# Data Envelopment Analysis of two-stage processes: An alternative (non-conventional) approach

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**Abstract**

Network data envelopment analysis (NDEA) is an extension of standard data envelopment analysis that models the efficiency assessment of DMUs by considering their internal structure. While in standard DEA the DMU is regarded as a single process, in NDEA the DMU is viewed as a network of interconnected sub-processes (stages, divisions), where the flow of the intermediate products (measures) is essential in the efficiency assessment. In the prevalent conventional methodological approach to NDEA, the sub-processes are assumed as distinct entities with distinct inputs and outputs. Thus, each sub-process has its own production possibility set (PPS), which can be derived axiomatically from a set of assumptions using the minimum extrapolation principle. The PPS of the overall system is defined as the composition of the individual PPSs. The conventional approach comprises all the methods, where the common characteristic is that the system and the divisional efficiencies are computed jointly in a single mathematical program. A fundamental property connecting the system with the divisional efficiencies is that a system is overall efficient if and only if its divisions are all efficient. However, real-word case studies have shown that there are cases where none of the DMUs is rendered overall efficient regardless of the NDEA method used. This is the main issue we discuss in this paper and our motivation to propose an alternative, non-conventional, approach to address it in the frame of two-stage processes. We consider the two-stage process as a system that can be viewed in two perspectives depending on the role of the intermediate measures: The system as producer and as consumer of the intermediates. As our approach is based on standard DEA, it acquires the basic desirable properties. The fundamental NDEA property, that the overall system is efficient if and only if both perspectives are efficient, is met. The efficient frontier of the system is explicitly defined by overall efficient observed DMUs. The inefficient DMUs are projected on the efficient frontier of the system. The models are equivalently expressed in both the multiplier and the envelopment forms due to strict primal-dual correspondence and are able to operate under both constant and variable returns-to-scale assumptions. We use the case of twenty-two automotive manufactures for the fiscal year 2019 as an example to illustrate our approach. Comparison with other NDEA methods is also provided.

**Keywords:** Data Envelopment Analysis, Network Data Envelopment Analysis, Two-stage processes.

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## 1. Introduction

Data envelopment analysis (DEA) is a widely used linear programming technique for evaluating the relative efficiency of peer decision making units (DMUs) that use multiple inputs to produce multiple outputs. The two fundamental DEA models [5] and [2] have become standards in performance measurement under the assumptions of constant (CRS) and variable returns-to-scale (VRS) respectively. Network data envelopment analysis (NDEA) has been introduced as an extension of DEA to model the assessment of DMUs by considering their internal structure. While in standard DEA the DMU is regarded as a single process, in NDEA the DMU is viewed as a network of interconnected sub-processes (stages, divisions), where the flow of the intermediate products (measures) is essential in the efficiency assessment. Thus, efficiency in NDEA is a multi-dimensional measure, as one accounts for both divisional efficiencies and overall (system) efficiency. Färe and Grosskopf [15] were among the first to deal with the efficiency assessment in such processes. Reviews of NDEA methods can be found in [3] and [17]. In [11], the methods developed for the efficiency assessment of two-stage processes are comparatively reviewed. In [17], the NDEA models are classified in nine types, which are mainly based on whether they are developed in multiplier or envelopment forms. All but the independent approach can be collectively grouped into the so-called holistic approach [14]. In the independent approach, the efficiencies of the individual sub-processes and of the entire DMU are assessed independently to each other [26], [24]. In the holistic approach, on the other hand, the efficiencies of the stages and the overall efficiency of the system are assessed jointly. In this case, the interdependency of the stages is taken into account by means of the flow of intermediate measures. The holistic paradigm comprises the game-theoretic approaches, such as the non-cooperative (leader-follower) and the cooperative (centralized) methods [21]. The centralized method is also known as the relational model or the multiplicative decomposition method [18]. The centralized method along with the additive method [8] are efficiency decomposition methods, in the sense that the overall efficiency of the DMU is evaluated first and the stage efficiencies derive subsequently as offspring. The holistic approach comprises a methodological stream that views the assessment of multistage processes as a multi-objective programming (MOP) problem. The common characteristics of these methods, which they are grouped collectively in the so-called "composition approach", is that they give priority to the assessment of stage efficiencies and that the overall DMU efficiency is calculated ex post [12], [13],[14], [1], [16], [23]. Several drawbacks have been reported in network DEA studies for various network DEA methods, such as non-uniqueness of efficiency decomposition, bias in efficiency estimates, and lack of primal-dual correspondence [1], [7], [12]. Nevertheless, most of these drawbacks have been partially addressed in the relevant literature.

In NDEA, the DMUs are assumed structurally homogeneous, i.e., they have the same types of sub-processes with the same interconnections. The sub-processes are assumed as distinct entities with distinct inputs and outputs. Thus, each sub-process has its own production possibility set (PPS), which can be derived axiomatically from a set of assumptions using the minimum extrapolation principle. The PPS of the overall system is defined as the composition of the individual PPSs [22]. This is the conventional and prevalent approach to NDEA, which includes all the methods of the holistic paradigm. When assessing the overall system (DMU) efficiency, the PPS incorporates in the same model the PPSs of all the sub-processes. This brings into the fore the fundamental property that the overall system is efficient if and only if all its sub-processes are efficient. In a pre-evaluation phase, it is possible to determine which DMUs have the potential to meet this property. Indeed, the independent efficiency scores of the sub-processes are upper bounds for their efficiency scores obtained by a holistic assessment. Thus, a necessary condition for a DMU to have the potential to be rendered efficient is that all its sub-processes are independently efficient. However, as the validity of such a condition is occasional, there may be cases without efficient units at the system level. This particular issue is mentioned in [19] for the relational model and it is attributed to an unfairness hidden in the efficiency evaluation. The eventual absence of overall efficient units is corroborated in real-world case studies where none of the DMUs was rendered overall efficient regardless of the NDEA method used [18], [8], [12], [13], [9]. As the efficiency of the system in NDEA is a relative measure obtained by benchmarking the evaluated DMU in relation to best practice observed units, the question that arises is where the best practice units are in the cases where none of the observed units is efficient. Failure in explicitly determining the efficient frontier of the system spanned from frontier (efficient) observations is a deviation from standard DEA and needs further consideration. This is the issue that we will discuss in the next section for simple two-stage processes and the motivation to propose an alternative non-conventional approach to address it. The underlying idea on which we build our approach is:

> *Whatever is produced within the system it is used by the system along with external inputs to produce the final outputs. As long as the intermediate measures play a dual (output/input) role in the operation of the system, the system can be viewed in two different ways, depending on the role of the intermediates:*
> *Perspective I (The system as producer of the intermediates)- the system uses the external inputs X to produce the intermediates Z and the final outputs Y.*
> *Perspective II (The system as consumer of the intermediates)- the system uses the inputs X along with the intermediates Z to produce the final outputs Y.*
> *The efficiency of the system is determined either by its less efficient perspective or as the squared geometric mean of the perspectives' efficiencies (conservative approach).*

Within the above conceptual framework, the proposed models acquire the following features: The fundamental property, that the overall system is efficient if and only if both perspectives are efficient, is met. The efficient frontier of the system is explicitly defined. The inefficient DMUs are projected on the efficient frontier of the system. The models are equivalently expressed in both the multiplier and the envelopment forms due to strict primal-dual correspondence. The models, originally presented for CRS technologies, can be straightforwardly adjusted to deal with VRS technologies.

The rest of the paper is organized as follows. In section two, we discuss the basic characteristics and the conventions made in the prevalent NDEA, on the basis of simple two-stage processes. Latent problems deriving from the conventions are discussed. In section three, we introduce an alternative non-conventional approach to measure the efficiency of DMUs in the presence of intermediate measures. The existence of overall efficient DMUs as well as the projection of the inefficient DMUs on the efficiency frontier are proved in this section. In section four we apply our method to evaluate the financial performance of twenty-two worldwide automotive manufacturers for the year 2019. We compare our efficiency results with those obtained by other prominent methods in the field. Concluding remarks are given in section five.

## 2. The conventions in the prevalent NDEA approaches

As mentioned in the previous section, there are applications of NDEA where the assessment results show that none of the evaluated units is efficient at the system level. Given the conventions made in NDEA, such an outcome is not unexpected. These conventions and their effect on the efficiency assessments are discussed in this section on the basis of the typical two-stage network structure depicted in Fig.1.

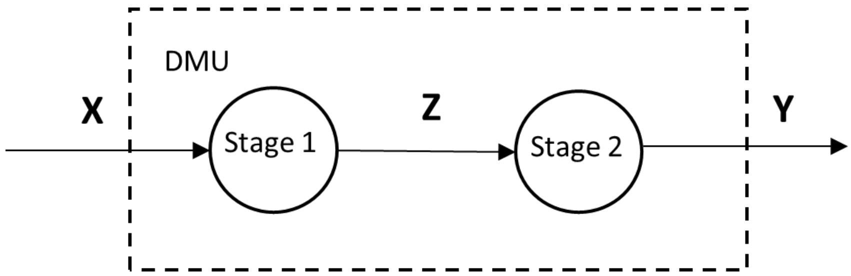


**Fig. 1**: A DMU as a two-stage process

*Notation*

$j \in J = \{1, \dots, n\}$: The index set of the *n* DMUs.
$j_0 \in J$: Denotes the evaluated DMU.
$X_j = (x_{ij}, i = 1, \dots, m)$: The vector of external inputs used by DMU$_j$.
$Z_j = (z_{pj}, p = 1, \dots, q)$: The vector of intermediate measures for DMU$_j$.
$Y_j = (y_{rj}, r = 1, \dots, s)$: The vector of final outputs produced by DMU$_j$.
$\eta = (\eta_1, \dots, \eta_m)$: The vector of weights for the external inputs.
$\varphi^1 = (\varphi_1^1, \dots, \varphi_q^1)$: The vector of weights for the intermediate measures considered as outputs of the first stage.
$\varphi^2 = (\varphi_1^2, \dots, \varphi_q^2)$: The vector of weights for the intermediate measures considered as inputs to the second stage.
$\omega = (\omega_1, \dots, \omega_s)$: The vector of weights for the final outputs.
$e_j^o$: The overall efficiency of DMU$_j$.
$e_j^1$: The efficiency of the first stage of DMU$_j$.
$e_j^2$: The efficiency of the second stage of DMU$_j$.
$E_j^1$: The independent efficiency score of the first stage of DMU$_j$.
$E_j^2$: The independent efficiency score of the first stage of DMU$_j$.

*2.1 The definition of the divisional efficiencies*

As a vehicle for the discussion, let us consider the relational model below, which maximizes the overall efficiency of the evaluated DMU subject to the constraints that the overall efficiency and the efficiencies of the two stages do not exceed one unit [18]. Model (1) is the multiplicative decomposition model, which is also known as the centralized cooperative model [21].

$$e_{j_0}^{o} = \max \frac{\omega Y_{j_0}}{\eta X_{j_0}}$$

$$\begin{aligned} &\omega Y_j - \eta X_j \le 0, \quad j = 1, \dots, n \\ &\varphi^1 Z_j - \eta X_j \le 0, \quad j = 1, \dots, n \\ &\omega Y_j - \varphi^2 Z_j \le 0, \quad j = 1, \dots, n \\ &\varphi^1 = \varphi^2 \\ &\eta \ge 0, \varphi^1 \ge 0, \varphi^2 \ge 0, \omega \ge 0 \end{aligned} \tag{1}$$

In model (1) it is assumed that the weights assigned to the intermediate measures Z are the same regardless of whether these measures are considered as outputs of the first stage or inputs to the second one, i.e., $\varphi^1 = \varphi^2$ . This is the dominant approach to form the link between the stages via the intermediate measures Z.

The definition of the divisional (stage) efficiencies is based on the convention that the divisions of the production process are distinct entities that are technically linked via the intermediate measures. Given the typical two-stage process that uses the external inputs X to produce the final outputs Y via the intermediate measures Z, the efficiencies of the first and the second stage of DMU $j \in J$ are respectively defined as

$$e_j^1 = \frac{\varphi^1 Z_j}{\eta X_j}, \quad e_j^2 = \frac{\omega Y_j}{\varphi^2 Z_j}$$

The DMU is efficient if and only if $e_j^1 = 1$ and $e_j^2 = 1$. However, for the stage efficiencies holds that $e_j^1 \le E_j^1$ and $e_j^2 \le E_j^2$, where $E_j^1$ and $E_j^2$ are the efficiency scores obtained by assessing the efficiencies of the stages independently to each other. Thus, a necessary condition for a DMU $j \in J$ to be overall efficient is ($E_j^1 = 1$ and $E_j^2 = 1$). The latter, however, holds only occasionally, as the stages are treated as autonomous entities assessed in different DEA problems (models).

*2.2 The link of the sub-processes*

There are three different approaches reported in the literature to form the link between the stages via the intermediate measures *Z*. The dominant one assumes that the weights of the intermediate measures are the same regardless of whether these measures are considered as outputs of the first stage or inputs to the second one, i.e., $\varphi^1 = \varphi^2$ . In the centralized model, this assumption leads to the definition of the DMU's overall efficiency as the squared geometric average of the stage efficiencies [18], [21]. Another approach to link the stages was introduced earlier in [25]. As in [24], the efficiencies of the individual stages are assessed independently to each other by means of standard DEA models. However, the overall efficiency of the evaluated DMU is determined by the efficiency of the first stage, with its intermediate measures adjusted to render the second stage efficient. If the second stage of the evaluated unit is originally (independently) efficient, the overall system efficiency is determined by the efficiency of the first stage. Therefore, for a unit to be efficient, it is necessary that it is independently efficient in both stages. However, as the validity of such a condition is occasional, there may be cases without efficient units at the system level. Recently, Chu and Zhu [9] introduced the concept of "total value flow equivalence" to link the two stages. Specifically, in their multiplier model, the total virtual intermediate measure created by the first stage is assumed equal to the corresponding one utilized in the second stage. In fact, their method is a multiplicative decomposition method. However, their proposed model differentiates from the centralized model [18], [21] as the constraints $\varphi^1 = \varphi^2$ are replaced by the single constraint $\varphi^1 Z_{j_o} = \varphi^2 Z_{j_o}$. As this constraint reflects the total flow equivalence only for the unit being evaluated, the total flow equivalence is lost for the other units. Therefore, the unit being evaluated is somehow disengaged from the other units. In terms of the envelopment model, the total flow equivalence assumption is interpreted as a "production scale matching", i.e., a proportional adjustment of the intermediate measures Z and the external inputs X of the evaluated unit by the same factor. However, given that constant returns-to-scale is assumed, changing proportionally the inputs and the outputs of a unit does not alter its efficiency score or the scores of the others. This implies that stage efficiencies are actually the independent efficiencies of the stages. Summarizing, the overall efficiency in [9] degenerates into the product of the independent efficiency scores of the two stages. Therefore, no linkage of the stages is in fact formed.

### *2.3 The incorporation of all the divisional PPSs in the same assessment model*

In NDEA, the DMUs are assumed structurally homogeneous, i.e., they have the same types of sub-processes with the same interconnections. So, the inter-DMU homogeneity holds. However, the sub-processes are assumed as distinct entities with distinct inputs and outputs. Each sub-process and the entire DMU as system have their own production possibility sets (PPSs). When assessing the overall system (DMU) efficiency, the PPS incorporates in the same model the PPSs of all the sub-processes. So, the assessment model exhibits an intra-DMU heterogeneity. In the independent approach, the overall system efficiency and the efficiencies of the sub-processes are assessed independently to each other. Although one can argue against this approach as it treats the system and its divisions as independent entities, it is the only one that employs standard DEA models to each division and to the entire system. In particular, the efficiency of each division is assessed against similar (homogeneous) entities, i.e., the corresponding divisions of the other DMUs. The same homogeneity property holds when the entire DMU is assessed against the other DMUs on the basis of comparable external inputs X and final outputs Y. Moving from the independent to the holistic approach, the intra-DMU homogeneity property is lost. Consider for example the centralized model (1). Given that the number of DMUs is $n$ and the number of the intermediate measures is $q$, model (1) has $3n+p$ constraints. Notice that the first set of constraints is redundant as it derives from the two sets of constraints related to the stages. However, we keep this set of constraints deliberately in the model for the sake of the discussion. The mathematical interpretation of the constraints is that, for all the DMUs, the efficiency at the system level as well as at the divisional levels should not exceed one. However, the typical interpretation of these constraints in a standard DEA configuration is that each one of them corresponds to a particular observed unit in the production possibility set. Therefore, there are $3n$ observations from which the PPS is built. These observations constitute three different instances of the *n* DMUs. The first *n* constraints are related to the performance of the DMUs $j = 1, \dots, n$ at the system level, with inputs $X_j$ and outputs $Y_j$. The next *n* constraints are associated with the performance of the DMUs' first division, with inputs $X_j$ and outputs $Z_j$. The last *n* constraints correspond to the performance of the second division of the *n* DMUs, with inputs $Z_j$ and outputs $Y_j$. Clearly, this is not a standard DEA setting because the observations that form the PPS are not homogeneous, i.e., they do not use the same inputs to yield the same outputs. Thus, the efficiency of the DMU at the system level is assessed in a heterogeneous intra-DMU environment. The inputs *X* are used in full capacity for both the system and the first division. The final outputs *Y* are considered in full capacity as outputs from both the system and the second division. However, as long as it has been decided to breakdown the system into divisions, *X* is used only by the first division to produce *Z*, and *Y* is exclusively produced by the second division using only *Z*. Notice that *Y* is not (cannot be) produced directly from *X*.

Model (1) implies that all $3n$ entities comprising the defining the PPS can use X and Z as inputs and can produce Z and Y as outputs, but selectively, they chose not to use or produce some of them. This is clarified by model (2), which is equivalent to model (1) in full dimensionality, and by the data structure in Table 1. Table 1 presents the rows of the coefficient matrix for the constraints of model (2), which correspond to a particular unit *j* with two external inputs $(X_1, X_2)$, two intermediate measures $(Z_1, Z_2)$ and two final outputs $(Y_1, Y_2)$.

$$e_{j_0}^{o} = \max \frac{\omega Y_{j_0} + \varphi^1 0}{\varphi^2 0 + \eta X_{j_0}}$$

$$\begin{aligned} &\omega Y_j + \varphi^1 0 - \varphi^2 0 - \eta X_j \le 0, \ \ j = 1, \dots, n \\ &0 Y_j + \varphi^1 Z_j - \varphi^2 0 - \eta X_j \le 0, \ \ j = 1, \dots, n \\ &\omega Y_j + \varphi^1 0 - \varphi^2 Z_j - \eta 0 \le 0, \ \ j = 1, \dots, n \\ &\varphi^1 = \varphi^2 \\ &\eta \ge 0, \varphi^1 \ge 0, \varphi^2 \ge 0, \omega \ge 0 \end{aligned} \tag{2}$$

**Table 1**: Coefficient matrix of the constraints of model (2) for unit *j*

| | Role: Inputs | | Role: Outputs | | Role: Inputs | | Role: Outputs | |
|---|---|---|---|---|---|---|---|---|
| | $X_1$ | $X_2$ | $Z_1$ | $Z_2$ | $Z_1$ | $Z_2$ | $Y_1$ | $Y_2$ |
| DMU*j* (System) | $-x_{j1}$ | $-x_{j2}$ | 0 | 0 | 0 | 0 | $y_{j1}$ | $y_{j2}$ |
| DMU*j* (Div.1) | $-x_{j1}$ | $-x_{j2}$ | $z_{j1}$ | $z_{j2}$ | 0 | 0 | 0 | 0 |
| DMU*j* (Div.2) | 0 | 0 | 0 | 0 | $-z_{j1}$ | $-z_{j2}$ | $y_{j1}$ | $y_{j2}$ |
| Variables | $\eta_1$ | $\eta_2$ | $\varphi_1^1$ | $\varphi_2^1$ | $\varphi_1^2$ | $\varphi_2^2$ | $\omega_1$ | $\omega_2$ |

What is described above, however, is rather an issue of inability of the entities to use (produce) all the inputs (outputs) than a case of missing data. Therefore, filling the gaps with zeros or any other values to homogenize heterogeneous entities is not a valid practice [10].

## 3. A non-conventional approach to measure the efficiency of two-stage processes

In this section, we introduce an alternative approach to assess the efficiency of two-stage processes. We consider the DMU as a system, which can be viewed from two different perspectives, i.e., as the two sides of the same coin. As the intermediate measures play a dual (output/input) role in the operation of the system, the system can be viewed in two different ways, depending on the role of the intermediates: Perspective I (The system as producer of the intermediates)- the system uses the external inputs X to produce the intermediates Z and the final outputs Y. Perspective II (The system as consumer of the intermediates)- the system uses the inputs X along with the intermediates Z to produce the final outputs Y. The efficiency of the system is determined either by its less efficient perspective or as the squared geometric mean of the perspectives' efficiencies (conservative approach).

### *3.1 Modelling the two perspectives of the system*

The models (3a) and (3b), in multiplier and envelopment form respectively, are standard DEA models for measuring the efficiency of the system's perspective I, according to which the external inputs X are used for the production of the intermediate measures Z and the final outputs Y.

| | |
|---|---|
| $\delta_{j_0}^{1(CRS)} = \max \frac{\omega Y_{j_0} + \varphi Z_{j_0}}{\eta X_{j_0}}$<br>$s.t$<br>$\omega Y_j + \varphi Z_j - \eta X_j \le 0, \;\; j = 1, \dots, n$<br>$\eta \ge 0, \varphi \ge 0, \omega \ge 0$ (3a) | $\min \delta_{j_0}^{1(CRS)}$<br>$s.t$<br>$-X\lambda + \delta_{j_0}^{1(CRS)} X_{j_0} \ge 0$<br>$Z\lambda \ge Z_{j_0}$<br>$Y\lambda \ge Y_{j_0}$<br>$\lambda \ge 0$ (3b) |
| $\delta_{j_0}^{2(CRS)} = \max \frac{\omega Y_{j_0}}{\eta X_{j_0} + \varphi Z_{j_0}}$<br>$s.t.$<br>$\omega Y_j - \eta X_j - \varphi Z_j \le 0, \;\; j = 1, \dots, n$<br>$\eta \ge 0, \varphi \ge 0, \omega \ge 0$ (4a) | $\min \delta_{j_0}^{2(CRS)}$<br>$s.t$<br>$-X\mu + \delta_{j_0}^{2(CRS)} X_{j_0} \ge 0$<br>$-Z\mu + \delta_{j_0}^{2(CRS)} Z_{j_0} \ge 0$<br>$Y\mu \ge Y_{j_0}$<br>$\mu \ge 0$ (4b) |

Correspondingly, the pair of models (4a) and (4b) are the standard DEA models that measure the efficiency of the system's perspective II. The internal measures Z are now used by the system as inputs, along with the external inputs X, to produce the final outputs Y. We define the overall efficiency of the DMU $j_0$ as $\delta_{j_0}^{o(CRS)} = \min\{\delta_{j_0}^{1(CRS)}, \delta_{j_0}^{2(CRS)}\}$. Alternatively, the overall efficiency of the DMU can be defined as $\delta_{j_0}^{o(CRS)} = \delta_{j_0}^{1(CRS)} \times \delta_{j_0}^{2(CRS)}$. The latter definition is more conservative than the former.

**Property 1**: A DMU $j_0$ is efficient if and only if $\delta_{j_0}^{1(CRS)} = 1$ and $\delta_{j_0}^{2(CRS)} = 1$.
The property is a direct implication of the definition of the overall efficiency of the DMU and the fact that the efficiencies $\delta_{j_0}^{1(CRS)} and\ \delta_{j_0}^{2(CRS)}$ are bounded within (0,1].

**Theorem 1**: There is at least one DMU efficient in both models (3a-3b) and (4a-4b) and, thus, efficient at the system level.
**Proof:** Refer to the proofs of theorem 1 and the corollary 1 in [6], pp. 477-478. Indeed, as X and Y are common inputs and outputs respectively in models (4a) and (5a), the DMU $k$ for which: $eY_k/eX_k = \max_j\{eY_j/eX_j\}$, is located on the CRS frontier of both models and, thus, it is overall efficient at the system level. Here, $e$ denotes the vector of appropriate dimensions with all its components set to 1. ■

The overall efficient units span the efficient frontier of the entire system (DMU). Apparently, our approach can be applied to a VRS technology by adjusting accordingly the DEA models (3a-3b) and (4a-4b)[1]. Obviously, Theorem 1 is valid under VRS assumption. Moreover, as in standard DEA, the VRS efficiency scores are not lower than their CRS counterparts, a property that does not hold generally in NDEA.

*3.2 Projecting the inefficient units on the efficient frontier of the system.*

Let $K \subset J$ be the set of the overall efficient DMUs. The projection of a unit in $J \backslash K$ is obtained iteratively by the following procedure:

**Procedure 1**

For a unit $j_o \in J \backslash K$:

1. Set t=0; $(X\lambda^{t*}, Z\lambda^{t*}, Y\lambda^{t*}) = (X\mu^{t*}, Z\mu^{t*}, Y\mu^{t*}) = (X_{j_o}, Z_{j_o}, Y_{j_o})$
2. Set $t = t + 1$. Solve model (3b)[2] and get the optimal solution $(\delta_{j_0}^{1t*}, \lambda^{t*})$ and the perspective's I projection $(X\lambda^{t*}, Z\lambda^{t*}, Y\lambda^{t*})$.
3. Solve model (4b) by replacing $(X\mu^{t-1*}, Z\mu^{t-1*}, Y\mu^{t-1*})$ with $(X\lambda^{t*}, Z\lambda^{t*}, Y\lambda^{t*})$ and get the optimal solution $(\delta_{j_0}^{2t*}, \mu^{t*})$. If $\delta_{j_0}^{2t*} = 1$, then store the final projection $(X\lambda^{t*}, Z\lambda^{t*}, Y\lambda^{t*})$ for the current unit $j_o$ and go to step 1 with the next inefficient unit $j_o \in J \backslash K$ . Otherwise, get the perspective's II modified projection $(X\mu^{t*}, Z\mu^{t*}, Y\mu^{t*})$ and go to step 2 by replacing in (3b) $(X\lambda^{t-1*}, Z\lambda^{t-1*}, Y\lambda^{t-1*})$ with $(X\mu^{t*}, Z\mu^{t*}, Y\mu^{t*})$.

[1] As the VRS variants of models (3a-3b) and (4a-4b) derive straightforwardly, they are omitted for economy of space.
[2] The iterative procedure can start from any of the perspectives (3b) or (4b). The projections may differ.

Let $X_{jt}^{1*}$ be the projected inputs obtained from perspective's I model (3b) in iteration t and $X_{jt}^{2*}$ the projected inputs obtained from perspective's II model (4b) in iteration *t*. Also, let $\delta_{jt}^{1*} \leq 1$ and $\delta_{jt}^{2*} \leq 1$ be respectively the optimal values of the objective functions of models (3b) and (4b) in iteration *t*. Let, also, $Y_{jt}^{1*}$ be the projected outputs obtained from perspective's I model (3b) in iteration *t* and $Y_{jt}^{2*}$ the projected outputs obtained from perspective's II model (4b) in iteration *t*. Then, the following holds:

**Theorem 2:** Procedure 1 generates a non-increasing sequence of input vectors $X_j \geq X_{j1}^{1*} \geq X_{j1}^{2*} \geq X_{j2}^{1*} \geq X_{j2}^{2*} \geq \cdots \geq X_{jt}^{1*} \geq X_{jt}^{2*}$ and a non-decreasing sequence of output vectors $Y_j \leq Y_{j1}^{1*} \leq Y_{j1}^{2*} \leq Y_{j2}^{1*} \leq Y_{j2}^{2*} \leq \cdots \leq Y_{jt}^{1*} \leq Y_{jt}^{2*}$.

**Proof:** Consider a unit *j* with a profile $(X_j, Z_j, Y_j)$.

*The case of the input vector* $X_j$: From model (3b) in iteration *t* we have $X_{jt}^{1*} = X\lambda^{t*} \leq \delta_{jt}^{1*} \times X_{jt-1}^{2*} \leq X_{jt-1}^{2*}$. For *t*=1, $X_{jt-1}^{2*} = x_j$. From model (4b) in iteration *t* we have $X_{jt}^{2*} = X\mu^{t*} \leq \delta_{jt}^{2*} \times X_{jt}^{1*} \leq X_{jt}^{1*}$. So, passing the projected $(X_j^*, Z_j^*, Y_j^*)$ from one model to the other iteratively, we get a non-increasing sequence of input vectors $X_j \geq X_{j1}^{1*} \geq X_{j1}^{2*} \geq X_{j2}^{1*} \geq X_{j2}^{2*} \geq \cdots \geq X_{jt}^{1*} \geq X_{jt}^{2*}$.

*The case of the output vector* $Y_j$: From model (3b) in iteration *t* we have $Y_{jt}^{1*} = Y\lambda^{t*} \geq Y_{jt-1}^{2*}$. For *t*=1, $Y_{jt-1}^{2*} = Y_j$. From model (4b) in iteration *t* we have $Y_{jt}^{2*} = Y\mu^{t*} \geq Y_{jt}^{1*}$. So, passing the projected $(X_j^*, Z_j^*, Y_j^*)$ from one model to the other iteratively, we get a non-decreasing sequence of output vectors $Y_j \leq Y_{j1}^{1*} \leq Y_{j1}^{2*} \leq Y_{j2}^{1*} \leq Y_{j2}^{2*} \leq \cdots \leq Y_{jt}^{1*} \leq Y_{jt}^{2*}$. ■

Theorem 2 shows that, applying Procedure 1, the unit *j* improves gradually its performance relatively to its peers in terms of the inputs *X* and the output *Y*, which are common in both models (3b) and (4b).

**Theorem 3:** The sequences of $\delta_{jt}^{1*}, t = 1,2, \ldots$ and $\delta_{jt}^{2*}, t = 1,2, \ldots$ converge to 1.

**Proof**: We know that $\delta_{jt}^{1*} \leq 1 \forall t$ and from Theorem 2 that $X_j \geq X_{j1}^{1*} \geq X_{j2}^{1*} \geq \cdots \geq X_{jt}^{1*}$. If $\lim_{t\to\infty} \delta_{jt}^{1*} < 1$ then $\lim_{t\to\infty} X_{jt}^{1*} = 0$, which is not true. Thus $\lim_{t\to\infty} \delta_{jt}^{1*} = 1$. Similarly, we have $\delta_{jt}^{2*} \leq 1 \forall t$ and from Theorem 2 $X_j \geq X_{j1}^{2*} \geq X_{j2}^{2*} \geq \cdots \geq X_{jt}^{2*}$. If $\lim_{t\to\infty} \delta_{jt}^{2*} < 1$ then $\lim_{t\to\infty} X_{jt}^{2*} = 0$, which is not true. Thus $\lim_{t\to\infty} \delta_{jt}^{2*} = 1$. ■

Unlike *X* and *Y*, which are common inputs and outputs respectively in both perspectives, the intermediates *Z* have opposite roles in the two perspectives. They are outputs in model (3b) and inputs in model (4b). Let $Z_{jt}^{1*}$ be the projected intermediates, as obtained from perspective's

I model (3b) in iteration t, and $Z_{jt}^{2*}$ the projected intermediates, as obtained from perspective's II model (4b) in iteration $t$. From model (3b) in iteration $t$ we have $Z_{jt}^{1*} = Z\lambda^{t*} \geq Z_{jt-1}^{2*}$. For $t$=1, $Z_{jt-1}^{2*} = Z_j$. From model (4b) in iteration $t$ we have $Z_{jt}^{2*} = Z\mu^{t*} \leq \delta_{jt}^{2*} \times Z_{jt}^{1*} \leq Z_{jt}^{1*}$. So, the values of intermediates may oscillate between two values. Theorems 2 and 3 guarantee that Procedure 1 will end up with a final projection on the system frontier, with the intermediates being stabilized at optimal levels after oscillating between two values.

## 4. Illustration

As an illustration, we apply our approach to assess the financial performance of twenty-two automotive manufacturers for the fiscal year 2019. Each manufacturer uses the external inputs $X_1$ (Number of employees) and $X_2$ (Total Assets) to position itself in the market by means of final outputs $Y_1$ (Market Capitalization) and $Y_2$ (Earnings Per Share). This transformation of external inputs to final outputs is carried out via the intermediate measures $Z_1$ (Revenues) and $Z_2$ (Earnings Before Interest and Taxes) [3].

Table 2 presents the data for the automotive manufacturing companies. The monetary values that were originally expressed in local currencies are converted to dollars using the annual average exchange rate for 2019. Table 3 exhibits the results obtained by applying models (3a-3b), (4a-4b) and their VRS variants on the data of Table 2. In particular, columns (2-4) present the perspective I, the perspective II and the overall system's efficiency scores under the CRS assumption. The corresponding VRS scores are given in columns (5)-(7). As shown in Table 3, our approach, under CRS assumption, evaluates three DMUs as overall efficient, i.e., the companies 9, 10, and 12. Under the VRS assumption, there are eight overall efficient DMUs, namely the companies 1, 4, 9, 10, 12, 17, 19 and 21. As expected, the VRS scores are not lower than their CRS counterparts and the number of efficient units has been increased.

Table 4 summarizes the stage and the overall efficiency scores obtained by applying three conventional NDEA approaches, namely the relational model (1) (columns 2-4), the Sexton & Lewis [25] method[4] (columns 5-7), and the Chu & Zhu [9] method (columns 8-10). Notice that the efficiency scores of the two stages obtained from the two latter methods are identical. This is expected, as both methods provide the independent efficiency scores for the two stages. Their overall efficiency scores are also identical, but this does not hold in general. The fact that none of the DMUs has both stages independently efficient, indicates that none of them has the potential to be overall efficient, regardless of which of the above three methods is employed. The absence of overall efficient DMUs in Table 4 confirms the above.

[3] Sources: for $X_1$ and $X_2$ the annual reports of the companies for the year 2019,
for $Z_1$ , $Z_2$ , $Y_1$ and $Y_2$ the statista.com platform (statista.com/study/70700/automotive-financial-kpi-benchmark; statista.com/study/9644/automotive-industry-statista-dossier).
We have excluded from our study the manufacturers that exhibit negative values or missing data.

[4] For comparison purposes we have used the input-oriented CRS variant of Sexton & Lewis (2003).

**Table 2**: Data of 22 automotive manufacturing companies

| | DMU | External inputs | | Intermediate measures | | Final outputs | |
|---|---|---|---|---|---|---|---|
| | | Employees - ($X_1$) | Total Assets - ($X_2$) million dollars | Revenues - ($Z_1$) million dollars | Earnings Before Interest and Taxes (EBIT) - ($Z_2$) million dollars | Market capitalization - ($Y_1$) million dollars | Earnings Per Share (EPS) - ($Y_2$) dollars |
| 1 | Toyota Motor Corp. | 370870 | 476450.802 | 277268.072 | 22476.204 | 168556.436 | 5.968 |
| 2 | Volkswagen AG | 671205 | 579729.003 | 282902.576 | 20557.671 | 97255.767 | 29.810 |
| 3 | Daimler AG (currently Mercedes-Benz Group) | 298655 | 338676.372 | 193443.449 | 6414.334 | 59146.585 | 2.486 |
| 4 | Ford Motor Co. | 190000 | 258537.000 | 155900.000 | 549.000 | 37860.300 | 0.010 |
| 5 | Fiat Chrysler Automobiles | 191752 | 110054.390 | 122963.210 | 6609.160 | 23215.400 | 2.040 |
| 6 | Bayerische Motoren Werke AG (BMW) | 133778 | 270619.261 | 116696.529 | 8098.544 | 53963.270 | 8.365 |
| 7 | Hyundai Motor | 122217 | 166863.447 | 90715.188 | 3102.198 | 27071.014 | 9.717 |
| 8 | Peugeot S.A. | 208780 | 78125.420 | 83685.330 | 7081.747 | 21344.905 | 4.009 |
| 9 | AUDI AG | 90640 | 77581.187 | 62351.624 | 4362.822 | 38521.837 | 100.258 |
| 10 | Suzuki Motor Corp. | 67721 | 31208.443 | 35515.705 | 2975.745 | 20728.846 | 3.626 |
| 11 | Mazda Motor Corporation | 49998 | 26398.182 | 32696.426 | 755.055 | 7155.053 | 0.920 |
| 12 | Subaru Corporation | 34200 | 29177.648 | 28953.380 | 1747.165 | 17743.193 | 1.692 |
| 13 | Mitsubishi Motors Corporation | 31314 | 18441.848 | 23067.977 | 1025.750 | 8028.108 | 0.819 |
| 14 | Isuzu Motors Ltd | 37263 | 19548.052 | 19715.691 | 1621.734 | 9837.260 | 1.378 |
| 15 | Nissan Shatai Co. Ltd. | 4032 | 2475.901 | 5530.622 | 71.362 | 1191.656 | 0.378 |
| 16 | Sanyang Motor Co. Ltd. | 2301 | 1330.902 | 1080.439 | 14.594 | 565.865 | 0.088 |
| 17 | IFAD Autos Limited | 1049 | 388.743 | 131.160 | 19.534 | 237.657 | 0.055 |
| 18 | GENERAL MOTORS | 164000 | 228037.000 | 137237.000 | 8393.000 | 52050.000 | 4.820 |
| 19 | VOLVO | 103985 | 55408.777 | 28985.619 | 1512.425 | 35423.496 | 1.865 |
| 20 | KIA | 52578 | 47477.859 | 49880.852 | 1724.290 | 15236.597 | 0.004 |
| 21 | SAIC | 216360 | 122956.400 | 122044.048 | 1658.500 | 40338.700 | 0.317 |
| 22 | Honda Motor | 219722 | 187317.646 | 145756.431 | 6,663.46 | 48344.066 | 3.174 |

**Table 3:** Results from models (3a-3b), (4a-4b) and their VRS variants

| DMU | $\delta_{j_0}^{1(CRS)}$ | $\delta_{j_0}^{2(CRS)}$ | $\delta_{j_0}^{o(CRS)}$ | $\delta_{j_0}^{1(VRS)}$ | $\delta_{j_0}^{2(VRS)}$ | $\delta_{j_0}^{o(VRS)}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0.9318 | 0.9318 | 1 | 1 | 1 |
| 2 | 0.5926 | 0.3796 | 0.2250 | 1 | 0.5723 | 0.5723 |
| 3 | 0.5763 | 0.5289 | 0.3048 | 0.8780 | 0.7115 | 0.6246 |
| 4 | 0.6238 | 1 | 0.6238 | 1 | 1 | 1 |
| 5 | 0.8508 | 0.3310 | 0.2816 | 1 | 0.3312 | 0.3312 |
| 6 | 1 | 0.7816 | 0.7816 | 1 | 0.8644 | 0.8644 |
| 7 | 0.6634 | 0.5751 | 0.3815 | 0.8641 | 0.5752 | 0.4970 |
| 8 | 0.9507 | 0.4143 | 0.3939 | 1 | 0.4143 | 0.4143 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 0.6312 | 0.4275 | 0.2699 | 1 | 0.4300 | 0.4300 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 |
| 13 | 0.8714 | 0.6865 | 0.5982 | 1 | 0.6879 | 0.6879 |
| 14 | 0.9628 | 0.7766 | 0.7477 | 0.9699 | 0.7777 | 0.7543 |
| 15 | 1 | 0.8462 | 0.8462 | 1 | 0.9297 | 0.9297 |
| 16 | 0.6813 | 1 | 0.6813 | 0.7806 | 1 | 0.7806 |
| 17 | 0.9277 | 1 | 0.9277 | 1 | 1 | 1 |
| 18 | 0.9480 | 0.6154 | 0.5834 | 1 | 0.7109 | 0.7109 |
| 19 | 0.9870 | 1 | 0.9870 | 1 | 1 | 1 |
| 20 | 0.8752 | 0.6745 | 0.5903 | 1 | 0.6763 | 0.6763 |
| 21 | 0.5850 | 0.7159 | 0.4188 | 1 | 1 | 1 |
| 22 | 0.6878 | 0.5269 | 0.3624 | 0.9574 | 0.5763 | 0.5517 |

**Table 4:** CRS scores obtained from different NDEA methods

| | Kao & Hwang [18] -model (1) | | | Sexton & Lewis [23] | | | Chu & Zhu [9] | | |
|---|---|---|---|---|---|---|---|---|---|
| DMU | $e_j^1$ | $e_j^2$ | $e_j^o$ | $\theta_j^1$ | $\theta_j^2$ | $\theta_j^o$ | $\theta_j^1$ | $\theta_j^2$ | $\theta_j^o$ |
| 1 | 0.9962 | 0.3951 | 0.3936 | 1 | 0.4348 | 0.4348 | 1 | 0.4348 | 0.4348 |
| 2 | 0.5486 | 0.2682 | 0.1472 | 0.5913 | 0.2790 | 0.1650 | 0.5913 | 0.2790 | 0.1650 |
| 3 | 0.5763 | 0.3016 | 0.1738 | 0.5763 | 0.3480 | 0.2005 | 0.5763 | 0.3480 | 0.2005 |
| 4 | 0.4861 | 0.3526 | 0.1714 | 0.5982 | 1 | 0.5982 | 0.5982 | 1 | 0.5982 |
| 5 | 0.8508 | 0.1548 | 0.1317 | 0.8508 | 0.1555 | 0.1323 | 0.8508 | 0.1555 | 0.1323 |
| 6 | 1 | 0.3530 | 0.3530 | 1 | 0.3579 | 0.3579 | 1 | 0.3579 | 0.3579 |
| 7 | 0.6541 | 0.3364 | 0.2200 | 0.6579 | 0.3852 | 0.2534 | 0.6579 | 0.3852 | 0.2534 |
| 8 | 0.9467 | 0.1732 | 0.1640 | 0.9507 | 0.1844 | 0.1753 | 0.9507 | 0.1844 | 0.1753 |
| 9 | 0.9338 | 1 | 0.9338 | 0.9338 | 1 | 0.9338 | 0.9338 | 1 | 0.9338 |
| 10 | 1 | 0.3962 | 0.3962 | 1 | 0.4214 | 0.4214 | 1 | 0.4214 | 0.4214 |
| 11 | 0.6048 | 0.2719 | 0.1644 | 0.6312 | 0.3172 | 0.2002 | 0.6312 | 0.3172 | 0.2002 |
| 12 | 1 | 0.4774 | 0.4774 | 1 | 0.4866 | 0.4866 | 1 | 0.4866 | 0.4866 |
| 13 | 0.8714 | 0.3189 | 0.2779 | 0.8714 | 0.3287 | 0.2864 | 0.8714 | 0.3287 | 0.2864 |
| 14 | 0.9488 | 0.3224 | 0.3059 | 0.9628 | 0.3576 | 0.3443 | 0.9628 | 0.3576 | 0.3443 |
| 15 | 1 | 0.3498 | 0.3498 | 1 | 0.5069 | 0.5069 | 1 | 0.5069 | 0.5069 |
| 16 | 0.3551 | 0.7773 | 0.2760 | 0.3664 | 1 | 0.3664 | 0.3664 | 1 | 0.3664 |
| 17 | 0.4301 | 0.8706 | 0.3745 | 0.5270 | 1 | 0.5270 | 0.5270 | 1 | 0.5270 |
| 18 | 0.9480 | 0.2930 | 0.2778 | 0.9480 | 0.2993 | 0.2837 | 0.9480 | 0.2993 | 0.2837 |
| 19 | 0.3748 | 1 | 0.3748 | 0.3748 | 1 | 0.3748 | 0.3748 | 1 | 0.3748 |
| 20 | 0.8752 | 0.2968 | 0.2598 | 0.8752 | 0.3379 | 0.2957 | 0.8752 | 0.3379 | 0.2957 |
| 21 | 0.4304 | 0.4643 | 0.1998 | 0.4485 | 0.6293 | 0.2822 | 0.4485 | 0.6293 | 0.2822 |
| 22 | 0.6878 | 0.2900 | 0.1995 | 0.6878 | 0.3009 | 0.2070 | 0.6878 | 0.3009 | 0.2070 |

Table 5 provides the CRS projections of the units on the frontier of the system, derived from the Procedure 1. The last column shows the number of linear programs solved to get the final projections. Table 6 exhibits the corresponding projections derived under the VRS assumption. In both cases, the projections derived by starting Procedure 1 from perspective I. In the Appendix, we provide the CRS and VRS projection when perspective II is the starting point in the procedure. Fig. 2, indicatively exhibits for DMU 1 the evolution of the inputs, the intermediates and the outputs until the converge to the final projections.

**Table 5:** CRS Projections starting from perspective I

| DMU | X1 | X2 | Z1 | Z2 | Y1 | Y2 | # LPs |
|---|---|---|---|---|---|---|---|
| 1 | 345023.16 | 275094.00 | 276276.25 | 17275.26 | 168556.44 | 17.27 | 4 |
| 2 | 359121.95 | 280247.76 | 280773.75 | 17836.85 | 171081.63 | 29.81 | 4 |
| 3 | 139537.33 | 102359.74 | 103498.21 | 6807.21 | 62794.36 | 13.19 | 4 |
| 4 | 101341.23 | 60193.48 | 63424.58 | 4683.83 | 37860.30 | 10.55 | 4 |
| 5 | 155693.93 | 89359.17 | 95856.69 | 7157.35 | 57053.21 | 9.83 | 4 |
| 6 | 104560.42 | 89213.87 | 88040.43 | 5332.73 | 53963.27 | 8.36 | 3 |
| 7 | 62926.37 | 46808.64 | 46630.73 | 3070.88 | 28328.67 | 9.72 | 4 |
| 8 | 161164.01 | 74270.58 | 84521.10 | 7081.75 | 49330.99 | 8.63 | 2 |
| 9 | 90640.00 | 77581.19 | 62351.62 | 4362.82 | 38521.84 | 100.26 | 2 |
| 10 | 67721.00 | 31208.44 | 35515.70 | 2975.74 | 20728.85 | 3.63 | 2 |
| 11 | 24396.55 | 13840.40 | 14757.85 | 1116.16 | 8775.51 | 2.39 | 4 |
| 12 | 34200.00 | 29177.65 | 28953.38 | 1747.17 | 17743.19 | 1.69 | 2 |
| 13 | 25358.57 | 14934.50 | 15899.16 | 1172.20 | 9483.47 | 1.77 | 4 |
| 14 | 33997.56 | 17835.01 | 19604.06 | 1533.27 | 11578.35 | 1.83 | 4 |
| 15 | 3347.34 | 1884.35 | 2005.31 | 152.74 | 1191.66 | 0.38 | 4 |
| 16 | 1532.01 | 886.12 | 950.12 | 70.58 | 565.87 | 0.09 | 4 |
| 17 | 771.37 | 360.62 | 406.66 | 33.95 | 237.66 | 0.06 | 2 |
| 18 | 136564.44 | 116515.42 | 115283.35 | 6970.47 | 70655.04 | 8.96 | 3 |
| 19 | 102635.35 | 54689.61 | 59895.67 | 4644.55 | 35423.50 | 5.42 | 2 |
| 20 | 41106.49 | 30210.48 | 30630.13 | 2008.13 | 18584.99 | 3.26 | 4 |
| 21 | 111545.43 | 63390.76 | 67829.06 | 5110.19 | 40338.70 | 9.21 | 4 |
| 22 | 144675.21 | 114632.05 | 114981.86 | 7225.62 | 70125.79 | 9.05 | 4 |

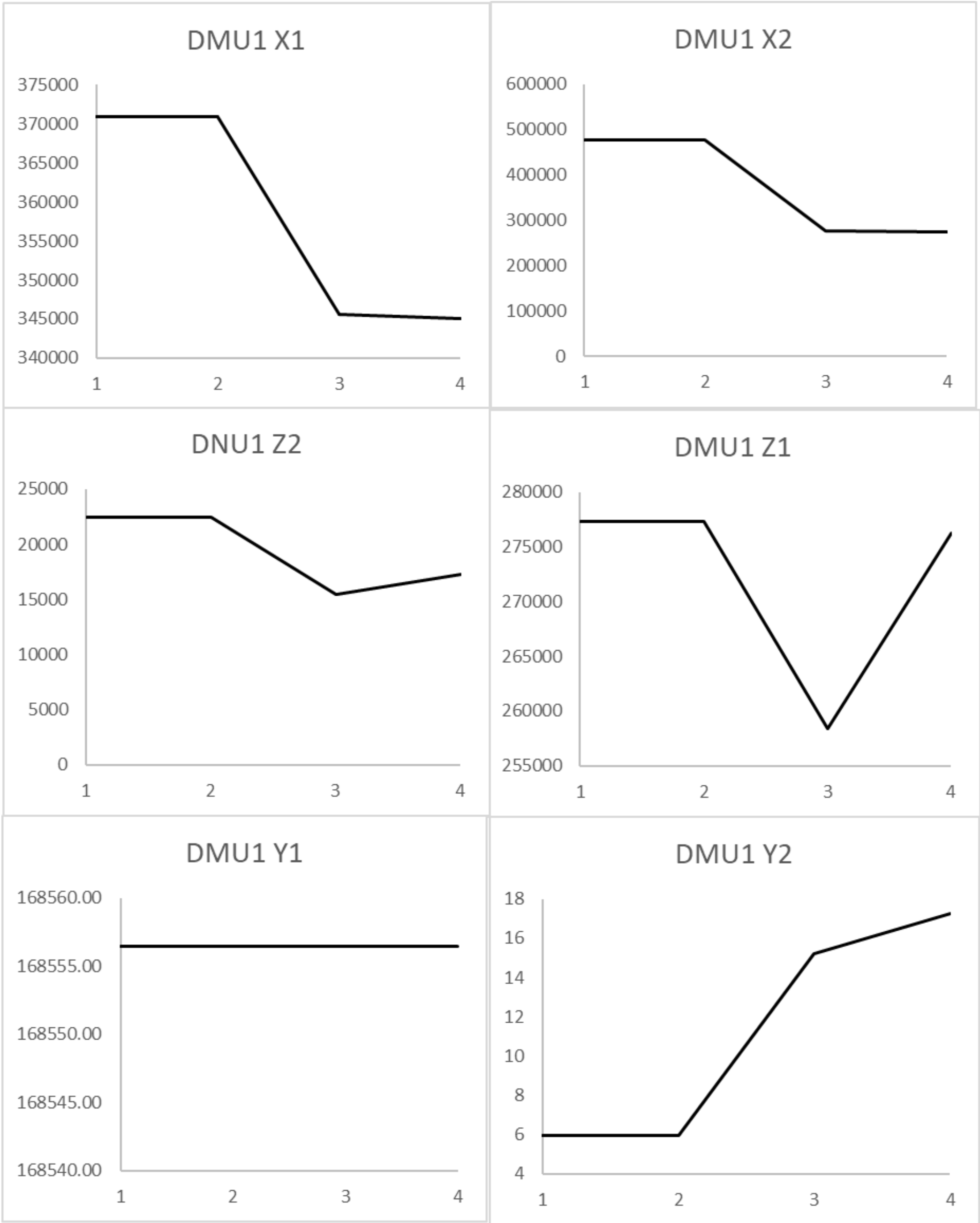


**Fig. 2:** The evolution of inputs, intermediates, and outputs untill the CRS projection of DMU1 at the system frontier.

**Table A6:** VRS Projections starting from perspective I

| DMU | X1 | X2 | Z1 | Z2 | Y1 | Y2 | # LPs |
|---|---|---|---|---|---|---|---|
| 1 | 370870.00 | 476450.80 | 277268.07 | 22476.20 | 168556.44 | 5.97 | 2 |
| 2 | 218059.94 | 257207.11 | 158231.66 | 12442.07 | 97255.77 | 53.42 | 5 |
| 3 | 191568.46 | 219454.18 | 137865.98 | 10725.59 | 84952.62 | 62.23 | 4 |
| 4 | 190000.00 | 258537.00 | 155900.00 | 549.00 | 37860.30 | 0.01 | 2 |
| 5 | 63501.85 | 36446.33 | 31286.93 | 2123.93 | 23215.40 | 2.42 | 3 |
| 6 | 115635.79 | 135835.19 | 88543.00 | 6711.71 | 53963.27 | 8.37 | 3 |
| 7 | 77001.51 | 75730.63 | 57329.79 | 4049.13 | 35169.27 | 40.19 | 4 |
| 8 | 68792.31 | 32345.34 | 35309.07 | 2909.82 | 21344.91 | 4.01 | 4 |
| 9 | 90640.00 | 77581.19 | 62351.62 | 4362.82 | 38521.84 | 100.26 | 2 |
| 10 | 67721.00 | 31208.44 | 35515.70 | 2975.74 | 20728.85 | 3.63 | 2 |
| 11 | 20714.97 | 11086.38 | 11908.37 | 922.27 | 7155.05 | 1.09 | 4 |
| 12 | 34200.00 | 29177.65 | 28953.38 | 1747.17 | 17743.19 | 1.69 | 2 |
| 13 | 21427.52 | 12619.37 | 13290.24 | 976.91 | 8028.11 | 1.12 | 4 |
| 14 | 34276.67 | 17981.43 | 19622.15 | 1535.46 | 11660.01 | 1.85 | 4 |
| 15 | 3190.05 | 1958.89 | 1717.57 | 124.88 | 1191.66 | 0.38 | 6 |
| 16 | 1766.52 | 921.81 | 676.85 | 55.14 | 565.87 | 0.11 | 4 |
| 17 | 1049.00 | 388.74 | 131.16 | 19.53 | 237.66 | 0.06 | 2 |
| 18 | 115541.37 | 124669.04 | 85054.70 | 6348.89 | 52050.00 | 49.01 | 4 |
| 19 | 103985.00 | 55408.78 | 28985.62 | 1512.43 | 35423.50 | 1.87 | 2 |
| 20 | 35188.20 | 24462.84 | 25165.23 | 1691.99 | 15236.60 | 1.80 | 4 |
| 21 | 216360.00 | 122956.40 | 122044.05 | 1658.50 | 40338.70 | 0.32 | 2 |
| 22 | 120150.63 | 102430.95 | 66815.50 | 4723.09 | 48344.07 | 51.20 | 3 |

## 5. Conclusion

We introduced in this paper an alternative approach to assess the efficiency of two-stage processes. We were motivated by the fact that in many real-world applications, none of the observed DMUs was rendered overall efficient, regardless of the NDEA method used. We attribute this phenomenon on the conventions made in the prevalent methodological approach, i.e., the definition of the divisional and the overall efficiencies, the ways used to link the divisions via the intermediate measures and the incorporation of all the divisional PPSs in the same assessment model. The proposed approach obviates from the conventional NDEA methodology, as it is based on an alternative conceptual framework, according to which the two-stage process (DMU) is considered as a system that can be viewed in two ways (perspectives), depending on the role of the intermediates measures: the DMU as producer and as consumer of the intermediates. As the assessments are performed with standard DEA, the proposed approach acquires the following desirable characteristics. The fundamental property, that the overall system is efficient if and only if both perspectives are efficient, is met. The efficient frontier of the system is explicitly defined and the inefficient DMUs are projected on this frontier. The models are equivalently expressed in both the multiplier and the envelopment forms due to strict primal-dual correspondence. The models can be straightforwardly adjusted to deal with VRS technologies. We prove that, among the observed DMUs, there is always at least one overall efficient, i.e., there are DMUs, which are efficient from whichever side one sees them. Moreover, we provide a convergent procedure to project the inefficient units on the frontier of the system. Technically, in our models (in multipliers form) the number of constraints is half of the number of constraints in the typical NDEA models. This is an advantage and a disadvantage. It is an advantage from a computational perspective, however, the highest degrees of freedom in our linear programs affect the discriminating power of our approach. Although our approach is straightforwardly extendable to more general two-stage structures, the problem of discrimination becomes more intensive if it is to deal with production processes with additional final outputs from the first stage and/or additional external inputs to the second stage. To maintain the discrimination capabilities of the models, the well-known rule-of-thumb relating the number of DMUs with the number of inputs and outputs must be considered in applications, as exactly happens in conventional DEA. The extension of our approach to more complex network structures is a subject of future research.

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**References**

[1] Ang, A., & Chen C.M. (2016). Pitfalls of decomposition weights in the additive multi-stage DEA model. Omega 58, 139-153.
[2] Banker, R.D., Charnes, & A., Cooper W.W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management Science 30: 1078-1092.
[3] Castelli, L., Pesenti, R., & Ukovich, W. (2010). A classification of DEA models when the internal structure of the Decision Making Units is considered. Annals of Operations Research 173: 207-235.
[4] Charnes, A., & Cooper, W.W. (1962). Programming with linear fractional functional. Naval Research Logistics 9:181-185.
[5] Charnes, A., Cooper, & W.W., Rhodes, E. (1978). Measuring the efficiency of decision-making units. European Journal of Operational Research 2: 429-444.
[6] Chen, Y., & Ali, AI. (2002). Output–input ratio analysis and DEA frontier. European Journal of Operational Research 142: 476–479
[7] Chen, Y., Cook, W.D., Kao, C., & Zhu, J. (2013). Network DEA pitfalls: Divisional efficiency and frontier projection under general network structures. European Journal of Operational Research, 226 (1), 507–515.
[8] Chen, Y., Cook, W.D., Li, N., & Zhu, J. (2009). Additive efficiency decomposition in two-stage DEA. European Journal of Operational Research: 1170-1176.
[9] Chu, J., & Zhu, J. (2021). Production scale-based two-stage network data envelopment analysis. European Journal of Operational Research, 294 (1), 283-294.
[10] Cook, W.D., Harrison, J., Imanirad, R., Rouse, P., & Zhu, J. (2013). Data Envelopments Analysis with Nonhomogeneous DMUs. Operations Research 3: 666-676.
[11] Cook, W.D., Liang, L., & Zhu, J. (2010). Measuring performance of two-stage network structures by DEA: A review and future perspective, Omega 38, 423-430.
[12] Despotis, D.K., Koronakos, G., & Sotiros, D. (2016a). Composition versus decomposition in two-stage network DEA: A reverse approach. Journal of Productivity Analysis, 45(1), 71-87.
[13] Despotis, D.K., Sotiros, D., & Koronakos, G. (2016b). A network DEA approach for series multi-stage processes. Omega, 61, 35-48.
[14] Despotis, D.K., Koronakos, G., & Sotiros, D. (2016c). The “weak-link” approach to network DEA for two-stage processes. European Journal of Operational Research, 254(2), 481-492.
[15] Färe R, & Grosskopf S. (1996). Productivity and intermediate products: a frontier approach. Economic Letters 50: 65–70.

[16] Guo, C., Shureshjani, R.A., Foroughi, A.A., & Zhu, J. (2017). Decomposition weights and overall efficiency in two-stage additive network DEA. European Journal of Operational Research, 257(3), 896-906.
[17] Kao, C. (2014). Network data envelopment analysis: A review. European Journal of Operational Research 239: 1–16.
[18] Kao C, & Hwang S.-N. (2008). Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. European Journal of Operational Research 185: 418–429.
[19] Li, Y. (2017). A fair evaluation of certain stage in a two-stage structure: revisiting the typical two-stage DEA approaches. Omega 68, 155-167.
[20] Li, Y., Chen, Y., Liang, L., & Xie, J. (2012). DEA models for extended two-stage network structures. Omega, 40(5), 611-618.
[21] Liang, L., Cook, W.D., & Zhu, J. (2008). DEA models for two-stage processes: Game approach and efficiency decomposition. Naval Research Logistics, 55(7), 643-653.
[22] Lozano, S. (2011). Scale and cost efficiency analysis of networks of processes. Expert Systems with Applications 38: 6612-6617.
[23] Sahoo B. K., Saleh H., Shafiee M., Tone K., & Zhu J. (2021). An Alternative Approach to Dealing with the Composition Approach for Series Network Production Processes. Asia-Pacific Journal of Operational Research, 38(06), 2150004.
[24] Seiford L.M. & Zhu J. (1999). Profitability and marketability of the top 55 US commercial banks. Manage Sci 45(9):1270–1288
[25] Sexton T.R. & Lewis H.F. (2003). Two-stage DEA: An application to Major League Baseball. Journal of Productivity Analysis 19, 227-249.
[26] Wang C.H., Gopal R.D. & Zionts S. (1997). Use of data envelopment analysis in assessing information technology impact on firm performance. Annals of Operations Research 73: 191–213

**Appendix**

**Table A1:** CRS Projections starting from perspective II

| DMU | X1 | X2 | Z1 | Z2 | Y1 | Y2 | # LPs |
|---|---|---|---|---|---|---|---|
| 1 | 345023.16 | 275094.00 | 276276.25 | 17275.26 | 168556.44 | 17.27 | 3 |
| 2 | 251273.44 | 156105.58 | 162323.15 | 11727.77 | 97255.77 | 29.81 | 3 |
| 3 | 156436.22 | 92863.63 | 99098.62 | 7252.34 | 59146.59 | 8.16 | 3 |
| 4 | 101341.23 | 60193.48 | 63424.58 | 4683.83 | 37860.30 | 10.55 | 5 |
| 5 | 63185.32 | 36264.66 | 39005.37 | 2906.61 | 23215.40 | 3.31 | 3 |
| 6 | 104560.42 | 89213.87 | 88040.43 | 5332.73 | 53963.27 | 8.37 | 2 |
| 7 | 69357.88 | 43754.63 | 45123.99 | 3245.03 | 27071.01 | 9.72 | 3 |
| 8 | 68689.98 | 32300.37 | 36502.38 | 3029.12 | 21344.91 | 4.01 | 3 |
| 9 | 90640.00 | 77581.19 | 62351.62 | 4362.82 | 38521.84 | 100.26 | 2 |
| 10 | 67721.00 | 31208.44 | 35515.71 | 2975.75 | 20728.85 | 3.63 | 2 |
| 11 | 20491.09 | 11071.41 | 12083.49 | 930.06 | 7155.05 | 1.08 | 3 |
| 12 | 34200.00 | 29177.65 | 28953.38 | 1747.17 | 17743.19 | 1.69 | 2 |
| 13 | 21377.14 | 12589.71 | 13459.64 | 989.21 | 8028.11 | 1.12 | 3 |
| 14 | 28876.86 | 15148.71 | 16656.09 | 1302.42 | 9837.26 | 1.53 | 3 |
| 15 | 3347.34 | 1884.35 | 2005.31 | 152.74 | 1191.66 | 0.38 | 3 |
| 16 | 1532.01 | 886.12 | 950.12 | 70.58 | 565.87 | 0.09 | 5 |
| 17 | 771.37 | 360.62 | 406.66 | 33.95 | 237.66 | 0.06 | 3 |
| 18 | 100908.08 | 85532.88 | 84970.71 | 5144.92 | 52050.00 | 5.00 | 3 |
| 19 | 102635.35 | 54689.61 | 59895.67 | 4644.55 | 35423.50 | 5.42 | 3 |
| 20 | 35280.16 | 24442.75 | 25223.00 | 1699.32 | 15236.60 | 1.80 | 3 |
| 21 | 110290.90 | 62960.93 | 67805.61 | 5067.34 | 40338.70 | 5.78 | 3 |
| 22 | 115065.35 | 77230.15 | 80220.08 | 5496.96 | 48344.07 | 5.91 | 3 |

**Table A2:** VRS Projections starting from perspective II

| DMU | X1 | X2 | Z1 | Z2 | Y1 | Y2 | # LPs |
|---|---|---|---|---|---|---|---|
| 1 | 370870.00 | 476450.80 | 277268.07 | 22476.20 | 168556.44 | 5.97 | 2 |
| 2 | 218059.94 | 257207.11 | 158231.66 | 12442.07 | 97255.77 | 53.42 | 4 |
| 3 | 132488.69 | 144262.51 | 96649.80 | 7297.89 | 59146.58 | 59.98 | 5 |
| 4 | 190000.00 | 258537.00 | 155900.00 | 549.00 | 37860.30 | 0.01 | 2 |
| 5 | 63501.85 | 36446.33 | 31286.93 | 2123.93 | 23215.40 | 2.42 | 2 |
| 6 | 115635.79 | 135835.19 | 88543.00 | 6711.71 | 53963.27 | 8.37 | 2 |
| 7 | 70180.05 | 44283.36 | 31986.98 | 1923.37 | 27071.01 | 9.72 | 3 |
| 8 | 68792.31 | 32345.34 | 35309.07 | 2909.82 | 21344.91 | 4.01 | 3 |
| 9 | 90640.00 | 77581.19 | 62351.62 | 4362.82 | 38521.84 | 100.26 | 2 |
| 10 | 67721.00 | 31208.44 | 35515.71 | 2975.75 | 20728.85 | 3.63 | 2 |
| 11 | 20714.97 | 11086.38 | 11908.37 | 922.27 | 7155.05 | 1.09 | 3 |
| 12 | 34200.00 | 29177.65 | 28953.38 | 1747.17 | 17743.19 | 1.69 | 2 |
| 13 | 21427.52 | 12619.37 | 13290.24 | 976.91 | 8028.11 | 1.12 | 3 |
| 14 | 28925.00 | 15173.96 | 16509.50 | 1291.90 | 9837.26 | 1.53 | 3 |
| 15 | 3190.05 | 1958.89 | 1717.57 | 124.88 | 1191.66 | 0.38 | 5 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 16 | 1766.52 | 921.81 | 676.85 | 55.14 | 565.87 | 0.11 | 5 |
| 17 | 1049.00 | 388.74 | 131.16 | 19.53 | 237.66 | 0.06 | 2 |
| 18 | 115541.37 | 124669.04 | 85054.70 | 6348.89 | 52050.00 | 49.01 | 3 |
| 19 | 103985.00 | 55408.78 | 28985.62 | 1512.43 | 35423.50 | 1.87 | 2 |
| 20 | 35188.20 | 24462.84 | 25165.23 | 1691.99 | 15236.60 | 1.80 | 3 |
| 21 | 216360.00 | 122956.40 | 122044.05 | 1658.50 | 40338.70 | 0.32 | 2 |
| 22 | 126112.35 | 98658.80 | 58405.28 | 4002.87 | 48344.07 | 21.23 | 3 |