

# Small polygons with large area

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## Abstract

A polygon is *small* if it has unit diameter. The maximal area of a small polygon with a fixed number of sides  $n$  is not known when  $n$  is even and  $n \geq 14$ . We determine an improved lower bound for the maximal area of a small  $n$ -gon for this case. The improvement affects the  $1/n^3$  term of an asymptotic expansion; prior advances affected less significant terms. This bound cannot be improved by more than  $O(1/n^3)$ . For  $n = 6, 8, 10$ , and  $12$ , the polygon we construct has maximal area.

**Keywords** Polygons, isodiametric problem, maximal area

## 1 Introduction

A polygon is said to be *small* if it has diameter 1. Reinhardt [1] first studied two extremal problems for small polygons a century ago: determining the maximal area for a small polygon with  $n$  sides, and determining the maximal perimeter for a small convex polygon with  $n$  sides. These are sometimes referred to as *isodiametric problems* for polygons in the literature. See [2, 3] for a survey of work on these problems and related ones. We focus on the area problem in this paper.

Reinhardt proved that the regular small polygon alone has maximal area when  $n$  is odd, and that this polygon is never optimal when  $n$  is even and  $n \geq 6$ . It is straightforward to show that there are infinitely many different small quadrilaterals with maximal area, including the square, and the optimal hexagon was first determined by Graham in 1975 [4]. The optimal octagon was established by Audet et al. in 2002 [5], and the cases  $n = 10$  and  $n = 12$  were resolved by Henrion and Messine in 2013 [6]. The problem remains open for larger  $n$ .

In 2006, Foster and Szabo [7] proved that the area  $A(P_n)$  of a small polygon  $P_n$  having an even number of sides  $n$  satisfies  $A(P_n) < \bar{A}_n$ , where

$$\bar{A}_n = \frac{n}{2} \sin \frac{\pi}{n} - \frac{n-1}{2} \tan \frac{\pi}{2n-2} = \frac{\pi}{4} - \frac{5\pi^3}{48n^2} - \frac{\pi^3}{24n^3} + O\left(\frac{1}{n^4}\right). \quad (1)$$

Since the area of the small regular polygon  $R_n$  with  $n$  even is given by  $\frac{n}{8} \sin \frac{2\pi}{n}$ , it follows easily that

$$\bar{A}_n - A(R_n) = \frac{\pi^3}{16n^2} + O\left(\frac{1}{n^3}\right).$$

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In 2005, the second author [8] described a construction for a polygon  $M_n$  with an even number of sides  $n$  for which

$$\bar{A}_n - A(M_n) = \frac{a_3\pi^3}{n^3} + O\left(\frac{1}{n^4}\right),$$

with

$$a_3 = \frac{5303 - 456\sqrt{114}}{5808} = 0.0747679609\dots$$

The first author [9] recently improved this, constructing a polygon  $B_n$  for each even  $n \geq 6$  satisfying

$$A(B_n) - A(M_n) = \frac{a_5\pi^3}{n^5} + O\left(\frac{1}{n^6}\right),$$

with  $a_5 = 0.25097\dots$  when  $n \equiv 2 \pmod{4}$  and  $a_5 = 0.35411\dots$  when  $n \equiv 0 \pmod{4}$ . In this paper, we generalize the latter construction to produce small polygons  $Q_n$  that exhibit an improvement in the  $1/n^3$  term for the area problem. We establish the following result.

**Theorem 1.** *Let  $n \geq 6$  be an even integer, let  $B_n$  denote the small  $n$ -gon from [9], and let  $\bar{A}_n$  denote the upper bound on the area of a small  $n$ -gon given by (1). There exists a small  $n$ -gon  $Q_n$  satisfying*

$$\bar{A}_n - A(Q_n) = \frac{\delta\pi^3}{n^3} + O\left(\frac{1}{n^4}\right) < \frac{8\pi^3}{109n^3} + O\left(\frac{1}{n^4}\right),$$

with  $\delta = 0.0733883168\dots$ , so

$$A(Q_n) - A(B_n) > \frac{\pi^3}{725n^3} + O\left(\frac{1}{n^4}\right).$$

Moreover,  $Q_n$  is the optimal small polygon for  $n \leq 12$ .

This article is organized in the following way. Section 2 establishes some notation and describes some prior constructions. Section 3 describes the new construction and proves Theorem 1. Section 4 reports on the computation of optimal small  $n$ -gons for a number of even  $n$  assuming of an axis of symmetry, and compares the results of our construction with these polygons.

## 2 Prior constructions

The *skeleton* of a small polygon  $P$  consists of the vertices of  $P$ , together with all of the line segments that connect two vertices of  $P$  at unit distance from one another. Let  $n \geq 6$  denote an even integer. From [4] it is known that the skeleton of an optimal  $n$ -gon  $P$  forms a connected graph, and a *linear thrackle*: each pair of line segments in the skeleton intersect one another, possibly at an endpoint. Foster and Szabo [7] proved that the skeleton of an optimal small polygon, considered as a graph, consists of an  $(n-1)$ -cycle, with a single additional pendant edge connected to the remaining vertex. It is conjectured that this additional pendant edge forms an axis of symmetry in the skeleton of the optimal polygon; this is in fact the case for  $n \leq 12$ . We thus consider polygons having a skeleton of the form shown in Figure 1: a star with  $n-1$  points on vertices  $v_0, \dots, v_{n-2}$ , an additional vertex  $v_{n-1}$  with distance 1 from  $v_0$ , and the line connecting  $v_0$  and  $v_{n-1}$  forming an axis of symmetry for the polygon.

We place  $v_0$  at the origin and  $v_1$  at the point  $(0, 1)$  in  $\mathbb{R}^2$ . Let  $\theta_0$  denote the angle  $\angle v_{n-1}v_0v_1$ , and for  $0 < k < n/2$  let  $\theta_k = \angle v_{k-1}v_kv_{k+1}$ . Due to the symmetry of the construction, and the fact that the star forms a closed path, we have

$$\sum_{j=0}^{\frac{n}{2}-1} \theta_j = \frac{\pi}{2}. \quad (2)$$

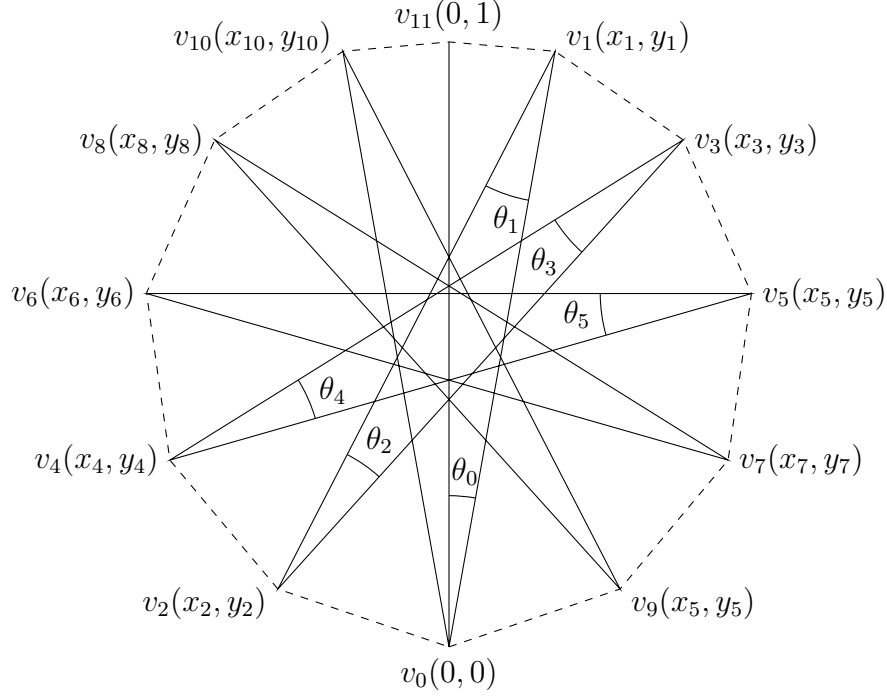


Figure 1: Polygon and skeleton for  $n = 12$

If we let  $(x_k, y_k)$  denote the coordinates of  $v_k$ , then

$$x_k = \sum_{j=0}^{k-1} (-1)^j \sin \left( \sum_{i=0}^j \theta_i \right), \quad y_k = \sum_{j=0}^{k-1} (-1)^j \cos \left( \sum_{i=0}^j \theta_i \right). \quad (3)$$

In addition, the vertices  $v_{n/2-1}$  and  $v_{n/2}$  are connected by a horizontal line in the skeleton, so

$$x_{n/2-1} = -x_{n/2} = \frac{(-1)^{\frac{n}{2}}}{2}. \quad (4)$$

We can compute the area  $A = A(\theta_0, \dots, \theta_{n/2-1})$  of such a polygon by determining the area of  $n/2 - 1$  triangles  $A_k$ :  $A_1$  is the area of the triangle  $\Delta v_0 v_{n-1} v_1$ , and  $A_k$  is the area of  $\Delta v_0 v_{k-1} v_{k+1}$  for  $2 \leq k < n/2$ . Then

$$A = 2 \sum_{k=1}^{\frac{n}{2}-1} A_k. \quad (5)$$

It follows that

$$\begin{aligned} 2A_1 &= x_0 = \sin \theta_0, \\ 2A_k &= x_{k+1}y_{k-1} - y_{k+1}x_{k-1} \\ &= \sin \theta_k + 2(-1)^k \left( x_k \sin \left( \frac{\theta_k}{2} + \sum_{j=0}^{k-1} \theta_j \right) + y_k \cos \left( \frac{\theta_k}{2} + \sum_{j=0}^{k-1} \theta_j \right) \right) \sin \frac{\theta_k}{2} \\ &= \sum_{i=0}^{k-2} (-1)^i \left( \sin \left( \sum_{j=0}^{i+1} \theta_{k-j} \right) - \sin \left( \sum_{j=1}^{i+1} \theta_{k-j} \right) \right) \end{aligned} \quad (6)$$

for  $2 \leq k < n/2$ . We thus obtain an expression for the area in terms of the  $n/2$  angles  $\theta_k$ .

In [8, 9], this formulation was simplified to employ just three variables,  $\alpha$ ,  $\beta$ , and  $\gamma$ , by taking

$$\begin{aligned}\theta_0 &= \alpha, \\ \theta_1 &= \beta + \gamma, \\ \theta_2 &= \beta - \gamma, \\ \theta_k &= \beta \quad \text{for } k \geq 3.\end{aligned}\tag{7}$$

This configuration was selected to mimic some of the pattern observed in [8] for the small  $n$ -gons with large area for  $n \leq 20$ , which were constructed using heuristic optimization methods over the parameters  $\theta_0, \dots, \theta_{n/2-1}$ . We extend these calculations in Section 4 for  $n \leq 120$  and note that this pattern continues: see Table 3. There, in each polygon constructed, the angles  $\theta_i$  show a pattern of damped oscillation, with the odd-indexed values for the constructed  $n$ -gon appearing to converge from above to a limiting value in  $(\frac{\pi}{n}, \frac{\pi}{n-1})$ , and the even-indexed angles (after  $\theta_0$ ) converging to the same value from below. Using  $\alpha$ ,  $\beta$ , and  $\gamma$  in this way then allowed approximating the largest variations one appears to expect in the sequence of angles  $\theta_i$ , while keeping the analysis tractable by using only three variables.

As in [9], we note certain constraints on  $\alpha$ ,  $\beta$ , and  $\gamma$  inherited by the geometry, namely

$$\alpha + \left(\frac{n}{2} - 1\right) \beta = \frac{\pi}{2},\tag{8}$$

$$\sin(\alpha + \beta + \gamma) = \sin \alpha + \frac{\sin(\alpha + 3\beta/2)}{2 \cos(\beta/2)}.\tag{9}$$

The former clearly follows from (2), and the latter is a consequence of this, combined with (3) and (4). After using these to eliminate  $\beta$  and  $\gamma$ , the expression for the area in [9] thus relied only on  $\alpha$ , and an asymptotic analysis was performed on this expression after setting  $\alpha = a\pi/n + b\pi/n^2 + c\pi/n^3$ . A similar analysis was performed in [8]. Both of those works found that

$$\bar{A}_n - A(P_n) = \frac{(5303 - 456\sqrt{114})\pi^3}{5808n^3} + \frac{(192107 - 17934\sqrt{114})\pi^3}{21296n^4} + O\left(\frac{1}{n^5}\right),\tag{10}$$

with  $P_n = M_n$  or  $B_n$  respectively, with the analysis in [9] producing an improvement in the  $1/n^5$  term.

### 3 Proof of Theorem 1

We obtain improved small polygons by generalizing the construction of Section 2, keeping some additional variables to allow for more variation in the sequence of angles  $\theta_i$ , beyond what is captured by (7). Let  $r$  denote a positive integer, and suppose  $n$  is even and  $n \geq 2r + 4$ . We describe a construction for a small  $n$ -gon which involves  $r + 2$  variables.

Assume first that  $r$  is even. Our variables are  $\alpha$ ,  $\beta$ ,  $\beta_1, \dots, \beta_{r/2}$ , and  $\gamma_1, \dots, \gamma_{r/2}$ , and we set

$$\begin{aligned}\theta_0 &= \alpha, \\ \theta_{2i-1} &= \beta_i + \gamma_i, \quad 1 \leq i \leq r/2, \\ \theta_{2i} &= \beta_i - \gamma_i, \quad 1 \leq i \leq r/2, \\ \theta_k &= \beta, \quad r < k < n/2.\end{aligned}$$

For convenience, let

$$\varphi_r = \alpha + 2 \sum_{i=1}^{r/2} \beta_i.$$

We derive an expression for the area of the small  $n$ -gon in terms of  $\alpha$ ,  $\beta$ , and the  $\beta_i$  and  $\gamma_i$ . For  $k > r$ , the coordinates  $(x_k, y_k)$  in (3) become

$$\begin{aligned} x_k &= x_r + \sum_{j=r}^{k-1} (-1)^j \sin(\varphi_r + (j-r)\beta) \\ &= x_r + \frac{\sin(\varphi_r - \frac{\beta}{2}) - (-1)^k \sin(\varphi_r + (2k-2r-1)\frac{\beta}{2})}{2 \cos(\beta/2)}, \\ y_k &= y_r + \sum_{j=r}^{k-1} (-1)^j \cos(\varphi_r + (j-r)\beta) \\ &= y_r + \frac{\cos(\varphi_r - \frac{\beta}{2}) - (-1)^k \cos(\varphi_r + (2k-2r-1)\frac{\beta}{2})}{2 \cos(\beta/2)} \end{aligned} \quad (11)$$

when  $r$  is even. The constraint (8) on the sum of the angles is now

$$\varphi_r + \left(\frac{n}{2} - r - 1\right) \beta = \frac{\pi}{2}, \quad (12)$$

and by combining this with (4) and (11), we deduce a generalization of (9):

$$x_r + \frac{\sin(\varphi_r - \frac{\beta}{2})}{2 \cos(\beta/2)} = 0. \quad (13)$$

Using (6) and (11), the area  $2A_k$  is then

$$\begin{aligned} 2A_k &= \sin \beta + 2(-1)^k \left( x_k \sin \left( \varphi_r + (2k-2r-1)\frac{\beta}{2} \right) \right. \\ &\quad \left. + y_k \cos \left( \varphi_r + (2k-2r-1)\frac{\beta}{2} \right) \right) \sin \frac{\beta}{2} \\ &= \sin \beta - \tan(\beta/2) + 2(-1)^k \left( x_r \sin \left( \varphi_r + (2k-2r-1)\frac{\beta}{2} \right) \right. \\ &\quad \left. + y_r \cos \left( \varphi_r + (2k-2r-1)\frac{\beta}{2} \right) + \frac{\cos((k-r)\beta)}{2 \cos(\beta/2)} \right) \sin \frac{\beta}{2} \end{aligned}$$

for  $r < k < n/2$ . Combining this with (12) and (13), it follows that

$$\sum_{k=r+1}^{\frac{n}{2}-1} 2A_k = \left(\frac{n}{2} - r - 1\right) \left( \sin \beta - \tan \frac{\beta}{2} \right) - \left( x_r \sin \varphi_r + y_r \cos \varphi_r + \frac{1}{2} \right) \tan \frac{\beta}{2}.$$

We thus obtain an expression for the area of our small  $n$ -gon having  $r+2$  variables, whose number of terms depends only on  $r$ :

$$A = \sum_{k=1}^r 2A_k + \left(\frac{n}{2} - r - 1\right) \left( \sin \beta - \tan \frac{\beta}{2} \right) - \left( x_r \sin \varphi_r + y_r \cos \varphi_r + \frac{1}{2} \right) \tan \frac{\beta}{2}. \quad (14)$$

We then eliminate  $\beta$  using (12) and  $\gamma_{r/2}$  with (13).

If  $r$  is odd, we employ the same strategy as with  $r+1$ , except we set  $\beta_{\frac{r+1}{2}} = \beta$ . The scheme (7) therefore corresponds to the case  $r=1$ .

For each even  $n \geq 6$  and  $r \geq 1$ , let  $Q_{n,r}$  denote the small polygon obtained by maximizing our area function (14) over the parameters  $\alpha, \beta_1, \dots, \beta_{\lfloor r/2 \rfloor}, \gamma_1, \dots, \gamma_{\lfloor r/2 \rfloor - 1}$ , for  $\alpha \in [\frac{\pi}{2n-2}, \frac{\pi}{n}]$ ,  $\beta_i \in [\frac{\pi}{n}, \frac{2\pi}{n}]$

for each  $i$ , and  $\gamma_i \in [0, \frac{\pi}{n}]$  for each  $i$ . An asymptotic analysis reveals that the area is maximized for large  $n$  when

$$\begin{aligned}\alpha(n) &= \frac{a\pi}{n} + O\left(\frac{1}{n^2}\right), \\ \beta_i(n) &= \frac{b_i\pi}{n} + O\left(\frac{1}{n^2}\right), \quad 1 \leq i \leq \lfloor r/2 \rfloor, \\ \gamma_i(n) &= \frac{c_i\pi}{n} + O\left(\frac{1}{n^2}\right), \quad 1 \leq i \leq \lceil r/2 \rceil - 1,\end{aligned}$$

where  $a, b_1, \dots, b_{\lfloor r/2 \rfloor}, c_1, \dots, c_{\lceil r/2 \rceil - 1}$  maximize a particular cubic polynomial in  $r$  variables. When  $r = 1$  this polynomial is

$$-\frac{\pi^3}{192}(88a^3 + 84a^2 - 222a + 107),$$

while at  $r = 2$  it is

$$-\frac{\pi^3}{192}(88a^3 + 12a^2(8b_1 - 1) - 6a(16b_1^2 + 21) + 128b_1^3 - 48b_1^2 - 216b_1 + 243),$$

and  $r = 3$  yields

$$\begin{aligned}-\frac{\pi^3}{192}(88a^3 + 12a^2(16b_1 - 12c_1 + 7) - 6a(32b_1^2 + 64b_1c_1 - 80c_1^2 + 56c_1 + 37) \\ + 128b_1^3 + 192b_1^2c_1 + 384b_1c_1^2 - 384c_1^3 + 336c_1^2 - 240b_1 + 204c_1 + 267).\end{aligned}$$

After removing the factor  $-\pi^3$ , we use the `NMinimize` function in Mathematica to determine the optimal value for these polynomials by numerical methods, requiring  $0 \leq a \leq 1$ ,  $0 \leq b_i \leq 2$  for each  $i$ , and  $0 \leq c_i \leq 1/3$  for each  $i$ . This produces an asymptotic estimate for  $A(Q_{n,r})$  having the form

$$A(Q_{n,r}) = \frac{\pi}{4} - \frac{5\pi^3}{48n^2} - \frac{q_r\pi^3}{n^3} + O\left(\frac{1}{n^4}\right). \quad (15)$$

For completeness we let  $Q_{n,0}$  denote the polygon created by selecting  $\alpha$  optimally in the  $n$ -gon with  $\theta_0 = \alpha$  and  $\theta_i = \beta$  for  $1 \leq i < n/2$ , subject to (8), so when  $\alpha = \pi/(2n-2)$  and  $\beta = \pi/(n-1)$ . This is the polygon created by simply adding a vertex at unit distance antipodal to one vertex of the regular small  $(n-1)$ -gon, as in Figure 1 for  $n = 12$ . The polygon  $Q_{n,0}$  then has

$$q_0 = \frac{7}{48} = 0.1458333333 \dots$$

For  $r = 1$ , as reported in [8,9], the optimal choice for  $a$  is

$$a = \frac{2\sqrt{114} - 7}{22} = 0.6524616592 \dots,$$

which produces

$$q_1 = \frac{5545 - 456\sqrt{114}}{5808} = 0.1164346275 \dots$$

At  $r = 2$ , we obtain

$$q_2 = 0.1156971503 \dots$$

In fact,  $q_2$  is a root of the polynomial

$$x^4 - \frac{70705}{15876}x^3 + \frac{269167127}{41150592}x^2 - \frac{3381027871}{987614208}x + \frac{737985313}{2341011456}.$$

The case  $r = 3$  yields a further improvement,

$$q_3 = 0.1150899130 \dots,$$

which is a root of a polynomial with rational coefficients and degree 8:

$$\begin{aligned} x^8 &- \frac{338067189760423194}{232662255261540774}x^7 + \frac{1980606171874180754147}{22335576505107914304}x^6 \\ &- \frac{158140620301705167575191}{536053836122589943296}x^5 + \frac{59647522303796634759434731}{102922336535537269112832}x^4 \\ &- \frac{836103610314364495378933003}{1235068038426447229353984}x^3 + \frac{52675103710698128327456883067}{118566531688938934017982464}x^2 \\ &- \frac{14538141342029184829034957803}{105392472612390163571539968}x + \frac{442235633612728385344035304147}{40470709483157822811471347712}. \end{aligned}$$

We improve this further with subsequent values of  $r$ : our results through  $r = 16$  are summarized in Table 1.

The polygons  $Q_{n,r}$  are small by construction, and for even  $n \geq 6$  we set

$$Q_n = \begin{cases} Q_{n,n/2-2} & n \leq 34, \\ Q_{n,16} & n \geq 36. \end{cases}$$

The first statement of Theorem 1 then follows by combining (15) at  $r = 16$  with (1) and (10). For the final statement, we calculate the areas of  $Q_{6,1}$ ,  $Q_{8,2}$ ,  $Q_{10,3}$ , and  $Q_{12,4}$  by optimizing over  $\alpha$  and the relevant  $\beta_i$  and  $\gamma_i$ . The values we obtain are consistent with the optimal areas for  $n = 6$  from [4],  $n = 8$  from [5], and  $n = 10$  and  $12$  from [6], and the polygons  $Q_n$  are optimal in these cases. Values for the area and angles of these polygons are recorded in Table 2.  $\square$

Additional small improvements can certainly be obtained using larger values for  $r$ . Of course, the procedure becomes more computationally onerous as  $r$  increases, due to the increasing complexity of the optimization procedure as the number of variables grows. Such improvements are likely to be minuscule, however, given the rapid convergence exhibited in the  $q_r$  values we compute.

## 4 Constructions for small $n$

We construct some small polygons with large area for particular values of  $n$  and display their values in two tables. First, for even integers  $n$  with  $6 \leq n \leq 120$ , we calculate the small  $n$ -gon with maximal area  $P_n^*$ , assuming the presence of an axis of symmetry. We employ the skeleton of Foster and Szabo as in Figure 1, and assume that the polygon is symmetric about the line connecting  $v_0$  and  $v_{n-1}$ . For each such  $n$ , using (5) and (6) we construct  $P_n^*$  by maximizing the area  $A$  over  $n/2$  variables  $\theta_0, \theta_1, \dots, \theta_{\frac{n}{2}-1}$ , subject to (2) and (4). More precisely,

$$\begin{aligned} A(P_n^*) &= \max_{\theta_0, \theta_1, \dots, \theta_{\frac{n}{2}-1}} \sin \theta_0 + \sum_{k=2}^{\frac{n}{2}-1} 2A_k(\theta_1, \theta_2, \dots, \theta_k) \\ \text{s. t. } &\sum_{k=0}^{\frac{n}{2}-1} \theta_k = \frac{\pi}{2}, \\ &\sum_{i=0}^{\frac{n}{2}-2} (-1)^i \sin \left( \sum_{j=0}^i \theta_j \right) = \frac{(-1)^{\frac{n}{2}}}{2}, \\ &0 \leq \theta_0 \leq \pi/6, \\ &0 \leq \theta_k \leq \pi/3, \quad 1 \leq k \leq n/2 - 1. \end{aligned} \tag{16}$$

Table 1: Optimal values of  $q_r$  in (15) for  $Q_{n,r}$ , together with values for the free parameters that produce this coefficient

$r$	$q_r$	$a$	$b_1, \dots, b_{\lfloor r/2 \rfloor}$	$c_1, \dots, c_{\lfloor r/2 \rfloor - 1}$
0	0.1458333333333333			
1	0.1164346275953378	0.6524616592755737		
2	0.1156971503834968	0.6554858160170336	1.022718374818576	
3	0.1150899130453658	0.6585214692355722	1.027063969740726	0.06794672543480737
4	0.1150687309140004	0.6586249743177490	1.027209190884176 1.003828109549754	0.06761473542307868
5	0.1150557470337394	0.6586900229138448	1.027300358228207 1.004461819824590	0.06740615925761905 0.01028318684355288
6	0.1150552764425733	0.6586923731079715	1.027303650624826 1.004484647927062 1.000570284354183	0.06739862447575472 0.01023212957791077
7	0.1150549998390641	0.6586937593365635	1.027305592742427 1.004498111789651 1.000662623065463	0.06739417976638237 0.01020201310839604 0.001509019841357918
8	0.1150549897593822	0.6586938098307351	1.027305663478150 1.004498602162219 1.000665984979524 1.000083456862159	0.06739401787457336 0.01020091616496605 0.001501502526879076
9	0.1150549838708233	0.6586938392297916	1.027305704817829 1.004498888799399 1.000667949966222 1.000096926042045	0.06739392319318871 0.01020027499791677 0.001497108650978320 0.0002203516777390261
10	0.1150549836560699	0.6586938403067916	1.027305706321311 1.004498899243624 1.000668021641018 1.000097417195005 1.000012181578676	0.06739391975105442 0.01020025160734295 0.001496948418422726 0.0002192534425102402
11	0.1150549835307224	0.6586938408223755	1.027305707189345 1.004498905348772 1.000668063456477 1.000097703894909 1.000014146646888	0.06739391767843365 0.01020023796370542 0.001496854886068534 0.0002186123676843603 0.00003215289654154845
12	0.1150549835261505	0.6586938408430183	1.027305707214692 1.004498905576070 1.000668064981190 1.000097714354882 1.000014218316758 1.000001777358190	0.06739391761102890 0.01020023745865263 0.001496851472990505 0.0002185889876955666 0.00003199263572187769
13	0.1150549835234823	0.6586938407444091	1.027305707207800 1.004498905698189 1.000668065875004 1.000097720453880 1.000014260156537 1.000002064060629	0.06739391752000037 0.01020023715826531 0.001496849480792517 0.0002185753423198028 0.00003189910564581757 0.000004691124054714891
14	0.1150549835233850	0.6586938407702582	1.027305707207799 1.004498905739078 1.000668065815119 1.000097720770217 1.000014261621179 1.000002074524450 1.000000259301370	0.06739391752829402 0.01020023716420131 0.001496849406827393 0.0002185748364450190 0.00003189570671658796 0.000004667741696766253
15	0.1150549835233282	0.6586938406842894	1.027305707182444 1.004498905675305 1.000668065987254 1.000097720721770 1.000014262600232 1.000002080718492 1.000000301103361	0.06739391752641309 0.01020023709721798 0.001496849396193236 0.0002185745506217027 0.00003189368791732473 0.000004654108249822218 0.0000006844384307984062
16	0.1150549835233261	0.6586938406334803	1.027305707190814 1.004498905736593 1.000668065901726 1.000097720834017 1.000014262573901 1.000002080858254 1.000000302668948 1.000000037787016	0.06739391746458313 0.01020023714027207 0.001496849368106921 0.0002185745455879002 0.00003189363350580009 0.000004653599949365328 0.0000006810134779486807



Table 2: Constructing optimal polygons for small  $n$ 

$n$	$A(Q_{n,n/2-2})$	$\alpha$	$\beta_1, \beta_2$	$\gamma_1$
6	0.6749814429301047	0.3509301888703616		
8	0.7268684827516268	0.2652408674910718	0.4379295350493946	
10	0.7491373458778303	0.2126101953284637	0.3433714044229845	0.02476000789351616
12	0.7607298734487962	0.1770854623284314	0.2827755557037131	0.01982894085863103
			0.2763754214389234	

Problem (16) was solved on the NEOS Server 6.0 using AMPL with the nonlinear programming solver Ipopt 3.13.4 [10]. The AMPL code is available in OPTIGON [11], a free package for extremal convex small polygons available on GitHub. For selected even  $n \leq 120$ , Table 3 shows the values  $\theta_i^*$  that we calculated for constructing  $P_n^*$ , and each value  $A(P_n^*)$  is displayed in Table 4. The areas shown here for  $n \leq 20$  agree with those from [8], and the values of  $A(P_n^*)$  for larger  $n$  in Table 4 match or slightly exceed the best value found in the literature [12–14]. Ipopt required less than 1 second to compute each value in Table 4.

Second, for selected even  $n \leq 120$  we determine the area of  $Q_{n,r}$  for  $0 \leq r \leq 4$  by optimizing (14) over the  $r$  parameters  $\alpha, \beta_1, \dots, \beta_{\lfloor r/2 \rfloor}, \gamma_1, \dots, \gamma_{\lfloor r/2 \rfloor - 1}$ . These areas are also displayed in Table 4, along with that of the regular  $n$ -gon  $R_n$  and the upper bound  $\bar{A}_n$ . Julia and MATLAB functions that give the coordinates of the vertices of all polygons presented in this work are provided in OPTIGON.

In all tables in this paper, each numerical value is rounded at the last displayed digit.

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Table 3: Angles  $\theta_0^*, \theta_1^*, \dots, \theta_{\frac{n}{2}-1}^*$  of  $P_n^*$ .

$n$	$i$	$\theta_{8i}^*$	$\theta_{8i+1}^*$	$\theta_{8i+2}^*$	$\theta_{8i+3}^*$	$\theta_{8i+4}^*$	$\theta_{8i+5}^*$	$\theta_{8i+6}^*$	$\theta_{8i+7}^*$
6	0	0.350930	0.653342	0.566524					
8	0	0.265241	0.470631	0.405228	0.429696				
10	0	0.212610	0.368131	0.318611	0.339137	0.332306			
12	0	0.177085	0.302604	0.262947	0.279461	0.273290	0.275409		
14	0	0.151583	0.257026	0.233904	0.237628	0.232444	0.234442	0.233769	
16	0	0.132428	0.223448	0.194967	0.206716	0.202285	0.204013	0.203359	0.203580
18	0	0.117533	0.197661	0.172654	0.182938	0.179070	0.180577	0.179999	0.180218
	1	0.180145							
20	0	0.105629	0.177228	0.154925	0.164076	0.160640	0.161977	0.161464	0.161661
	1	0.161586	0.161611						
22	0	0.0959016	0.160633	0.140496	0.148745	0.145652	0.146854	0.146393	0.146571
	1	0.146503	0.146528	0.146520					
24	0	0.0878067	0.146886	0.128526	0.136037	0.133224	0.134316	0.133898	0.134059
	1	0.133997	0.134021	0.134012	0.134015				
30	0	0.0700443	0.116891	0.102361	0.108291	0.106075	0.106933	0.106605	0.106731
	1	0.106683	0.106701	0.106694	0.106697	0.106696	0.106696	0.106696	
40	0	0.0523626	0.0872236	0.0764267	0.0808253	0.0791841	0.0798182	0.0795763	0.0796689
	1	0.0796334	0.0796470	0.0796418	0.0796437	0.0796429	0.0796432	0.0796431	0.0796431
	2	0.0796431	0.0796431	0.0796431	0.0796431				
50	0	0.0418008	0.0695718	0.0609765	0.0644752	0.0631706	0.0636742	0.0634822	0.0635556
	1	0.0635275	0.0635382	0.0635340	0.0635355	0.0635349	0.0635351	0.0635350	0.0635350
	2	0.0635350	0.0635350	0.0635349	0.0635349	0.0635349	0.0635349	0.0635349	0.0635349
	3	0.0635349							
60	0	0.0347820	0.0578637	0.0507223	0.0536280	0.0525448	0.0529626	0.0528033	0.0528642
	1	0.0528408	0.0528496	0.0528462	0.0528474	0.0528469	0.0528470	0.0528469	0.0528469
	2	0.0528468	0.0528468	0.0528468	0.0528467	0.0528467	0.0528467	0.0528467	0.0528467
	3	0.0528467	0.0528466	0.0528466	0.0528466	0.0528466	0.0528466		
70	0	0.0297804	0.0495296	0.0434206	0.0459055	0.0449793	0.0453364	0.0452002	0.0452521
	1	0.0452321	0.0452396	0.0452366	0.0452376	0.0452371	0.0452372	0.0452371	0.0452370
	2	0.0452370	0.0452369	0.0452369	0.0452369	0.0452368	0.0452368	0.0452368	0.0452367
	3	0.0452367	0.0452367	0.0452367	0.0452367	0.0452366	0.0452366	0.0452366	0.0452366
	4	0.0452366	0.0452366	0.0452366					
80	0	0.0260361	0.0432947	0.0379568	0.0401276	0.0393185	0.0396302	0.0395112	0.0395565
	1	0.0395390	0.0395454	0.0395428	0.0395437	0.0395432	0.0395432	0.0395431	0.0395430
	2	0.0395430	0.0395429	0.0395429	0.0395428	0.0395428	0.0395427	0.0395427	0.0395426
	3	0.0395426	0.0395426	0.0395425	0.0395425	0.0395425	0.0395424	0.0395424	0.0395424
	4	0.0395424	0.0395424	0.0395424	0.0395424	0.0395424	0.0395423	0.0395423	0.0395423
90	0	0.0231282	0.0384545	0.0337147	0.0356421	0.0349236	0.0352001	0.0350945	0.0351345
	1	0.0351190	0.0351247	0.0351223	0.0351230	0.0351226	0.0351225	0.0351224	0.0351223
	2	0.0351222	0.0351221	0.0351221	0.0351220	0.0351219	0.0351219	0.0351218	0.0351218
	3	0.0351217	0.0351217	0.0351216	0.0351216	0.0351215	0.0351215	0.0351215	0.0351214
	4	0.0351214	0.0351214	0.0351214	0.0351213	0.0351213	0.0351213	0.0351213	0.0351213
	5	0.0351213	0.0351213	0.0351213	0.0351213	0.0351213			
100	0	0.0208046	0.0345883	0.0303258	0.0320589	0.0314127	0.0316612	0.0315662	0.0316021
	1	0.0315881	0.0315931	0.0315909	0.0315915	0.0315911	0.0315910	0.0315909	0.0315907
	2	0.0315906	0.0315905	0.0315904	0.0315903	0.0315903	0.0315902	0.0315901	0.0315900
	3	0.0315900	0.0315899	0.0315898	0.0315898	0.0315897	0.0315897	0.0315897	0.0315896
	4	0.0315896	0.0315895	0.0315895	0.0315895	0.0315894	0.0315894	0.0315894	0.0315894
	5	0.0315893	0.0315893	0.0315893	0.0315893	0.0315893	0.0315893	0.0315893	0.0315893
	6	0.0315893	0.0315893						
110	0	0.0189055	0.0314290	0.0275563	0.0291308	0.0285436	0.0287692	0.0286828	0.0287153
	1	0.0287025	0.0287070	0.0287050	0.0287054	0.0287050	0.0287049	0.0287048	0.0287046
	2	0.0287045	0.0287043	0.0287042	0.0287041	0.0287040	0.0287039	0.0287038	0.0287037
	3	0.0287037	0.0287036	0.0287035	0.0287034	0.0287034	0.0287033	0.0287033	0.0287032
	4	0.0287031	0.0287031	0.0287031	0.0287030	0.0287030	0.0287029	0.0287029	0.0287029
	5	0.0287028	0.0287028	0.0287028	0.0287028	0.0287027	0.0287027	0.0287027	0.0287027
	6	0.0287027	0.0287027	0.0287026	0.0287026	0.0287026	0.0287026	0.0287026	
120	0	0.0173243	0.0287991	0.0252508	0.0266933	0.0261551	0.0263616	0.0262824	0.0263121
	1	0.0263003	0.0263043	0.0263024	0.0263028	0.0263023	0.0263022	0.0263020	0.0263018
	2	0.0263017	0.0263015	0.0263014	0.0263012	0.0263011	0.0263010	0.0263009	0.0263008
	3	0.0263007	0.0263006	0.0263005	0.0263004	0.0263003	0.0263003	0.0263002	0.0263001
	4	0.0263001	0.0263000	0.0262999	0.0262999	0.0262998	0.0262998	0.0262997	0.0262997
	5	0.0262996	0.0262996	0.0262996	0.0262995	0.0262995	0.0262995	0.0262994	0.0262994
	6	0.0262994	0.0262994	0.0262994	0.0262993	0.0262993	0.0262993	0.0262993	0.0262993
	7	0.0262993	0.0262993	0.0262993	0.0262993				

Table 4: Comparing areas of small polygons

$n$	$A(R_n)$	$A(Q_{n,0})$	$A(Q_{n,1})$	$A(Q_{n,2})$	$A(Q_{n,3})$	$A(Q_{n,4})$	$A(P_n^*)$	$\bar{A}_n$
6	0.6495190528	0.6722882584	0.6749814429	–	–	–	0.6749814429	0.6877007594
8	0.7071067812	0.7253199909	0.7268542719	0.7268684828	–	–	0.7268684828	0.7318815691
10	0.7347315654	0.7482573378	0.7491189262	0.7491297887	0.7491373459	–	0.7491373459	0.7516135587
12	0.7500000000	0.7601970055	0.7607153082	0.7607228359	0.7607297471	0.7607298734	0.7607298734	0.7621336536
14	0.7592965435	0.7671877750	0.7675203660	0.7675256353	0.7675308404	0.7675309615	0.7675310111	0.7684036467
16	0.7653668647	0.7716285345	0.7718535572	0.7718573456	0.7718611688	0.7718612660	0.7718613220	0.7724408116
18	0.7695453225	0.7746235089	0.7747824059	0.7747852057	0.7747880405	0.7747881160	0.7747881651	0.7751926059
20	0.7725424859	0.7767382147	0.7768543958	0.7768565173	0.7768586570	0.7768587158	0.7768587560	0.7771522071
22	0.7747645313	0.7782865351	0.7783739622	0.7783756055	0.7783772514	0.7783772976	0.7783773302	0.7785970008
24	0.7764571353	0.7794540033	0.7795213955	0.7795226929	0.7795239821	0.7795240189	0.7795240452	0.7796927566
30	0.7796688406	0.7816380102	0.7816725130	0.7816732130	0.7816738921	0.7816739122	0.7816739269	0.7817597927
40	0.7821723252	0.7833076096	0.7833221318	0.7833224422	0.7833227341	0.7833227431	0.7833227495	0.7833587784
50	0.7833327098	0.7840695435	0.7840769608	0.7840771244	0.7840772750	0.7840772797	0.7840772830	0.7840956746
60	0.7839634745	0.7844798073	0.7844840910	0.7844841875	0.7844842749	0.7844842777	0.7844842796	0.7844949027
70	0.7843439529	0.7847256986	0.7847283918	0.7847284534	0.7847285085	0.7847285103	0.7847285115	0.7847351925
80	0.7845909573	0.7848845934	0.7848863952	0.7848864368	0.7848864738	0.7848864750	0.7848864758	0.7848909473
90	0.7847603296	0.7849931681	0.7849944322	0.7849944617	0.7849944876	0.7849944885	0.7849944890	0.7849976272
100	0.7848814941	0.7850706272	0.7850715479	0.7850715695	0.7850715884	0.7850715890	0.7850715895	0.7850738759
110	0.7849711494	0.7851278167	0.7851285079	0.7851285242	0.7851285384	0.7851285389	0.7851285392	0.7851302562
120	0.7850393436	0.7851712379	0.7851717699	0.7851717826	0.7851717935	0.7851717939	0.7851717941	0.7851731162

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