A Ramsey-Type Equilibrium Model with Spatially Dispersed Agents

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Abstract. We present a spatial and time-continuous Ramsey-type equilibrium model for households and firms that interact on a spatial domain to model labor mobility in the presence of commuting costs. After discretization in space and time, we obtain a mixed complementarity problem that represents the spatial equilibrium model. We prove existence of equilibria using the theory of finite-dimensional variational inequalities and derive a tailored diagonalization method to solve the resulting large-scale instances. Finally, we present a case study that highlights the influence of commuting costs and show that the model allows to analyze transitory effects of industrial agglomeration that emerge and vanish over time as in the real economy.

1. Introduction

In an ideal economic world, production factors move freely: people migrate or commute to get employed and entrepreneurs invest where returns are highest. Everything is in flux. But, who moves first, and how fast? These questions matter. A mismatch between locally available labor on the one hand and invested capital on the other hand implies inefficiency and disequilibrium. Capital follows labor and labor follows capital, either by migration and commuting or as cross-regional direct investments. The economic landscape is shaped by these factor movements. Agglomeration patterns emerge and disappear such as the rise and decline of industrial belts or the prosperity and impoverishment of border regions.

Workers may prefer to commute rather than to relocate. We have in mind a fund manager working in London but living in Bern. Or the engineer working in Luxembourg but living in Trier. Depending on the commuting costs and wage-differential, people are willing to separate their where-to-live and where-to-work decisions. However, any positive commuting costs are a persistent attraction to seek employment close to the residency. This affects the long-run direction of all factor movements, including capital. Workers may move first but they have an incentive to work close to their residence. The strength of this incentive depends on commuting costs. We focus first on the welfare implications and second on the agglomeration patterns that result from this residential home bias.

We are interested in spatial pattern formation with a focus on commuting. A spatial dynamic general equilibrium (SCGE) model serves as a laboratory to investigate the role of commuting as a surrogate of permanent migration. The starting point of our analysis is an almost white map. A number of firms is distributed on this map, each of which has a certain spatial outreach within which it can produce. This outreach defines the boundaries of a region.

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Private households are also spatially distributed, each being assigned a point on the map as permanent residence. They earn income from labor and capital assets. The latter is spent on a single consumption good or saved for future consumption, following the traditional Ramsey growth model. The decision of where-to-invest and where-to-work is subject to household choice, too. However, the decision about where-to-work can involve commuting costs. In particular, commuting takes time and therefore reduces welfare. This means that occupational choice has a home-bias. The decision about residence results from a broader set of socio-demographic factors that are not subject of our analysis. Therefore, we take it as given.

The consistent way we model a home-bias of workers and commuting adds a new element to classic SCGE modeling. Such models are widely used in policy consulting and applied economic research. The RHOMOLO model used by the European Commission in policy is a prime example; see [11]. These models, however, assume either perfect or no spatial mobility of labor. This neglects a key feature of modern societies. Within the European Union, 6 % of all employees are classified as cross-regional commuters within their countries; see, e.g., [4]. In Lorraine (France), even 13 % commute into a different country; see [5].

Our model is a first step to integrate commuting decisions in an SCGE model. It turns out that even in a highly stylized setting, a rich variety of spatial patterns emerges. By making such aspects numerically visible, our paper also contributes to the literature on spatial economic patterns, which has a long tradition in economic modeling; see, e.g., [13] for an overview. In the "new economic geography" literature, see, e.g., [9, 10], it is mainly the trade-off between economies of scale on the one hand and immobile consumers on the other hand that generates interesting patterns. High transportation costs matter as they favor local produce-and-buy decisions instead of long-distance shipments, thus counteracting economies of scale. The resulting dynamics are well understood and agglomeration occurs when transportation costs fall below certain threshold values.

The mathematical modeling of the setup discussed so far leads to a very challenging problem that can be seen as a generalized Nash equilibrium problem (GNEP). Our main mathematical tools for tackling these GNEPs are mixed complementarity problems (MCPs) that combine the optimality conditions of all players with suitably chosen equilibrating conditions. In this paper, we prove the existence of equilibria using the classic theory of finite-dimensional variational inequalities. Since the resulting MCPs are of large scale due to necessary discretizations of space and time, we additionally develop a tailored diagonalization method that is shown to clearly outperform classic MCP approaches in terms of running times.

In Section 2, we describe the model in detail. In particular, we introduce an innovative way of regularization that prevents single-point clusters. Afterward, in Section 3, we prove the existence of a spatial equilibrium. In Section 4, we present our computational experiments. In particular, we discuss the effects of a scenario with high commuting costs and one with low commuting costs. Section 5 summarizes our results.

2. Modeling

2.1. The Time-Continuous Model. The model's spatial interaction domain is a compact set $X \subset \mathbb{R}^2$. We consider finite sets of households $\mathcal{H}$ and firms $\mathcal{F}$ that are located on the domain X. The firm $f \in \mathcal{F} = \{1, \dots, n_{\mathcal{F}}\}$ is modeled by a catchment area from which it obtains the households' labor and asset supply. Each household $h \in \mathcal{H}$ with residence $x_h \in X$ is free in its choice on where they supply their labor and asset on the domain X. However, supplying labor in larger distance to the residence allows less leisure and thus gets penalized in the corresponding objective function. Further, each household $h \in \mathcal{H} = \{1, \dots, n_{\mathcal{H}}\}$ obtains revenues through labor l supplied to firm f at prices w_f and on asset a supplied to firm f at prices r_f. The optimization problem of household $h \in \mathcal{H}$ on the time horizon $[0, T]$ is given by

$$
\begin{aligned}
\max_{\substack{c_h(\cdot), l_h(\cdot,\cdot),\\ a_h(\cdot,\cdot)}} \quad & \int_{[0,T]} \Bigg(\left(\omega_h \, u_h(c_h(t)) + (1-\omega_h) \int_X l_h(x,t) v_h(x) \, \mathrm{d}x \right) \\
& \qquad - \frac{\lambda_h}{2} \left(\left\| \frac{\partial}{\partial t} a_h(x,t) \right\|_2^2 + \left\| \frac{\partial}{\partial t} l_h(x,t) \right\|_2^2 \right) \Bigg) e^{-\gamma_h t} \, \mathrm{d}t \\
\text{s.t.} \quad & c_h(t) \geq 0 \quad \text{for all} \quad t \in [0,T], \\
& a_h(x,t) \geq 0 \quad \text{for all} \quad x \in X, \ t \in (0,T], \\
& l_h(x,t) \geq 0 \quad \text{for all} \quad x \in X, \ t \in (0,T], \\
& a_h(x,0) = a_h^0(x) \quad \text{for all} \quad x \in X, \\
& l_h(x,0) = l_h^0(x) \quad \text{for all} \quad x \in X, \\
& \int_X a_h(x,T) \, \mathrm{d}x \geq a_h^T, \\
& \int_X l_h(x,t) \, \mathrm{d}x = l_h(t) \quad \text{for all} \quad t \in [0,T], \\
& \int_X a_h(x,t) \, \mathrm{d}x = a_h(t) \quad \text{for all} \quad t \in [0,T], \\
& \frac{\mathrm{d}a_h(t)}{\mathrm{d}t} = \int_X \tilde{r}(x,t) a_h(x,t) \, \mathrm{d}x + \int_X \tilde{w}(x,t) l_h(x,t) \, \mathrm{d}x - c_h(t),
\end{aligned} \tag{1}
$$

where

$$\tilde{r}(x,t) = \sum_{f \in \mathcal{F}} (r_f(t) - \delta) g_f(x), \quad \tilde{w}(x,t) = \sum_{f \in \mathcal{F}} w_f(t) g_f(x)$$

are the average returns on capital and labor. Here and in what follows, $g_f(x)$ models the catchment area of firm f, i.e., the household's h labor or asset supply at location x, $l_h(x,t)$ or $a_h(x,t)$, is given to firm f at ratio $g_f(x) \in [0,1]$. The other parameters are the time discount rate $\gamma_h > 0$, the utility weight $\omega_h \in (0,1]$, the instantaneous utility function u_h of CRRA type, the distant work penalty v_h given by

$$v_h(x) = e^{-\vartheta_h \|x - x_h\|_2^2}$$

with scaling factor $\vartheta_h > 0$, the initial spatial distribution of assets $a_h^0 \colon X \to \mathbb{R}_{\geq 0}$ and labor $l_h^0 \colon X \to \mathbb{R}_{\geq 0}$, as well as the depreciation rate $\delta \geq 0$. Furthermore, we have the scaling factor $\lambda_h \in \mathbb{R}_{\geq 0}$ for the regularization terms that penalize large time derivatives of asset and labor. The latter are incorporated to obtain smooth solutions over time. Moreover, individual labor endowment is strictly positive and exogenously given, i.e., $l_h(t) > 0$ for all $t \in [0,T]$ and we assume that $a_h^0(x) \geq 0$ and $\int_X \sum_{h \in \mathcal{H}} a_h^0(x) \, \mathrm{d}x > 0$ holds. For the regularization terms we use the L^2-norm

$$\|\varphi\|_2 = \left(\int_X \varphi(x)^2 \, \mathrm{d}x \right)^{1/2}$$

of a given function φ. In what follows, we assume that the utility functions u_h of all households $h \in \mathcal{H}$ can be different but all satisfy the following standard assumptions; see, e.g., Chapter 8.1 in [1].

Assumption 1. *All utility functions u are twice differentiable and it holds $u' > 0$, $u'' < 0$, i.e., utilities are concave and strictly increasing. Moreover, we suppose that* *the conditions*

$$\lim_{x\to\infty} u'(x) = 0 \quad \text{and} \quad \lim_{x\to 0} u'(x) = \infty$$

hold.

Each firm $f \in \mathcal{F}$ is maximizing its profit in each point of time $t \in [0, T]$, i.e.,

$$\begin{aligned} \max_{K_f(t), L_f(t)} \quad & F_f(\mathcal{A}_f(t), K_f(t), L_f(t)) - r_f(t)K_f(t) - w_f(t)L_f(t) \\ \text{s.t.} \quad & K_f(t) \geq 0, \quad L_f(t) \geq 0, \end{aligned} \tag{2}$$

with a production function F_f. Here, $\mathcal{A}_f(t) > 0$, $f \in \mathcal{F}$, is an exogenously given productivity factor, $K_f(t)$ is the engaged capital at given price $r_f(t)$ and $L_f(t)$ is the engaged labor at given wage rate $w_f(t)$. Furthermore, we require that all production functions satisfy the Inada conditions.

Assumption 2. *All production functions*

$$F(\mathcal{A}, K, L)\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},$$

are twice continuously differentiable. We assume $F(\mathcal{A}, K, L)$ is linear in $\mathcal{A}$. We have that $F(\mathcal{A}, K, L) = 0$ implies $K = 0$ or $L = 0$, i.e., asset and labor are essential production goods. Furthermore, all production functions are homogeneous of degree 1 in K and L, i.e., $F(\mathcal{A}, \alpha K, \alpha L) = \alpha F(\mathcal{A}, K, L)$ for $\alpha \in \mathbb{R}$. All production functions are concave and strictly increasing in K and L. Lastly, they satisfy the so-called Inada conditions, i.e.,

$$\begin{aligned} &\lim_{K\to 0} F'_K(\mathcal{A}, K, L) \to \infty, && \lim_{L\to 0} F'_L(\mathcal{A}, K, L) \to \infty, \\ &\lim_{K\to\infty} F'_K(\mathcal{A}, K, L) \to 0, && \lim_{L\to\infty} F'_L(\mathcal{A}, K, L) \to 0. \end{aligned}$$

For our numerical experiments we later specify the specific production functions that satisfy these assumptions.

To complete our equilibrium model, we need further equilibrating conditions in the sense of market clearing conditions stating that each firm can use at most the households aggregated capital holding and labor according to the firms catchment area. Hence, we impose the equilibrium conditions

$$\begin{aligned} 0 \leq \sum_{h\in\mathcal{H}} \int_X g_f(x) a_h(x,t) \,\mathrm{d}x - K_f(t) \perp r_f(t) \geq 0, \\ 0 \leq \sum_{h\in\mathcal{H}} \int_X g_f(x) l_h(x,t) \,\mathrm{d}x - L_f(t) \perp w_f(t) \geq 0, \end{aligned} \tag{3}$$

for all $f \in \mathcal{F}$ and $t \in [0, T]$.

The individual catchment areas $\hat{g}_f\colon X \to [0, \infty)$ of the firms are exogenously given, the normalized catchment areas g_f are given by

$$g_f(x) = \frac{\hat{g}_f(x)}{\sum_{f'\in\mathcal{F}} \hat{g}_{f'}(x)},$$

and we assume that $\sum_{f\in\mathcal{F}} \hat{g}_f > 0$ holds. Here, $g_f(x)$ specifies how much of the household's asset or labor is used at location x by company $f \in \mathcal{F}$. Accordingly, the household also receives the corresponding returns of the respective company.

Next, we will show some basic properties of these catchment area functions.

Lemma 1. *Assume $r_f(t) > 0$ holds for all $f \in \mathcal{F}$. Then,*

$$\sum_{f\in\mathcal{F}} K_f(t) = \sum_{h\in\mathcal{H}} a_h(t)$$

holds, i.e., the market clears at each point in time.

Proof. Using (3) and $r_f(t) > 0$, we can conclude

$$\begin{aligned}\sum_{f\in\mathcal{F}} K_f(t) &= \sum_{f\in\mathcal{F}}\sum_{h\in\mathcal{H}}\int_X g_f(x)a_h(x,t)\,\mathrm{d}x\\ &= \sum_{h\in\mathcal{H}}\int_X \sum_{f\in\mathcal{F}} g_f(x)a_h(x,t)\,\mathrm{d}x\\ &= \sum_{h\in\mathcal{H}}\int_X a_h(x,t)\,\mathrm{d}x\\ &= \sum_{h\in\mathcal{H}} a_h(t),\end{aligned}$$

where we use the normalization of g_f in the third equation. □

Finally, the spatial Ramsey equilibrium problem is to find a solution of

households (1), firms (2), and equilibrating conditions (3).

2.2. **Discretization.** For a discretization of the given equilibrium problem we assume a finite termination time $T \in \mathbb{R}_{\geq 0}$. We discretize in time using n_t intervals given by

$$0 = t_0 < t_1 < \cdots < t_{n_t-1} < t_{n_t} = T$$

with interval lengths $\tau_k = t_{k+1} - t_k$ for all $k = 0,\dots,n_t-1$. For the ease of presentation, we consider the unit square domain $X = [0,1]^2$ and discretize using $(n_x+1)(n_y+1)$ grid points (x_i, y_j) for $i = 0,\dots,n_x$ and $j = 0,\dots,n_y$ given by

$$\begin{aligned}&0 = x_0 < x_1 < \cdots < x_{n_x-1} < x_{n_x} = 1,\\ &0 = y_0 < y_1 < \cdots < y_{n_y-1} < y_{n_y} = 1\end{aligned}$$

with interval lengths $\xi_k = x_{k+1} - x_k$ for $k = 0,\dots,n_x-1$ and $\mu_k = y_{k+1} - y_k$ for $k = 0,\dots,n_y-1$. We denote the set containing the indices of the grid points with Σ. Furthermore, we use the abbreviation $a_{h,i,j,k} = a_h(x_{i,j}, t_k)$ and $a_h = (a_{h,i,j,k})$ denotes the vector with all entries for $i = 1,\dots,n_x$, $j = 1,\dots,n_y$, as well as $k = 0,\dots,n_t$. The same abbreviation is used for the other discretized functions. Using the implicit Euler scheme for the discretizing the differential equation leads to the finite-dimensional problem

$$\begin{aligned}\max_{\substack{c_h,l_h\\a_h}}\quad & \sum_{k=0}^{n_t-1}\tau_k\Bigg(\omega_h\,u_h(c_{h,k+1}) + (1-\omega_h)\sum_{(i,j)\in\Sigma}\tilde{l}_{h,i,j,k+1}v_{h,i,j}\\ &\qquad -\frac{\lambda_h}{2}\sum_{(i,j)\in\Sigma}\Bigg(\left(\tilde{a}_{h,i,j,k+1}-\tilde{a}_{h,i,j,k}\right)^2\\ &\qquad\qquad +\left(\tilde{l}_{h,i,j,k+1}-\tilde{l}_{h,i,j,k}\right)^2\Bigg)\Bigg)e^{-\gamma_h t_{k+1}}\\ \text{s.t.}\quad & c_{h,k+1}\geq 0 \quad\text{for all}\quad k=0,\dots,n_t-1,\\ & a_{h,i,j,k}\geq 0 \quad\text{for all}\quad (i,j)\in\Sigma,\ k=1,\dots,n_t,\\ & l_{h,i,j,k}\geq 0 \quad\text{for all}\quad (i,j)\in\Sigma,\ k=1,\dots,n_t,\\ & a_{h,i,j,0} = a^0_{h,i,j} \quad\text{for all}\quad (i,j)\in\Sigma,\\ & l_{h,i,j,0} = l^0_{h,i,j} \quad\text{for all}\quad (i,j)\in\Sigma,\\ & \sum_{(i,j)\in\Sigma}\tilde{a}_{h,i,j,n_t}\geq a^T_h,\end{aligned} \tag{4}$$

$$\sum_{(i,j)\in\Sigma} \tilde{l}_{h,i,j,k} = l_{h,k} \quad \text{for all} \quad k = 1, \dots, n_t,$$
$$\frac{1}{\tau_k} \sum_{(i,j)\in\Sigma} (\tilde{a}_{h,i,j,k+1} - \tilde{a}_{h,i,j,k}) = \sum_{(i,j)\in\Sigma} \left(\tilde{r}_{i,j,k+1}\tilde{a}_{h,i,j,k+1} + \tilde{w}_{i,j,k+1}\tilde{l}_{h,i,j,k+1}\right) - c_{h,k+1} \quad \text{for all} \quad k = 0, \dots, n_t - 1,$$

for household $h \in \mathcal{H}$. Here we used the abbreviations

$$\tilde{r}_{i,j,k} = \sum_{f\in\mathcal{F}} (r_{f,k} - \delta) g_{f,i,j}, \quad \tilde{w}_{i,j,k} = \sum_{f\in\mathcal{F}} w_{f,k} g_{f,i,j}$$

and

$$\tilde{l}_{h,i,j,k} = \mathbb{D}_{i,j}\, l_{h,i,j,k}, \quad \tilde{a}_{h,i,j,k} = \mathbb{D}_{i,j}\, a_{h,i,j,k},$$

where $\mathbb{D}\colon \Sigma \to [0,1]$ with

$$\mathbb{D}_{i,j} = \begin{cases} \frac{1}{4}\left(\xi_0\mu_0\right), & (i,j) = (0,0), \\ \frac{1}{4}\left(\xi_0\mu_{n_y-1}\right), & (i,j) = (0,n_y), \\ \frac{1}{4}\left(\xi_{n_x-1}\mu_0\right), & (i,j) = (n_x,0), \\ \frac{1}{4}\left(\xi_{n_x-1}\mu_{n_y-1}\right), & (i,j) = (n_x,n_y), \\ \frac{1}{4}\left(\xi_i\mu_j + \xi_{i-1}\mu_j\right), & i = 1,\dots,n_x-1,\ j = 0, \\ \frac{1}{4}\left(\xi_{i-1}\mu_{i-1} + \xi_i\mu_{i-1}\right), & i = 1,\dots,n_x-1,\ j = n_y, \\ \frac{1}{4}\left(\xi_i\mu_j + \xi_i\mu_{j-1}\right), & i = 0,\ j = 1,\dots,n_y-1, \\ \frac{1}{4}\left(\xi_{i-1}\mu_{j-1} + \xi_{i-1}\mu_j\right), & i = n_x,\ j = 1,\dots,n_y-1, \\ \frac{1}{4}\left(\xi_{i-1}\mu_{j-1} + \xi_{i-1}\mu_j + \xi_i\mu_{j-1} + \xi_i\mu_j\right), & \text{else}, \end{cases}$$

is an auxiliary function for the discretization of the integrals in space.

The discretized distant work penalty $v_{h,i,j}$ reads

$$v_{h,i,j} = e^{-\vartheta_h \|x_{i,j} - x_h\|_2^2}$$

and the normalized catchment areas are given by

$$g_{f,i,j} = \frac{\hat{g}_{f,i,j}}{\sum_{f'\in\mathcal{F}} \hat{g}_{f',i,j}},$$

for all $f \in \mathcal{F}$ and $(i,j) \in \Sigma$. A discrete version of Lemma 1 still holds for the discretized catchment areas.

The discretized optimization problem of the firms reads

$$\begin{aligned} \max_{K_{f,k}, L_{f,k}} \quad & F_f(\mathcal{A}_{f,k}, K_{f,k}, L_{f,k}) - r_{f,k}K_{f,k} - w_{f,k}L_{f,k}, \\ \text{s.t.} \quad & K_{f,k} \geq 0, \quad L_{f,k} \geq 0 \end{aligned} \tag{5}$$

for all $f \in \mathcal{F}$ and $k = 1, \dots, n_t$. Note that we do not have to include the baseline time period $k = 0$ due to our discretization. Finally, we have to discretize the equilibrium conditions, leading to

$$\begin{aligned} 0 \leq \sum_{h\in\mathcal{H}} \sum_{(i,j)\in\Sigma} g_{f,i,j}\tilde{a}_{h,i,j,k} - K_{f,k} \perp r_{f,k} \geq 0, \\ 0 \leq \sum_{h\in\mathcal{H}} \sum_{(i,j)\in\Sigma} g_{f,i,j}\tilde{l}_{h,i,j,k} - L_{f,k} \perp w_{f,k} \geq 0, \end{aligned} \tag{6}$$

for all $f \in \mathcal{F}$ and $k = 1, \dots, n_t$.

Since we only face concave objective functions and affine-linear constraints, the Karush–Kuhn–Tucker (KKT) conditions of Problem (4) and (5) are both necessary and sufficient. The KKT conditions of the households, already in the form of a mixed complementarity problem (MCP), are given by

$$
\begin{aligned}
0 &\leq -\tau_k \omega_h u_h'(c_{h,k+1}) e^{-\gamma_h t_{k+1}} + \theta_{h,k} \perp c_{h,k+1} \geq 0, \\
\text{free } &\lambda_h \tau_0 \mathbb{D}_{i,j} \left(\tilde{a}_{h,i,j,1} - \tilde{a}_{h,i,j,0}\right) e^{-\gamma_h t_1} + \frac{\theta_{h,0}}{\tau_0} \mathbb{D}_{i,j} \perp a_{h,i,j,0} - a^0_{h,i,j} = 0, \\
0 &\leq \lambda_h \tau_{n_t-1} \mathbb{D}_{i,j} \left(\tilde{a}_{h,i,j,n_t} - \tilde{a}_{h,i,j,n_t-1}\right) e^{-\gamma_h t_{n_t}} \\
&\quad -\varepsilon_h \mathbb{D}_{i,j} + \theta_{h,n_t-1} \mathbb{D}_{i,j} \left(\frac{1}{\tau_{n_t-1}} - \tilde{r}_{i,j,n_t}\right) \perp a_{h,i,j,n_t} \geq 0, \\
\text{free } &\lambda_h \tau_0 \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,1} - \tilde{l}_{h,i,j,0}\right) e^{-\gamma_h t_1} \perp l_{h,i,j,0} - l^0_{h,i,j} = 0, \\
0 &\leq -\tau_{n_t-1}(1-\omega_h) \mathbb{D}_{i,j} v_{h,i,j} e^{-\gamma_h t_{n_t}} \\
&\quad +\lambda_h \tau_{n_t-1} \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,n_t} - \tilde{l}_{h,i,j,n_t-1}\right) e^{-\gamma_h t_{n_t}} \\
&\quad +\zeta_{h,n_t} \mathbb{D}_{i,j} - \theta_{h,n_t-1} \mathbb{D}_{i,j} \tilde{w}_{i,j,n_t} \perp l_{h,i,j,n_t} \geq 0 \\
0 &\leq \sum_{(i,j)\in\Sigma} \tilde{a}_{h,i,j,n_t} - a^T_h \perp \varepsilon_h \geq 0
\end{aligned}
\tag{7}
$$

as well as

$$
\begin{aligned}
0 &\leq -\lambda_h \tau_k \mathbb{D}_{i,j} \left(\tilde{a}_{h,i,j,k+1} - \tilde{a}_{h,i,j,k}\right) e^{-\gamma_h t_{k+1}} \\
&\quad +\lambda_h \tau_{k-1} \mathbb{D}_{i,j} \left(\tilde{a}_{h,i,j,k} - \tilde{a}_{h,i,j,k-1}\right) e^{-\gamma_h t_k} \\
&\quad -\frac{\theta_{h,k}}{\tau_k} \mathbb{D}_{i,j} + \theta_{h,k-1} \mathbb{D}_{i,j} \left(\frac{1}{\tau_{k-1}} - \tilde{r}_{i,j,k}\right) \perp a_{h,i,j,k} \geq 0, \\
0 &\leq -\tau_{k-1}(1-\omega_h) \mathbb{D}_{i,j} v_{h,i,j} e^{-\gamma_h t_k} \\
&\quad -\lambda_h \tau_k \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,k+1} - \tilde{l}_{h,i,j,k}\right) e^{-\gamma_h t_{k+1}} \\
&\quad +\lambda_h \tau_{k-1} \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,k} - \tilde{l}_{h,i,j,k-1}\right) e^{-\gamma_h t_k} \\
&\quad +\zeta_{h,k} \mathbb{D}_{i,j} - \theta_{h,k-1} \mathbb{D}_{i,j} \tilde{w}_{i,j,k} \perp l_{h,i,j,k} \geq 0
\end{aligned}
\tag{8}
$$

for $k = 1, \ldots, n_t - 1$ and

$$
0 = \sum_{(i,j)\in\Sigma} \tilde{l}_{h,i,j,k} - l_{h,k} \perp \zeta_{h,k} \text{ free} \tag{9}
$$

for $k = 1, \ldots, n_t$ and, finally,

$$
\begin{aligned}
0 &= \frac{1}{\tau_k} \sum_{(i,j)\in\Sigma} \left(\tilde{a}_{h,i,j,k+1} - \tilde{a}_{h,i,j,k}\right) \\
&\quad - \sum_{(i,j)\in\Sigma} \left(\tilde{r}_{i,j,k+1} \tilde{a}_{h,i,j,k+1} + \tilde{w}_{i,j,k+1} \tilde{l}_{h,i,j,k+1}\right) + c_{h,k+1} \perp \theta_{h,k} \text{ free}
\end{aligned}
\tag{10}
$$

for $k = 0, \ldots, n_t - 1$. For better reading we did not mention in the above formulas that we have all conditions including spatially varying quantities such as $a_{h,i,j,k}$ or $l_{h,i,j,k}$ for all $(i,j) \in \Sigma$.

Lastly, the KKT conditions of the firms' problem (5) in MCP form are given by

$$
\begin{aligned}
0 &\leq r_{f,k} - \frac{\partial}{\partial K} F_f(\mathcal{A}_{f,k}, K_{f,k}, L_{f,k}) \perp K_{f,k} \geq 0, \\
0 &\leq w_{f,k} - \frac{\partial}{\partial L} F_f(\mathcal{A}_{f,k}, K_{f,k}, L_{f,k}) \perp L_{f,k} \geq 0,
\end{aligned}
\tag{11}
$$

for all $f \in \mathcal{F}$ and $k = 1, \dots, n_t$. The discretized spatial Ramsey equilibrium problem is then to find a solution of

$$\text{households (7)–(10)}, \qquad \text{firms (11)}, \qquad \text{and equilibrating conditions (6)}. \tag{12}$$

3. Existence of Equilibria

We now prove the existence of a solution under mild assumptions. To ensure that the KKT conditions of (12) are well-defined and that production levels are finite, we make the following standard assumption.

Assumption 3. *There exist constants $m > 0$ and $M < \infty$ so that $K_{f,k}, L_{f,k} \geq m$ and $K_{f,k}, L_{f,k} \leq M$ for all $f \in \mathcal{F}$ and $k = 0, \dots, n_t$.*

In the following we use the classic theory of variational inequalities (VIs), see, e.g., [6], to show the existence of equilibria. To this end, we re-state (12) as the VI

$$F(x)^\top (y - x) \geq 0 \quad \text{for all} \quad y \in Z.$$

In what follows, we use the abbreviations $\alpha = |\mathcal{F}| n_t$ as well as $\beta = |\mathcal{H}| n_t$ and specify the VI using

$$\begin{aligned}
X &= \mathbb{R}^{\beta}_{\geq 0} \times Z_2 \times Z_3 \times \mathbb{R}^{|\mathcal{H}|}_{\geq 0} \times \mathbb{R}^{\beta} \times \mathbb{R}^{\beta} \times \mathbb{R}^{\alpha}_{\geq 0} \times \mathbb{R}^{\alpha}_{\geq 0} \times \mathbb{R}^{\alpha}_{\geq 0} \times \mathbb{R}^{\alpha}_{\geq 0},\\
Z_2 &= \prod_{h \in \mathcal{H}} \prod_{(i,j) \in \Sigma} \left(\{a^0_{h,i,j}\} \times \mathbb{R}^{n_t}_{\geq 0} \right),\\
Z_3 &= \prod_{h \in \mathcal{H}} \prod_{(i,j) \in \Sigma} \left(\{l^0_{h,i,j}\} \times \mathbb{R}^{n_t}_{\geq 0} \right),
\end{aligned}$$

and

$$F(x) = (F_\ell(x))_{\ell=1}^{10}.$$

The variable vector x is given by[1]

$$x = (c, a, l, \varepsilon, \zeta, \theta, K, L, r, w)$$

and the entries of F read

$$F_1(x) = \left(-\tau_k \omega_h u_h'(c_{h,k+1}) e^{-\gamma_h t_{k+1}} + \theta_{h,k} \right)_{k=0,\dots,n_t-1,\ h \in \mathcal{H}}$$

$$F_2(x) = \begin{pmatrix}
\lambda_h \tau_0 \mathbb{D}_{i,j} (\tilde{a}_{h,i,j,1} - \tilde{a}_{h,i,j,0}) e^{-\gamma_h t_1} + \frac{\theta_{h,0}}{\tau_0} \mathbb{D}_{i,j} \\
\begin{pmatrix}
-\lambda_h \tau_k \mathbb{D}_{i,j} (\tilde{a}_{h,i,j,k+1} - \tilde{a}_{h,i,j,k}) e^{-\gamma_h t_{k+1}} \\
+\lambda_h \tau_{k-1} \mathbb{D}_{i,j} (\tilde{a}_{h,i,j,k} - \tilde{a}_{h,i,j,k-1}) e^{-\gamma_h t_k} \\
-\frac{\theta_{h,k}}{\tau_k} \mathbb{D}_{i,j} + \theta_{h,k-1} \mathbb{D}_{i,j} \left(\frac{1}{\tau_{k-1}} - \tilde{r}_{i,j,k} \right)
\end{pmatrix}_{k=1,\dots,n_t-1} \\
\begin{pmatrix}
\lambda_h \tau_{n_t-1} \mathbb{D}_{i,j} \left(\tilde{a}_{h,i,j,n_t} - \tilde{a}_{h,i,j,n_t-1} \right) e^{-\gamma_h t_{n_t}} \\
-\varepsilon_h \mathbb{D}_{i,j} + \theta_{h,n_t-1} \mathbb{D}_{i,j} \left(\frac{1}{\tau_{n_t-1}} - \tilde{r}_{i,j,n_t} \right)
\end{pmatrix}
\end{pmatrix}_{(i,j) \in \Sigma,\ h \in \mathcal{H}},$$

$$F_3(x) = \begin{pmatrix}
\lambda_h \tau_0 \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,1} - \tilde{l}_{h,i,j,0} \right) e^{-\gamma_h t_1} \\
\begin{pmatrix}
-\tau_{k-1} (1 - \omega_h) \mathbb{D}_{i,j} v_{h,i,j} e^{-\gamma_h t_k} \\
-\lambda_h \tau_k \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,k+1} - \tilde{l}_{h,i,j,k} \right) e^{-\gamma_h t_{k+1}} \\
+\lambda_h \tau_{k-1} \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,k} - \tilde{l}_{h,i,j,k-1} \right) e^{-\gamma_h t_k} \\
+\zeta_{h,k} \mathbb{D}_{i,j} - \theta_{h,k-1} \mathbb{D}_{i,j} \tilde{w}_{i,j,k}
\end{pmatrix}_{k=1,\dots,n_t-1} \\
\begin{pmatrix}
-\tau_{n_t-1} (1 - \omega_h) \mathbb{D}_{i,j} v_{h,i,j} e^{-\gamma_h t_{n_t}} \\
+\lambda_h \tau_{n_t-1} \mathbb{D}_{i,j} \left(\tilde{l}_{h,i,j,n_t} - \tilde{l}_{h,i,j,n_t-1} \right) e^{-\gamma_h t_{n_t}} + \\
\zeta_{h,n_t} \mathbb{D}_{i,j} - \theta_{h,n_t-1} \mathbb{D}_{i,j} \tilde{w}_{i,j,n_t}
\end{pmatrix}
\end{pmatrix}_{(i,j) \in \Sigma,\ h \in \mathcal{H}},$$

$$F_4(x) = \left(\sum_{(i,j) \in \Sigma} \tilde{a}_{h,i,j,n_t} - a_h^T \right)_{h \in \mathcal{H}},$$

[1] We omit the transposition of vectors here for better reading.

$$F_5(x) = \left(\sum_{(i,j)\in\Sigma} \tilde{l}_{h,i,j,k} - l_{h,k} \right)_{k=1,\dots,n_t,\ h\in\mathcal{H}},$$

$$F_6(x) = \left(\begin{array}{c} \frac{1}{\tau_k} \sum_{(i,j)\in\Sigma}(\tilde{a}_{h,i,j,k+1} - \tilde{a}_{h,i,j,k}) \\ -\sum_{(i,j)\in\Sigma}\big(\tilde{r}_{i,j,k+1}\tilde{a}_{h,i,j,k+1} + \tilde{w}_{i,j,k+1}\tilde{l}_{h,i,j,k+1}\big) + c_{h,k+1} \end{array} \right)_{k=0,\dots,n_{t-1},\ h\in\mathcal{H}},$$

$$F_7(x) = \left(r_{f,k} - \frac{\partial}{\partial K} F_f(\mathcal{A}_{f,k}, K_{f,k}, L_{f,k}) \right)_{k=1,\dots,n_t,\ f\in\mathcal{F}},$$

$$F_8(x) = \left(w_{f,k} - \frac{\partial}{\partial L} F_f(\mathcal{A}_{f,k}, K_{f,k}, L_{f,k}) \right)_{k=1,\dots,n_t,\ f\in\mathcal{F}},$$

$$F_9(x) = \left(\sum_{h\in\mathcal{H}} \sum_{(i,j)\in\Sigma} g_{f,i,j,k}\tilde{a}_{h,i,j,k} - K_{f,k} \right)_{k=1,\dots,n_t,\ f\in\mathcal{F}},$$

$$F_{10}(x) = \left(\sum_{h\in\mathcal{H}} \sum_{(i,j)\in\Sigma} g_{f,i,j,k}\tilde{l}_{h,i,j,k} - L_{f,k} \right)_{k=1,\dots,n_t,\ f\in\mathcal{F}}.$$

It is easy to see that the Jacobian of F is not symmetric on Z. For instance,

$$\frac{\partial}{\partial r_{f,k}} F_7(x)_{f,k} = 1 \neq -1 = \frac{\partial}{\partial K_{f,k}} F_9(x)_{f,k}$$

holds. Thus, there is no function f with $\nabla f = F$, i.e., it is not possible to solve a properly chosen optimization problem for solving the VI(Z, F); see, e.g., Theorem 1.3.1 in [6].

To show the existence of solutions later on we have to ensure that the final asset constraint is binding. However, this is only the case under certain assumptions on the discretization of the MCP, which leads to an a-priori criterion for the time discretization being reasonable.

Proposition 1. *Suppose that Assumption 3 holds, that x^* is a solution of VI(Z,F), $\delta = 0$, $\lambda_h = 0$, and that $\tau_{n-1} < 1/\bar{r}$ holds with*

$$\bar{r} := \max_{f\in\mathcal{F}} \frac{\partial F_f(\mathcal{A}_k, m, M)}{\partial K}.$$

Then, $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,n_t} = a_h^T$ holds for all $h \in \mathcal{H}$.

Proof. Since the Inada conditions hold, i.e., $\partial F_f/\partial K > 0$ and $\partial^2 F_f/\partial K^2 < 0$, we obtain

$$\begin{aligned} r^*_{f,n} &= \frac{\partial}{\partial K} F_f \left(\mathcal{A}_k, \sum_{h\in\mathcal{H}} \sum_{(i,j)\in\Sigma} g_{f,i,j,n} a^*_{h,i,j,n}, \sum_{h\in\mathcal{H}} \sum_{(i,j)\in\Sigma} g_{f,i,j,n} l^*_{h,i,j,n} \right) \\ &\le \max_{f'\in\mathcal{F}} \frac{\partial}{\partial K} F_{f'}\left(\mathcal{A}_k, m, M\right) = \bar{r}. \end{aligned}$$

We prove the statement via contradiction. Hence, we assume that $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,n} > a_h^T$ holds for a household $h \in \mathcal{H}$. From complementarity and $F_2(x^*) \ge 0$ it follows that there exists $(i', j') \in \Sigma$ with

$$\theta_{h,n_t-1}\mathbb{D}_{i',j'} \left(\frac{1}{\tau_{n_t-1}} - \tilde{r}_{i',j',n_t} \right) = 0.$$

Thus, either $\theta^*_{i,n_t-1} = 0$ holds, leading to

$$0 = \theta^*_{h,n_t-1} \ge u'_h(c^*_{h,n_t}) e^{-\gamma_h T} \tau_{n_t-1} > 0,$$

which contradicts the properties of the chosen utility function, or $1/\tau_{n_t-1} - (\tilde{r}^*_{i',j',n_t}) = 0$ needs to hold, which yields

$$0 = \frac{1}{\tau_{n-1}} - \tilde{r}^*_{i',j',n_t} = \frac{1}{\tau_{n-1}} - \sum_{f\in\mathcal{F}} g_{f,i',j',n_t} r^*_{f,n_t} > \frac{1}{\tau_{n-1}} - \bar{r}$$

by using $\sum_{f\in\mathcal{F}} g_{f,i',j',n_t} = 1$. However, since τ_{n-1} is chosen so that $1/\tau_{n-1} - \bar{r} > 0$ holds, we also obtain a contradiction in this case as well. □

Our overall strategy now is to prove the existence of an equilibrium by exploiting the following classical existence result for VIs.

Theorem 1. *[6, Corollary 2.2.5] Let $Z \subseteq \mathbb{R}^n$ be a nonempty, convex, and compact set and let $F\colon Z \to \mathbb{R}^n$ be a continuous function. Then, the VI(Z, F) has a solution.*

The VI function F is obviously continuous in our setting. However, the feasible set Z is not compact but Assumption 3 can be used to show the existence of a compact and convex subset including all solutions of the original VI so that the last theorem can still be applied.

Theorem 2. *Suppose that Assumptions 1, 2, and 3 hold and that $\sum_{(i,j)\in\Sigma} \tilde{a}_{h,i,j,0} \geq a^T_h$ holds for all $h \in \mathcal{H}$ as well as that $\tau_k = \tau$ is chosen sufficiently small. Moreover, suppose $\delta = 0$ and $\lambda_h = 0$. Then, there exists a convex and compact subset $\tilde{Z}$ such that the solutions sets of VI(Z, F) and VI$(\tilde{Z}, F)$ coincide.*

Proof. Let $x^* = (c^*, a^*, l^*, \varepsilon^*, \zeta^*, \theta^*, K^*, L^*, r^*, w^*)$ be a solution of VI(Z, F). We will show that this x^* is contained in and bounded subset $\tilde{Z}$.

Boundedness of r^* and w^*: Assumption 3 implies that K^* and L^* are bounded away component-wise by a constant from 0 and ∞. By complementarity, $F_7(x) = F_8(x) = 0$ follows and, hence, the prices are given by the derivatives of F_f. By using again the boundedness of K^* and L^* and Assumption 2, i.e., that F_f is strictly increasing and that K and L are essential, it follows that there exists $0 < r^-, w^-$ and $r^+, w^+ < \infty$ such that r^* and w^* are bounded, i.e.,

$$r^- \leq r^*_{f,k} \leq r^+ \quad \text{and} \quad w^- \leq w^*_{f,k} \leq w^+$$

holds for all $f \in \mathcal{F}$ and all $k = 1, \dots, n_t$.

Boundedness of a^* and l^*: Since $r^*_{f,k} \geq 0$ is complementary to $F_9(x^*)_{f,k} \geq 0$ and $w^*_{f,k} \geq 0$ to $F_{10}(x^*)_{f,k} \geq 0$, we can conclude that the individual asset holding is bounded by the bounds of K, i.e.,

$$0 \leq a^*_{h,i,j,k} \leq \sum_{(i',j')\in\Sigma} a^*_{h,i',j',k} \leq \sum_{f\in\mathcal{F}} K^*_{f,k}$$

is bounded by the sum of the upper bounds of $K^*_{f,k}$ for all $(i,j) \in \Sigma, k = 1, \dots, n_t$. Analogously, we get that $l^*_{h,i,j,k}$ is bounded by the upper bound of the sums of $L^*_{f,k}$.

Boundedness of c^* and θ^*: From the latter bounds it follows by the discretized differential equation $(F_6(x^*) = 0)$ and by utilizing a telescope sum that $c^*_{h,k}$ is bounded from above by

$$\begin{aligned}
&\sum_{k=0}^{n_t-1} c^*_{h,k+1} \\
&= \sum_{k=0}^{n_t-1} \left(\sum_{(i,j)\in\Sigma} \left(\tilde{r}^*_{i,j,k+1} \tilde{a}^*_{h,i,j,k+1} + \tilde{w}^*_{i,j,k+1} \tilde{l}^*_{h,i,j,k+1} \right) - \frac{1}{\tau} \sum_{(i,j)\in\Sigma} (\tilde{a}^*_{h,i,j,k+1} - \tilde{a}^*_{h,i,j,k}) \right) \\
&= \left(\sum_{k=0}^{n_t-1} \sum_{(i,j)\in\Sigma} \left(\tilde{r}^*_{i,j,k+1} \tilde{a}^*_{h,i,j,k+1} + \tilde{w}^*_{i,j,k+1} \tilde{l}^*_{h,i,j,k+1} \right) \right) + \frac{1}{\tau} \sum_{(i,j)\in\Sigma} \tilde{a}^*_{h,i,j,0} - \tilde{a}^*_{h,i,j,n_t} \\
&= \left(\sum_{k=0}^{n_t-1} \sum_{(i,j)\in\Sigma} \sum_{f\in\mathcal{F}} g_{f,i,j,k+1} \left((r^*_{f,k+1} - \delta) a^*_{h,i,j,k+1} + w^*_{f,k+1} \tilde{l}^*_{h,i,j,k+1} \right) \right) \\
&\quad + \frac{1}{\tau} \sum_{(i,j)\in\Sigma} \left(\tilde{a}^*_{h,i,j,0} - \tilde{a}^*_{h,i,j,n_t} \right) \\
&\le n_t (r^+ + w^+) M + M,
\end{aligned}$$

with r^+, w^+, M being bounds on r^*, w^* and a^*, l^*, respectively. Next, we show that $c^*_{h,k}$ is also bounded from below. By utilizing once again a telescope sum, by denoting the gap between initial capital stock and final capital stock with $a_{h,\text{diff}}$, and by using that the final capital stock condition is binding due to Proposition 1, we get

$$\begin{aligned}
\sum_{k=0}^{n_t-1} c^*_{h,k+1} &= \left(\sum_{k=0}^{n_t-1} \sum_{(i,j)\in\Sigma} \sum_{f\in\mathcal{F}} g_{f,i,j,k+1} \left((r^*_{f,k+1} - \delta) a^*_{h,i,j,k+1} + w^*_{f,k+1} \tilde{l}^*_{h,i,j,k+1} \right) \right) \\
&\quad + \frac{1}{\tau} \sum_{(i,j)\in\Sigma} \left(\tilde{a}^*_{h,i,j,0} - \tilde{a}^*_{h,i,j,n_t} \right) \\
&\ge \frac{1}{\tau} \sum_{(i,j)\in\Sigma} \left(\tilde{a}^*_{h,i,j,0} - \tilde{a}^*_{h,i,j,n_t} \right) =: a_{\text{diff}} > 0,
\end{aligned}$$

where we use that $(r^*_{f,k+1} - \delta)$ is non-negative since $\delta = 0$. Hence, for each household $h \in \mathcal{H}$, there is an index $k' \in \{1, \ldots, n_t\}$ such that

$$c^*_{h,k'} \ge a_{\text{diff}}/n_t > 0 \tag{13}$$

holds. Thus,

$$\omega_h \, u'_h(c^*_{h,k'}) e^{-\gamma_h \sum_{j=0}^{k'} \tau_j} \tau_{k-1} = \theta^*_{h,k-1}$$

is satisfied due to the complementarity of $c^*_{h,k'} > 0$ and $F_1(x^*)_{h,k'} = 0$. This shows that we have a strictly positive lower bound for this time period k' on $c^*_{h,k'}$ and on $\theta^*_{h,k'-1}$ by the monotonicity of u'_h due to the Inada conditions of u_h, too.

In the following we are considering the two cases $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,k'} = 0$ and $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,k'} > 0$. Let us start with the case that for k' it holds $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,k'} = 0$. By rewriting the discretized differential equation

$(F_6(x^*)_{h,k'-1} = 0)$ it follows again by using $\delta = 0$ that

$$\begin{aligned} c^*_{h,k'} &= \sum_{(i,j)\in\Sigma} \left(\tilde{a}^*_{h,i,j,k'} \tilde{r}^*_{i,j,k'} + \tilde{l}^*_{h,i,j,k'} w^*_{i,j,k'} \right) - \frac{1}{\tau} \sum_{(i,j)\in\Sigma} (\tilde{a}^*_{h,i,j,k'} - \tilde{a}^*_{h,i,j,k'-1}) \\ &= \sum_{(i,j)\in\Sigma} \tilde{l}^*_{h,i,j,k'} w^*_{f,k'} + \frac{1}{\tau} \sum_{(i,j)\in\Sigma} \tilde{a}^*_{h,i,j,k'-1} \\ &\geq \min_{f\in\mathcal{F}} w^*_{f,k'} l_{h,k'} > 0 \end{aligned}$$

holds. For the other case, i.e., k' with $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,k'} > 0$, there exists $(i',j') \in \Sigma$ such that $a^*_{h,i',j',k'} > 0$ holds since a^* is non-negative. By complementarity of $a^*_{h,i',j',k'} > 0$ and $F_2(x^*)_{h,i',j',k'} = 0$, we obtain

$$-\frac{\theta^*_{h,k'}}{\tau} \mathbb{D}_{i,j} + \theta^*_{h,k'-1} \mathbb{D}_{i,j} \left(\frac{1}{\tau} - \tilde{r}^*_{i,j,k'} \right) = 0.$$

Rearranging leads to

$$\theta^*_{h,k'-1} = \frac{\theta^*_{h,k'}}{1 - \tau \tilde{r}^*_{i,j,k'}}$$

and $\theta^*_{h,k'-1}$ is bounded by the the term on the right-hand side, whereas $1 - \tau\tilde{r}^*_{i,j,k'} > 0$ holds for sufficiently small τ, e.g., $\tau < 1/r^+$. Now, by utilizing $F_1(x^*)_{h,k'} \geq 0$ it follows that

$$\theta^*_{h,k'-1} \geq \tau \omega_h \, u'_h(c^*_{h,k'}) e^{-\gamma_h t_{k'}}$$

holds. From this and the Inada conditions in Assumption 1 ($u'_h(c) \to \infty$ for $c \to 0$) we get that $c^*_{h,k'}$ is bounded from 0 by a constant that depends on $\theta^*_{h,k'-1}$. In summary, for k', the value of $c_{h,k'}$ is constrained either directly from below for $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,k'} = 0$ or via $\theta_{h,k'-1}$ or $\theta_{h,k'+1}$ for the case $\sum_{(i,j)\in\Sigma} a^*_{h,i,j,k'} > 0$. Since we showed the boundedness for at least one k' in (13) and because only the two aforementioned cases occur we have showed the boundedness for all times k. The complementarity of $c^* > 0$ and $F_1(x^*) = 0$ then yields the boundedness of θ^*.

Boundedness of ζ^* and ε^*: Because $F_5(x^*) = 0$ holds we know that for all $h \in \mathcal{H}$ and k', there exists $(i',j') \in \Sigma$ with $l^*_{h,i',j',k'} > 0$ such that by complementarity, $F_3(x^*)_{h,i',j',k'} = 0$ holds. Rearranging this equation leads to

$$\zeta^*_{h,k'} = \tau(1 - \omega_h) v_{h,i',j'} e^{-\gamma_h t_{k'}} + \theta^*_{h,k'-1} \tilde{w}^*_{i',j',k'}.$$

Hence, we obtain the boundedness of $\zeta^*_{h,k'}$. Analogously, from $F_4(x^*) \geq 0$ it follows for all $h \in \mathcal{H}$ that there exists $(i',j') \in \Sigma$ with $a^*_{h,i',j',n_t} > 0$ such that, by complementarity, $F_2(x^*)_{h,i',j',n_t} = 0$ holds. Rearranging this equation leads to

$$\varepsilon^*_h = \theta^*_{h,n_t-1} \left(\frac{1}{\tau} - \tilde{r}^*_{i',j',n_t} \right)$$

and we, thus, finally obtain the boundedness of ε^*_h. □

Since we showed the boundedness of all occurring variables in Theorem 2, by utilizing Theorem 1 we can state the following existence result.

Corollary 1. *Assume the setting of Theorem 2 holds. Then, there exists a solution x^* of the VI(Z,F), i.e., there exists a solution for the discretized spatial Ramsey equilibrium problem (12).*

4. Computational Experiments

In this section, we introduce a problem-tailored parallel diagonalization method and present the calibration of our model. Afterward, we discuss the initialization of our algorithm and compare the method with a state-of-the-art solver for MCPs. Finally, we perform a case study for two different scenarios and analyze the impact of different travel costs on the economic outcomes.

4.1. **Solution Approach.** The considered class of equilibrium problems is numerically challenging and is usually solved as follows. For special problem classes, e.g., for convex optimization problems of the agents that are part of the equilibrium problem (as it is the case in our models), the KKT conditions are necessary and sufficient if a suitable constraint qualification is satisfied. Concatenating these KKT conditions with the equilibrating constraints leads to the MCP (12). Alternatively, we can also state the equilibrium problem as a VI as it is done in the last section. The state-of-the-art solver for such problems is PATH; see, e.g., [3]. Since our model is highly nonlinear and of large scale, it presents a significant challenge for solvers such as PATH. Furthermore, the complexity of both the MCP and the VI approach increases in terms of the granularity of the used discretizations and in terms of the number of players.

As a computational remedy, we now present a problem-tailored diagonalization method. Diagonalization methods are widely used in the field of generalized Nash equilibrium problems (GNEPs), which fits to our context since the equilibrium problem (12) can be seen as a finite-dimensional GNEP. Although rigorous convergence results are only available for special cases, see, e.g., [7], these methods often work well in practice even if more general cases are considered.

An advantageous property of the method presented in Algorithm 1 is that the optimization problems of all households and all firms are decoupled, i.e., the problems of all households and, afterward, the problems of all firms can be solved in parallel. Thus, increasing the number of players is not drastically harming the performance of the method. In contrast to that, this has been the case for the PATH solver applied to the MCP formulation in our preliminary numerical experiments. Hence, in our implementation of Algorithm 1, we solve the problems of all households in parallel for given interest and wage rates. From the resulting asset and labor distribution of the households we then compute the new capital and labor holding of the firms by using the equilibrium conditions. Afterward, we use the optimality conditions of the firms to obtain new values for the interest and wage rates. The firms-related computations are not parallelized in our actual implementation since the respective running times are negligible.

Algorithm 1 either stops because the iteration limit is reached or yields an approximate equilibrium, which follows from that the finite-dimensional problems of all households and all firms are uniquely solvable. This is the case, since the problems of the households and firms are strictly concave maximization problems. For the households, the assumptions on the utility function and the regularization terms in the objective guarantee strictly concave problems, while for the firms, the assumptions on the production function guarantee a strictly concave problem as well.

Let us finally comment in more detail on the initialization step of the algorithm in which we solve the optimization problem of an aggregated household in which we insert an aggregated firms' problem while assuming that markets clear. Our preliminary tests showed that the overall performance of the algorithm significantly benefits from this initialization. We obtain the aggregated data as follows. The values

$$F_{\bar{f}}(\mathcal{A}, K, L) = |\mathcal{F}|^{-1} \sum_{f \in \mathcal{F}} F_f(\mathcal{A}, K, L), \quad \mathcal{A}_{\bar{f}} = |\mathcal{F}|^{-1} \sum_{f \in \mathcal{F}} \mathcal{A}_f, \quad \tilde{g}_{\bar{f},i,j} = 1$$

serve as the data of the aggregated firm and

$$a^0_{\bar{h},i,j} = \sum_{h \in \mathcal{H}} a^0_{h,i,j}, \quad a^T_{\bar{h},i,j} = \sum_{h \in \mathcal{H}} a^T_h, \quad l^0_{\bar{h},i,j} = \sum_{h \in \mathcal{H}} l^0_{h,i,j}, \quad l^0_{\bar{h},k} = \sum_{h \in \mathcal{H}} l_{h,k}$$

serve as the data of the aggregated household. All other parameters of the aggregated household are simply chosen as the averages of the corresponding data of the separate households.

The aggregated problem can be formulated as a nonlinear optimization problem by replacing $r_{\bar{f}}$ and $w_{\bar{f}}$ with the first-order conditions of the firm. Moreover, we directly insert the equilibrating conditions, i.e., the aggregated firm's asset $K_{\bar{f},k}$ is the aggregated asset of the households and analogously for labor. The interest and wage rates resulting from the solution of the aggregated problem are used as initial rates in Algorithm 1, where we set

$$r_{f,k} = r_{\bar{f},k}, \quad w_{f,k} = w_{\bar{f},k}$$

for all $f \in \mathcal{F}$ and all k.

Algorithm 1 Diagonalization Method for the Spatial Ramsey Model

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Require: Update parameter $\beta \in (0, 1]$, maximum iteration number N, tolerance $\varepsilon > 0$, equidistant grid width τ

Initialize the iteration counter: $\ell \leftarrow 0$.

Solve the aggregated equilibrium problem to obtain prices $r^\ell_{f,k}$ and $w^\ell_{f,k}$ for all f and all k.

repeat

  for $h \in \mathcal{H}$ **do**

    Compute new asset holdings a^ℓ_h and labor distributions l^ℓ_h by solving the optimization problem (4) of household h with given prices $r^\ell_{f,k}$ and $w^\ell_{f,k}$.

  end for

  for $f \in \mathcal{F}$ **do**

    Compute asset $K^{\ell+1}_{f,k}$ and labor holding $L^{\ell+1}_{f,k}$ by assuming strictly positive interest and wage rates and by solving the equilibrating conditions (6) for $K^{\ell+1}_{f,k}$ and $L^{\ell+1}_{f,k}$ with given $g_{f,i,j}$, a^ℓ_h, and l^ℓ_h.

    Compute new prices $\hat{r}^{\ell+1}_{f,k}$ and $\hat{w}^{\ell+1}_{f,k}$ for given $K^{\ell+1}_{f,k}$ and $L^{\ell+1}_{f,k}$ by assuming $K^{\ell+1}_{f,k}, L^{\ell+1}_{f,k} > 0$ and by solving the firms' first-order conditions (11) for $\hat{r}^{\ell+1}_{f,k}$ and $\hat{w}^{\ell+1}_{f,k}$ with given $K^{\ell+1}_{f,k}$ and $L^{\ell+1}_{f,k}$.

  end for

  Compute the new error

$$\varepsilon_\ell = \frac{\sqrt{\tau} \|(\hat{r}^\top, \hat{w}^\top)^{\ell+1} - (\hat{r}^\top, \hat{w}^\top)^\ell\|_2}{\beta \|(\hat{r}^\top, \hat{w}^\top)^{\ell+1}\|_2}.$$

  Update the interest and wage rates via

$$r^{\ell+1}_{f,k} = (1-\beta) r^\ell_{f,k} + \beta\, \hat{r}^{\ell+1}_{f,k}, \quad w^{\ell+1}_{f,k} = (1-\beta) w^\ell_{f,k} + \beta\, \hat{w}^{\ell+1}_{f,k}.$$

  Set $\ell \leftarrow \ell + 1$.

until $\varepsilon_\ell < \varepsilon$ **or** $\ell > N$

return Prices $\hat{r}^\ell, \hat{w}^\ell$, asset holdings a^ℓ_h, and labor distributions l^ℓ_h.

4.2. **Computational Setup.** The numerical experiments have been carried out on a compute cluster with 755 GiB of memory and with an Intel(R) Xeon(R) CPU E5-2699 CPU. The operating system is Ubuntu 18.04.6. The instances are created by implementing Algorithm 1 in Python 3.6.9, modeling the underlying optimization problems with Pyomo 6.1.1 as well as with GAMS 37.1.0. We solve all occurring problems using CONOPT4 (version 4.12) with its default settings. As the initial point for the households' problems we use the solutions of the iteration before as a warm start. In the first iteration, we use as the initial point the solution of the aggregated problem and for the aggregated problem, we use CONOPT4's default initial point. The parallelization is implemented in Python by using the multiprocessing library. We solve the households' problems in parallel by running multiple instances of GAMS in parallel. When all instances are finished, we apply the update step on the firms, and compute the new interest and wage rates.

4.3. **Calibration.** The used calibration is more of a structural type rather than of matching real-world data. The model is endowed with a simple spatial structure and rather stylized. Four type of households $\mathcal{H} = \{1, 2, 3, 4\}$ and three firms $\mathcal{F} = \{1, 2, 3\}$ populate a white map, which is the unit square $X = [0, 1]^2$. One may think of Germany as South, West, and East, for example, when it comes to the location of the firms.

Households are identical with one exception: they have chosen different residences, which they stick to. We place household 1 at $(0.25, 0.75)^\top$, the second one at $(0.15, 0.25)^\top$, the third one at $(0.75, 0.75)^\top$, and the fourth one at $(0.75, 0.25)^\top$. Their instantaneous utility is of CRRA-type as it is standard in growth economics; see [2]. We choose $\eta_h = 1.45$ as inter-temporal elasticity of substitution as proposed by Nordhaus in his DICE-13 model; see [12, p. 336]. Time is discounted using $\gamma_h = 0.03$. We set $\omega_h = 0.99$. Every household is equally endowed with asset and labor: $a^0_{h,i,j} = 281.25$ and $l^0_{h,i,j} = 52.5$, adding up to $K(0) = 1125$ and $L(0) = 210$. Labor is constant over time, hence $l_{h,k} = 52.5$. Households are free to choose where to work, facing a trade-off between travel costs and higher wages. The travel-cost scaling factor ϑ_h is either high (0.5) or low (0.05), depending on the scenario.

Firms $\mathcal{F}$ produce a homogeneous output, which aggregates to the gross domestic product (GDP). Their spatial dimension relates to their outreach on factor markets. Within the exogenously given factor-market area (catchment area) they can match capital freely with labor to produce their output. Therefore, any type of spatial structure or clustering of production sites is feasible as long as the firms operate within their initially assigned matching area.

Technology is of Cobb–Douglas type with $Y_{f,k} = F(A_{f,k}, K_{f,k}, L_{f,k})$ and $\alpha_f = r_{f,0}K_{f,0}/Y_{f,0}$ for $f \in \mathcal{F}$ at time k. GDP is normed in baseline and we set the initial production to 100 for the firms, i.e., $Y_{f,0} = 100$, gross interest is given by $r_{f,0} = 0.08$, and the wage rate index is $w_{f,0} = 1$. All factor prices are expressed in units of output, i.e., we take GDP as numeraire. Hence, we obtain $K_f(0) = 375$ and $L_f(0) = 70$. Given these numbers, $A_{f,0} = 0.863$ holds. To initiate a reasonable economic growth rate, we increase this value and fix it to $A_{f,k} \leftarrow 2$ for the entire time horizon.

We set the utility weight ω to 0.99, ϑ_h to 0.5, and λ_h to 10^{-4}. To choose the update parameter β in Algorithm 1 for our numerical tests, we run Algorithm 1 with the calibrated model for all $\beta \in \{0.025\ell \colon \ell = 1, \dots, 40\}$ with an iteration limit of 500 and a tolerance of 10^{-5}. We observed convergence for $\beta \in \{0.025\ell \colon \ell = 1, \dots, 9\}$, whereas the number of iterations and the total running time was the lowest for $\beta = 0.2$. Therefore we fix β to 0.2. Taking a closer look at the values $\beta > 0.225$ shows a significant longer running time per iteration. We conjecture that the longer

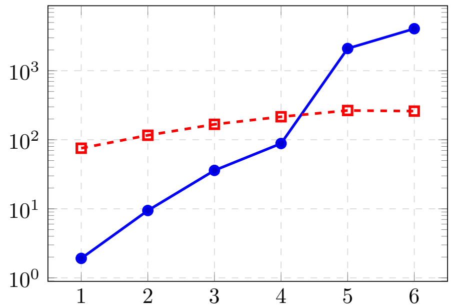

FIGURE 1. Running time in seconds (y-axis) vs. number of households (x-axis). Solid blue: MCP. Dashed red: Diagonalization.

running time per iteration is due to that if we choose β too large, the warm-starting is harmed significantly.

4.4. **Computational Comparison of the MCP and the Diagonalization Approach.** Next, we compare the performance of Algorithm 1 with the approach that solves the problem formulated as an MCP using the PATH solver [3]. Note that other MCP solvers such as KNITRO are also available. However, our preliminary tests showed no significant advantage of KNITRO over PATH, which is why we compare Algorithm 1 with PATH in the following.

The MCP formulation already fails to solve the calibrated model for instances with $n_t|\Sigma| \approx 2000$ discretization points, whereas with Algorithm 1, we can successfully solve instances of size $n_t|\Sigma| \approx 20\,000$. Accordingly, we limit our comparative analysis to smaller instances with $n_t|\Sigma| \approx 1000$.

Moreover, since Algorithm 1 serves as a heuristic because we have no convergence guarantees, we have to verify ex post that it converged to an equilibrium point. To this end, we ran numerous examples with random parameter settings and with the convergence tolerance set to $\varepsilon = 10^{-6}$ and the iteration limit set to 200. We ran the same tests for the MCP as well. All instances are solvable by the MCP approach and by Algorithm 1 within the iteration limit and we obtained the same equilibria.

Finally, we exemplarily analyze the numerical behavior of the MCP and the diagonalization approach for the case of varying numbers of households. To this end, we set $T = 100$ and discretize using an equidistant grid in time and space with $\tau = 5$ and $\xi_k = \mu_k = 1/10$. We increase the number of identical households, calibrated as in Section 4.3, from 1 up to 6. Figure 1 displays the running time compared for both approaches. We clearly see an exponential increase in running time for the MCP approach, whereas the running time of Algorithm 1 only shows a moderate increase for the larger instances.

4.5. **Impact of the Spatial Dimension.** We now compare two scenarios to study the implications of the spatial dimension on economic growth and welfare. They differ in travel costs ϑ_h only. The first scenario, HTC, has high travel costs ($\vartheta_h = 0.5$), whereas the second one, LTC, has low travel costs ($\vartheta_h = 0.05$).

The initial situation ($t = 0$) in HTC is as follows: Each household supplies each firm with a third of its initial asset holding. Hence, there is no home-bias in capital markets. Furthermore, the households' initial labor endowment is homogeneously distributed in a 0.1-ball around the household's residence x_h. Thus, at the beginning, household 1 supplies all labor to firm 1 (red), household 2 to firm 2 (green), household 3 to firm 3 (blue), and household 4 to firm 2 (green); see Figure 2. Given this allocation, $r_{1,0} = r_{3,0} < r_{2,0}$ and $w_{1,0} = w_{3,0} > w_{2,0}$ holds; see the bottom plots in Figure 3 and 4. Note that this initial spatial allocation of labor and capital is not

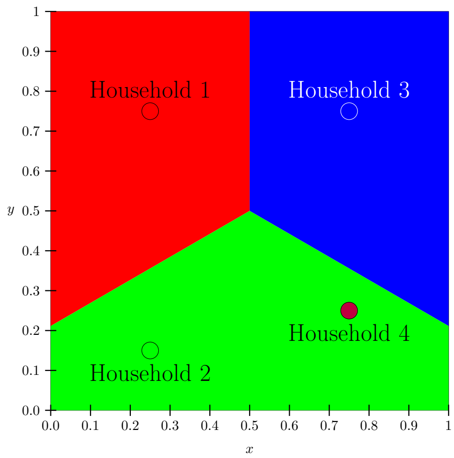

FIGURE 2. The red area is the catchment area of firm 1, the green area the one of firm 2, and the blue area the one of firm 3. Household 2 is farther away from firm 1 (red) than household 4 from firm 3 (blue). Firm 1 (red) and firm 3 (blue) as well as household 1 and household 3, respectively, are symmetrically located.

efficient. Due to constant returns to scale, efficiency would prevail if one third of the total initial asset holdings would be allocated to firm 1 and 3, and two thirds to firm 2. Otherwise, there is an initial tension on factor markets, as firm 2 attracts more capital and less labor than firms 1 and 3.

The scenarios are initialized below the steady-state in a heuristic way. We choose the initial asset values such that the economy grows without external drivers like population growth or technological progress. In our preliminary numerical experiments, we observed that the transition period into a steady-state takes roughly the same time in a spatial setting as in the standard non-spatial growth model; cf. [8]. After roughly 100 time periods, all variables are quite close to their steady-state values. Within the transition phase, however, we observe very interesting spatial patterns.

We first stick to the key economic indicators of the HTC scenario; see Figure 3. Households 1 and 3 enjoy higher consumption than households 2 and 4 due to higher wages that they earn if working for the firms in their proximity. Firms "red" and "blue" can match more capital with labor and, hence, pay higher wages than the firm "green" does. This initial disadvantage of households 2 and 4 fades out over time but it does not vanish completely. Households 2 and 4 are investing slightly more than 1 and 3, resulting in slightly higher asset holdings in the long run.

The outcomes of the LTC scenario are different. Figure 4 shows that household 1 slightly outperforms household 3 in terms of consumption. This results from slightly higher wages paid by the firm "red". Labor migration from household 2 into the "red" firm's area is slightly slower then 4's migration into the "blue" firms area. Hence, the wage difference in red is preserved longer, yielding higher income for household 1. However, as Figure 5 shows, both households 1 and 3 suffer from labor migration. Their consumption declines relative to the HTC scenario. Overall, the total efficiency gain due to low travel costs is small; a drastic drop in travel costs is almost a zero-sum game when looking on consumption patterns.

To figure out the dynamics in the transition phase, we present a series of contour plots in Figure 6. They show how household 2 allocates its labor force in the LTC scenario. Recall that this household has a strong incentive to work in the "red" firm's

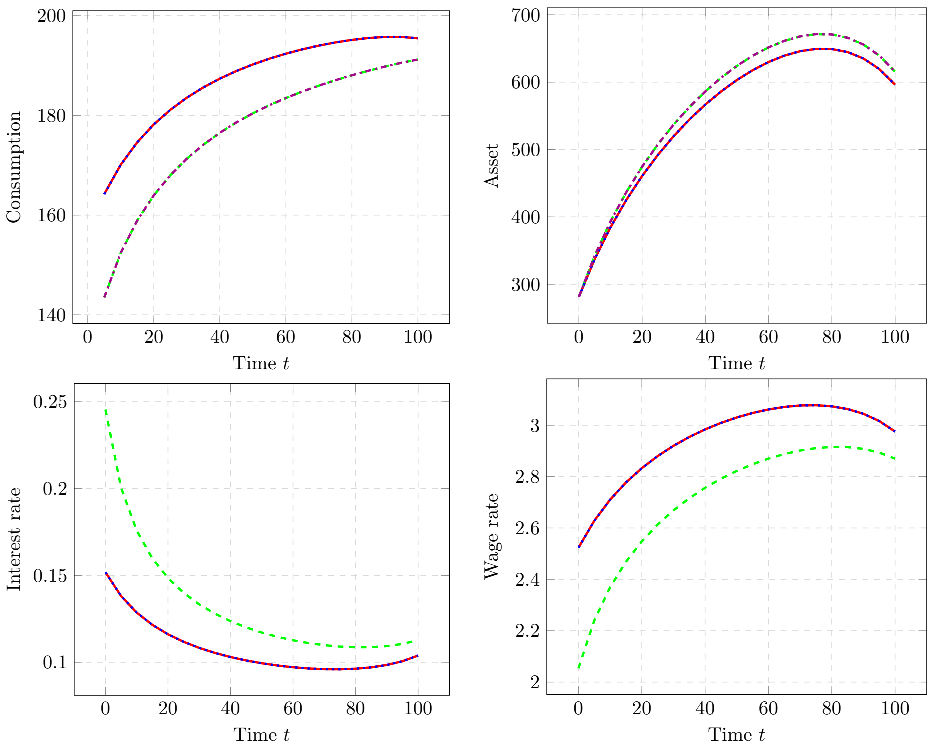

FIGURE 3. The setting of homogeneous households (with high travel costs) and firms placed according to Figure 2. Top: Consumption and asset holding for household 1 (red, solid), 2 (green, dashed), 3 (blue, dotted), and 4 (purple, dash-dotted). The red and blue curves as well as the green and purple curves overlap. Bottom: Interest and wage rate for firm 1 (red, solid), 2 (green, dashed), and 3 (blue, dotted). Here, the red and blue curves overlap.

area since wages are higher there. The household and the firm match at the border as the firm can move without costs but the household faces costs. An industry along the border between red and green emerges, which is only of transitory nature. In a modeling scenario without regularization terms in the households' objective function, a household h would supply its work to a firm f at the shortest distance due to the travel costs. See for example Figure 7, household 2's shortest distance to firm 1 (red) is at the projection of household 2's residence on the domain of firm 1 (red), which is marked as 6. Since the regularization term consists of taking the L^2-norm (in space) of the time derivative, we observe a smoothing effect both in time and space. Therefore, if a household has a preference to work (partially) at a firm, labor moves towards the projection of this firm with a smooth transition through time and space. Hence, labor does not accumulate in point 6 in our study, but in a smooth decaying area around it during this transition phase. As the capital moves toward higher interest rates, the firm in the green area gets more capital-intensive and can offer higher wage rates. In the long run, interest must be equal across regions, implying that the capital-to-labor ratio must be equal across regions. This, in turn, implies equal wage rates, too. Consequently, this economy will center production around the residences of the agents in the long run.

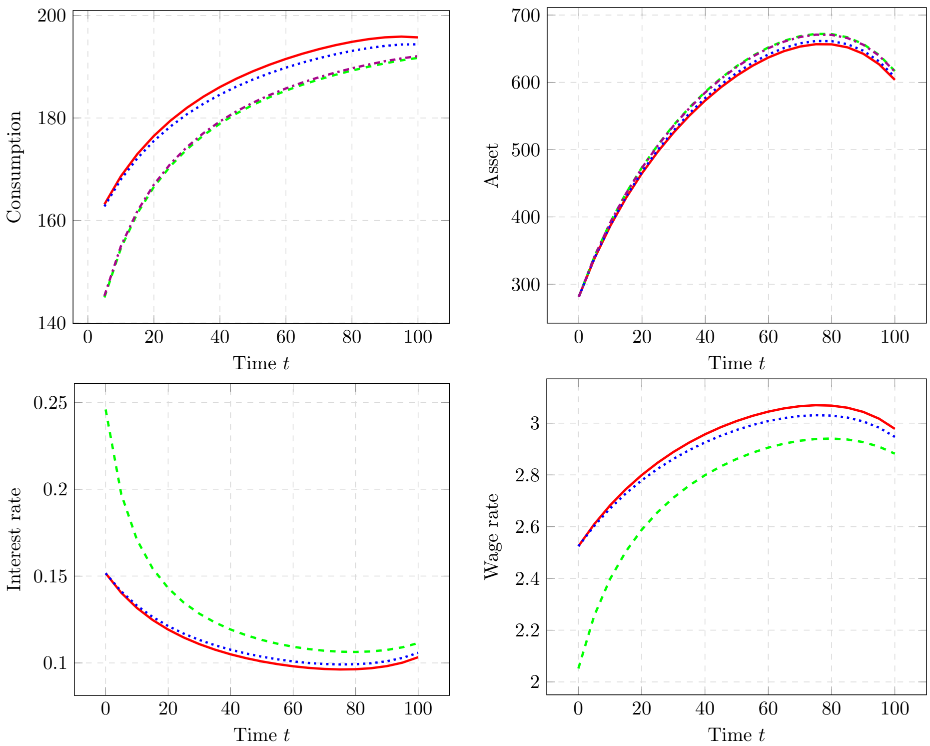

FIGURE 4. The setting of homogeneous households (with low travel costs) and firms placed according to Figure 2. Top: Consumption and asset holding for household 1 (red, solid), 2 (green, dashed), 3 (blue, dotted), and 4 (purple, dash-dotted). Bottom: Interest and wage rate for firm 1 (red, solid), 2 (green, dashed), and 3 (blue, dotted).

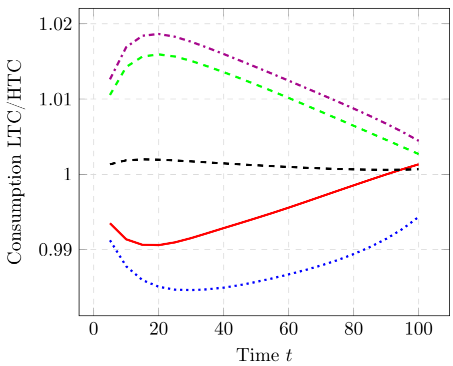

FIGURE 5. Consumption paths for the LTC scenario relative to the one for the HTC scenario per household over time. The dashed black shows the relative path of the total consumption.

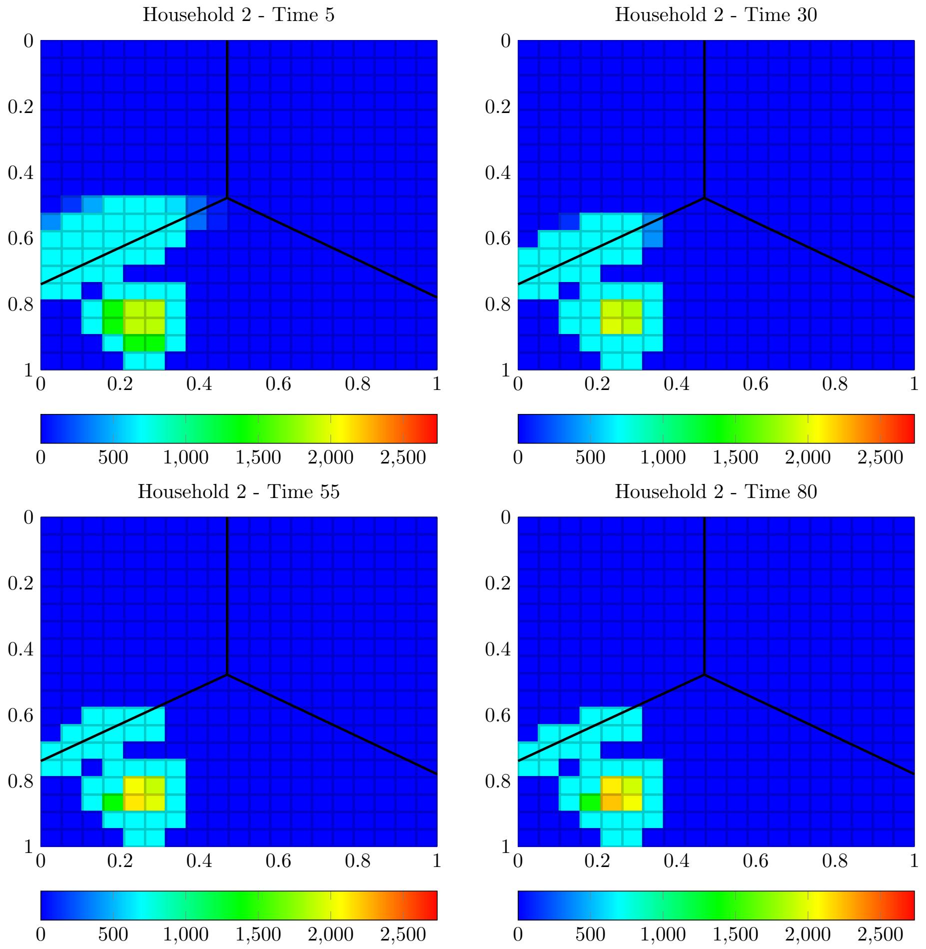

FIGURE 6. Labor allocation of household 2 in time period 5, 30, 55, and 80 for the LTC scenario.

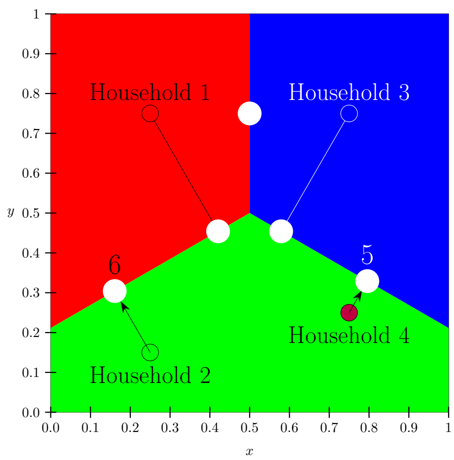

FIGURE 7. Production sites 5 and 6 are seen only during transition.

5. Conclusion

In this paper we developed a Ramsey-type equilibrium model to study the time-dependent and spatial interaction of utility-maximizing households and profit-maximizing firms. After discretization in space and time we obtain a finite-dimensional generalized Nash equilibrium problem that we model using a mixed complementarity problem. We prove existence of equilibria by the classic theory of variational inequalities and develop a tailored diagonalization method that clearly outperforms standard approaches.

Our numerical case study illustrates that our model allows to study effects of industrial agglomeration that is driven by the two opposed aspects of high wage rates and commuting costs. It is shown that these aspects are only transitory, i.e., they emerge and vanish over time so that the obtained equilibria tend towards local live-and-work economies.

Despite the developments in this paper, there is still room for improvements and future research works. Let us briefly sketch two of them. From the mathematical point of view, it is still open to study whether the resulting spatial equilibria are unique. From the point of view of economics, endogenous catchment areas of the firms would probably lead to even more realistic results but also puts a large burden on the computational tractability of the model.

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