Integral Global Optimality Conditions and an Algorithm for Multiobjective Problems

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Abstract

In this work, we propose integral global optimality conditions for multiobjective problems not necessarily differentiable. The integral characterization, already known for single objective problems, are extended to multiobjective problems by weighted sum and Chebyshev weighted scalarizations. Using this last scalarization, we propose an algorithm for obtaining an approximation of the weak Pareto front whose effectiveness is illustrated by solving a collection of multiobjective test problems.

Keywords: Multiobjective optimization; Pareto front; Weighted sum scalarization; Chebyshev weighted scalarization; Integral global optimality conditions.

AMS Classification: 90C29, 65K05, 49M37.

1 Introduction

The multiobjective optimization addresses problems of Decision Making which are characterized by multiple and possibly conflicting objective functions to be optimized simultaneously on a set of feasible decisions. Examples of these problems appear in several applications, for instance, Finance [6], Biology [37], Management Science [24], Game Theory [33], Engineering [18], among other fields.

The first results in multiobjective optimization are due to V. Pareto, who, in his famous work *"Cours d'Economie Politique"* [34] introduced the concept of an efficient solution. This notion of optimality has been widely used in Economics because it is closely related to the Theory of Social Welfare. After the Second World War (a time that coincides with the apogee of Operational Research), numerous studies appeared in this field. Necessary and sufficient conditions for the determination of efficient points were studied. Since then, these problems have been extensively studied in the literature, being treated both from theoretical and applied point of view. For more historical information about this theme, see [38].

Formally, a multiobjective problem admits the following formulation:

$$\begin{array}{l} \text{minimize } F(x) = (f_1(x), \cdots, f_r(x)) \\ \text{subject to } x \in X, \end{array} \tag{MOP}$$

where $f_\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell = 1, \cdots, r$, are given functions and X is a nonempty subset of $\mathbb{R}^n$.

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Due to the conflicting nature of the objectives, an optimal solution that simultaneously minimizes all the objectives is usually not available. For vectorial functions, the minimum can be defined in terms of efficient solutions. In this paper we use the following notions of optimality:

- weak Pareto optimality: a point $\overline{x} \in X$ is a weak Pareto optimal (or weakly efficient) solution of the problem (MOP) if there is no other feasible point $x \in X$ such that $f_\ell(x) < f_\ell(\overline{x})$ for all $\ell = 1, \ldots, r$.
- Pareto optimality: a point $\overline{x} \in X$ is a Pareto optimal (or efficient) solution of (MOP) if there is no other feasible point $x \in X$ such that $f_\ell(x) \leq f_\ell(\overline{x})$ for all $\ell = 1, \ldots, r$, with strict inequality valid for some ℓ_0.

The set of the values of all Pareto optimal solutions to (MOP) forms the so-called Pareto front. In this work we present integral global optimality conditions to the problem (MOP) and based on these we propose an algorithm to compute an approximation of the weak Pareto front. Other optimality characterization for multiobjective problems are discussed in several works. For the differentiable case, necessary first order conditions can be found in [5, 7, 29]; second order conditions are discussed in [3, 16, 19, 22, 23, 25, 35, 36, 41]; sufficient conditions under generalized convexity assumptions are proposed in [20, 28, 31, 32]. Optimality conditions for non-smooth problems can be found in [2, 8, 30, 39], for instance. Following a different approach, we present a characterization of optimality via integration, inspired by Falk [15] who proposed it in 1973 for single objective problems. As this approach only requires the continuity of the objective function and the compactness of the feasible set, it can be applied to a larger variety of problems. In the context of single objective problems, the works [10, 21, 27, 42–48] also use integration techniques, with weakness hypotheses of continuity and compactness. The characterization of optimality occurs through the concepts of mean value and variance on the level sets of the objective function. From this characterization, the authors of [21] proposed an algorithm to obtain global minimizers of single objective problems and some numerical tests were carried out to illustrate the performance of the method.

We apply these ideas to the problem (MOP) by applying scalarization techniques to transform the multiobjective problem into a single objective (scalar) problem, in a such way that the solutions of the multiobjective problem can be obtained by solving a classical nonlinear programming problem. There are several techniques for scalarization of multiobjective problems. Among these methods, perhaps the best known is the weighted sum scalarization. This technique was introduced by Gass and Saaty [17] in 1955 and it is probably the most used due to its simplicity. The weighted sum technique is a simple way to generate different Pareto optimal solutions. The failure of this method is that not all Pareto optimal points can be found if the problem is nonconvex. Another scalarization method is the weighted Chebyshev technique, introduced by Bowman [4] in 1976, which allows us to ensure that any weak Pareto optimal solution of the multiobjective problem (MOP) is solution of the weighted Chebyshev problem for some choice of weights. This fact is central for our results related to global optimality conditions for multiobjective problems.

Such results are obtained by applying these weighted scalarization techniques to the problem (MOP) and using integral global optimality conditions obtained by Cui, Wang and Zheng [10], Hong and Zheng [21], Wu, Cui and Zheng [42], Zheng [43–47] and Zheng and Zhuang [48] to the scalarized problem. In addition, based on the integral characterization of optimality, we extend to multiobjective problems, the algorithm proposed by Hong and Zheng in [21] for single objective problems. We perform numerical experiments to illustrate the effectiveness of the proposed algorithm for solving multiobjective problems.

The paper is organized as follows. Sec. 2 recalls integral optimality conditions for scalar problems and extends them to multiobjective problems. Based on these conditions, we propose in Sec. 3 an algorithm to solve multiobjective problems and prove its global convergence. Sec. 4 is dedicated to numerical experiments to illustrate the performance of the algorithm. Some conclusions are presented in Sec. 5.

2 Integral Global Optimality Conditions

In this section we present integral global optimality conditions for multiobjective problems. First, we recall integral optimality conditions for single objective problems and then we extend it to multiobjective problems, one of the main contributions of this paper.

2.1 Single objective problem

Consider the following single objective (scalar) optimization problem:

$$\begin{array}{l}\text{minimize } f(x)\\ \text{subject to } x \in X\end{array} \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given function and X is a nonempty subset of $\mathbb{R}^n$. In the sequence, we evoke some results on the integral characterization of global optimality for the scalar problem (P). First, we present a result proposed originally by Falk [15] for maximization problems and rewritten now for our context.

Theorem 2.1. *Consider $X \subset \mathbb{R}^n$ a compact set with nonempty interior, $f : X \to (-\infty, 0)$ a continuous function and $\overline{x} \in X$ such that $f(\overline{x}) = -1$. The integral $\Upsilon(t) = \int_X [-f(x)]^t \, dx$ converges, when $t \to \infty$ if, and only if, $\overline{x}$ is a global solution of the problem* (P)*.*

We are interested on the global minimization of functions, not necessarily continuous. In this context, the concepts of level sets and robustness are essential. The level set of the function f is defined, for each real number c, by

$$H_c = \{x \in \mathbb{R}^n \mid f(x) \leq c\}.$$

The concept of robustness is a generalization of that of openness. A set $D \subset \mathbb{R}^n$ is robust if its closure coincides with the closure of its interior, $\mathrm{cl}(D) = \mathrm{cl}(\mathrm{int}\, D)$. Clearly, any open set G is robust since $G = \mathrm{int}\, G$. On the other hand, a closed set may be nonrobust. In fact, the set with a single point is closed in $\mathbb{R}^n$, but it is nonrobust. Furthermore, the concept of the robustness of a set is closely related to its topological structure. For instance, the set $D = \{1, 2\}$ is nonrobust on $\mathbb{R}^1$, but it is robust in $\mathbb{Z}$ with the discrete topology, [21, 47]. Next, we have some useful properties of robust sets.

Remark 2.2 (Q. Zheng [47])**.** *The following properties hold for robust sets:*

1. *The union of robust sets is robust,*
2. *The intersection of a robust set and an open set is robust,*
3. *If D is robust then, its closure $cl(D)$ is also robust.*

From these concepts, we say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is upper robust over X if, and only if, the set $\{x \in X \mid f(x) < c\}$ is robust, for each real number c. For more details on robustness, see Q. Zheng [45]- [47]. From now on, we assume the following assumptions on the problem (P):

A1. X is robust,

A2. The function $f : X \to \mathbb{R}$ is lower semicontinuous and upper robust,

A3. There exists $c \in \mathbb{R}$ such that $H_c \cap X$ is a compact set.

Under these assumptions, we present some definitions that are fundamental for the sequence of the work.

Definition 2.3. *[47, Def. 5.1] Suppose that Assumptions* **A1**, **A2** *and* **A3** *hold. Consider* $\bar{c} = \min_{x\in X} f(x)$ *and let* $c > \bar{c}$. *We define the mean value, variance and modified variance of the function* f *over* $H_c \cap X$, *respectively, as follows:*

$$
\begin{aligned}
M(f,c,X) &= \frac{1}{\mu(H_c \cap X)} \int_{H_c\cap X} f(x)\, d\mu, && (1)\\
V(f,c,X) &= \frac{1}{\mu(H_c \cap X)} \int_{H_c\cap X} (f(x) - M(f,c,X))^2\, d\mu, && (2)\\
V_1(f,c,X) &= \frac{1}{\mu(H_c \cap X)} \int_{H_c\cap X} (f(x) - c)^2\, d\mu, && (3)
\end{aligned}
$$

where μ *denotes the Lebesgue measure in* $\mathbb{R}^n$.

Under Assumptions **A1**, **A2** and **A3**, it can be proved that $\mu(H_c \cap X) > 0$, for $c > \bar{c}$ and the function f is measurable on $H_c \cap X$ (see [47, Lemma 5.1]). Therefore, in this case, the mean value, variance and modified variance are well defined. Furthermore, for $c = \bar{c}$, these definitions can be extended by a limit process as follows.

Definition 2.4. *[47, Def. 5.2] Under the assumptions of Definition 2.3, we can extend it to* $c \geq \bar{c}$ *by:*

$$
\begin{aligned}
M(f,c,X) &= \lim_{c_k \downarrow c} \frac{1}{\mu(H_{c_k} \cap X)} \int_{H_{c_k}\cap X} f(x)\, d\mu,\\
V(f,c,X) &= \lim_{c_k \downarrow c} \frac{1}{\mu(H_{c_k} \cap X)} \int_{H_{c_k}\cap X} (f(x) - M(f,c,X))^2\, d\mu,\\
V_1(f,c,X) &= \lim_{c_k \downarrow c} \frac{1}{\mu(H_{c_k} \cap X)} \int_{H_{c_k}\cap X} (f(x) - c)^2\, d\mu.
\end{aligned}
$$

According to [21], under the assumptions, these limits exist and are independent of choices of the decreasing sequence $\{c_k\}$. With these concepts we can characterize the integral global optimality for the problem (P) as follows.

Theorem 2.5. *[47, Thm. 5.1] Suppose that Assumptions* **A1**, **A2** *and* **A3** *hold. The following statements are equivalent:*

(i) $\bar{x} \in X$ *is a global minimizer of* (P) *and* $\bar{c} = f(\bar{x})$ *is the global minimum value of* f *over* X,

(ii) $M(f, \overline{c}, X) = \overline{c}$,

(iii) $V(f, \overline{c}, X) = 0$,

(iv) $V_1(f, \overline{c}, X) = 0$.

Next, we will extend these integral characterizations for global optimality to the multiobjective problem (MOP).

2.2 Multiobjective Problem

In this section we return our attention to the multiobjective problem (MOP) to extend the results seen in last section. We assume that F is a continuous function and $X \subset \mathbb{R}^n$ is a compact set with nonempty interior.

First, we recall some results regarding the scalarization of multiobjective problems. Define the sets of weighting vectors

$$W = \{w \in \mathbb{R}^r \mid w_\ell \geq 0,\ \ell = 1, \ldots, r \text{ and } \|w\|_1 = 1\} \tag{4}$$

and

$$W^* = \{w \in \mathbb{R}^r \mid w_\ell > 0,\ \ell = 1, \ldots, r \text{ and } \|w\|_1 = 1\}, \tag{5}$$

where $\|w\|_1 = \sum_{\ell=1}^{r} |w_\ell|$. For each $w \in W$, we define the weighted sum scalarization function $\Phi_w : \mathbb{R}^n \to \mathbb{R}$ by

$$\Phi_w(x) = \sum_{\ell=1}^{r} w_\ell f_\ell(x) \tag{6}$$

and we consider the following weighted sum problem:

$$\begin{array}{l} \text{minimize } \Phi_w(x) \\ \text{subject to } x \in X. \end{array} \tag{WS_w}$$

The connections between the solutions of the weighted sum problem (WS_w) and the (weak) Pareto optimal solutions of the problem (MOP) are given in the following theorems.

Theorem 2.6. *[29, Thm. 3.1.1 and 3.1.2] If there exists $w \in W$ (respectively, $w \in W^*$) such that $\overline{x} \in X$ is a solution of (WS_w) then $\overline{x}$ is a weak Pareto optimal solution (respectively, Pareto optimal solution) of* (MOP).

Now we will define the weighted Chebyshev scalarization. For that, let $F^* \in \mathbb{R}^r$ be the ideal objective vector, where its components f_ℓ^* are obtained by minimizing each objective function individually subject to the constraints, that is, for each $\ell = 1, \ldots, r$,

$$f_\ell^* = \min_{x \in X} f_\ell(x). \tag{7}$$

If there exists $\overline{x} \in X$, such that $F(\overline{x}) = F^*$, then $\overline{x}$ would be a solution of the multiobjective problem (MOP) and the Pareto optimal set would be reduced to it. In general, the ideal objective vector can be used as a lower bound for the objective function at the Pareto optimal set. Now, given $\xi \in \mathbb{R}^r_+$

with small positive components, we consider the utopian objective vector $u^* = F^* - \xi$ and for $w \in W$, we define the weighted Chebyshev scalar function $\Psi_w : \mathbb{R}^n \to \mathbb{R}$ by

$$\Psi_w(x) = \max_{\ell=1,\ldots,r} \{w_\ell(f_\ell(x) - u^*_\ell)\} \tag{8}$$

and we solve the following problem:

$$\begin{array}{l} \text{minimize } \Psi_w(x) \\ \text{subject to } x \in X. \end{array} \tag{WCS_w}$$

The convexity (or generalized convexity) of the multiobjective optimization problem (MOP) is sufficient to ensure that all Pareto optimal solutions can be found using the weighted sum scalarization. See Theorem 3.1.4 in [29], Lemma 2 in [41] and Theorems 3.2 and 3.3 in [32]. On the other hand, next theorem shows that all weak Pareto optimal solutions can be found by the weighted Chebyshev technique, without any additional hypotheses.

Theorem 2.7. *[29, Thm. 3.4.2 and 3.4.5] The point $\overline{x} \in X$ is a weak Pareto optimal solution of the multiobjective problem* (MOP) *if, and only if, $\overline{x}$ is a solution of (WCS_w) for some weighting vector $w \in W^*$.*

It is interesting to note that if, for $w \in W^*$, the problem (WCS_w) has a unique solution, then it will be a Pareto optimal point [29, Cor. 3.4.4]. In addition, if the set of Pareto solutions is uniformly dominant[1], then every Pareto point can be obtained through the Chebyshev scalarization [4, Thm. 3 and 4]. For more details on scalarization methods, see Chankong and Haimes [7], Jahn [26] and Miettinen [29].

Now, we will present integral characterizations of global optimality for multiobjective problems (MOP) from these scalarization techniques. As F is a continuous function on the compact set X, the functions Φ_w and Ψ_w, defined by (6) and (8), respectively, are continuous. From Weierstrass Theorem, it follows that there exist constants M_1 and M_2 such that $\Phi_w(x) < M_1$ and $\Psi_w(x) < M_2$ for all $x \in X$. Define the functions, for $x \in \mathbb{R}^n$, by

$$\begin{aligned} \widetilde{\Phi}_w(x) &= \Phi_w(x) - M_1 \\ \widetilde{\Psi}_w(x) &= \Psi_w(x) - M_2. \end{aligned}$$

These functions are continuous on X and $\widetilde{\Phi}_w(x), \widetilde{\Psi}_w(x) < 0$ for all $x \in X$.

Remark 2.8. *As a consequence, a point $\overline{x} \in X$ is a global minimizer of Φ_w over X if, and only if, $\overline{x}$ minimizes the function $x \mapsto -\frac{\widetilde{\Phi}_w(x)}{\widetilde{\Phi}_w(\overline{x})}$ on X. A similar result holds for the function Ψ_w.*

Now we state the results inherited from last section by the application of the weighted scalarization techniques to the problem (MOP).

Theorem 2.9. *Consider $\overline{x} \in X$ and $w \in W$ (respectively, $w \in W^*$). If $\Upsilon_w(t) = \displaystyle\int_X \left[\frac{\widetilde{\Phi}_w(x)}{\widetilde{\Phi}_w(\overline{x})}\right]^t d\mu$ converges as $t \to \infty$, then $\overline{x}$ is a weak Pareto optimal solution (respectively, Pareto optimal solution) for the problem* (MOP).

[1] The efficient set is uniformly dominant if for every non-efficient point x' there exists an efficient point x^* such that $f_\ell(x') > f_\ell(x^*)$ for all $\ell = 1, \ldots, r$.

Proof. The set X is compact and the function $x \mapsto -\frac{\widetilde{\Phi}_w(x)}{\widetilde{\Phi}_w(\overline{x})}$ satisfies the hypotheses of Theorem 2.1. Thus, $\overline{x} \in X$ is a global minimizer of this function over X, and consequently, a global solution of the problem (WS_w), by Remark 2.8. So, the result follows from Theorem 2.6. $\square$

Theorem 2.10. *A point $\overline{x} \in X$ is a weak Pareto optimal solution of* (MOP) *if, and only if, there exists $w \in W^*$ such that the function defined by $\Upsilon_w(t) = \int_X \left[\frac{\widetilde{\Psi}_w(x)}{\widetilde{\Psi}_w(\overline{x})}\right]^t d\mu$ converges when $t \to \infty$.*

Proof. By Theorem 2.7, $\overline{x} \in X$ is a weak Pareto solution of (MOP) if, and only if, there exists $\overline{w} \in W^*$ such that $\overline{x}$ is a solution of (WCS_w), which is equivalent to

$$\widetilde{\Psi}_{\overline{w}}(\overline{x}) \leq \widetilde{\Psi}_{\overline{w}}(x) \text{ for all } x \in X. \tag{9}$$

Since $\widetilde{\Psi}_{\overline{w}}(x) < 0$, for all $x \in X$, (9) is equivalent to say that $\overline{x}$ is a global minimizer, in X, of the function $x \mapsto -\frac{\widetilde{\Psi}_{\overline{w}}(x)}{\widetilde{\Psi}_{\overline{w}}(\overline{x})}$. As this function satisfies the hypotheses of Theorem 2.1, the proof is concluded. $\square$

Now we will discuss optimality conditions for the multiobjective problem (MOP) from the concepts of mean, variance and modified variance. In particular, the next theorem establishes global optimality necessary conditions to the problem (MOP) using the weighted sum scalarization.

Theorem 2.11. *Assume that Assumption* **A1** *holds. Suppose that there exists $w \in W$ (respectively, $w \in W^*$) such that the function Φ_w satisfies Assumptions* **A2** *and* **A3**. *Consider $\overline{x} \in X$ and $\overline{c} = \Phi_w(\overline{x})$. Then the following conditions are equivalent:*

(i) $\overline{x} \in X$ *is a solution of the problem* (WS_w),

(ii) $M(\Phi_w, \overline{c}, X) = \overline{c}$,

(iii) $V(\Phi_w, \overline{c}, X) = 0$,

(iv) $V_1(\Phi_w, \overline{c}, X) = 0$,

where M, V and V_1 are, respectively, the mean value, variance and modified variance of Φ_w. Moreover, in these equivalent situations, $\overline{x}$ is a weak Pareto optimal solution (respectively, Pareto optimal solution) of (MOP).

Proof. The result is an immediate consequence of Theorems 2.5 and 2.6. $\square$

Analogous result holds for the weighted Chebyshev scalarization (8). However, for this scalarization, we have stronger global optimality conditions by considering the following assumptions:

A1′. X is a robust and closed set,

A2′. The functions f_ℓ, $\ell = 1, \cdots, r$, are continuous,

A3′. There exist an index ℓ_0 and $c_0 \in \mathbb{R}$ such that the set $\{x \in X \mid f_{\ell_0}(x) \leq c_0\}$ is compact.

Next proposition ensures that if the problem (MOP) satisfies these assumptions, then **A1, A2** and **A3** hold for weighted sum problem (WS_w) and weighted Chebyshev problem (WCS_w), for all $w \in W^*$.

Proposition 2.12. *Suppose that* $\mathbf{A1'}$, $\mathbf{A2'}$ *and* $\mathbf{A3'}$ *hold. Then,* $\mathbf{A1}$ *holds and for all* $w \in W^*$*, the functions* Φ_w *and* Ψ_w *satisfies Assumptions* $\mathbf{A2}$ *and* $\mathbf{A3}$*.*

Proof. Assumption $\mathbf{A1}$ follows trivially from $\mathbf{A1'}$. Consider $w \in W^*$. Using the Assumption $\mathbf{A2'}$, the functions Φ_w and Ψ_w are continuous. Furthermore, for each $c \in \mathbb{R}$, the sets $\{x \in \mathbb{R}^n \mid \Phi_w(x) < c\}$ and $\{x \in \mathbb{R}^n \mid \Psi_w(x) < c\}$ are open. By Assumption $\mathbf{A1'}$ and Remark 2.2, their intersections with X are robust. Consequently, Φ_w and Ψ_w are upper robust functions and $\mathbf{A2}$ holds.

For each $c \in \mathbb{R}$, consider the level set $H_c = \{x \in \mathbb{R}^n \mid \Phi_w(x) \leq c\}$. Assumption $\mathbf{A1'}$ implies that $H_c \cap X$ is a closed set. Furthermore, as $w \in W^*$, we have, in particular to ℓ_0 given in Assumption $\mathbf{A3'}$, that

$$H_c \cap X = \left\{ x \in X \mid \sum_{\ell=1}^{r} w_\ell f_\ell(x) \leq c \right\} = \left\{ x \in X \mid f_{\ell_0}(x) \leq \frac{1}{w_{\ell_0}} \left(c - \sum_{\substack{\ell=1 \\ \ell \neq \ell_0}}^{r} w_\ell f_\ell(x) \right) \right\}.$$

Taking $c = w_{\ell_0} c_0 + \sum_{\substack{\ell=1 \\ \ell \neq \ell_0}}^{r} w_\ell f_\ell(x)$, with c_0 given in Assumption $\mathbf{A3'}$, the set $H_c \cap X$ is compact.

Analogously, for each $c \in \mathbb{R}$, consider the level set $H_c = \{x \in \mathbb{R}^n \mid \Psi_w(x) \leq c\}$. Assumption $\mathbf{A1'}$ implies that $H_c \cap X$ is a closed set. Furthermore, as $w \in W^*$, we have, in particular to ℓ_0 given in Assumption $\mathbf{A3'}$, that

$$H_c \cap X \subset \{x \in X \mid w_{\ell_0}(f_{\ell_0}(x) - u^*_{\ell_0}) \leq c\} = \left\{ x \in X \mid f_{\ell_0}(x) \leq \frac{c + w_{\ell_0} u^*_{\ell_0}}{w_{\ell_0}} \right\}.$$

Taking $c = w_{\ell_0}(c_0 - u^*_{\ell_0})$, with c_0 given in Assumption $\mathbf{A3'}$, the set $H_c \cap X$ is compact, which proves $\mathbf{A3}$ for both functions and concludes the proof. □

Next theorem ensures necessary and sufficient global optimality conditions of (MOP) using the weighted Chebyshev scalarization (8), while Theorem 2.11 establishes only necessary conditions for the weighted sum scalarization.

Theorem 2.13. *Suppose that the problem* (MOP) *satisfies* $\mathbf{A1'}$, $\mathbf{A2'}$ *and* $\mathbf{A3'}$*. Consider* $\overline{x} \in X$*. Then, the following conditions are equivalent:*

(i) $\overline{x}$ *is a weak Pareto optimal solution of (MOP),*

(ii) there exists $w \in W^*$ *such that* $\overline{x}$ *minimizes* Ψ_w *over* X *and* $\overline{c} = \Psi_w(\overline{x})$*,*

(iii) there exists $w \in W^*$ *such that* $M(\Psi_w, \overline{c}, X) = \overline{c}$*, with* $\overline{c} = \Psi_w(\overline{x})$*,*

(iv) there exists $w \in W^*$ *such that* $V(\Psi_w, \overline{c}, X) = 0$*, with* $\overline{c} = \Psi_w(\overline{x})$*,*

(v) there exists $w \in W^*$ *such that* $V_1(\Psi_w, \overline{c}, X) = 0$*, with* $\overline{c} = \Psi_w(\overline{x})$*,*

where M*,* V *and* V_1 *are, respectively, the mean value, variance and modified variance of* Ψ_w*.*

Proof. The result is an immediate consequence of Proposition 2.12 and Theorems 2.5 and 2.7. □

It is important to note that the result of Theorem 2.13 holds under more general conditions. In fact, it is enough that X is robust and Ψ_w satisfies $\mathbf{A2}$ and $\mathbf{A3}$, for $w \in W^*$. Based on Theorem 2.13, we extend the algorithm proposed in [21] (originally to solve single objective problems) for obtaining an approximation of the weak Pareto front of the multiobjective problem (MOP).

3 The algorithm

Now, inspired by [21], we state an algorithm based on the mean value of level sets for multiobjective problems and we discuss its global convergence regarding the scalarized problem (WCS_w).

Algorithm 1. Mean Value of Level Sets for Multiobjective Problems – MVLSM

Data: $\varepsilon \geq 0$, $\overline{w} \in \mathbb{R}^r_+$ with $\overline{w}_i \in (0,1)$ for all $i = 1, \ldots, r$ and $\xi \in \mathbb{R}^r_+ \backslash \{0\}$.

Scalarization

Compute $f^*_\ell = \min_{x \in X}\{f_\ell(x)\}$, for each $\ell \in \{1, \ldots, r\}$ and define $u^* = F^* - \xi$.

Consider the weighting vector $w \in \mathbb{R}^r$ such that $w_i = \dfrac{\overline{w}_i}{\|\overline{w}\|_1}$, for $i = 1, \ldots, r$, and the scalarized function $\Psi_w(x) = \max_\ell\{w_\ell(f_\ell(x) - u^*_\ell)\}$.

Initialization

Take $c_0 \in \mathbb{R}$ such that $H_{c_0} = \{x \in \mathbb{R}^n \mid \Psi_w(x) \leq c_0\} \neq \emptyset$.

Set $k := 0$.

Iterations

REPEAT

Let $H_{c_k} = \{x \in \mathbb{R}^n \mid \Psi_w(x) \leq c_k\}$.

Compute $VF = V_1(\Psi_w, c_k, X) = \dfrac{1}{\mu(H_{c_k} \cap X)} \displaystyle\int_{H_{c_k} \cap X} (\Psi_w(x) - c_k)^2 \, d\mu.$

Compute $c_{k+1} = M(\Psi_w, c_k, X) = \dfrac{1}{\mu(H_{c_k} \cap X)} \displaystyle\int_{H_{c_k} \cap X} \Psi_w(x) \, d\mu.$

$k := k + 1$

UNTIL $VF < \varepsilon$

$\overline{c} = c_k$ and $\overline{H} = H_{\overline{c}}$

The scalar c_0 can be chosen as any real such that the set $H_{c_0} \cap X$ is nonempty. So, it can be set as a sufficiently large real or as $c_0 = \Psi_w(x_0)$, for a given initial point $x_0 \in X$. The stopping criterion of the algorithm is justified by Theorem 2.13, item (v). From now on, assume that $\varepsilon = 0$ and the algorithm generates an infinite sequence $\{c_k\}$. Next theorem ensures that this sequence converges to the global minimum value of the scalarized function Ψ_w over X.

Theorem 3.1. *Suppose that the problem* (MOP) *satisfies* **A1′, A2′** *and* **A3′**. *Given a weighting vector $w \in W^*$, consider the sequence $\{c_k\}$ generated by Algorithm 1. Then, this sequence is convergent and the limit $\overline{c} = \lim_{k \to \infty} c_k$ is the global minimum value of Ψ_w over X. Furthermore, $H_{\overline{c}} \cap X$ is the set of its global minimizers and consequently a subset of weak Pareto optimal solutions of* (MOP).

Proof. Let $w \in W^*$, $\hat{c} = \min_{x \in X} \Psi_w(x)$ and the sequence $\{c_k\}$ generated by the algorithm from c_0 such that $H_{c_0} = \{x \in \mathbb{R}^n \mid \Psi_w(x) \leq c_0\} \neq \emptyset$. If $c_0 = \hat{c}$, then $VF = V_1(\Psi_w, c_0, X) = 0$ and the algorithm stops. Now, consider $c_0 > \hat{c}$. In this case, for all $x \in H_{c_0} \cap X$, $\hat{c} \leq \Psi_w(x) \leq c_0$. Integrating this expression and using the definition of c_1 and the fact that, by [47, Lemma 5.1], $\mu(H_{c_0} \cap X) > 0$, we have $\hat{c} \leq c_1 \leq c_0$. Following a similar reasoning we can conclude that $\hat{c} \leq c_{k+1} \leq c_k$ for all $k \geq 0$.

If there exists $k_0 \in \mathbb{N}$ such that $c_{k_0} = M(\Psi_w, c_{k_0}, X)$, then, by Theorem 2.13, $V_1(\Psi, c_{k_0}, X) = 0$ and the algorithm stops with $\hat{c} = c_{k_0}$. In this case, $H_{c_{k_0}} \cap X$ is the set of global minimizers of Ψ_w and, by Theorem 2.7, it is a subset of Pareto optimal solutions of (MOP).

Otherwise, the sequence $\{c_k\}$ of mean values is decreasing and bounded below, and consequently convergent, say to $\overline{c} \geq \hat{c}$. Thus, by the convergence of the sequence $\{c_k\}$ and the continuity of the function M with respect to the second argument (more details about the continuity of function M can be found in [21, Prop. 1.3]), we have

$$\overline{c} = M(\Psi_w, \overline{c}, X).$$

Applying Theorem 2.13, we conclude that $\overline{c} = \hat{c}$ is the global minimum value of Ψ_w over X. On the other hand, as the sequence $\{c_k\}$ is decreasing and bounded below by $\overline{c}$, it follows that the sequence $\{H_{c_k}\}$ of level sets satisfies the following condition

$$(H_{c_0} \cap X) \supset (H_{c_1} \cap X) \supset \ldots \supset (H_{c_k} \cap X) \supset (H_{c_{k+1}} \cap X) \supset \ldots \supset (H_{\overline{c}} \cap X).$$

This fact implies

$$\bigcap_{k=1}^{\infty} (H_{c_k} \cap X) = H_{\overline{c}} \cap X = \{x \in X \mid \Psi_w(x) = \overline{c}\},$$

which proves $H_{\overline{c}} \cap X$ is the set of global minimizers of Ψ_w over X. Furthermore, by Theorem 2.13, this set is a subset of weak Pareto optimal solutions of (MOP), completing the proof. $\square$

For each vector $w \in \mathbb{R}^r$, we obtain a subset of weak Pareto optimal solutions of (MOP). So, to determine an approximation of the weak Pareto front of the multiobjective problem (MOP), the algorithm should be run several times using different weights.

4 Numerical experiments

In this section, we describe numerical experiments to illustrate the computational performance of Algorithm 1. The tests were performed in a high performance workstation MARKOV: 2*CPU: Intel® Xeon® Processor E5-2650 v3 (10 Cores 25M Cache, 2.30 GHz), 160GB RAM 2,133GHz, using `Matlab` 2018b. The set of test problems consists of **all** 26 unconstrained and box-constrained multiobjective problems with continuous variable of dimension at most 4 presented in [7,12,13,40].

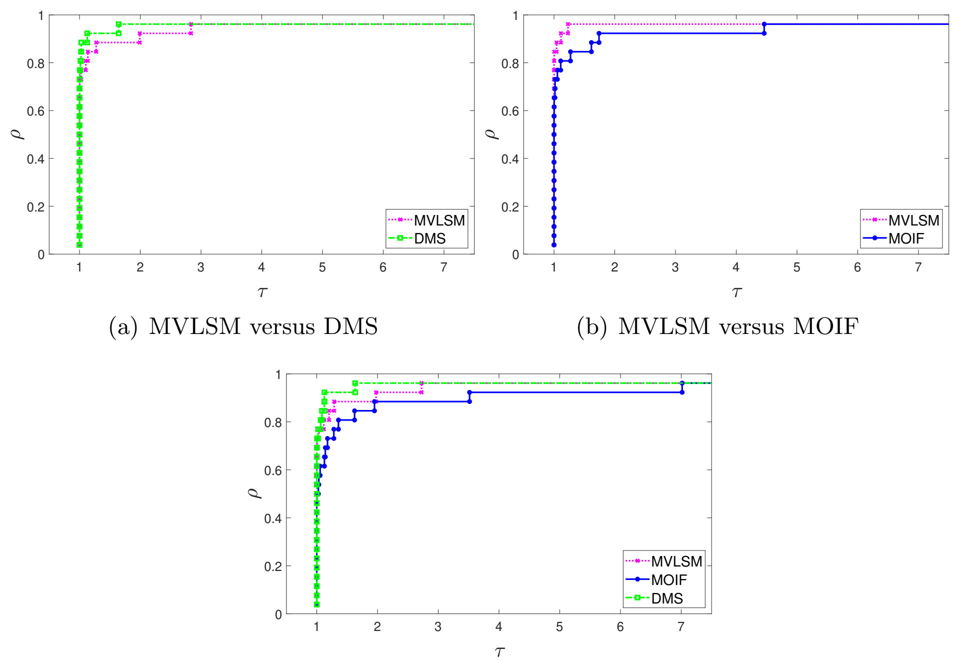
In Algorithm 1, the random weighting vector $\bar{w} \in \mathbb{R}^r$ has been computed by the RAND MATLAB routine, the initial mean value has been set as $c_0 = 10^8$, the stopping tolerance as $\varepsilon = 10^{-8}$ and $\xi_\ell = 10^{-4}$ for all $\ell = 1, \ldots, r$. The multiple integrals in the modified variance VF and in the mean value c_k were computed by nested commands of the TRAPZ MATLAB routine. The domain of integration was discretized in 10000 points uniformly distributed.

Initially, we run 3000 times Algorithm 1 considering different weighting random vectors for solving each problem. Tables 1 - 7 show the results where the first column displays the data of the problems such as references, dimension n, number r of objectives and some results as the average T of the CPU time and the average $\bar{k}$ of the number of iterations among the 3000 runs for each problem. As the dimension of the problems presented in Tables 1 - 5 is less than 3, we show, in the second column, the graph of the objective functions. The last column of all tables presents the approximation of the weak Pareto front generated from the total of runs of the algorithm and the exact Pareto front is shown whenever its analytical expression is available. These figures illustrate the good performance of Algorithm 1 that found a good approximation of the weak Pareto front for all 26 problems spending in average $T = 0.0518$ sec and 25 iterations. The longest CPU time was 0.3112 sec and the largest number of iterations was, 129 spent for solving [7, Example 9] and [7, Problem 4.7], respectively.

Secondly, we compared the performance of MVLSM (Mean Value of Level Sets for Multiobjective Problems) as described in Algorithm 1 for solving the 26 problems with two solvers from the literature, namely:

- DMS (Direct Multisearch) proposed by Custódio, Madeira, Vaz, and Vicente in [11] and freely available at *http://www.mat.uc.pt/dms*;
- MOIF (Multiobjecticve Implicit Filtering) proposed by Cocchi, Liuzzi, Papini, and Sciandrone in [9] and freely available at *http://www.dis.uniroma1.it/lucidi/DFL*.

The solvers DMS and MOIF have been tested using their default parameters. In these experiments, we fixed the maximum function evaluations as 20000 for each algorithm for solving each problem. Figure 1 shows the performance profile [14] using the purity metric [1] which compares the quality of Pareto fronts obtained by different solvers. Although the solver DMS is slightly more efficient, the three algorithms are competitive.

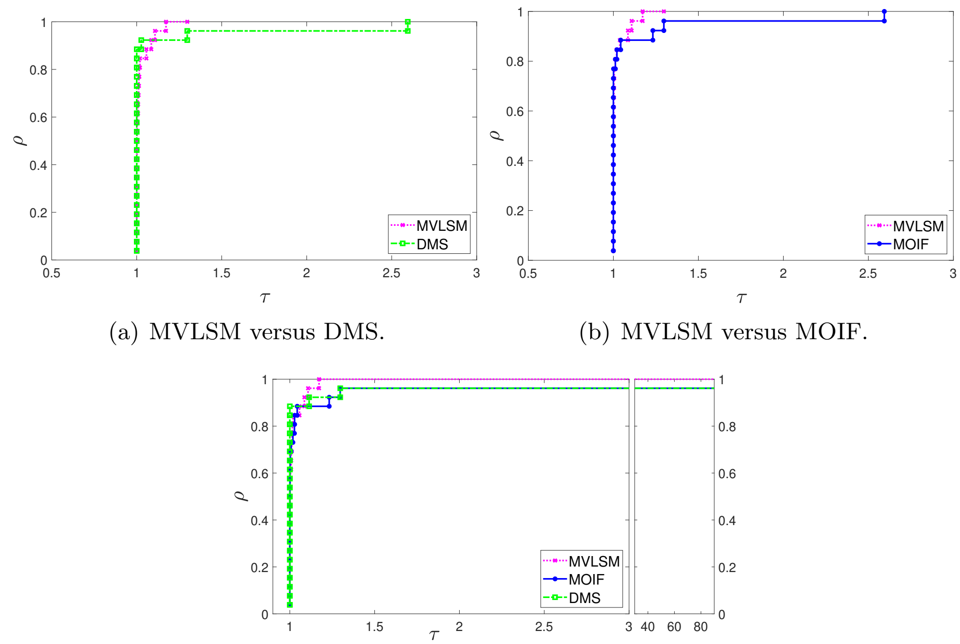


(a) MVLSM versus DMS (b) MVLSM versus MOIF

(c) All

Figure 1: Comparing the MVLSM with DMS and MOIF based on purity performance profiles for multiobjective problems.

Figure 2 shows the performance profile using the hypervolume indicator which represents the volume in the objective space dominated by a Pareto front approximation Y_N and delimited above by an objective vector $v \in \mathbb{R}^r$ such that for all $y \in Y_N$, we have that $y < v$, as explained in [49]. According to this figure, DMS is more efficient than the other solvers. However, MVLSM is the most robust ones.



(a) MVLSM versus DMS.

(b) MVLSM versus MOIF.

(c) All.

Figure 2: Comparing the MVLSM with DMS and MOIF based on hypervolume performance profiles for multiobjective problems.

5 Conclusion

In this paper, integral global optimality conditions are extended to multiobjective optimization problems from single objective case by using weighted scalarization techniques. These conditions of optimality via integration can be a powerful tool to deal with several optimization problems of practical nature that appear in diverse areas of knowledge. Based on the theoretical results using Chebyshev scalarization, we proposed an algorithm to build an approximation of the weak Pareto front. The algorithm proposed was implemented in MATLAB and its good performance was illustrated by solving a set of unconstrained and box constrained problems with continuous variables and dimension at most 4.

The integral optimality conditions are interesting, among other reasons, because they can be applied even in the non-smooth case since no kind of derivative (or sub-derivative) is used. On the other hand, these conditions are stated in terms of multiple integrals, which may narrow applying such a theory to problems with many variables. Future research topics include implementing efficient methods to compute integrals with many variables and smarter choices of the weights to get points well spread in the (weak) Pareto front approximation. Also, we intend to study integral optimality conditions using other scalarization techniques.

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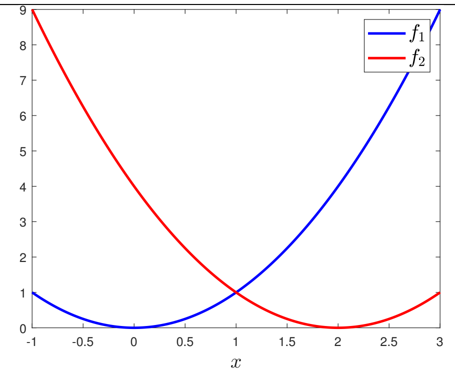
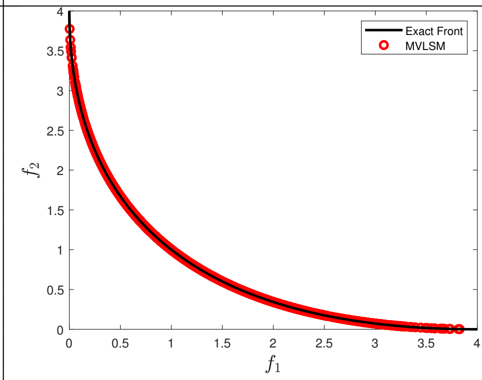
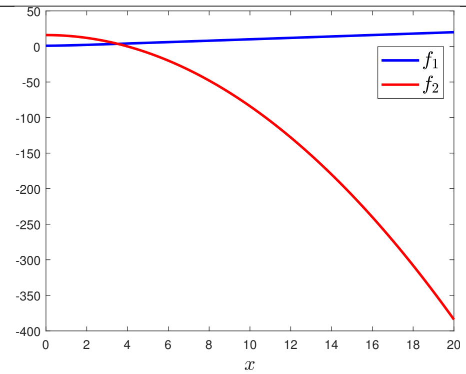
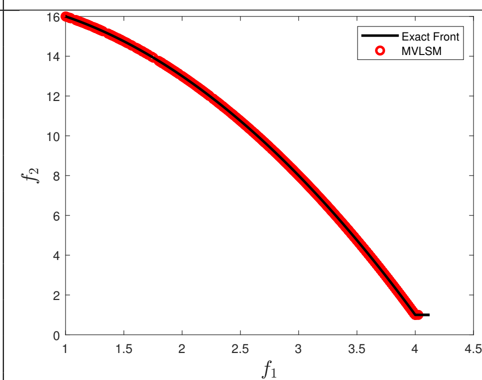
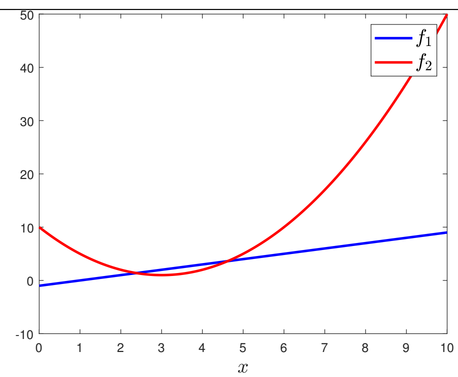
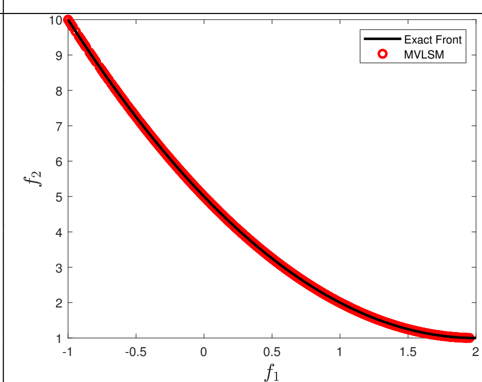
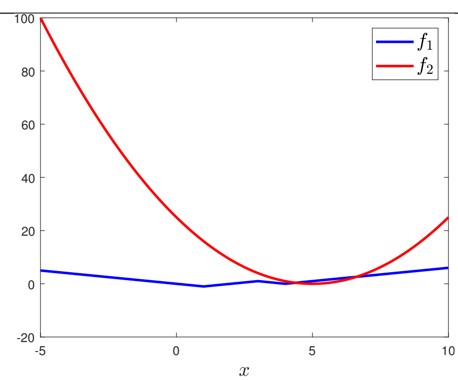
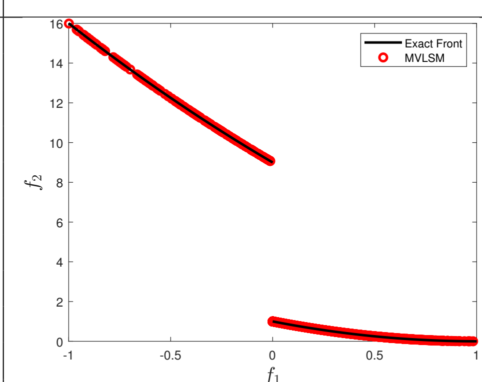
Problem	Objective functions	Pareto front
[40, MOP 13] [13, SCH1] $n = 1$ $r = 2$ $T = 0.0973$ sec $\overline{k} = 50.4$		
[7, Example 9] $n = 1$ $r = 2$ $T = 0.1972$ sec $\overline{k} = 97.5$		
[7, Problem 4.7] $n = 1$ $r = 2$ $T = 0.1653$ sec $\overline{k} = 85.0$		
[40, MOP 14] $n = 1$ $r = 2$ $T = 0.0867$ sec $\overline{k} = 45.7$		

Table 1: Results for problems with dimension $n = 1$.

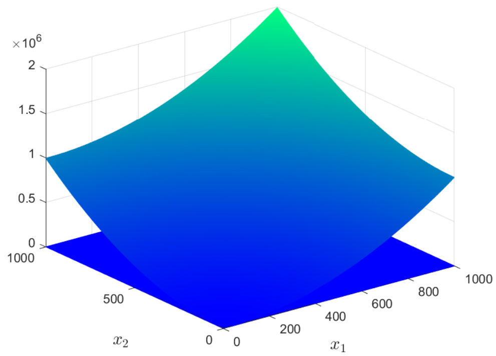
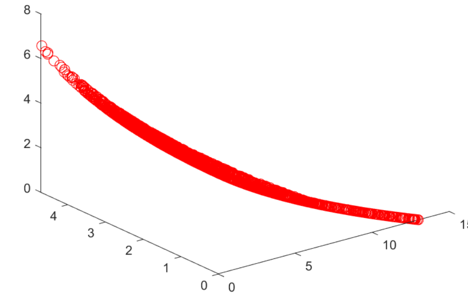
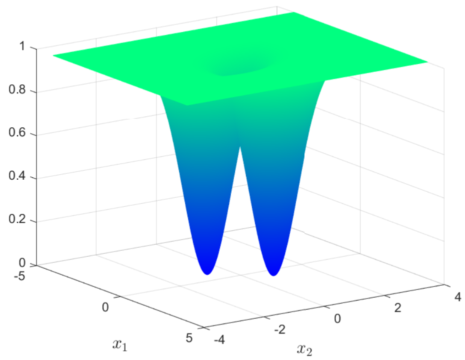
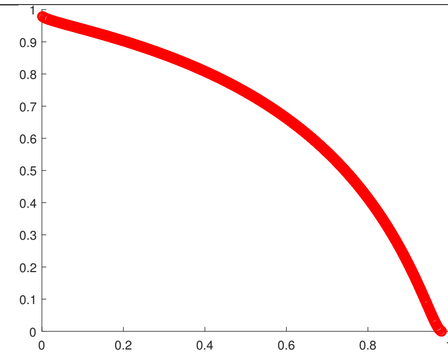
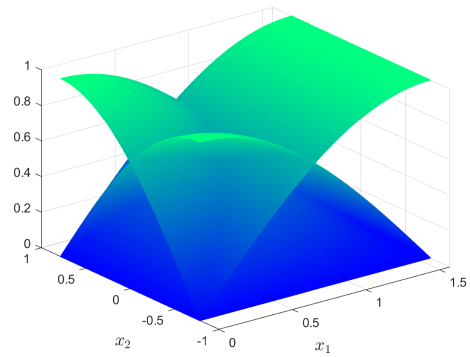
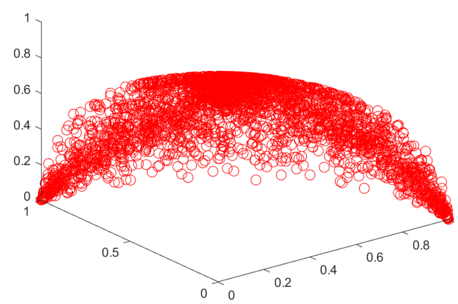
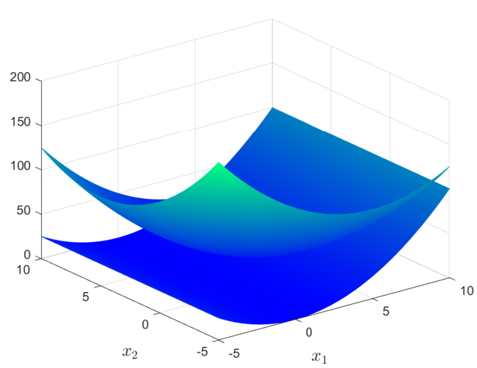
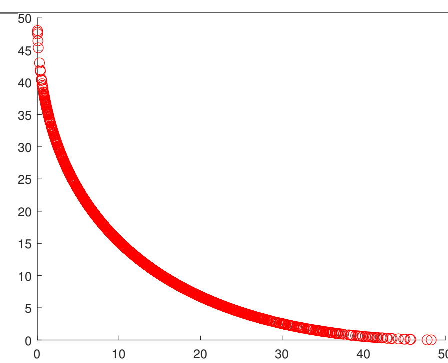
problems	Objective functions	Pareto front
[7, Example 4.3.6] $n = 2$ $r = 3$ $T = 0.0610$ sec $\overline{k} = 32.1$		
[13, (6.1)], [40, MOP 4] $n = 2$ $r = 2$ $T = 0.0391$ sec $\overline{k} = 20.4$		
[13, (6.2)] $n = 2$ $r = 3$ $T = 0.0427$ sec $\overline{k} = 20.4$		
[40, MOP 1] $n = 2$ $r = 2$ $T = 0.0427$ sec $\overline{k} = 23.1$		

Table 2: Results for problems with dimension $n = 2$.

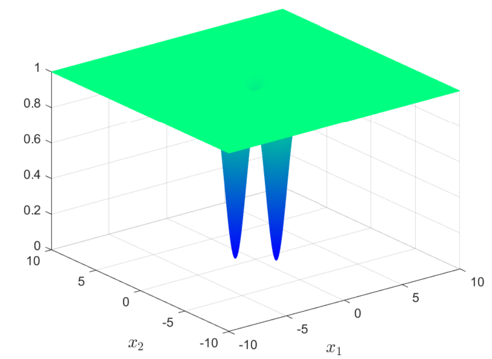
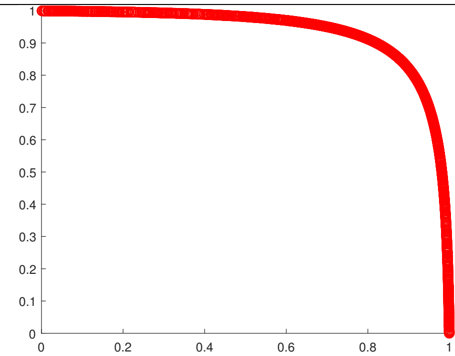
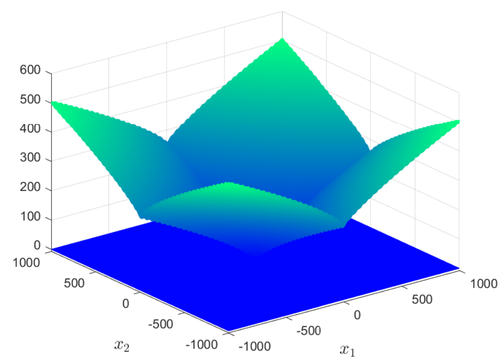
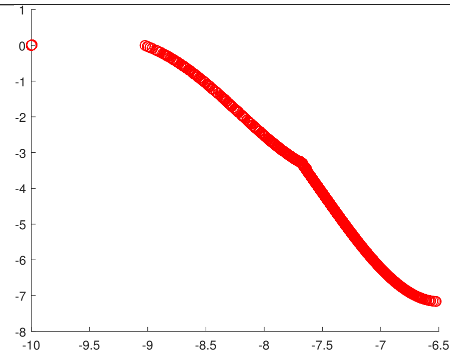
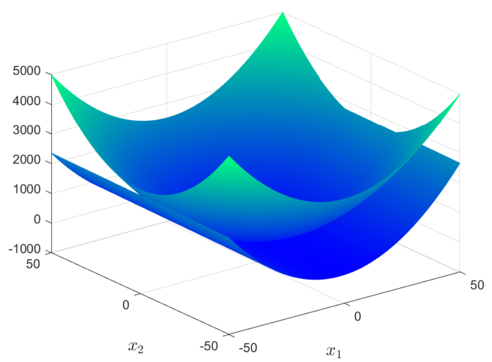
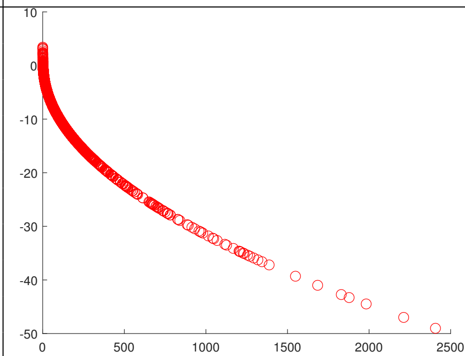
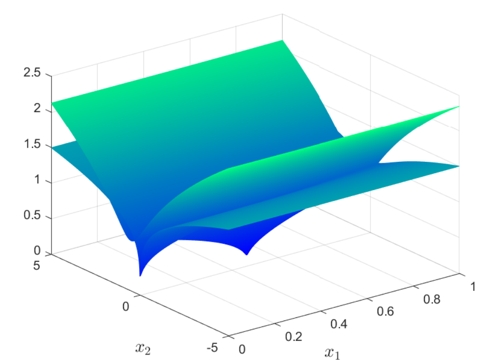
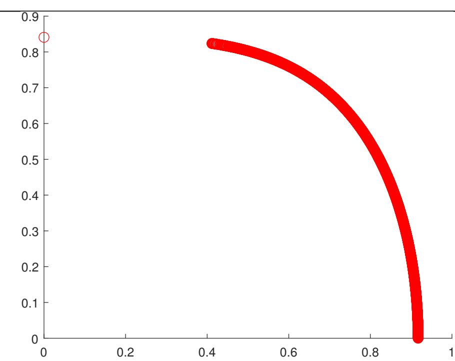
Problem	Objective functions	Pareto front
[40, MOP 3] $n = 2$ $r = 2$ $T = 0.0283$ sec $\overline{k} = 14.0$		
[40, MOP 5] $n = 2$ $r = 2$ $T = 0.0829$ sec $\overline{k} = 32.6$		
[40, MOP 6] $n = 2$ $r = 2$ $T = 0.0511$ sec $\overline{k} = 28.0$		
[40, MOP 7] $n = 2$ $r = 2$ $T = 0.0503$ sec $\overline{k} = 21.1$		

Table 3: Results for problems with dimension $n = 2$ of reference [40].

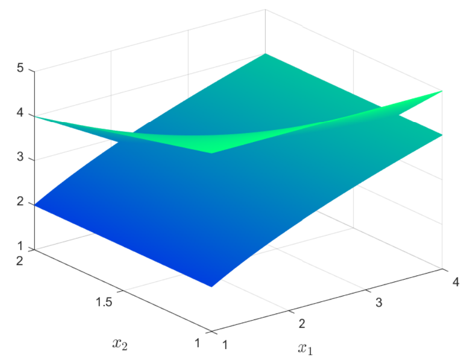
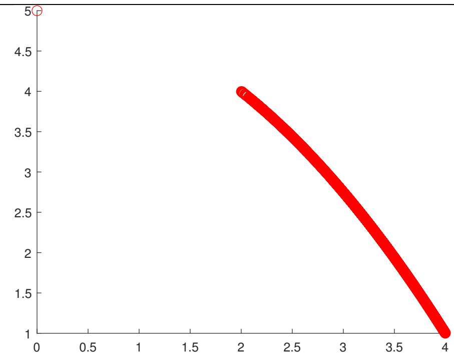
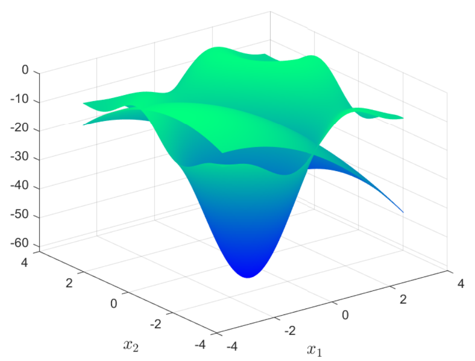
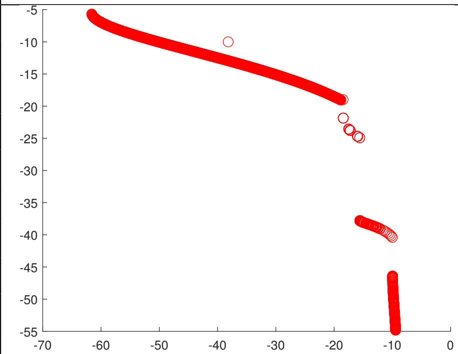
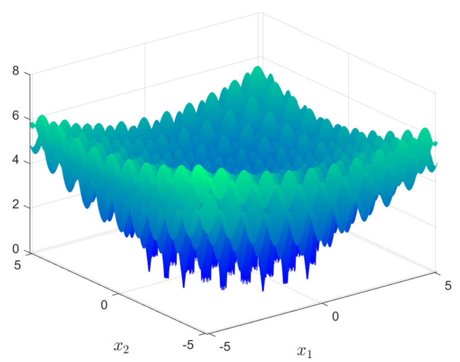
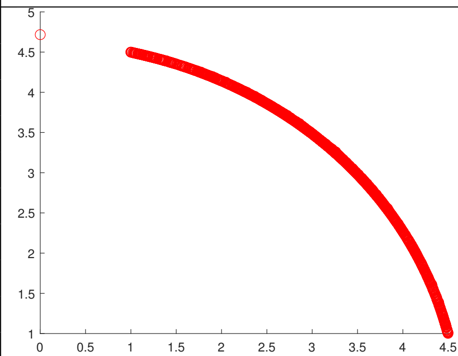
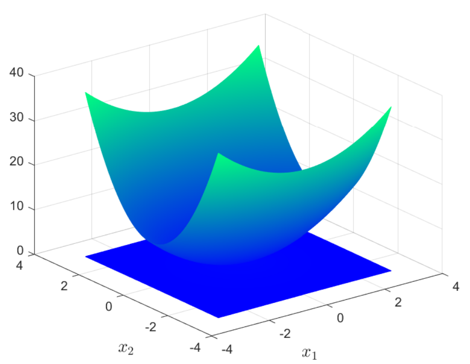
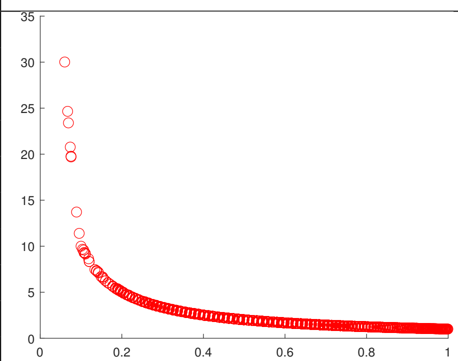
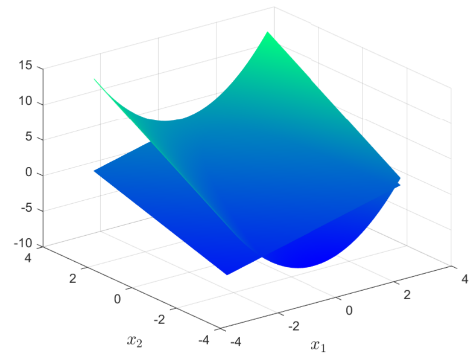
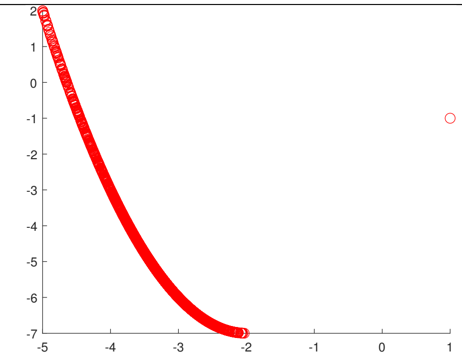
Problem	Objective functions	Pareto front
[40, MOP 8] $n = 2$ $r = 2$ $T = 0.0196$ sec $\overline{k} = 8.7$		
Problem [40, MOP 9] $n = 2$ $r = 2$ $T = 0.0339$ sec $\overline{k} = 14.3$		
Problem [40, MOP 10] $n = 2$ $r = 2$ $T = 0.0323$ sec $\overline{k} = 13.2$		
[40, MOP 11] $n = 2$ $r = 2$ $T = 0.0242$ sec $\overline{k} = 12.1$		

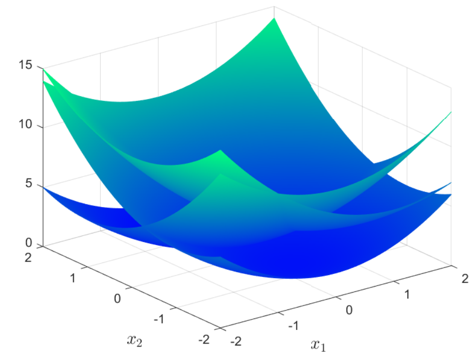
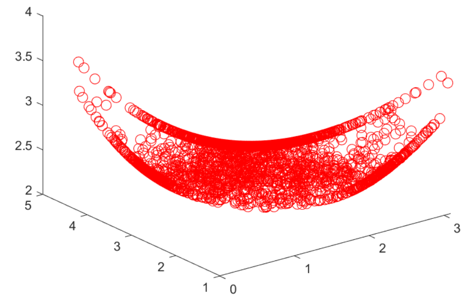
Table 4: Results for problems with dimension $n = 2$ of reference [40].

<table>
<tr><th>Problem</th><th>Objective functions</th><th>Pareto front</th></tr>
<tr><td>[40, MOP 12]

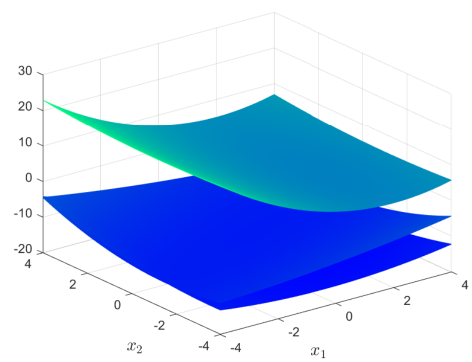
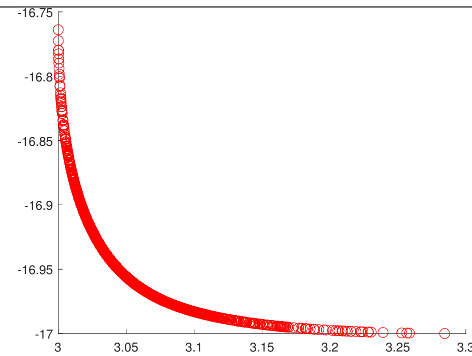
$n = 2$
$r = 2$

$T = 0.0232$ sec
$\overline{k} = 12.1$</td><td>
</td><td>
</td></tr>
<tr><td>[40, MOP 16]

$n = 2$
$r = 3$

$T = 0.0581$ sec
$\overline{k} = 29.7$</td><td>
</td><td>
</td></tr>
<tr><td>[40, MOP 17]

$n = 2$
$r = 3$

$T = 0.0538$ sec
$\overline{k} = 26.3$</td><td>
</td><td>
</td></tr>
<tr><td>[40, MOP 18]

$n = 2$
$r = 3$

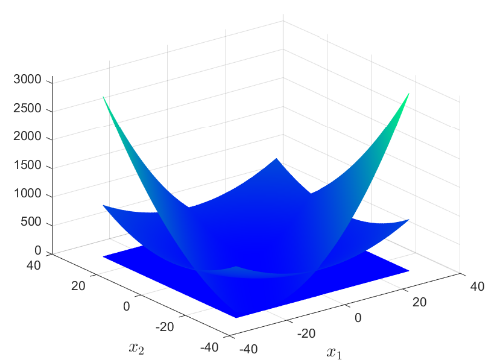
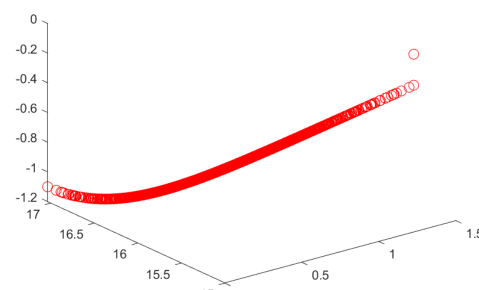
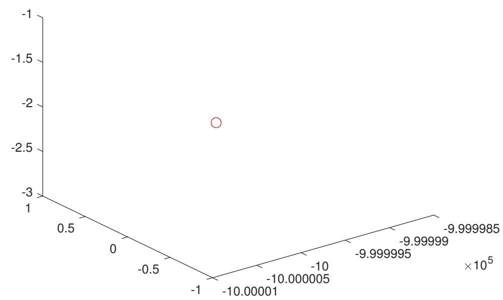
$T = 0.0606$ sec
$\overline{k} = 30.0$</td><td>
</td><td>
</td></tr>
</table>

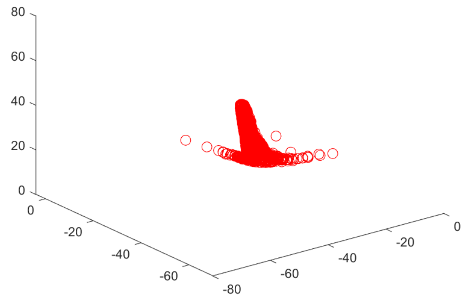
Table 5: Results for problems with dimension $n = 2$ of reference [40].

<table>
<tr><th>Problem</th><th>Pareto front</th></tr>
<tr><td>[40, MOP 2]

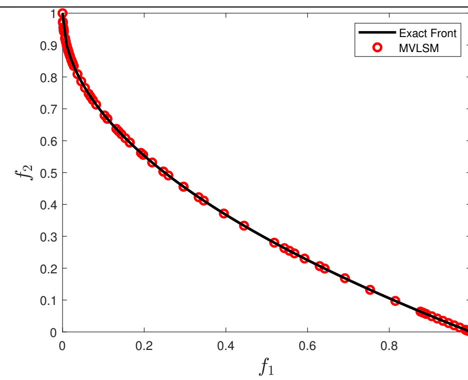
$n = 2$
$r = 3$

$T = 0.0032$ sec
$\overline{k} = 1$</td><td></td></tr>
<tr><td>[13, (6.28)]

$n = 3$
$r = 3$

$T = 0.0200$ sec
$\overline{k} = 6.3$</td><td></td></tr>
<tr><td>[12, ZDT 1]

$n = 4$
$r = 2$

$T = 0.0116$ sec
$\overline{k} = 3.5$</td><td>
</td></tr>
<tr><td>[12, ZDT 2]

$n = 4$
$r = 2$

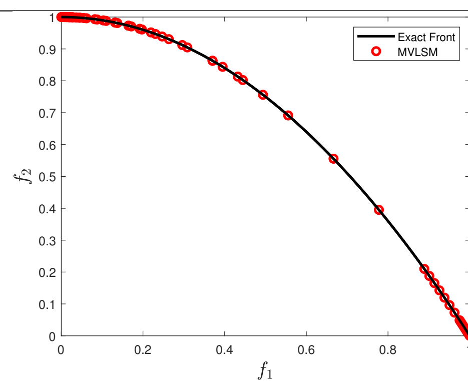
$T = 0.0110$ sec
$\overline{k} = 3.3$</td><td>
</td></tr>
</table>

Table 6: Results for problems with dimension $n \geq 3$ or $r = 3$.

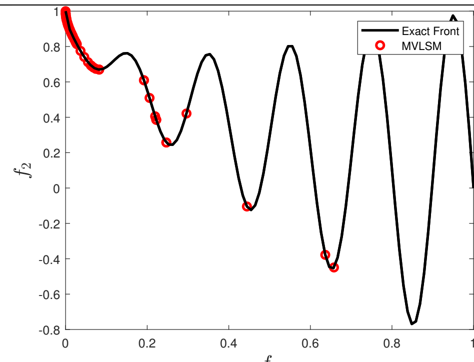
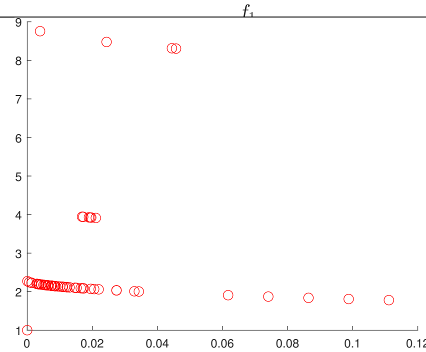
Problem	Pareto front
[12, ZDT 3] $n = 4$ $r = 2$ $T = 0.0123$ sec $\overline{k} = 3.7$	
[12, ZDT 4] $n = 4$ $r = 2$ $T = 0.0392$ sec $\overline{k} = 7.9$	

Table 7: Results for problems with dimension $n = 4$.