2=\{\bar{y}\in E\mid \exists k'\in K^+\setminus\ell(K^+),c'\in C^+:\ \inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c')\geq k'(\bar{y})\}.$$

It is easy to check that, under the validity of (R3), we have

$$\begin{aligned}\Psi^1&=\{\bar{y}\in E\mid \exists k'\in K^{\&}:\ \sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c')\geq k'(\bar{y})\}\\&=\{\bar{y}\in E\mid \exists k'\in K^{\&}:\ \inf_{x\in\Omega}(k'\circ f)(x)\geq k'(\bar{y})\},\end{aligned}$$

and under the validity of (R4),

$$\begin{aligned}\Psi^2&=\{\bar{y}\in E\mid \exists k'\in K^+\setminus\ell(K^+):\ \sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c')\geq k'(\bar{y})\}\\&=\{\bar{y}\in E\mid \exists k'\in K^+\setminus\ell(K^+):\ \inf_{x\in\Omega}(k'\circ f)(x)\geq k'(\bar{y})\}.\end{aligned}$$

Theorem 4.8. *Assume that the regularity condition (R3) holds true. Define*

$$\begin{aligned}\mathscr{M}&:=\{\bar{y}\in E\mid \exists k'\in K^{\&}:\ \sup_{c'\in C^+}\inf_{x\in\widehat{\Omega}}L_{k'\circ f}(x,c')=k'(\bar{y})\}\\&=\{\bar{y}\in E\mid \exists k'\in K^{\&}:\ \inf_{x\in\Omega}(k'\circ f)(x)=k'(\bar{y})\}.\end{aligned}$$

Then, the following assertions hold:

1° $(f[\Omega]+K)\cap\mathscr{M}\subseteq\mathrm{MAX}(\Psi^1,K)\subseteq\mathscr{M}$.

2° $f[\Omega]\cap\mathscr{M}=f[\Omega]\cap\mathrm{MAX}(\Psi^1,K)$.

3° *If $f[\Omega]+K$ is τ_c-closed and convex, then* $\mathrm{MAX}(\Psi^1,K)=(f[\Omega]+K)\cap\mathscr{M}$.

Proof. 1° Take $\bar{y} \in \mathrm{MAX}(\Psi^1, K)$. Assume that

$$\beta := \sup_{c' \in C^+} \inf_{x \in \widehat{\Omega}} L_{k' \circ f}(x, c') > k'(\bar{y}) \tag{4.12}$$

for some $k' \in K^{\&}$. Then, there is $k \in K \setminus \ell(K)$ with $k'(k) = \beta - k'(\bar{y}) > 0$, hence $k'(\bar{y} + k) = k'(\bar{y}) + k'(k) = \beta$. This shows that $\bar{y} + k \in \Psi^1$. Observing $\bar{y} + k \in \bar{y} + (K \setminus \ell(K))$, we get a contradiction to $\bar{y} \in \mathrm{MAX}(\Psi^1, K)$. Thus, $\beta \leq k'(\bar{y})$ for all $k' \in K^{\&}$. Due to $\bar{y} \in \Psi^1$, there is $k' \in K^{\&}$ with $\beta = k'(\bar{y})$.

The remaining inclusion $(f[\Omega] + K) \cap \mathscr{M} \subseteq \mathrm{MAX}(\Psi^1, K)$ follows by Theorem 4.2 (2°).

2° It is a direct consequence of 1°.

3° The assertion follows by 1° taking into account that $\mathrm{MAX}(\Psi^1, K) \subseteq f[\Omega] + K$ by Lemma 4.3 (1°) and Lemma 3.3 (2°). □

Remark 4.9. Assume that the regularity condition (R3) holds true and that $\inf_{x \in \Omega} (k' \circ f)(x)$ with $k' \in K^{\&}$ has a solution. Now, the aim could be first to solve the scalar dual problem

$$\beta := \sup_{c' \in C^+} \inf_{x \in \widehat{\Omega}} L_{k' \circ f}(x, c')$$

and then to compute an element $\bar{x} \in \Omega$ with

$$\beta = \inf_{x \in \Omega} (k' \circ f)(x) = k'(f(\bar{x})).$$

By Theorem 4.8, we conclude $f(\bar{x}) \in (f[\Omega] + K) \cap \mathscr{M} \subseteq \mathrm{MAX}(\Psi^1, K)$. Notice that

$$f[\Omega] \cap \mathrm{MAX}(\Psi^1, K) = f[\Omega] \cap \mathscr{M} = \{y \in f[\Omega] \mid \exists k' \in K^{\&} : \inf_{x \in \Omega} (k' \circ f)(x) = k'(y)\} = f[\Omega] \cap \Psi^1.$$

Theorem 4.9. *Assume that the regularity condition (R4) holds true. Define*

$$\begin{aligned} \mathscr{M}_w &:= \{\bar{y} \in E \mid \exists k' \in K^+ \setminus \ell(K^+) : \sup_{c' \in C^+} \inf_{x \in \widehat{\Omega}} L_{k' \circ f}(x, c') = k'(\bar{y})\} \\ &= \{\bar{y} \in E \mid \exists k' \in K^+ \setminus \ell(K^+) : \inf_{x \in \Omega} (k' \circ f)(x) = k'(\bar{y})\}. \end{aligned}$$

Then, the following assertions hold:

1° $(f[\Omega] + K) \cap \mathscr{M}_w \subseteq \mathrm{WMAX}(\Psi^2, K) \subseteq \mathscr{M}_w$.
2° $f[\Omega] \cap \mathscr{M} = f[\Omega] \cap \mathrm{WMAX}(\Psi^2, K)$.
3° *If* $f[\Omega] + K$ *is* τ_c*-closed and convex, then* $\mathrm{WMAX}(\Psi^2, K) = (f[\Omega] + K) \cap \mathscr{M}_w$.

Proof. 1° Consider $\bar{y} \in \mathrm{WMAX}(\Psi^2, K)$. Suppose that (4.12) is valid for some $k' \in K^+ \setminus \ell(K^+)$. Then, there is $k \in \mathrm{icor}\, K$ with $k'(k) = \beta - k'(\bar{y}) > 0$, hence $k'(\bar{y} + k) = k'(\bar{y}) + k'(k) = \beta$. We conclude that $\bar{y} + k \in \Psi^2$ and $\bar{y} + k \in \bar{y} + \mathrm{icor}\, K$, a contradiction to $\bar{y} \in \mathrm{WMAX}(\Psi^2, K)$. Consequently, there is $k' \in K^+ \setminus \ell(K^+)$ with $\beta = k'(\bar{y})$.

Theorem 4.2 (4°) provides the remaining inclusion $(f[\Omega] + K) \cap \mathscr{M}_w \subseteq \mathrm{WMAX}(\Psi^2, K)$.

2° This assertion is a direct consequence of 1°.

3° Since $\mathrm{WMAX}(\Psi^2, K) \subseteq f[\Omega] + K$ by Lemma 4.3 (2°) and Lemma 3.3 (2°), the assertion follows by 1°.

□

Remark 4.10. Assume that the regularity condition (R4) holds true and that $\inf_{x\in\Omega}(k'\circ f)(x)$ with $k'\in K^+\setminus \ell(K^+)$ has a solution. As already described in Remark 4.9 (for the concept of maximality), the aim could be first to solve the scalar dual problem

$$\beta := \sup_{c'\in C^+} \inf_{x\in\widehat{\Omega}} L_{k'\circ f}(x,c')$$

and then to compute an element $\bar{x}\in\Omega$ with

$$\beta = \inf_{x\in\Omega}(k'\circ f)(x) = k'(f(\bar{x})).$$

By Theorem 4.9 (1°), we conclude $f(\bar{x})\in (f[\Omega]+K)\cap\mathscr{M}_w \subseteq \mathrm{WMAX}(\Psi^2,K)$. Notice that

$$f[\Omega]\cap\mathrm{WMAX}(\Psi^2,K) = f[\Omega]\cap\mathscr{M}_w = \{y\in f[\Omega]\mid \exists k'\in K^+\setminus\ell(K^+):\ \inf_{x\in\Omega}(k'\circ f)(x) = k'(y)\} = f[\Omega]\cap\Psi^2.$$

In our vector duality approach, we neither imposed a pointedness assumption nor a solidness assumption for the convex cone K involved in the solution concept of the vector optimization problem (P). In the concluding example, we like to illustrate our duality results in a finite-dimensional setting with a convex cone K which is neither pointed nor solid (but relatively solid).

Example 4.1. Consider the Euclidean spaces $X := \mathbb{R}^n$, $E := \mathbb{R}^m$, $V := \mathbb{R}^q$, $n,m,q\in\mathbb{N}$, the convex set $\widehat{\Omega} := \mathbb{R}^n_+$, the convex cone

$$K := \mathbb{R}^{m_1}_+ \times \mathbb{R}^{m_2}\times\{0\}^{m_3} \qquad \text{with } m = m_1+m_2+m_3,\ \ m_1,m_2,m_3\in\mathbb{N},$$

and its dual cone

$$K^+ = \mathbb{R}^{m_1}_+\times\{0\}^{m_2}\times\mathbb{R}^{m_3}.$$

For the Euclidean inner product defined on $E=\mathbb{R}^m$ (respectively, on $V=\mathbb{R}^q$) we will use the notation $\langle\cdot,\cdot\rangle$. For K we have the following properties

$$\begin{aligned} \mathrm{icor}\,K &= (\mathrm{int}\,\mathbb{R}^{m_1}_+)\times\mathbb{R}^{m_2}\times\{0\}^{m_3}, \text{i.e., } K \text{ is relatively solid,}\\ \mathrm{int}\,K &= \mathrm{cor}\,K = \emptyset,\ \text{i.e., } K \text{ is not solid,}\\ \ell(K) &= \{0\}^{m_1}\times\mathbb{R}^{m_2}\times\{0\}^{m_3},\ \text{i.e., } K \text{ is not pointed,}\\ K\setminus\ell(K) &= (\mathbb{R}^{m_1}_+\setminus\{0\})\times\mathbb{R}^{m_2}\times\{0\}^{m_3}, \end{aligned}$$

while for K^+ we have

$$\begin{aligned} \mathrm{icor}\,K^+ &= K^{\&} = (\mathrm{int}\,\mathbb{R}^{m_1}_+)\times\{0\}^{m_2}\times\mathbb{R}^{m_3},\ \text{i.e., } K^+ \text{ is relatively solid,}\\ \mathrm{int}\,K^+ &= \mathrm{cor}\,K^+ = \emptyset,\ \text{i.e., } K^+ \text{ is not solid,}\\ \ell(K^+) &= \{0\}^{m_1}\times\{0\}^{m_2}\times\mathbb{R}^{m_3},\ \text{i.e., } K^+ \text{ is not pointed,}\\ K^+\setminus\ell(K^+) &= (\mathbb{R}^{m_1}_+\setminus\{0\})\times\{0\}^{m_2}\times\mathbb{R}^{m_2}. \end{aligned}$$

Thus, K and K are neither pointed nor solid (but relatively solid).

We like to consider the primal vector problem (P) with inequality constraints (i.e., $C:=\mathbb{R}^q_+$, $C^+ = \mathbb{R}^q_+$) or with equality constraints (i.e., $C:=\{0\}$, $C^+=\mathbb{R}^q$). According to our presented duality approach, we are interested in the following instances of our dual vector problem (D):

- Compute elements of $\mathrm{MAX}(\Psi^1,K)$, where

$$\Psi^1 = \{\bar{y}\in\mathbb{R}^m \mid \exists k'\in\mathrm{icor}\,K^+, c'\in C^+\,\forall x\in\mathbb{R}^n_+:\ \langle k',f(x)\rangle + \langle c',g(x)\rangle \geq \langle k',\bar{y}\rangle\};$$

- Compute elements of $\mathrm{WMAX}(\Psi^2, K)$, where

$$\Psi^2 = \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in K^+ \setminus \ell(K^+), c' \in C^+ \, \forall x \in \mathbb{R}^n_+ : \; \langle k', f(x)\rangle + \langle c', g(x)\rangle \geq \langle k', \bar{y}\rangle\}.$$

We assume that the regularity condition

(R4): For any $k' \in K^+ \setminus \ell(K^+)$ there is $\bar{c}' \in C^+$ such that

$$\inf_{x\in\Omega} \langle k', f(x)\rangle = \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, \bar{c}') = \sup_{c'\in C^+} \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, c').$$

is satisfied, hence

(R3): For any $k' \in \operatorname{icor} K^+$ there is $\bar{c}' \in C^+$ such that

$$\inf_{x\in\Omega} \langle k', f(x)\rangle = \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, \bar{c}') = \sup_{c'\in C^+} \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, c')$$

is valid. The dual sets Ψ^1 and Ψ^2 have the following representations:

$$\begin{aligned}
\Psi^1 &= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in \operatorname{icor} K^+ : \; \sup_{c'\in C^+} \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, c') \geq \langle k', \bar{y}\rangle\} \\
&= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in \operatorname{icor} K^+ : \; \inf_{x\in\Omega} \langle k', f(x)\rangle \geq \langle k', \bar{y}\rangle\}, \\
\Psi^2 &= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in K^+ \setminus \ell(K^+) : \; \sup_{c'\in C^+} \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, c') \geq \langle k', \bar{y}\rangle\} \\
&= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in K^+ \setminus \ell(K^+) : \; \inf_{x\in\Omega} \langle k', f(x)\rangle \geq \langle k', \bar{y}\rangle\}.
\end{aligned}$$

Moreover, the sets

$$\begin{aligned}
\mathscr{M} &= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in \operatorname{icor} K^+ : \; \sup_{c'\in C^+} \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, c') = \langle k', \bar{y}\rangle\} \\
&= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in \operatorname{icor} K^+ : \; \inf_{x\in\Omega} \langle k', f(x)\rangle = \langle k', \bar{y}\rangle\}, \\
\mathscr{M}_w &= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in K^+ \setminus \ell(K^+) : \; \sup_{c'\in C^+} \inf_{x\in\mathbb{R}^n_+} L_{\langle k', f(\cdot)\rangle}(x, c') = \langle k', \bar{y}\rangle\} \\
&= \{\bar{y} \in \mathbb{R}^m \mid \exists k' \in K^+ \setminus \ell(K^+) : \; \inf_{x\in\Omega} \langle k', f(x)\rangle = \langle k', \bar{y}\rangle\}.
\end{aligned}$$

are of special interest. Let us summarize our main duality statements:

Assume that $f[\Omega] + K$ is convex and algebraically closed. Then, the following assertions hold:

1° If $\mathrm{WMIN}(f[\Omega], K) + \operatorname{icor} K \subseteq \operatorname{icor}(f[\Omega] + K)$, then

$$\mathrm{WMIN}(f[\Omega], K) = f[\Omega] \cap \mathrm{WMAX}(\Psi^2, K) = f[\Omega] \cap \mathscr{M}_w.$$

2° If $\mathrm{WMIN}(f[\Omega] + K, K) + \operatorname{icor} K \subseteq \operatorname{icor}(f[\Omega] + K)$, then

$$\mathrm{WMIN}(f[\Omega] + K, K) = \mathrm{WMAX}(\Psi^2, K) = (f[\Omega] + K) \cap \mathscr{M}_w.$$

3°

$$\mathrm{PMIN}_c(f[\Omega], K) = f[\Omega] \cap \mathrm{MAX}(\Psi^1, K) = f[\Omega] \cap \mathscr{M}.$$

4°

$$\mathrm{PMIN}_c(f[\Omega] + K, K) = \mathrm{MAX}(\Psi^1, K) = (f[\Omega] + K) \cap \mathscr{M}.$$

5° If $\mathrm{PMIN}(f[\Omega], K) + \operatorname{icor} K \subseteq \operatorname{icor}(f[\Omega] + K)$, then

$$\mathrm{PMIN}(f[\Omega], K) = f[\Omega] \cap \mathrm{MAX}(\Psi^1, K) = f[\Omega] \cap \mathscr{M}.$$

6° If $\mathrm{PMIN}(f[\Omega]+K,K)+\mathrm{icor}\,K \subseteq \mathrm{icor}(f[\Omega]+K)$, then

$$\mathrm{PMIN}(f[\Omega]+K,K)=\mathrm{MAX}(\Psi^1,K)=(f[\Omega]+K)\cap\mathscr{M}.$$

5. Conclusions

Following basically the duality approach proposed by Jahn [19], [21, Sec. 8.2], we have shown new duality statements for (weakly, properly) minimal elements of vector optimization problems in real linear spaces. We are employing scalarizations by linear functionals using recent results concerning the concept of relative solidness for convex cones (i.e., convex cones with nonempty intrinsic cores). It is important to mention that we do not suppose the pointedness and not the solidness of the convex cones K involved in the solution concept of the vector optimization problem. The concept of weak minimality of the abstract vector optimization problem with a linear image space is defined using the intrinsic cone and two concepts of proper minimality are defined by generalized dilating cones where the intrinsic cone of K and the lineality space $\ell(K)$ of K are involved.

Furthermore, we have shown our duality statements under weak convexity assumptions; namely, we consider a K-convexlike primal objective function f.

We have studied certain types of surrogate scalarized problems to the primal vector optimization problem, where the scalarizing linear functionals belong to subsets of the dual cone K^+ ($K^{\&}$ and $K^+\setminus\ell(K^+)$). Taking into account the structure of the scalarized primal problems, we have introduced corresponding dual vector optimization problems where $K^{\&}$ and $K^+\setminus\ell(K^+)$ are involved. Then, we have derived weak and (under additional regularity assumptions) strong duality assertions for (weakly, properly) minimal / maximal solutions.

In a forthcoming paper, we will apply our duality statements to vector-valued portfolio problems and to entropy problems in order to derive useful dual problems in consideration of the special structure of the objective function and the corresponding Lagrangian.

Furthermore, we will apply the duality assertions for deriving algorithms to solve vector optimization problems, especially to get estimations of lower bounds for the primal problem and stopping criteria in primal-dual algorithms.

Because numerical procedures usually generate approximate solutions, it would be interesting to derive corresponding duality assertions for approximate solutions of the primal and dual vector optimization problem.

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