Convergence to a second-order critical point of composite nonsmooth problems by a trust region method

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Abstract

An algorithm for finding a first-order and second-order critical point of composite nonsmooth problems is proposed in this paper. For smooth problems, algorithms for searching such a point usually utilize the so called negative-curvature directions. In this paper, the method recently proposed for nonlinear semidefinite problems by the current author is extended for solving general composite nonsmooth problems. Acceleration by Newton-like method is also proposed, where near a solution, the active functions are identified, and solving a set of linear equations is suffice to give the local quadratic convergence. It is also shown that by further solving another set of linear equations, the second-order correction is possible to avoid the Maratos effect.

Key words. nonlinear optimization; nonsmooth optimization; trust-region method; negative-curvature direction; second-order optimality

1 Introduction

Minimization of a nonsmooth function is common in practice. Even though the continuity of the derivatives is present in most of the functions that describe the real world processes, non-negligible number of advanced applications of optimization contain inherent nonsmoothness. Nonsmooth functions

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can be found in many applied fields, for example in optimal control, machine learning, economics, and computational chemistry and physics. See Part II of [1] for example.

If the structure of the nonsmooth function is not available, we usually rely on subgradient methods (single-point methods) or bundle methods (multi-point methods). The subgradient method was mainly developed by Shor and others in the Soviet Union. Basic references on subgradient methods include [14, 12, 2]. The basic idea behind the subgradient methods is to generalize the smooth methods by adopting a subgradient at the current point instead of the gradient for smooth problems. Because of its simple structure, the subgradient methods are widely used in nonsmooth optimization. However, they suffer from some serious drawbacks again due to its simple structure. We refer to the above books and follow-up works for possible remedies. Instead of relying on a single subgradient, the bundle methods first proposed in [3] consist in the simultaneous use of the information provided by many subgradients which are gathered from past iterations. Detailed description and further development in this area are described in [11, 8].

For a polyhedral convex composite objective function, trust-region methods were proposed (Fletcher [6], Yamakawa, Fukushima and Ibaraki [17], and Wright [16]). For more general class of the objective function, that is, for $F(x) : \mathbb{R}^n \to \mathbb{R}$ which is locally Lipschitz continuous and regular on $\mathbb{R}^n$, Dennis, Li, and Tapia proposed a trust-region method [5]. This method and its extension is described in [4]. In these references and more recent works, issues related to the global and local convergence to a first-order critical point, and the second-order correction step for avoiding the Maratos effect are discussed. However there is no work related to the convergence to a second-order critical point as far as the author is aware. One exception is the recent work [18] in which the current author proves the convergence to a first-order and second-order critical point by an interior point method for nonlinear semidefinite problems with an l_1-penalty merit function.

In this paper, it is shown that the convergence to a first-order and second-order critical point of the composite nonsmooth function by a trust region method which incorporates searches along negative-curvature directions is possible. Also it is shown that, by solving appropriate sets of linear equations near a KKT point, it is possible to obtain a quadratic convergence and a second-order correction step for avoiding Maratos effect.

In many practical cases, the nonsmooth function under consideration is highly structured and its form is available. Therefore, we consider the following composite nonsmooth optimization problem:

$$\text{minimize}\, F(x) = f(x) + h(c(x)),\ x \in \mathbb{R}^n, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$ are smooth, and $h : \mathbb{R}^m \to \mathbb{R}$ may be nonsmooth but convex. We assume that the structure of the nonsmooth convex function h is known.

In the following, we consider the functions f and c in a bounded set $\Omega \subset \mathbb{R}^n$ which will be specified later. Thus $c(x), x \in \Omega$ is in a bounded set $\Omega' \subset \mathbb{R}^m$. Since h is convex, it is Lipschitz continuous in Ω', and then differentiable almost everywhere in Ω' from Rademacher's theorem.

If $x^* \in \mathbb{R}^n$ is a local minimizer of $F(x)$, the first-order necessary condition (KKT condition) is given by

$$0 \in \partial F(x^*) = \nabla f(x^*) + \nabla c(x^*)\partial h(c(x^*)), \tag{2}$$

where $\partial F(x^*)$ and $\partial h(c(x^*))$ are the generalized gradients of F and h at x^* and $c(x^*)$ respectively, and

$$\nabla c(x) = (\nabla c_1(x), ..., \nabla c_m(x)) \in \mathbb{R}^{n\times m}.$$

Equivalently, we have

$$\nabla_x L(x^*, y^*) = \nabla f(x^*) + \nabla c(x^*)y^* = 0, y^* \in \partial h(c(x^*)), \tag{3}$$

where

$$L(x, y) = f(x) + c(x)^T y$$

is the Lagrangian function.

In order to discuss the second-order optimality condition for our nonsmooth problem, we need to characterize the behavior of $h(c(x))$ near a KKT point x^*. For this purpose we basically follow the framework given in 14.2 of Fletcher's book [7]. See also [13] for related information under slightly different assumptions. Let the dimension of $\partial h(c)$ be $l(\leq m)$. We will denote this by $l = \dim\partial h(c)$. Let $y \in \partial h(c)$, and $V(c) = (v_1, ..., v_l) \in \mathbb{R}^{m\times l}$ be such that $\partial h(c) - y \subset \operatorname{span}\{v_1, ..., v_l\}$, i.e., vectors $v_1, ..., v_l \in \mathbb{R}^m$ form a basis of the set $\partial h(c) - y$. Thus we can write $\partial h(c)$ as

$$\partial h(c) = \left\{ y' \in \mathbb{R}^m \,\middle|\, y' = y + V(c)w, w \in W(c, y) \subset \mathbb{R}^l \right\}, \tag{4}$$

where $W(c, y)$ is a convex and compact set. We call $V(c)$ as a basis matrix of $\partial h(c) - y$. We assume that a basis in $V(c)$ satisfies $\|V(c)\|_F <$

$\infty, \forall c \in \Omega'$, where $\|\cdot\|_F$ denotes the Frobenius norm for matrices. Let $\mathcal{V}(c) = \text{span}\{v_1, ..., v_l\}$, and $\mathcal{U}(c)$ be the linear subspace of $\mathbb{R}^m$ orthogonal to $\mathcal{V}(c)$. The subspaces $\mathcal{U}(c)$ and $\mathcal{V}(c)$ correspond to the $\mathcal{VU}$ decomposition proposed in [9].

Definition 1. We say that the nondegeneracy condition is satisfied at a KKT point x^*, if $\nabla c(x^*)V^*$, $V^* = (v_1, ..., v_l)$ has rank l, where $l = \dim\partial h(c(x^*))$, and $V^* = V(c(x^*))$ is a basis matrix of $\partial h(c(x^*)) - y^*$.

Similarly to (4), we use the following representation for $\partial h(c(x^*))$ in the following:

$$\partial h(c(x^*)) = \left\{y \in \mathbb{R}^m \,\middle|\, y = y^* + V^* w, w \in W^* \subset \mathbb{R}^l\right\}. \tag{5}$$

If the nondegeneracy condition holds, $y^* \in \partial h(c(x^*))$ is unique.

Definition 2. $h(c)$ is said to be locally linear at $c^* = c(x^*)$, if there exists an open neighborhood Π^* of c^* such that

$$h(c) = h(c^*) + \max_{y \in \partial h(c^*)} (c - c^*)^T y, \forall c \in \Pi^*.$$

Let the critical cone at x^* be defined by

$$\mathcal{C}(x^*) = \left\{ d \in \mathbb{R}^n \,\middle|\, \max_{y \in \partial h(c(x^*))} d^T \left(\nabla f(x^*) + \nabla c(x^*) y\right) = 0 \right\}.$$

If x^* is a KKT point that satisfies the nondegeneracy condition, and if $h(c)$ is locally linear at $c(x^*)$, the second-order necessary condition for optimality

$$d^T \nabla^2_{xx} L(x^*, y^*) d \geq 0, \forall d \in \mathcal{C}(x^*) \tag{6}$$

holds. The second-order sufficient condition for strict and isolated local optimality at a KKT point x^* is

$$d^T \nabla^2_{xx} L(x^*, y^*) d > 0, \forall d \in \mathcal{C}(x^*) \backslash \{0\}. \tag{7}$$

If (7) holds at a KKT pair (x^*, y^*), then the following quadratic-growth condition holds at all x in a neighborhood of x^* with $\nu > 0$

$$F(x) \geq F(x^*) + \nu \|x - x^*\|^2. \tag{8}$$

Let the manifold $\mathcal{M}(\hat{c})$ be defined by

$$\mathcal{M}(\hat{c}) = \{x \in \mathbb{R}^n \,| c(x) - \hat{c} \in \mathcal{U}(\hat{c})\},$$

for $\hat{c} \in \mathbb{R}^m$, and $V(\hat{c}) = (\hat{v}_1, ..., \hat{v}_l)$ be a basis matrix at $\hat{c}$. Then we have

$$\mathcal{M}(\hat{c}) = \left\{x \in \mathbb{R}^n \left| V(\hat{c})^T (c(x) - \hat{c}) = 0\right.\right\},$$

since $\mathcal{V}(\hat{c}) = \text{span}\{\hat{v}_1, ..., \hat{v}_l\}$. If $h(c)$ is locally linear at c^*, we have

$$h(c(x)) = h(c^*) + y^{*T}(c(x) - c^*),$$

for $x \in \mathcal{M}(c^*)$ sufficiently close to x^*, where $\partial h(c^*)$ is represented by (5). We see that F is smooth on $\mathcal{M}(c^*)$ in a neighborhood of x^*. This property is the "restricted smoothness" in [10]. We note that the tangent space of $\mathcal{M}(\hat{c})$ at $x \in \mathcal{M}(\hat{c})$ is given by

$$T_{\mathcal{M}(\hat{c})}(x) = \left\{d \in \mathbb{R}^n \left| V(\hat{c})^T \nabla c(x)^T d = 0\right.\right\}.$$

Definition 3. Under the nondegeneracy condition, the strict complementarity condition means that

$$0 \in \text{int}W^*, \tag{9}$$

where W^* is defined in (5), and $\text{int}W^*$ denotes the interior of W^*.

The strict complementarity condition is equivalent to $0 \in \text{ri}\partial F(x^*)$, or $y^* \in \text{ri}\partial h(c^*)$, where $\text{ri}\partial F(x^*)$ and $\text{ri}\partial h(c^*)$ denote the relative interior of $\partial F(x^*)$ and $\partial h(c^*)$ respectively.

If the nondegeneracy condition and the strict complementarity condition holds at x^*, then $\mathcal{C}(x^*) = T_{\mathcal{M}(c^*)}(x^*)$. In the following, we assume the regularity conditions for a KKT point that are defined as follows.

Definition 4. A KKT point x^* satisfies the regularity conditions, if (i) $h(c)$ is locally linear at $c^* = c(x^*)$, (ii) the nondegeneracy condition holds at x^*, and (iii) the strict complementarity condition holds at x^*.

Thus, in this paper, we consider an algorithm for obtaining a KKT point x^* that satisfies the following second order necessary condition

$$d^T \nabla^2_{xx} L(x^*, y^*) d \geq 0, \forall d \in T_{\mathcal{M}(c^*)}(x^*), \tag{10}$$

assuming x^* satisfies the regularity conditions.

In the following algorithm, we basically adopt a trust-region search with a standard descent step generated by a linear-quadratic approximation of the problem [4, 7]. If it is possible to estimate an approximation of the tangent space $T_{\mathcal{M}(c^*)}(x^*)$ at the current approximation (x,y) of (x^*,y^*), and if $\nabla^2_{xx}L(x,y)$ is not positive semidefinite on this approximate tangent space, we may generate a negative-curvature direction along this tangent space. The Lagrangian may decrease along this direction, but we cannot expect that the objective function decreases as well in general. Therefore we project a point on the tangent space onto the manifold where the behaviors of the objective function and the Lagrangian coincide approximately, and may obtain a decreasing value of the objective function. In this way, we escape from the negative-curvature region without losing the global convergence property.

Before discussing the details of our algorithm, well known examples of the function $h(c)$ are given here for later reference [4].

Example 5. The first example is

$$h(c) = \left\| c^+ \right\|,$$

where

$$c_i^+ = \begin{cases} c_i, & i \in \mathcal{E} \\ \max\{c_i, 0\} & i \in \mathcal{I} \end{cases},$$

and $\mathcal{E} \cup \mathcal{I} = \{1, ..., m\}, \mathcal{E} \cap \mathcal{I} = \varnothing$. If $\|\cdot\|$ is monotonic ($|a| \le |b| \Rightarrow \|a\| \le \|b\|$), then

$$\partial h(c) = \left\{ y \in \mathbb{R}^m \,\middle|\, c^T y = h(c),\ y_i \ge 0 \text{ for } i \in \mathcal{I},\ \text{and } \|y\|_D \le 1 \right\},$$

where $\|\cdot\|_D$ denotes the dual norm of $\|\cdot\|$ ($\|a\|_D = \sup_{\|b\| \le 1} a^T b$). If $c_i < 0, i \in \mathcal{I}$ then $y_i = 0$, and if $h(c) > 0$, then $\|y\|_D = 1$. We list the more specific examples in the following.

(i) If $\|c^+\| = \|c^+\|_1$, then $\|y\|_D = \|y\|_\infty$, and

$$\partial h(c) = \left\{ y \in \mathbb{R}^m \,\middle|\, y_i = \begin{cases} 1 & \text{if } c_i > 0, \\ \in [-1, 1] & \text{if } c_i = 0, i \in \mathcal{E}, \\ -1 & \text{if } c_i < 0, i \in \mathcal{E}, \\ \in [0, 1] & \text{if } c_i = 0, i \in \mathcal{I}, \\ 0 & \text{if } c_i < 0, i \in \mathcal{I}. \end{cases} \right\}.$$

Let the active function set be defined by $\mathcal{A} = \{i \,|\, c_i = 0, i = 1, ..., m\}$. Then a basis matrix $V = (v_1, ..., v_l)$, where $l = \dim \partial h(c) = |\mathcal{A}|$, is composed of $e_i, i \in \mathcal{A}$, where $(e_1, ..., e_m) = I$.

(ii) If $\|c^+\| = \|c^+\|_\infty$, then $\|y\|_D = \|y\|_1$, and

$$\partial h(c) = \left\{ y \in \mathbb{R}^m \middle| y_i \left\{ \begin{array}{ll} \geq 0 & \text{if } c_i = h(c), \\ \leq 0 & \text{if } c_i = -h(c) < 0, i \in \mathcal{E}, \\ = 0 & \text{if } |c_i| < h(c), i \in \mathcal{E}, \\ = 0 & \text{if } c_i < h(c), i \in \mathcal{I}, \end{array} \right. \right.$$
$$\left. \text{and} \sum_{i=1}^{m} |y_i| \left\{ \begin{array}{ll} = 1 & \text{if } h(c) > 0 \\ \leq 1 & \text{if } h(c) = 0 \end{array} \right. \right\}.$$

If $h(c) = 0$, then $\mathcal{A} = \{i \,| c_i = 0, i = 1, ..., m\}$, and V may be composed of $e_i, i \in \mathcal{A}$. If $h(c) > 0$, then $\mathcal{A} = \{i \,||c_i| = h(c), i \in \mathcal{E} \text{ and } c_i = h(c), i \in \mathcal{I}\}$, and V is composed of $\text{sign}(c_i)e_i - \text{sign}(c_q)e_q, i \in \mathcal{E} \cap (\mathcal{A} \backslash q)$ and $e_i - e_q, i \in \mathcal{I} \cap (\mathcal{A} \backslash q)$, where $q \in \mathcal{A}$ is arbitrary, and $\text{sign}(\cdot)$ means a sign of the corresponding quantity.

(iii) If $\|c^+\| = \|c^+\|_2$, then $\|y\|_D = \|y\|_2$, and

$$\partial h(c) = \left\{ y \in \mathbb{R}^m \middle| y_i \left\{ \begin{array}{ll} = \frac{c^+}{\|c^+\|_2} & \text{if } c^+ \neq 0, \\ \geq 0 & \text{if } i \in \mathcal{I}, \\ = 0 & \text{if } c_i < 0, i \in \mathcal{I}, \end{array} \right. \quad \text{and } \|y\|_2 \leq 1 \right\}.$$

If $c^+ \neq 0$, then $\|c^+\|_2$ is differentiable, and $l = \dim \partial h(c) = 0$. If $c^+ = 0$, then $\mathcal{A} = \{i \,| c_i = 0, i = 1, ..., m\}$, and V may be composed of $e_i, i \in \mathcal{A}$.

Example 6. The second example is a polyhedral convex function of the form

$$h(c) = \max_i a_i^T c + b_i, \tag{11}$$

where $a_i \in \mathbb{R}^m, i = 1, ..., m$ and $b_i \in \mathbb{R}, i = 1, ..., m$ are given constants. We note that the above functions $h(c) = \|c^+\|_1$ and $h(c) = \|c^+\|_\infty$ also belong to this class. We have

$$\partial h(c) = \text{co}\{a_i\}, i \in \mathcal{A} = \left\{ i \left| a_i^T c + b_i = h(c), i = 1, ..., m \right. \right\},$$

where $\text{co}\{\cdot\}$ means a convex hull. Columns of V are composed of any basis from $\text{span}\{a_i - a_q\}, i \in \mathcal{A} \backslash q$, where $q \in \mathcal{A}$ is arbitrary. A special case of this example is given by

$$h(c(x)) = \max_i c_i.$$

Then

$$\partial h(c) = \left\{ y \in \mathbb{R}^m \,\middle|\, y \geq 0,\ y_i = 0 \text{ if } c_i < h(c) \text{ and } \sum_{i=1}^{m} y_i = 1 \right\},$$

and V is composed of $e_i - e_q, i \in \mathcal{A} \backslash q$, where $q \in \mathcal{A}$ is arbitrary.

We note that the local linearity of $h(c)$ at $c(x^*)$ holds for the polyhedral convex function (11), and thus it also holds for $h(c) = \|c^+\|_1$ and $h(c) = \|c^+\|_\infty$. It is also true for $h(c) = \|c^+\|_2$ if $c^+(x^*) = 0$.

In this paper we assume the followings for the functions f, c and h. We again note that, whenever we refer to the variable $x \in \mathbb{R}^n$, we understand that $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a compact set.

Assumptions

(A.I) The functions f, $c_i, i = 1, ..., m$ are twice continuously differentiable, and the function h is convex.

(A.II) The second derivatives of the functions f, $c_i, i = 1, ..., m$ are Lipschitz continuous in a convex set $\bar{\Omega} \supset \Omega$. □

2 Trust-region method

In this section, we describe the proposed algorithm for finding a point that satisfies the first-order and second-order optimality conditions.

2.1 Linear and quadratic approximations

We will use linear and quadratic approximations of the objective function F in the proposed algorithm. Firstly, the model linear function at $x + s$ is defined by

$$\begin{aligned} F_l(x; s) &= f(x) + \nabla f(x)^T s + h(c(x) + \nabla c(x)^T s) \\ &= f_l(x; s) + h(c_l(x; s)), \end{aligned}$$

where

$$f_l(x; s) = f(x) + \nabla f(x)^T s \ \text{ and } \ c_l(x; s) = c(x) + \nabla c(x)^T s.$$

The difference of the functions $F_l(x;s)$ and $F(x)$ will play the crucial role in the following:

$$\Delta F_l(x;s) = F_l(x;s) - F(x) = \nabla f(x)^T s + h(c_l(x;s)) - h(c(x)). \qquad (12)$$

Secondly, the model quadratic function is

$$\begin{aligned} F_q(x;s) &= f(x) + \nabla f(x)^T s + \frac{1}{2} s^T \nabla^2 f(x) s + h(c(x) + \nabla c(x)^T s + \frac{1}{2} s^T \nabla^2 c(x) s) \\ &= f_q(x;s) + h(c_q(x;s)), \end{aligned}$$

where

$$f_q(x;s) = f_l(x;s) + \frac{1}{2} s^T \nabla^2 f(x) s$$

and

$$c_q(x;s) = c_l(x;s) + \frac{1}{2} s^T \nabla^2 c(x) s.$$

The difference of $F_q(x;s)$ and $F(x)$ is

$$\begin{aligned} \Delta F_q(x;s) &= F_q(x;s) - F(x) \\ &= \nabla f(x)^T s + \frac{1}{2} s^T \nabla^2 f(x) s + h(c_q(x;s)) - h(c(x)) \\ &= \Delta F_l(x;s) + \frac{1}{2} Q(x;s), \qquad (13) \end{aligned}$$

where

$$\frac{1}{2} Q(x;s) = F_q(x;s) - F_l(x;s) = \frac{1}{2} s^T \nabla^2 f(x) s + h(c_q(x;s)) - h(c_l(x;s)).$$

Since h is Lipschitz continuous, and $\nabla^2 f(x)$ and $\nabla^2 c_i(x), i = 1, ..., m$ are also Lipschitz continuous, we have

$$h\left(c(x+s)\right) - h\left(c_q(x;s)\right) = \mathrm{O}\left(\|c(x+s) - c_q(x;s)\|\right) = \mathrm{O}\left(\|s\|^3\right)$$

and

$$f(x+s) - f_q(x;s) = \mathrm{O}\left(\|s\|^3\right),$$

and thus

$$\Delta F(x;s) = F(x+s) - F(x) = \Delta F_q(x;s) + \mathrm{O}\left(\|s\|^3\right), \qquad (14)$$

where $a = \mathrm{O}(b), b \geq 0$ means that there exists a positive constant ξ such that the norm of a is bounded by ξb.

2.2 Subproblem for step calculation

In order to calculate steps for the trust region method which will be described later, we solve subproblems as follows. The first subproblem which is crucial for the global convergence is

$$\text{minimize} \quad \psi_D(d) = \tfrac{1}{2}d^T Dd + \nabla f(x)^T d + h(c_l(x;d)), d \in \mathbb{R}^n \tag{15}$$

where $D \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Its solution (d_D, y_D) satisfies

$$Dd_D + \nabla f(x) + \nabla c(x) y_D = 0, y_D \in \partial h(c_l(x; d_D)). \tag{16}$$

An obvious choice of the matrix D is a positive diagonal matrix. Another practical choice is a positive definite approximation to the Hessian of the Lagrangian of F. It is also to be noted that the terms in $\psi_D(d)$ can be interpreted as a linear approximation of the objective function $\nabla f(x)^T d + h(c_l(x;d))$ plus a stabilizing term $\frac{1}{2}d^T Dd$ which assures the existence of a solution d_D. Since the Hessian of the Lagrangian of ψ_D is D, the second-order sufficient condition for problem (15) is satisfied at a solution d_D. We will show that the direction d_D is a descent direction of the objective function $F(x)$ in Lemma 7. We also note that if $d_D = 0$, x is a KKT point.

Another subproblem is

$$\text{minimize} \quad \psi_G(d) = \tfrac{1}{2}d^T Gd + \nabla f(x)^T d + h(c_l(x;d)), d \in \mathbb{R}^n \tag{17}$$

where $G \in \mathbb{R}^{n\times n}$ is the Hessian of the Lagrangian. G may be indefinite. The KKT condition of the problem is given by

$$Gd_N + \nabla f(x) + \nabla c(x) y_N = 0, y_N \in \partial h(c_l(x; d_N)).$$

The subscript N in d_N stands for the Newton direction. This direction will be used later for the acceleration of the proposed algorithm. A problem in which $h(c_l(x;d))$ in (17) is replaced by $h(c_q(x;d))$ will be considered for the second-order correction for avoiding Maratos effect in 4.2.

Usual approaches for solving these subproblems rely on transformations to the corresponding equivalent quadratic programs whose forms depend on $h(c_l(x;d))$. See [7] and [4] for example.

Lemma 7. *There holds*

$$\Delta F_l(x; d_D) \le -\frac{1}{2} d_D^T D d_D. \tag{18}$$

Proof. Since

$$\psi_D(d_D) \le \psi_D(0) = h(c(x)),$$

it follows

$$\frac{1}{2} d_D^T D d_D + \nabla f(x)^T d_D + h(c_l(x; d_D)) - h(c(x)) \le 0.$$

Thus we have

$$\Delta F_l(x; d_D) = \nabla f(x)^T d_D + h(c_l(x; d_D)) - h(c(x)) \le -\frac{1}{2} d_D^T D d_D.$$

□

In the algorithm described later, it is necessary to minimize the model quadratic function $F_q(x; \alpha s)$ in the region $\alpha \in [0, 1]$. Since $F_q(x; \alpha s)$ is not smooth, its minimum cannot be expressed in a closed form. Instead, we rely on a smooth quadratic function $\bar{F}_q(x; \alpha, s)$ which approximates $F_q(x; \alpha s)$ from the above.

Lemma 8. *Let*

$$\bar{F}_q(x; \alpha, s) = F(x) + \alpha \Delta F_l(x; s) + \frac{1}{2} \alpha^2 Q(x; s).$$

Then for $\alpha \in [0, 1]$, *there hold*

$$\Delta F_l(x; \alpha s) \le \alpha \Delta F_l(x; s), \tag{19}$$

$$F_q(x; \alpha s) \le \bar{F}_q(x; \alpha, s), \tag{20}$$

and

$$F_q(x; 0) = F(x) = \bar{F}_q(x; 0, s), \;\; F_q(x; s) = \bar{F}_q(x; 1, s). \tag{21}$$

Proof. Since h is convex, we have

$$\begin{aligned} h(c_l(x; \alpha s)) &= h(\alpha c(x) + \alpha \nabla c(x) s + (1 - \alpha) c(x)) \\ &\le \alpha h(c_l(x; s)) + (1 - \alpha) h(c(x)). \end{aligned} \tag{22}$$

Thus from (12), it follows

$$\begin{aligned} \Delta F_l(x; \alpha s) &= \alpha \nabla f(x)^T s + h(c_l(x; \alpha s)) - h(c(x)) \\ &\le \alpha \nabla f(x)^T s + \alpha \left(h(c_l(x; s)) - h(c(x)) \right), \end{aligned}$$

and then we have (19). Similarly we have

$$\begin{aligned}
h(c_q(x;\alpha s)) &= h(\alpha c_l(x;\alpha s) + \frac{\alpha^2}{2} s^T \nabla^2 c(x) s + (1-\alpha) c_l(x;\alpha s)) \\
&\leq \alpha h(c_l(x;\alpha s) + \frac{\alpha}{2} s^T \nabla^2 c(x) s) + (1-\alpha) h(c_l(x;\alpha s)) \\
&= \alpha h(\alpha c(x) + \alpha \nabla c(x) s + \frac{\alpha}{2} s^T \nabla^2 c(x) s \\
&+ (1-\alpha) c(x)) + (1-\alpha) h(c_l(x;\alpha s)) \\
&\leq \alpha^2 h(c_q(x;s)) + (1-\alpha)\alpha h(c(x)) + (1-\alpha) h(c_l(x;\alpha s)).
\end{aligned}$$

Then from (22), we have

$$\begin{aligned}
h(c_q(x;\alpha s)) &\leq \alpha^2 h(c_q(x;s)) + (1-\alpha)\alpha h(c(x)) \\
&+ (1-\alpha)(\alpha h(c_l(x;s)) + (1-\alpha) h(c(x))) \\
&\leq \alpha^2 \left(h(c_q(x;s)) - h(c_l(x;s))\right) + \alpha \left(h(c_l(x;s)) - h(c(x))\right) + h(c(x)).
\end{aligned}$$

Now, we have

$$\begin{aligned}
F_q(x;\alpha s) &= f(x) + \alpha \nabla f(x)^T s + \frac{\alpha^2}{2} s^T \nabla^2 f(x) s + h(c_q(x;\alpha s)) \\
&\leq f(x) + h(c(x)) + \alpha \left(\nabla f(x)^T s + h(c_l(x;s)) - h(c(x))\right) \\
&+ \alpha^2 \left(\frac{1}{2} s^T \nabla^2 f(x) s + h(c_q(x;s)) - h(c_l(x;s))\right),
\end{aligned}$$

and (20) follows. Relations in (21) are obvious. □

Let

$$\alpha^*(x;d) = \operatorname{argmin}\left\{\bar{F}_q(x;\alpha,d) \,|\alpha \in [0,1], \|\alpha d\| \leq \delta\right\}, \tag{23}$$

where $\delta > 0$ is a trust-region radius.

Lemma 9. *Assume that $\Delta F_l(x;d) < 0$. Then the step size defined by (23) can be expressed as*

$$\alpha^*(x;d) = \min\left\{1, \frac{\delta}{\|d\|}, -\frac{\Delta F_l(x;d)}{\max\{Q(x;d), 0\}}\right\} \tag{24}$$

where the last term in the braces in the right-hand side is assumed to give the value ∞ if the value of the denominator is 0. Furthermore it holds that

$$\Delta F_q(x;\alpha^*(x;d)d) \leq \frac{1}{2}\alpha^*(x;d)\Delta F_l(x;d). \tag{25}$$

Proof. If $Q(x;d) \le 0$, we have

$$\alpha^*(x;d) = \min\left\{1, \frac{\delta}{\|d\|}\right\} = \min\left\{1, \frac{\delta}{\|d\|}, -\frac{\Delta F_l(x;d)}{\max\{Q(x;d), 0\}}\right\}.$$

Then from (20) and $\Delta F_l(x;d) < 0$,

$$\begin{aligned}\Delta F_q(x;\alpha^*(x;d)d) &\le \alpha^*(x;d)\Delta F_l(x;d) + \frac{1}{2}\alpha^*(x;d)^2 Q(x;d)\\ &\le \frac{1}{2}\alpha^*(x;d)\Delta F_l(x;d).\end{aligned}$$

Therefore Lemma is proved in this case.

If $Q(x;d) > 0$, the unconstrained minimum $\hat{\alpha}$ of $\bar{F}_q(x;\alpha,d)$ is

$$\hat{\alpha} = -\frac{\Delta F_l(x;d)}{Q(x;d)}.$$

Thus it follows

$$\alpha^*(x;d) = \min\left\{1, \frac{\delta}{\|d\|}, -\frac{\Delta F_l(x;d)}{Q(x;d)}\right\}. \tag{26}$$

From (26),

$$-\Delta F_l(x;d) \ge \alpha^*(x;d)Q(x;d),$$

and thus

$$\begin{aligned}\Delta F_q(x;\alpha^*(x;d)d) &\le \alpha^*(x;d)\Delta F_l(x;d) + \frac{1}{2}\alpha^*(x;d)^2 Q(x;d)\\ &\le \alpha^*(x;d)\Delta F_l(x;d) - \frac{1}{2}\alpha^*(x;d)\Delta F_l(x;d)\\ &= \frac{1}{2}\alpha^*(x;d)\Delta F_l(x;d).\end{aligned}$$

Therefore Lemma is also proved in this case. □

2.3 Direction of negative curvature

In this subsection we consider how to generate directions of negative curvature for obtaining a second-order critical point. If the objective function F is smooth at x, and if the Hessian matrix of F is not positive semidefinite, the negative-curvature direction d may be defined as an eigenvector of $\nabla^2 F$ which gives the most negative eigenvalue:

$$\nabla^2 F(x)d = \chi(x)d,\ \chi(x) = \lambda_{\min}(\nabla^2 F(x)) < 0,\ \nabla F(x)^T d \le 0,$$

where $\lambda_{\min}(M)$ is the smallest eigenvalue of the matrix M. Moving along the direction d may give a decreasing value of F for sufficiently small value of $\|\alpha d\|$, because

$$\Delta F_q(x;\alpha d) = \alpha \nabla F(x)^T d + \frac{\alpha^2}{2} d^T \nabla^2 F(x) d < 0,\ \alpha > 0.$$

If the function F is not smooth at x, a move along the negative-curvature direction is more complicated as explained below.

Let

$$\Gamma(x,y) = \left\{ s \in \mathbb{R}^n \left| h(c_q(x;s)) = h(c(x)) + (c_q(x;s) - c(x))^T y \right. \right\}, y \in \mathbb{R}^m. \tag{27}$$

For $s \in \Gamma(x,y)$, we have

$$h(c_q(x;s)) = h(c(x)) + (s^T \nabla c(x) + \frac{1}{2} s^T \nabla^2 c(x) s)^T y.$$

Thus it follows

$$\Delta F_q(x;s) = \nabla L(x,y)^T s + \frac{1}{2} s^T \nabla^2_{xx} L(x,y) s,\ \text{for } s \in \Gamma(x,y). \tag{28}$$

The relation above shows that the value of the model quadratic function can be represented by the first and second derivatives of the Lagrangian if $s \in \Gamma(x,y)$. In view of (10), if we have an estimate $\mathcal{T}$ of the tangent space $\mathcal{T}_{\mathcal{M}(c^*)}(x)$, a desired negative-curvature direction at x may be an eigenvector d of $P_{\mathcal{T}} \nabla^2_{xx} L(x,y) P_{\mathcal{T}}$ which gives the most negative eigenvalue where $P_{\mathcal{T}}$ is a projection matrix onto $\mathcal{T}$. Then a small move along d, and a projection onto $\Gamma(x,y)$ may give a desired negative-curvature move with a decreasing value of the objective function. In the following, we consider how the value of $h(c_q(x;s))$ behaves when $s \notin \Gamma(x,y)$. This analysis is necessary when we try to project the current point onto $\Gamma(x,y)$ approximately.

We firstly note the following property of $\partial h(c(x))$ around a KKT point x^*, where $\partial h(c(x^*))$ is represented by (5).

Lemma 10. *Assume that $h(c)$ is locally linear at $c^* = c(x^*)$, where x^* is a KKT point. Then there holds*

$$\partial h(c) \subseteq \partial h(c^*)$$

for any $c \in \Pi^*$. Furthermore, if

$$V^{*T}(c^* - c) = 0,$$

for $c \in \Pi^*$, then

$$\partial h(c) = \partial h(c^*).$$

A proof of this lemma is given in Lemma 14.4.1 and Lemma 14.4.2 of Fletcher's book [7]. The following Lemma shows that solving problem (15) near a KKT point x^* gives the basis matrix V^* and the value of $V^{*T}c^*$ which is necessary for forming $\mathcal{M}(c^*)$ or $\mathcal{T}_{\mathcal{M}(c^*)}(x)$.

Lemma 11. *Assume that a KKT point x^* satisfies the regularity conditions. Let $x \in \mathbb{R}^n$ be sufficiently close to x^*, then there exists a unique KKT pair (d_D, y_D) of problem (15) that satisfies (16), and furthermore*

$$V^{*T}(c^* - c_l(x; d_D)) = 0 \tag{29}$$

and

$$\partial h(c(x)) \subseteq \partial h(c^*) = \partial h(c_l(x; d_D)). \tag{30}$$

Proof. Since the second-order sufficient condition for problem (15) holds at a KKT point d_D, d_D is an isolated global minimum of problem (15).

Consider the following linear equation for $(d, w) \in \mathbb{R}^n \times \mathbb{R}^l$:

$$\begin{pmatrix} D & \nabla c(x)V^* \\ V^{*T}\nabla c(x)^T & 0 \end{pmatrix} \begin{pmatrix} d \\ w \end{pmatrix} = \begin{pmatrix} -\nabla f(x) - \nabla c(x)y^* \\ V^{*T}(c^* - c(x)) \end{pmatrix}. \tag{31}$$

From the fact that D is positive definite, and from the nondegeneracy assumption, the coefficient matrix of this linear equation at $x = x^*$ is nonsingular. Therefore the coefficient matrix at x is also nonsingular when x is sufficiently close to x^*. Then (d, w) is uniquely determined, and

$$\left\| \begin{pmatrix} d \\ w \end{pmatrix} \right\| = \mathrm{O}\left(\left\| \begin{pmatrix} \nabla f(x) + \nabla c(x)y^* \\ V^{*T}(c^* - c(x)) \end{pmatrix} \right\| \right).$$

At x^*, the solution of (31) is $(d, w) = (0, 0)$. Since $0 \in \text{int}W^*$ from the strict complementarity (see (9)), we have $w \in \text{int}W^*$ which means $y^* + V^*w \in \partial h(c^*)$. The first equation of (31) is the equality part of the KKT condition of problem (15) for $(d_D, y_D) = (d, y^* + V^*w)$, and the second equation leads to (29) which means uniquely determined d_D satisfies the relation (29). We can assume that $c(x) \in \Pi^*$, and then $c_l(x; d) \in \Pi^*$. Thus $\partial h(c(x)) \subseteq \partial h(c^*) = \partial h(c_l(x; d))$ from Lemma 10, and $y^* + V^*w \in \partial h(c^*)$ yields $y_D = y^* + V^*w \in \partial h(c_l(x; d))$. Since $\nabla c(x)V^*$ has rank $l = \text{dim}\partial h(c_l(x; d))$, the nondegeneracy condition holds for problem (15), and thus y_D is uniquely determined. Therefore $(d_D, y_D) = (d, y^* + V^*w)$ is a unique KKT pair of problem (15). $\square$

The following lemma describes the behavior of $h(c(x + s))$ near x under appropriate conditions.

Lemma 12. *Let $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ be given, and let*

$$\partial h(\hat{c}) = \left\{ y \in \mathbb{R}^m \,\middle|\, y = \hat{y} + V(\hat{c})w, w \in W(\hat{c}, \hat{y}) \subset \mathbb{R}^l \right\}, \hat{c} \in \mathbb{R}^m \tag{32}$$

where $\hat{y} \in \partial h(\hat{c})$, and $V(\hat{c})$ is a basis matrix of $\partial h(\hat{c}) - \hat{y}$. If $\partial h(c') \subseteq \partial h(\hat{c}), \forall c' \in [c(x + s), c(x)]$, then

$$h(c(x + s)) = h(c(x)) + (c(x + s) - c(x))^T \hat{y} + \mathrm{O}\left(\left\| V(\hat{c})^T (c(x + s) - c(x)) \right\| \right). \tag{33}$$

Proof. If $c(x+s) = c(x)$, the lemma holds. Therefore, assume that $c(x+s) \neq c(x)$. From the mean value theorem (see for example, Theorem 3.18 of [1]), there exists $\bar{c} \in (c(x + s), c(x))$ such that

$$\begin{aligned} h(c(x + s)) - h(c(x)) &\in \partial h(\bar{c})^T (c(x + s) - c(x)) \\ &\subseteq \partial h(\hat{c})^T (c(x + s) - c(x)), \end{aligned} \tag{34}$$

where the last line follows from the assumption $\partial h(c') \subseteq \partial h(\hat{c}), \forall c' \in [c(x + s), c(x)]$. Substituting $\partial h(\hat{c})$ from (32) into (34) gives

$$h(c(x+s)) \in h(c(x)) + (c(x+s) - c(x))^T \left\{ y \in \mathbb{R}^m \,\middle|\, y = \hat{y} + V(\hat{c})w, w \in W(\hat{c}, \hat{y}) \subset \mathbb{R}^l \right\}.$$

This yields

$$h(c(x + s)) = h(c(x)) + (c(x + s) - c(x))^T \hat{y} + \mathrm{O}\left(\left\| V(\hat{c})^T (c(x + s) - c(x)) \right\| \right),$$

where we use the fact that $W(\hat{c}, \hat{y})$ is compact. $\square$

Lemma 12 shows that if an active function which is not contained in $\partial h(\hat{c})$ does not appear in the line segment $[c(x+s), c(x)]$, (33) holds. The next lemma describes the behavior of $h(c(x+s))$ near a KKT point.

Lemma 13. *Under the assumption of Lemma 11,*

$$h(c(x+s)) = h(c(x)) + (c(x+s) - c(x))^T y_D + \mathrm{O}\left(\left\|V^{*T}(c(x+s) - c(x))\right\|\right) \tag{35}$$

for sufficiently small $\|s\|$.

Proof. If x is sufficiently close to x^*, and if $\|s\|$ is sufficiently small, the line segment $[c(x+s), c(x)]$ is in Π^*. Therefore $\partial h(c') \subseteq \partial h(c^*) = \partial h(c_l(x; d_D))$ for all $c' \in [c(x+s), c(x)]$ from Lemma 11. Thus from Lemma 12, setting $\hat{c} = c_l(x; d_D)$, $\hat{y} = y_D$ and $V(\hat{c}) = V^*$ yield (35). □

We note that Lemma 12 and Lemma 13 hold for $c(x+s)$ replaced by $c_q(x; s)$. Let the assumptions of Lemma 12 hold with $\hat{c} = c_l(x; d_D)$, where $c(x+s)$ replaced by $c_q(x; s)$, and let

$$\begin{aligned}\partial h(\hat{c}) &= \partial h(c_l(x; d_D)) \\ &= \left\{y \in \mathbb{R}^m \,\middle|\, y = y_D + V(c_l(x; d_D))w, w \in W(c_l(x; d_D), y_D) \subset \mathbb{R}^l\right\},\end{aligned}$$

and

$$\mathcal{M}_q(c(x)) = \left\{s \in \mathbb{R}^n \,\middle|\, V(c_l(x; d_D))^T (c_q(x; s) - c(x)) = 0\right\}.$$

For notational convenience, we set $V_D(x) = V(c_l(x; d_D))$ in the following. Then (33) becomes

$$h(c_q(x; s)) = h(c(x)) + (c_q(x; s) - c(x))^T y_D + \mathrm{O}\left(\left\|V_D(x)^T (c_q(x; s) - c(x))\right\|\right). \tag{36}$$

From (27) and (36), $s \in \mathcal{M}_q(c(x))$ yields $s \in \Gamma(x, y_D)$. Let $P(x)$ be a projection matrix onto the tangent space $T_{\mathcal{M}_q}(x)$ of $\mathcal{M}_q(c(x))$ at x. Since $T_{\mathcal{M}_q}(x) = \left\{d \in \mathbb{R}^n \,\middle|\, V_D(x)^T \nabla c(x)^T d = 0\right\}$, we have

$$P(x) = I - \nabla c(x) V_D(x) \left(V_D(x)^T \nabla c(x)^T \nabla c(x) V_D(x)\right)^{-1} V_D(x)^T \nabla c(x).$$

Then compute the smallest eigenvalue $\chi(x, y_D)$ of the matrix $P(x)\nabla^2_{xx} L(x, y_D) P(x)$. If

$$\chi(x, y_D) = \lambda_{\min}\left(P(x)\nabla^2_{xx} L(x, y_D) P(x)\right) < 0,$$

the corresponding eigenvector $d \in \mathbb{R}^n$ may be used as a desired negative-curvature direction. Let

$$P(x)\nabla^2_{xx}L(x,y_D)P(x)d = \chi(x,y_D)d,\ \chi(x,y_D) < 0,\ \nabla_x L(x,y_D)^T d \le 0. \tag{37}$$

We note $d \in T_{\mathcal{M}_q}(x)$. A simple move along the direction d may not give a desired smaller value of F_q. In view of (28), if there exists a negative eigenvalue of $\nabla^2_{xx}L(x,y_D)$ in the tangent space $T_{\mathcal{M}_q}(x)$ of $\mathcal{M}_q(c(x))$, we try to move along the negative-curvature direction $d \in T_{\mathcal{M}_q}(x)$, and then to restore to $\Gamma(x,y_D)$. With these procedure, a point $s \in \Gamma(x,y_D)$ that gives a smaller value of $F_q(x;s)$ is sought.

Let us describe the above procedure more precisely. Let $d \in T_{\mathcal{M}_q}(x)$. Then approximately project d onto $\mathcal{M}_q(c(x))$. That is, we want to have $p = d + v \in \mathcal{M}_q(c(x))$, where v is a solution of

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|v\|^2, v \in \mathbb{R}^n \\ \text{subjectto} & V_D(x)^T\left(c_q(x;d+v) - c(x)\right) = 0. \end{array}$$

Replacing $c_q(x;d+v)$ by $c_q(x;d) + \nabla c(x)^T v$, we obtain a solvable approximation:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|v\|^2, v \in \mathbb{R}^n \\ \text{subjectto} & V_D(x)^T\left(\nabla c(x)^T v + c_q(x;d) - c(x)\right) = 0. \end{array} \tag{38}$$

Its solution is given by

$$v = \nabla c(x)V_D(x)\left(V_D(x)^T\nabla c(x)^T\nabla c(x)V_D(x)\right)^{-1}V_D(x)^T\left(c(x) - c_q(x;d)\right), \tag{39}$$

if $\nabla c(x)V_D(x)$ is of full column rank. We denote the above approximate projection procedure to $\mathcal{M}_q(c(x))$ symbolically as

$$p = d + v = \mathcal{P}(x)d.$$

Lemma 14. *Assume that $\nabla c(x)V_D(x)$ is of full column rank. Then a solution of problem (38) given by (39) is well defined for sufficiently small $\|d\|$, and there hold*

$$\|v\| \le \kappa(x)\|d\|^2 \text{ and } p = \Theta(d), \tag{40}$$

where $p = d + v$, and $\kappa(x)$ is defined by

$$\kappa(x) = \left\|\nabla c(x)V_D(x)\left(V_D(x)^T\nabla c(x)^T\nabla c(x)V_D(x)\right)^{-1}\right\|_F \|V_D(x)\|_F \left\|\nabla^2 c(x)\right\|, \tag{41}$$

and

$$\left\|\nabla^2 c(x)\right\| = \left(\sum_{i,j,k}\left(\frac{\partial^2 c_i(x)}{\partial x_j \partial x_k}\right)^2\right)^{1/2}.$$

Furthermore,

$$\left\|V_D(x)^T\left(c_q(x;p)-c(x)\right)\right\| \le 2\kappa(x)\left\|V_D(x)\right\|_F\left\|\nabla^2 c(x)\right\|\|d\|^3, \tag{42}$$

for sufficiently small $\|d\|$.

Proof. Since the matrix $\nabla c(x)V_D(x)$ is of full column rank, it follows that

$$\begin{aligned}\|v\| \le& \left\|\nabla c(x)V_D(x)\left(V_D(x)^T\nabla c(x)^T\nabla c(x)V_D(x)\right)^{-1}\right\|_F \\ &\cdot\left\|V_D(x)^T\left(c(x)-c_q(x;d)\right)\right\| \\ =& \left\|\nabla c(x)V_D(x)\left(V_D(x)^T\nabla c(x)^T\nabla c(x)V_D(x)\right)^{-1}\right\|\left\|V_D(x)^T d^T\nabla^2 c(x)d\right\|\end{aligned}$$

where the relation

$$\begin{aligned}V_D(x)^T\left(c(x)-c_q(x;d)\right) &= -V_D(x)^T\left(\nabla c(x)^T d + \frac{1}{2}d^T\nabla^2 c(x)d\right) \\ &= -\frac{1}{2}V_D(x)^T d^T\nabla^2 c(x)d\end{aligned}$$

is used. Since

$$\left\|V_D(x)^T d^T\nabla^2 c(x)d\right\| \le \|V_D(x)\|_F\left\|d^T\nabla^2 c(x)d\right\| \le \|V_D(x)\|_F\left\|\nabla^2 c(x)\right\|\|d\|^2,$$

where we used the relation

$$\begin{aligned}\left\|d^T\nabla^2 c(x)d\right\|^2 &= \sum_i\left(\sum_{j,k} d_j\frac{\partial^2 c_i(x)}{\partial x_j\partial x_k}d_k\right)^2 \le \|d\|^2\sum_{i,j}\left(\sum_k\frac{\partial^2 c_i(x)}{\partial x_j\partial x_k}d_k\right)^2 \\ &\le \|d\|^4\sum_{i,j,k}\left(\frac{\partial^2 c_i(x)}{\partial x_j\partial x_k}\right)^2,\end{aligned}$$

in the last inequality, we obtain (40).

Next, from the definition of v, we have

$$
\begin{aligned}
&\left\|V_D(x)^T\left(c_q(x;d+v)-c(x)\right)\right\| \\
&=\left\|V_D(x)^T\left(c_q(x;d)+\nabla c(x)^T v+d^T\nabla^2 c(x)v+\frac{1}{2}v^T\nabla^2 c(x)v-c(x)\right)\right\| \\
&=\left\|V_D(x)^T\left(d^T\nabla^2 c(x)v+\frac{1}{2}v^T\nabla^2 c(x)v\right)\right\| \\
&\le\left\|V_D(x)^T d^T\nabla^2 c(x)v\right\|+\frac{1}{2}\left\|V_D(x)^T v^T\nabla^2 c(x)v\right\| \\
&\le 2\left\|V_D(x)^T d^T\nabla^2 c(x)v\right\| \\
&\le 2\left\|V_D(x)\right\|_F\left\|\nabla^2 c(x)\right\|\|d\|\|v\|,
\end{aligned}
$$

for sufficiently small $\|d\|$. Thus from (40), we have (42). □

Lemma 15. *Let the assumptions of Lemma 12 hold with $\hat{c}=c_l(x;d_D)$ and $c(x+s)$ replaced by $c_q(x;s)$, and the assumptions of Lemma 14 hold. If $d\in T_{\mathcal{M}_q}(x)$, $p=\mathcal{P}(x)d$ and $v=p-d$, then*

$$
\Delta F_q(x;p)=\nabla_x L(x,y_D)^T d+\frac{1}{2}d^T\nabla^2_{xx}L(x,y_D)d+\nabla_x L(x,y_D)^T v+\mathrm{O}(\|d\|^3),
$$

for sufficiently small $\|d\|$.

Proof. From (13), (33), we have

$$
\begin{aligned}
\Delta F_q(x;p)&=\nabla f(x)^T d+\frac{1}{2}d^T\nabla^2 f(x)d+h(c_q(x;p))-h(c(x)) \\
&=\nabla f(x)^T d+\frac{1}{2}d^T\nabla^2 f(x)d+(c_q(x;p)-c(x))^T y_D \\
&+\mathrm{O}\left(\left\|V_D(x)^T(c_q(x;p)-c(x))\right\|\right).
\end{aligned}
$$

Then from (42), the above yields

$$
\begin{aligned}
\Delta F_q(x;p)&=\nabla_x L(x,y_D)^T p+\frac{1}{2}p^T\nabla^2_{xx}L(x,y_D)p+\mathrm{O}\left(\left\|V_D(x)^T\left(c_q(x;p)-c(x)\right)\right\|\right) \\
&=\nabla_x L(x,y_D)^T(d+v)+\frac{1}{2}(d+v)^T\nabla^2_{xx}L(x,y_D)(d+v)+\mathrm{O}\left(\|d\|^3\right) \\
&=\nabla_x L(x,y_D)^T d+\frac{1}{2}d^T\nabla^2_{xx}L(x,y_D)d+\nabla_x L(x,y_D)^T v+\mathrm{O}\left(\|d\|^3\right).
\end{aligned}
$$

The lemma is proved. □

The following lemma shows that a move along the negative-curvature direction and an approximate projection can give a decreasing value of the model quadratic function F_q.

Lemma 16. *Let the assumptions of Lemma 15 hold, and assume that a negative-curvature direction d that satisfies (37) and $p = \mathcal{P}(x)d$ be given. If*

$$-\frac{1}{4}\chi(x, y_D) \geq \kappa(x) \|\nabla_x L(x, y_D)\|,$$

then, for sufficiently small $\|d\|$,

$$\Delta F_q(x; p) \leq \frac{1}{8}\chi(x, y_D) \|d\|^2. \tag{43}$$

Proof. From Lemma 15, Lemma 14 and (37), we have

$$\begin{aligned}
\Delta F_q(x; d) &= \frac{1}{2}d^T \nabla^2_{xx} L(x, y_D)d + \nabla_x L(x, y_D)v + \mathrm{O}\left(\|d\|^3\right) \\
&\leq \frac{1}{2}d^T \nabla^2_{xx} L(x, y_D)d + \|\nabla_x L(x, y_D)\| \|v\| + \mathrm{O}\left(\|d\|^3\right) \\
&\leq \frac{1}{2}d^T \nabla^2_{xx} L(x, y_D)d + \kappa(x) \|\nabla_x L(x, y_D)\| \|d\|^2 + \mathrm{O}\left(\|d\|^3\right) \\
&\leq \frac{1}{2}d^T \nabla^2_{xx} L(x, y_D)d - \frac{1}{4}\chi(x, y_D) \|d\|^2 + \mathrm{O}\left(\|d\|^3\right) \\
&= \frac{1}{2}\chi(x, y_D) \|d\|^2 - \frac{1}{4}\chi(x, y_D) \|d\|^2 + \mathrm{O}\left(\|d\|^3\right) \\
&= \frac{1}{4}\chi(x, y_D) \|d\|^2 + \mathrm{O}\left(\|d\|^3\right).
\end{aligned} \tag{44}$$

Thus (43) holds for sufficiently small $\|d\|$. □

It is shown by the following lemma that Lemma 16 holds if x is sufficiently close to a KKT point x^* under appropriate assumptions.

Lemma 17. *If the assumptions of Lemma 11 hold, Lemma 16 holds.*

Proof. It is proved in Lemma 13, that the assumptions of Lemma 12 holds under the assumptions of Lemma 11. The assumption of Lemma 14 holds by the nodegeneracy condition. Thus the assumptions of Lemma 16 are satisfied, and the conclusion of the lemma holds. □

2.4 Algorithm

In this section, we describe the proposed algorithm. Basic iteration consists of a descent search within the trust region. At each iteration k, it is tried to obtain a lesser model quadratic function value than the one given by the search along the direction d_{Dk} which is generated by solving problem (15). This requirement gives global convergence of the algorithm to a KKT point. Acceleration of this procedure by using Newton direction will be discussed later.

The direction of negative curvature is tried if it is considered to be possible. From Lemma 17 , if the current iterate x_k is sufficiently close to a KKT point x^*, where it is assumed that x^* satisfies the regularity conditions, then under the condition

$$-\frac{1}{4}\chi(x_k, y_{Dk+1}) \geq \kappa(x_k) \left\| \nabla_x L(x_k, y_{Dk+1}) \right\|, \tag{45}$$

where $y_{Dk+1} \in \partial h(c_l(x_k; d_{Dk}))$ is the Lagrange multiplier of problem (15) at x_k, we can expect

$$\Delta F_q(x_k; p_k) \leq \frac{1}{8}\chi(x_k, y_{Dk+1}) \left\| d_k \right\|^2 \tag{46}$$

holds for sufficiently small $\|d_k\|$ where d_k is a negative-curvature direction that satisfies (37) with $d = d_k$ and $y_D = y_{Dk+1}$, and $p_k = \mathcal{P}(x_k)d_k$. In order to obtain p_k that satisfies (46), we start from $d_k = \bar{d}_k$, $\left\|\bar{d}_k\right\| = \delta_k$, where δ_k is a trust region radius, and compute $p_k = \mathcal{P}(x_k)d_k$. If p_k does not exist, or (46) is not satisfied, the magnitude of d_k is reduced by the factor $\beta \in (0, 1)$, and recompute p_k. This Armijo-type procedure may be repeated till condition (46) is satisfied. However, we do not know whether the current point x_k is close enough to a KKT point x^* or not. Therefore we have to rely on some criteria for when to start and stop the above procedure.

In the following algorithm, we prove the convergence of the generated sequence $\{x_k, y_{k+1}\}, k \in K \subset \{0, 1, ...\}$ to a KKT pair $\{x^*, y^*\}$, and $d_{Dk} \to 0, \nabla_x L(x_k, y_{Dk+1}) \to 0, k \in K$. Therefore, the values of $\|d_{Dk}\|$ and $\|\nabla_x L(x_k, y_{Dk+1})\|$ can be measures for how close the iterate to a KKT point. Therefore, as an obvious criterion, we check whether $\|\nabla_x L(x_k, y_{Dk+1})\| \leq \varepsilon, \varepsilon > 0$, or not. The value of ε may be large or small depending on the chosen strategy. If $\|\nabla_x L(x_k, y_{Dk+1})\| \leq \varepsilon$, (45) and $\partial h(x_k) \subseteq \partial h(c_l(x_k; d_{Dk})$ are satisfied, and if $\nabla c(x_k)V_D(x_k)$ is of full column rank, then we compute $p_k = \mathcal{P}(x_k)d_k$ where

$d_k = \bar{d}_k, \left\|\bar{d}_k\right\| = \delta_k$. If p_k exists, check if (46) holds or not. If p_k does not exist, or (46) does not hold, let $d_k \leftarrow \beta d_k$, and computation is repeated while $\|d_k\| \geq \gamma \min\{\delta_k, \|d_{Dk}\|\}, \gamma \in (0,1)$ holds.

Algorithm NSTR

Step 0. Set an initial point $w_0 = (x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$, the maximum trust-region radius and an initial trust-region radius $\delta_{\max} > \delta_0 > 0$. Set a positive definite matrix D_0. Set $\beta \in (0,1), \gamma \in (0,1)$ and $\varepsilon > 0$. Set $k = 0$.

Step 1. Solve problem (15) at x_k, and calculate d_{Dk} and y_{Dk+1} that satisfy

$$D_k d_{Dk} + \nabla f(x_k) + \nabla c(x_k) y_{Dk+1} = 0,\ y_{Dk+1} \in \partial h(c_l(x_k; d_{Dk})).$$

Step 2. Calculate the smallest eigenvalue $\chi(x_k, y_{Dk+1})$ of $P(x_k)\nabla^2_{xx}L(x_k, y_{Dk+1})P(x_k)$, and the corresponding eigenvector $\bar{d}_k$, with $\left\|\bar{d}_k\right\| = \delta_k, \nabla_x L(x_k, y_{Dk+1})^T \bar{d}_k \leq 0$.

Step 3. Termination criteria:

(i) If $\|\nabla_x L(x_k, y_{Dk+1})\| = 0$ and $\chi(x_k, y_{Dk+1}) \geq 0$, then stop.

(ii) If $\|\nabla_x L(x_k, y_{Dk+1})\| = 0$ and x_k does not satisfy the nondegeneracy condition, then stop.

Step 4. Step calculation:

(i) If $\chi(x_k, y_{Dk+1}) \geq 0$, calculate the step $s_k \in \mathbb{R}^n$ that satisfies the conditions:

$$\begin{aligned} \|s_k\| &\leq \delta_k, \\ \Delta F_q(x_k; s_k) &\leq \frac{1}{2}\Delta F_q(x_k; \alpha^*(x_k, d_{Dk})d_{Dk}). \end{aligned}$$

(ii) Otherwise, i.e., if $\chi(x_k, y_{Dk+1}) < 0$, execute the following:

(ii-a) If $\|\nabla_x L(x_k, y_{Dk+1})\| \leq \varepsilon$ and $\nabla c(x_k) V_D(x_k)$ is of full column rank, and if $\partial h(x_k) \subseteq \partial h(x_k, c_l(x_k; d_{Dk}))$ and $\frac{1}{4}|\chi(x_k, y_{Dk+1})| \geq \kappa(x_k)\|\nabla_x L(x_k, y_{Dk+1})\|$ are satisfied, where $\kappa(\cdot)$ is defined in (41), repeat the following while

$$\|d_k\| \geq \gamma \min\{\delta_k, \|d_{Dk}\|\} \tag{47}$$

holds. Find the smallest nonnegative integer j_k such that $d_k = \beta^{j_k}\bar{d}_k$ and $p_k = \mathcal{P}(x_k)d_k$ satisfy

$$\Delta F_q(x_k; p_k) \leq \frac{1}{8}\chi(x_k, y_{Dk+1}) \|d_k\|^2 . \tag{48}$$

If d_k is obtained, and if

$$\Delta F_q(x_k; p_k) \leq \frac{1}{2}\Delta F_q(x_k; \alpha^*_{\delta_k}(x_k, d_{Dk})d_{Dk}), \tag{49}$$

set $s_k = p_k$.

(ii-b) Otherwise, set $s_k = \alpha^*_{\delta_k}(x_k, d_{Dk})d_{Dk}$.

Step 5. δ_{k+1} is given by

$$\delta_{k+1} = \begin{cases} \frac{1}{2}\delta_k & \text{if } \Delta F(x_k; s_k) > \frac{1}{4}\Delta F_q(x_k; s_k) \\ 2\delta_k & \text{if } \Delta F(x_k; s_k) \leq \frac{3}{4}\Delta F_q(x_k; s_k) \\ \delta_k & \text{otherwise.} \end{cases}$$

If $\delta_{k+1} > \delta_{\max}$, set $\delta_{k+1} = \delta_{\max}$.

Step 6. If $\Delta F(x_k; s_k) \leq 0$, then set $x_{k+1} = x_k + s_k$ and $y_{k+1} = y_{Dk+1}$. Otherwise set $(x_{k+1}, y_{k+1}) = (x_k, y_k)$.

Step 7. Compute a positive definite matrix D_{k+1}, set $k := k + 1$ and go to Step 1.

Remark 18. At Step 3, if x_k satisfies the KKT condition and $\chi(x_k, y_{Dk+1}) \geq 0$, the algorithm terminates. If a KKT point x_k is degenerate, the algorithm terminates even if $\chi(x_k, y_{Dk+1}) < 0$, because the negative-curvature search of this paper cannot be applied in this case. If a KKT point x_k with $\chi(x_k, y_{Dk+1}) < 0$ is nondegenerate, the algorithm continues, and the negative-curvature search is performed in Step 4 (ii). In this case, the negative-curvature search gives p_k that satisfies (48), and (49) is satisfied with the right-hand side set to zero ($d_{Dk} = 0$).

Remark 19. At iteration k, assume that $\chi(x_k, y_{Dk+1}) < 0$ and $\frac{1}{4}|\chi(x_k, y_{Dk+1})| \geq \kappa(x_k) \|\nabla_x L(x_k, y_{Dk+1})\|$ hold, and that Step 4 (ii-a) gives $j_k > 0$ that satisfies (48). Let $p'_k = \mathcal{P}(x_k)\beta^{-1}d_k$. Thus we have

$$\Delta F_q(x_k; p'_k) > \frac{1}{8}\beta^{2(j_k-1)}\chi(x_k, y_{Dk+1}) \left\|\bar{d}_k\right\|^2 . \tag{50}$$

Then from (44) and (50), it follows

$$\frac{1}{8}\beta^{2(j_k-1)}\chi(x_k, y_{Dk+1}) \left\|\bar{d}_k\right\|^2 < \Delta F_q(x_k; p'_k)$$
$$= \frac{1}{4}\beta^{2(j_k-1)}\chi(x_k, y_{Dk+1}) \left\|\bar{d}_k\right\|^2 + \beta^{3(j_k-1)}\mathrm{O}(\left\|\bar{d}_k\right\|^3)$$

which gives

$$0 < -\frac{1}{8}\chi(x_k, y_{Dk+1}) < \beta^{j_k-1}\mathrm{O}(\left\|\bar{d}_k\right\|). \tag{51}$$

Thus it follows that j_k cannot be arbitrary large with given $\left\|\bar{d}_k\right\| = \delta_k$ independently of (47).

3 Convergence

In this section, we prove that the proposed algorithm converges to a second-order critical point as well as to a first-order critical point. For this purpose, we assume the following in addition to the previous assumptions (A.I) and (A.II).

Assumptions

(A.III) The generated sequence $\{x_k\}$ remains in a compact set Ω.

(A.IV) The matrix D_k is uniformly positive definite and uniformly bounded for $k = 0, 1, ...$ □

Define subsequences $K_1 \subset \{0, 1, \cdots\}$ and $K_2 \subset \{0, 1, \cdots\}$ by

$$\Delta F(u_k; s_k) > \frac{1}{4}\Delta F_q(u_k; s_k), \quad k \in K_1, \tag{52}$$

$$\Delta F(u_k; s_k) \leq \frac{1}{4}\Delta F_q(u_k; s_k), \quad k \in K_2. \tag{53}$$

Then we have

$$K_1 \cup K_2 = \{0, 1, \cdots\} \quad \text{and} \quad K_1 \cap K_2 = \emptyset.$$

From Step 5 of Algorithm NSTR, $\delta_{k+1} = \frac{1}{2}\delta_k$, $k \in K_1$ and $\delta_{k+1} \geq \delta_k$, $k \in K_2$. Therefore, if $\liminf_{k\to\infty, k\in K_1} \delta_k > 0$ or K_1 is finite, we have $\liminf_{k\to\infty, k\in K_2} \delta_k > 0$, otherwise we have $\liminf_{k\to\infty, k\in K_1} \delta_k = 0$. Thus, it is sufficient to consider that either

(i) $\liminf_{k\to\infty, k\in K_1} \delta_k = 0$, or

(ii) $\liminf_{k\to\infty, k\in K_2} \delta_k > 0$ holds.

Theorem 20. *Let an infinite sequence $\{x_k, y_k\}$ be generated by Algorithm NSTR. Then there exists at least one accumulation point that satisfies the KKT condition. Specifically, every accumulation point of the following subsequences satisfies the KKT condition, and there exists at least one such accumulation point:*
(i) subsequences of K_1 in which $\lim_{k\to\infty} \delta_k = 0$,
(ii) subsequences of K_2 in which $\liminf_{k\to\infty} \delta_k > 0$.

Proof. By Step 4 of Algorithm NSTR, (24) and (25) we have

$$\Delta F_q(x_k; s_k) \le \frac{1}{4}\Delta F_l(x_k; d_{Dk}) \min\left\{ 1, \ \frac{\delta_k}{\|d_{Dk}\|}, -\frac{\Delta F_l(x_k; d_{Dk})}{\max\{Q(x_k; d_{Dk}), 0\}} \right\}, \tag{54}$$

if $d_{Dk} \neq 0$. We first consider a subsequence $K_1' \subset K_1$ in which $\delta_k \to 0$. Since $\|s_k\| \le \delta_k$, we have $s_k \to 0, k \in K_1'$. To show $\lim_{k\to\infty, k\in K_1'} \|d_{Dk}\| = 0$, we assume that

$$\limsup_{k\to\infty,\ k\in K_1'} \|d_{Dk}\| > 0,$$

and therefore assume that there exist a subsequence $K_1'' \subset K_1'$ and $c_1 > 0$ such that

$$\lim_{k\to\infty,\ k\in K_1''} \|d_{Dk}\| = c_1 > 0. \tag{55}$$

Then from (18), we can assume that there exists $c_2 > 0$ such that

$$\lim_{k\to\infty, k\in K_1''} |\Delta F_l(x_k; d_{Dk})| = c_2 > 0, \tag{56}$$

without loss of generality. From (14), we note

$$\Delta F(x_k; s_k) = \Delta F_q(x_k; s_k) + \mathrm{O}\left(\|s_k\|^3\right), \quad k \in K_1',$$

and thus from (52),

$$0 < \Delta F(x_k; s_k) - \frac{1}{4}\Delta F_q(x_k; s_k) = \frac{3}{4}\Delta F_q(x_k; s_k) + \mathrm{O}(\|s_k\|^3), \quad k \in K_1'. \tag{57}$$

It follows from (57) that

$$-\Delta F_q(x_k; s_k) = \mathrm{O}\left(\|s_k\|^3\right), \quad k \in K_1'. \tag{58}$$

From (55) and (56),

$$\min\left\{1, \frac{\delta_k}{\|d_{SDk}\|}, -\frac{\Delta F_l(x_k; d_{Dk})}{\max\{Q(x_k; d_{Dk}), 0\}}\right\} = \frac{\delta_k}{\|d_{Dk}\|}$$

holds for sufficiently large $k \in K_1''$, and then (54) gives the relation

$$-\Delta F_q(x_k; s_k) \geq \frac{|\Delta F_l(x_k; d_{Dk})|}{4\|d_{Dk}\|}\|s_k\| \geq \frac{c_2}{8c_1}\|s_k\|,$$

for sufficiently large $k \in K_1''$. This contradicts (58). Thus we have $\lim_{k\to\infty, k\in K_1'} \|d_{Dk}\| = 0$.

We next consider a subsequence $K_2' \subset K_2$ in which $\liminf_{k\to\infty} \delta_k > 0$. Since $\{F(x_k)\}$ is bounded below and non-increasing, we have

$$F(x_{k+1}) - F(x_k) = \Delta F(x_k; s_k) \to 0, \quad k \in K_2$$

and thus $\Delta F_q(x_k; s_k) \to 0,\ k \in K_2$, from (53) and $\Delta F_q(x_k; s_k) \leq 0$. Therefore, we have $\Delta F_l(x_k; d_{Dk}) \to 0,\ k \in K_2'$ from (54) which holds when $d_{Dk} \neq 0$, and thus $\|d_{Dk}\| \to 0,\ k \in K_2'$ from (18).

Therefore, for every subsequence $K \subset \{0, 1, \cdots\}$ that satisfies the condition (i) or (ii), we have

$$\lim_{k\to\infty, k\in K} \|d_{Dk}\| = 0.$$

Then from (16), we have

$$\lim_{k\to\infty, k\in K} \nabla f(x_k) + \nabla c(x_k) y_{Dk+1} = 0,\ y_{Dk+1} \in \partial h(c(x_k) + \nabla c(x_k)^T d_{Dk}),$$

and by (A.III), we can conclude that there exists $\hat{x} \in \Omega$ and $\hat{y} \in \partial h(c(\hat{x}))$ such that

$$\lim_{k\to\infty, k\in K} x_k = \hat{x}, \lim_{k\to\infty, k\in K} y_{Dk+1} = \hat{y} \quad \text{and} \quad \nabla f(\hat{x}) + \nabla c(\hat{x})\hat{y} = 0. \tag{59}$$

without loss of generality.

From the fact that there exists a subsequence that satisfies either of the condition (i) or (ii), we conclude that there exists at least one accumulation point that satisfies this result. Therefore, the theorem is proved. □

The following theorem proves that there exists at least one accumulation point that satisfies the KKT condition and the second-order necessary condition.

Theorem 21. *Let an infinite sequence* $\{x_k, y_{k+1}\}$ *be generated by Algorithm NSTR, and* $\{x_k, y_{k+1}\} \to \{x^*, y^*\}, k \in K \subset \{0, 1, ...\}$ *where* $\{x^*, y^*\}$ *is a KKT pair, and* K *is either of*
(i) subsequence of K_1 *in which* $\lim_{k\to\infty} \delta_k = 0$*, or*
(ii) subsequence of K_2 *in which* $\liminf_{k\to\infty} \delta_k > 0$*.*
If x^* *satisfies the regularity conditions, then* $\chi(x^*, y^*) \geq 0$*.*

Proof. Assume that $\chi(x^*, y^*) < 0$. Then we can assume that there exists $\eta < 0$ such that $\chi(x_k, y_{k+1}) \leq \eta$ for all $k \in K$ without loss of generality.

Since $\{x_k, y_{k+1}\} \to (x^*, y^*), k \in K$, it follows $\|\nabla_x L(x_k, y_{k+1})\| \to 0, k \in K$ and $\|d_{Dk}\| \to 0, k \in K$. Since, for sufficiently large $k \in K$, $\nabla c(x_k)V_D(x_k)$ is of full column rank, $\partial h(x_k) \subseteq \partial h(x_k, c_l(x_k; d_{Dk}))$ from Lemma 11, and $\frac{1}{4}|\chi(x_k, y_{Dk+1})| \geq \kappa(x_k)\|\nabla_x L(x_k, y_{Dk+1})\|$, all the conditions in Step 4 (ii-a) are satisfied, and the negative curvature search is performed. Then it follows from Lemma 16 that there exist p_k and j_k that satisfy

$$\Delta F_q(x_k; p_k) \leq \frac{1}{8}\beta^{2j_k}\chi(x_k, y_{k+1})\left\|\bar{d}_k\right\|^2,$$

for sufficiently large $k \in K$, since the reduction of $\|d_k\|$ is repeated if necessary while

$$\|d_k\| \geq \gamma \min\{\delta_k, \|d_{Dk}\|\}$$

holds, and the right-hand side of the above inequality tends to 0 for $k \in K$. Therefore from Step 4, the following relation is satisfied for sufficiently large $k \in K$:

$$\Delta F_q(x_k; s_k) \leq \min\left\{\frac{1}{2}\Delta F_q(x_k; \alpha^*_{\delta_k}(x_k, d_{Dk})d_{Dk}), \frac{1}{8}\beta^{2j_k}\chi(x_k, y_{k+1})\left\|\bar{d}_k\right\|^2\right\}. \tag{60}$$

We first consider the subsequence K that satisfies the condition (i). Thus assume that $\delta_k \to 0, k \in K \subset K_1$. Then we have $s_k \to 0, k \in K$. From (14) and (52),we have

$$0 < \Delta F(x_k; s_k) - \frac{1}{4}\Delta F_q(x_k; s_k) = \frac{3}{4}\Delta F_q(x_k; s_k) + \mathrm{O}(\|s_k\|^3), k \in K. \tag{61}$$

Then (60) and (61) yield

$$0 < \beta^{2j_k}\chi(x_k, y_{k+1}) \left\|\bar{d}_k\right\|^2 + \mathrm{O}(\|s_k\|^3),$$

for sufficiently large $k \in K$. The above inequality gives

$$-\eta\beta^{2j_k} \leq \frac{1}{\left\|\bar{d}_k\right\|^2}\mathrm{O}(\|s_k\|^3) = \frac{1}{\delta_k^2}\mathrm{O}(\|s_k\|^3) = \mathrm{O}(\delta_k) \to 0, k \in K. \tag{62}$$

Thus we have $j_k \to \infty, k \in K$. On the other hand, we have $-\frac{1}{8}\eta < \beta^{j_k-1}\mathrm{O}(\left\|\bar{d}_k\right\|)$ from (51), and this gives a contradiction when $j_k \to \infty, k \in K$.

Next suppose $\liminf_{k\to\infty} \delta_k > 0, k \in K \subset K_2$. In this case, from (53) and (60), we have

$$\Delta F(x_k; s_k) \leq \frac{1}{4}\Delta F_q(x_k; s_k) \leq \frac{1}{32}\beta^{2j_k}\chi(x_k, y_{k+1}) \left\|\bar{d}_k\right\|^2 = \frac{1}{32}\beta^{2j_k}\chi(x_k, y_{k+1})\delta_k^2. \tag{63}$$

Since $\{F(x_k)\}$ is bounded below and non-increasing, it follows $\Delta F(x_k; s_k) \to 0, k \in K$. Thus we conclude $j_k \to \infty, k \in K$. As above, this contradicts (51) since $\left\|\bar{d}_k\right\| = \delta_k \leq \delta_{\max}$.

In both cases, we have a contradiction if we assume $\chi(x^*, y^*) < 0$. Therefore the theorem is proved. □

4 Acceleration by Newton direction

In this section we discuss how to incorporate Newton-like iteration in the algorithm discussed above. One obvious way is to solve problem (17) near a KKT point, and the quadratic convergence of the resulting iteration is shown in [15]. See also [7]. Another viewpoint which should be noted is that the matrix D_k in the above algorithm can be some sort of positive definite approximation to the Hessian of the Lagrangian, and the resulting algorithm may benefit from this feature if properly done. In this section, we pursue another method which can be performed with lesser cost than solving (17). The method proposed below is based on identifying an active set of functions by solving problem (15) if necessary, and then solving a set of linear equations for the acceleration. Similar method adopted in the SQP framework appeared in [19].

4.1 Quadratic convergence

Let the current iterate (x_k, y_k) be sufficiently close to a KKT pair (x^*, y^*) where x^* satisfies the regularity conditions. Assume further that the second-order sufficient condition holds at x^*. Let a KKT pair of problem (15) at x_k be (d_{Dk}, y_{Dk+1}). Then from Lemma 11, we have

$$\begin{aligned}\partial h(c(x_k)) &\subseteq \partial h(c(x^*)) \\ &= \partial h(c_l(x_k; d_{Dk})) = \left\{ y \in \mathbb{R}^m \,\middle|\, y = y_{Dk+1} + V^* w, w \in W_*(y_{Dk+1}) \subset \mathbb{R}^l \right\},\end{aligned}$$

where $V^* = V(c_l(x_k; d_{Dk}))$. Thus near a solution, instead of solving (17), we solve the following set of linear equations for (d_{Nk}, y_{Nk+1})

$$\begin{aligned} G_k d_{Nk} + \nabla f(x_k) + \nabla c(x_k) y_{Nk+1} &= 0, \\ V^{*T} \left(c_l(x_k; d_{Dk}) - c_l(x_k; d_{Nk}) \right) &= 0, \end{aligned}$$

where $G_k = \nabla^2_{xx} L(x_k, y_k)$, and $V^{*T} c_l(x_k; d_{Dk}) = V^{*T} c(x^*)$ from Lemma 11. The first equation is the equality part of the optimality condition for problem (17) at x_k, and the second equation is the condition for $\partial h(c(x^*)) = \partial h(c_l(x_k; d_{Dk})) = \partial h(c_l(x_k; d_{Nk}))$ to hold. See Lemma 10 and Lemma 11. If we set $y_{Nk+1} = y_k + V^* w_{Nk+1}$, the above set of equations can be rewritten as

$$\begin{pmatrix} G_k & \nabla c(x_k) V^* \\ (\nabla c(x_k) V^*)^T & 0 \end{pmatrix} \begin{pmatrix} d_{Nk} \\ w_{Nk+1} \end{pmatrix} = \begin{pmatrix} -\nabla f(x_k) - \nabla c(x_k) y_k \\ V^{*T} \nabla c(x_k) d_{Dk} \end{pmatrix}. \tag{64}$$

Theorem 22. *Assume that a KKT point x^* satisfies the regularity conditions, and that the second-order sufficient condition holds at a KKT pair (x^*, y^*). Assume further that the current iterate (x_k, y_k) is sufficiently close to (x^*, y^*). Let a KKT pair (d_{Dk}, y_{Dk+1}) of problem (15) and $y_k \in \partial h(c_l(x_k; d_{Dk})) = \partial h(c(x^*))$ are given. Then there exists a unique solution of (64), and with*

$$x_{k+1} = x_k + d_{Nk}, \tag{65}$$

$$y_{k+1} = y_{Nk+1}, \tag{66}$$

there holds

$$y_{k+1} \in \partial h(c_l(x_k; d_{Nk})) = \partial h(c(x^*)), \tag{67}$$

and

$$\left\| \begin{pmatrix} x^* - x_{k+1} \\ y^* - y_{k+1} \end{pmatrix} \right\| = \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \, \|y^* - y_k\|). \tag{68}$$

Proof. From Lemma 11, we have $V^{*T}(c(x^*) - c_l(x_k; d_{Dk})) = 0$. Then Newton iteration (64) becomes

$$\begin{pmatrix} G_k & \nabla c(x_k)V^* \\ (\nabla c(x_k)V^*)^T & 0 \end{pmatrix} \begin{pmatrix} d_{Nk} \\ w_{Nk+1} \end{pmatrix} = \begin{pmatrix} -\nabla f(x_k) - \nabla c(x_k)y_k \\ V^{*T}\left(c(x^*) - c(x_k)\right) \end{pmatrix}. \tag{69}$$

From the assumption that (x_k, y_k) is sufficiently close to (x^*, y^*), the regularity conditions and the second-order sufficiency condition, the coefficient matrix of (69) is nonsingular. Thus there exists a unique solution (d_{Nk}, w_{Nk+1}) of (69). We also have

$$\left\| \begin{pmatrix} d_{Nk} \\ w_{Nk+1} \end{pmatrix} \right\| = \mathrm{O}\left(\left\| \begin{pmatrix} -\nabla f(x_k) - \nabla c(x_k)y_k \\ V^{*T}\left(c(x^*) - c(x_k)\right) \end{pmatrix} \right\| \right).$$

From similar reasoning as in the proof of Lemma 11, we have $\partial h(c(x^*)) = \partial h(c_l(x_k; d_{Nk}))$ and $y_{k+1} \in \partial h(c_l(x_k; d_{Nk}))$.

Now we have

$$\begin{aligned}
0 = \nabla_x L(x^*, y^*) &= \nabla_x L(x_k, y^*) + \nabla^2_{xx} L(x_k, y^*)(x^* - x_k) + \mathrm{O}(\|x^* - x_k\|^2) \\
&= \nabla_x L(x_k, y_k) + \nabla c(x_k)(y^* - y_k) + \nabla^2_{xx} L(x_k, y_k)(x^* - x_k) \\
&+ \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \, \|y^* - y_k\|) \\
&= \nabla_x L(x_k, y_k) + \nabla^2_{xx} L(x_k, y_k)(x_{k+1} - x_k) + \nabla c(x_k)(y_{k+1} - y_k) \\
&+ \nabla^2_{xx} L(x_k, y_k)(x^* - x_{k+1}) + \nabla c(x)(y^* - y_{k+1}) \\
&+ \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \, \|y^* - y_k\|) \\
&= \nabla_x L(x_k, y_k) + \nabla^2_{xx} L(x_k, y_k) d_{Nk} + \nabla c(x_k) V^* w_{Nk+1} \\
&+ \nabla^2_{xx} L(x_k, y_k)(x^* - x_{k+1}) + \nabla c(x)(y^* - y_{k+1}) \\
&+ \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \, \|y^* - y_k\|) \\
&= \nabla^2_{xx} L(x_k, y_k)(x^* - x_{k+1}) + \nabla c(x)(y^* - y_{k+1}) \\
&+ \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \, \|y^* - y_k\|),
\end{aligned} \tag{70}$$

where in the last equality, we use the fact $\nabla_x L(x_k, y_k) + \nabla^2_x L(x_k, y_k) d_{Nk} + \nabla c(x_k) V^* w_{Nk+1} = 0$ from (64). On the other hand, we have

$$\begin{aligned}
V^{*T} c(x^*) &= V^{*T}\left(c(x_k) + \nabla c(x_k)^T (x^* - x_k)\right) + \mathrm{O}(\|x^* - x_k\|^2) \\
&= V^{*T}\left(c(x_k) + \nabla c(x_k)^T (x_{k+1} - x_k)\right) + V^{*T} \nabla c(x_k)^T (x^* - x_{k+1}) \\
&+ \mathrm{O}(\|x^* - x_k\|^2).
\end{aligned} \tag{71}$$

Then from the second equation of (69), we have

$$V^{*T}(c(x^*) - c_l(x_k; d_{Nk})) = 0, \tag{72}$$

and (71) and (72) lead to

$$V^{*T}\nabla c(x_k)^T(x^* - x_{k+1}) = \mathrm{O}(\|x^* - x_k\|^2). \tag{73}$$

Now equations (70) and (73) yield

$$\begin{pmatrix} G_k & \nabla c(x_k)V^* \\ (\nabla c(x_k)V^*)^T & 0 \end{pmatrix} \begin{pmatrix} x^* - x_{k+1} \\ y^* - y_{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \, \|y^* - y_k\|) \\ \mathrm{O}(\|x^* - x_k\|^2) \end{pmatrix},$$

and then (68) follows. □

We note that a natural candidate of y_k is y_{Dk+1}. However if the above Newton iteration is once successful, then in the forthcoming iterations, we may choose $y_k = y_{Nk}$. I.e., the iteration can be

$$\begin{aligned} G_k d_{Nk} + \nabla f(x_k) + \nabla c(x_k) y_{Nk+1} &= 0, \\ V^{*T}\left(c_l(x_{k-1}; d_{Nk-1}) - c_l(x_k; d_{Nk})\right) &= 0, \end{aligned} \tag{74}$$

or

$$\begin{pmatrix} G_k & \nabla c(x_k)V^* \\ (\nabla c(x_k)V^*)^T & 0 \end{pmatrix} \begin{pmatrix} d_{Nk} \\ w_{Nk+1} \end{pmatrix} = \begin{pmatrix} -\nabla f(x_k) - \nabla c(x_k) y_k \\ V^{*T}\left(c_l(x_{k-1}; d_{Nk-1}) - c(x_k)\right) \end{pmatrix}, \tag{75}$$

$$y_{k+1} = y_k + V^* w_{Nk+1}, \tag{76}$$

where instead of $V_{Dk}^T\left(c_l(x_k; d_{Dk}) - c_l(x_k; d_{Nk})\right) = 0$, we use the requirement (74). Here $V^* = V_{Nk-1}$ is a basis matrix of $\partial h(c_l(x_{k-1}; d_{Nk-1}))$. Thus the iterations based on problem (15) may not be necessary hereafter. We restate the contents of Theorem 22 using the above iteration.

Theorem 23. *Assume that a KKT point x^* satisfies the regularity conditions, and that the second-order sufficient condition holds at a KKT pair (x^*, y^*). Assume further that the current iterate (x_k, y_k) is sufficiently close* *to* (x^*, y^*)*, where* $y_k \in \partial h(c(x^*)) = \partial h(c_l(x_k; d_{Nk-1}))$*. Then there exists a unique solution of (75), and with*

$$x_{k+1} = x_k + d_{Nk}, \tag{77}$$

and (76), there holds

$$y_{k+1} \in \partial h(c(x^*)) = \partial h(c_l(x_k; d_{Nk})), \tag{78}$$

and

$$\left\| \begin{pmatrix} x^* - x_{k+1} \\ y^* - y_{k+1} \end{pmatrix} \right\| = \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \, \|y^* - y_k\|). \tag{79}$$

4.2 Second-order correction

In the last subsection, it is proved that the fast local convergence is obtained if we use Newton like iteration defined by (64), (65) and (66), or (75), (76) and (77), and if we ignore the objective function value change. It is well known that this kind of procedure gives the Maratos effect, and that the objective function value may not decrease. In this subsection we describe how to avoid Maratos effect by using the second-order correction in our problem. Given a solution pair (d_{Nk}, y_{Nk+1}) of problem (17) near a KKT pair (x^*, y^*), modify the problem to

$$\begin{array}{ll} \text{minimize} & \psi'_{G_k}(d_{Nk} + d_C) \\ & = \frac{1}{2}(d_{Nk} + d_C)^T G_k (d_{Nk} + d_C) + \nabla f(x_k)^T (d_{Nk} + d_C) \\ & + h(c_q(x_k; d_{Nk} + d_C)), d_C \in \mathbb{R}^n \end{array} \tag{80}$$

where $G_k = \nabla^2_{xx} L(x_k, y_k)$. and d_N and $h(c_l(x_k; d_N))$ in (17) at x_k are replaced by $d_{Nk} + d_C$ and $h(c_q(x_k; d_{Nk} + d_C))$ respectively. The KKT condition of this problem is

$$\begin{array}{l} G_k(d_{Nk} + d_{Ck}) + \nabla f(x_k) + \nabla c(x_k) y_{NCk+1} = 0, \\ \qquad\qquad\qquad\qquad y_{NCk+1} \in \partial h(c_q(x_k; d_{Nk} + d_{Ck})). \end{array}$$

Actually, this problem is not tractable, so we modify $c_q(x_k; d_{Nk} + d_{Ck})$ to $c_q(x_k; d_{Nk}) + \nabla c(x_k)^T d_{Ck}$ ignoring the terms of orders $\|d_{Ck}\|^2$ and $\|d_{Ck}\| \, \|d_{Nk}\|$. Then the above KKT condition becomes

$$\begin{array}{l} G_k(d_{Nk} + d_{Ck}) + \nabla f(x_k) + \nabla c(x_k) y_{NCk+1} = 0, \\ \qquad\qquad\qquad\qquad y_{NCk+1} \in \partial h(c_q(x_k; d_{Nk}) + \nabla c(x_k)^T d_{Ck}). \end{array}$$

As in the Newton iteration of the previous subsection, we require

$$\begin{aligned}\partial h(c(x^*)) &= \partial h(c_l(x_k; d_{Nk-1})) = \partial h(c_l(x_k; d_{Nk})) \\ &= \partial h(c_q(x_k; d_{Nk}) + \nabla c(x_k)^T d_{Ck}).\end{aligned}$$

Thus we solve

$$G_k(d_{Nk} + d_{Ck}) + \nabla f(x_k) + \nabla c(x_k) y_{NCk+1} = 0, \tag{81}$$

$$V^{*T}\left(c_l(x_k; d_{Nk}) - c_q(x_k; d_{Nk}) - \nabla c(x)^T d_{Ck}\right) = 0, \tag{82}$$

which we rewrite to

$$\begin{pmatrix} G_k & \nabla c(x_k) V^* \\ (\nabla c(x_k) V^*)^T & 0 \end{pmatrix} \begin{pmatrix} d_{Ck} \\ w_{Ck+1} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} V^{*T} d_{Nk}^T \nabla^2 c(x) d_{Nk} \end{pmatrix}, \tag{83}$$

$$y_{NCk+1} = y_{Nk+1} + V^* w_{Ck+1}.$$

From Theorem 23, we have

$$\|d_{Nk}\| \le \|x_k + d_{Nk} - x^*\| + \|x_k - x^*\| = \mathrm{O}(\|x^* - x_k\|), \tag{84}$$

and then from (83), we obtain

$$\left\| \begin{pmatrix} d_{Ck} \\ w_{Ck+1} \end{pmatrix} \right\| = \mathrm{O}\left(\|d_{Nk}\|^2\right) = \mathrm{O}(\|x^* - x_k\|^2). \tag{85}$$

The following lemma gives an estimate of the decrease of the functions F_q and ψ_{G_k} near a KKT point.

Lemma 24. *Under the assumption of Theorem 23, the iteration defined by (81) and (82) gives*

$$\begin{aligned}&\Delta F_q(x_k; d_{Nk} + d_{Ck}) \\ &= \psi_{G_k}(d_{Nk}) - \psi_{G_k}(0) + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|),\end{aligned} \tag{86}$$

$$\begin{aligned}&F(x^*) - F(x_k) \\ &= \psi_{G_k}(d_{Nk}) - \psi_{G_k}(0) + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|).\end{aligned} \tag{87}$$

Proof. Similarly to the proof of Lemma 13, we have

$$[c_q(x_k; d_{Nk}) + \nabla c(x_k) d_{Ck}, c_l(x_k; d_{Nk})] \subseteq \Pi^*,$$

where we note $\|d_{Nk}\| = \mathrm{O}(\|x^* - x_k\|)$ and $\|d_{Ck}\| = \mathrm{O}(\|x^* - x_k\|^2)$ from (84) and (85). Then it follows

$$\partial h(c') \subseteq \partial h(c(x^*)) = \partial h(c_l(x_k; d_{Nk})) = \partial h(c_q(x_k; d_{Nk}) + \nabla c(x_k)^T d_{Ck}),$$

for all $c' \in [c_q(x_k; d_{Nk}) + \nabla c(x_k) d_{Ck}, c_l(x_k; d_{Nk})]$ from Lemma 10, (78) and (82). Thus from Lemma 12, (81) and (82), we have

$$\begin{aligned}
&h(c_q(x_k; d_{Nk}) + \nabla c(x_k) d_{Ck}) \\
&= h(c_l(x_k; d_{Nk})) + \left(c_q(x_k; d_{Nk}) + \nabla c(x_k)^T d_{Ck} - c_l(x_k; d_{Nk})\right)^T y_k \\
&+ \mathrm{O}(\left\| V^{*T} \left(c_q(x_k; d_{Nk}) + \nabla c(x_k)^T d_{Ck} - c_l(x_k; d_{Nk})\right) \right\|) \\
&= h(c_l(x_k; d_{Nk})) + \left(\frac{1}{2} d_{Nk} \nabla^2 c(x_k) d_{Nk} + \nabla c(x_k)^T d_{Ck}\right)^T y_k. \qquad (88)
\end{aligned}$$

Then from the definition of F_q, (85) and (88), we have

$$\begin{aligned}
&F_q(x_k; d_{Nk} + d_{Ck}) \\
&= f(x_k) + \nabla f(x_k)^T (d_{Nk} + d_{Ck}) + \frac{1}{2}(d_{Nk} + d_{Ck})^T \nabla^2 f(x_k)(d_{Nk} + d_{Ck}) \\
&+ h(c_q(x_k; d_{Nk} + d_{Ck})) \\
&= f(x_k) + \nabla f(x_k)^T (d_{Nk} + d_{Ck}) + \frac{1}{2}(d_{Nk} + d_{Ck})^T \nabla^2 f(x_k)(d_{Nk} + d_{Ck}) \\
&+ h(c_q(x_k; d_{Nk}) + \nabla c(x_k)^T d_{Ck}) + \mathrm{O}(\|d_{Ck}\|^2) + \mathrm{O}(\|d_{Nk}\| \, \|d_{Ck}\|) \\
&= f(x_k) + h(c_l(x_k; d_{Nk})) + \nabla f(x_k)^T d_{Nk} + \frac{1}{2} d_{Nk}^T G_k d_{Nk} \\
&+ \nabla_x L(x_k, y_k)^T d_{Ck} + \mathrm{O}(\|x^* - x_k\|^3),
\end{aligned}$$

where in the last equality, (85) is used. Since

$$\nabla_x L(x_k, y_k) = \mathrm{O}(\|x^* - x_k\|) + \mathrm{O}(\|y^* - y_k\|)$$

from Taylor expansion, we have

$$\begin{aligned}
F_q(x_k; d_{Nk} + d_{Ck}) &= f(x_k) + h(c_l(x_k; d_{Nk})) + \nabla f(x_k)^T d_{Nk} + \frac{1}{2} d_{Nk}^T G_k d_{Nk} \\
&+ \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \, \|y^* - y_k\|) \\
&= f(x_k) + \psi_{G_k}(d_{Nk}) + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \, \|y^* - y_k\|).
\end{aligned}$$

Then noting $\psi_{G_k}(0) = h(c(x_k))$ gives (86).

Next, we evaluate the value of $F(x^*)$. Firstly we evaluate $h(c(x^*))$ based on the value of $h(c(x_k))$. Since $h(c)$ is locally linear at $c(x^*)$, we have

$$\begin{aligned} h(c_l(x_k; d_{Nk})) &= h(c(x^*)) + \max_{y \in \partial h(c(x^*))} (c_l(x_k; d_{Nk}) - c(x^*))^T y \\ &= h(c(x^*)) + (c_l(x_k; d_{Nk}) - c(x^*))^T y_k + \mathrm{O}(\left\| V^{*T}(c(x^*) - c_l(x_k; d_{Nk})) \right\|) \\ &= h(c(x^*)) + (c_l(x_k; d_{Nk}) - c_q(x_k; x^* - x_k))^T y_k + \mathrm{O}(\|x^* - x_k\|^3), \end{aligned} \tag{89}$$

where we use the facts that we assumed $y_k \in \partial h(c(x^*))$, that

$$c(x^*) = c_q(x_k; x^* - x_k) + \mathrm{O}(\|x^* - x_k\|^3),$$

and that (72) holds. Then from (89), we have

$$\begin{aligned} h(c(x^*)) &= h(c_l(x_k; d_{Nk})) + (x^* - x_k - d_{Nk})^T \nabla c(x_k) y_k \\ &\quad + \frac{1}{2} y_k^T (x^* - x_k)^T \nabla^2 c(x_k)(x^* - x_k) + \mathrm{O}(\|x^* - x_k\|^3) \\ &= h(c_l(x_k; d_{Nk})) + (x^* - x_k - d_{Nk})^T \nabla c(x_k) y_{Nk+1} \\ &\quad + \frac{1}{2} y_k^T (x^* - x_k)^T \nabla^2 c(x_k)(x^* - x_k) \\ &\quad + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|), \end{aligned} \tag{90}$$

where in the last equality, we use the relation

$$\begin{aligned} \|y_{Nk+1} - y_k\| &= \|y_{Nk+1} - y^* + y^* - y_k\| \le \|y_{Nk+1} - y^*\| + \|y^* - y_k\| \\ &= \mathrm{O}(\|y^* - y_k\|). \end{aligned}$$

Then from (90),

$$\begin{aligned} F(x^*) &= f(x^*) + h(c(x^*)) \\ &= f(x_k) + \nabla f(x_k)^T (x^* - x_k) + \frac{1}{2}(x^* - x_k)^T \nabla^2 f(x_k)(x^* - x_k) \\ &\quad + h(c_l(x_k; d_{N_k})) + (x^* - x_k - d_{Nk})^T \nabla c(x_k) y_{Nk+1} \\ &\quad + \frac{1}{2} y_k^T (x^* - x_k)^T \nabla^2 c(x_k)(x^* - x_k) + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|) \\ &= f(x_k) + \nabla f(x_k)^T (x^* - x_k) + \frac{1}{2}(x^* - x_k)^T G_k (x^* - x_k) \\ &\quad + h(c_l(x_k; d_{N_k})) + (x^* - x_k - d_{Nk})^T \left(-G_k d_{Nk} - \nabla f(x_k)\right) \\ &\quad + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|), \end{aligned}$$

where in the last equality we use the KKT condition of problem (17): $G_k d_{Nk} + \nabla f(x_k) + \nabla c(x_k) y_{Nk+1} = 0$. Then we have

$$\begin{aligned}
F(x^*) &= f(x_k) + h(c_l(x_k; d_{N_k})) + \nabla f(x_k)^T d_{Nk} + \frac{1}{2} d_{Nk}^T G_k d_{Nk} \\
&\quad + \frac{1}{2}(x^* - x_k - d_{Nk})^T G_k (x^* - x_k - d_{Nk}) \\
&\quad + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|) \\
&= f(x_k) + h(c_l(x_k; d_{Nk})) + \nabla f(x_k)^T d_{Nk} + \frac{1}{2} d_{Nk}^T G_k d_{Nk} \\
&\quad + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|).
\end{aligned}$$

This gives (87). □

The above lemma leads to the following theorem.

Theorem 25. *Under the assumption of Theorem 23, there exists* $\nu > 0$ *such that*

$$\Delta F(x_k; d_{Nk} + d_{Ck}) \leq -\frac{1}{2}\nu \|x^* - x_k\|^2,$$

and furthermore

$$\left\| \begin{pmatrix} x^* - x_k - d_{Nk} - d_{Ck} \\ y^* - y_{NCk+1} \end{pmatrix} \right\| = \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \|y^* - y_k\|).$$

Proof. From (8), there exists $\nu > 0$ such that

$$F(x_k) \geq F(x^*) + \nu \|x^* - x_k\|^2$$

for x_k sufficiently close to x^*. Thus from Lemma 24, we have

$$\begin{aligned}
&\Delta F_q(x_k; d_{Nk} + d_{Ck}) \\
&= F(x^*) - F(x_k) + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|) \\
&\leq -\nu \|x^* - x_k\|^2 + \mathrm{O}(\|x^* - x_k\|^3) + \mathrm{O}(\|x^* - x_k\|^2 \|y^* - y_k\|).
\end{aligned}$$

On the other hand, from (14), (84) and (85),

$$\begin{aligned}
\Delta F(x_k; d_{Nk} + d_{Ck}) &= \Delta F_q(x_k; d_{Nk} + d_{Ck}) + \mathrm{O}(\|d_{Nk} + d_{Ck}\|^3) \\
&= \Delta F_q(x_k; d_{Nk} + d_{Ck}) + \mathrm{O}(\|x^* - x_k\|^3).
\end{aligned}$$

Then we have

$$\begin{aligned}\Delta F(x_k; d_{Nk} + d_{Ck}) &\leq -\nu \left\| x^* - x_k \right\|^2 + \mathrm{O}(\| x^* - x_k \|^3) + \mathrm{O}(\| x^* - x_k \|^2 \left\| y^* - y_k \right\|) \\ &\leq -\frac{1}{2}\nu \left\| x^* - x_k \right\|^2\end{aligned}$$

for sufficiently small $\|x^* - x_k\|$ and $\|y^* - y_k\|$. Now from (68) and (85),

$$\begin{aligned}\left\| \begin{pmatrix} x^* - x_k - d_{Nk} - d_{Ck} \\ y^* - y_{NCk+1} \end{pmatrix} \right\| &\leq \left\| \begin{pmatrix} x^* - x_k - d_{Nk} \\ y^* - y_{Nk+1} \end{pmatrix} \right\| + \left\| \begin{pmatrix} d_{Ck} \\ V^* w_{Ck+1} \end{pmatrix} \right\| . \\ &= \mathrm{O}(\|x^* - x_k\|^2) + \mathrm{O}(\|x^* - x_k\| \left\| y^* - y_k \right\|)\end{aligned}$$

□

5 Conclusion

In this paper, we proposed a method for finding a KKT point of structured nonsmooth problems that satisfies the second-order optimality condition by a trust region method. The global convergence to a KKT point is assured by the trust-region searches along the directions generated by subproblems that give descent directions. The negative-curvature searches for the second-order critical point approximately follow the manifold that give a second-order approximation of the objective function difference by the corresponding Lagrangian terms. It is also shown that it is possible to have an acceleration to the quadratic convergence by solving a set of linear equations. It is also shown that the second-order correction for avoiding the Maratosh effect can be done by solving another set of linear equations.

Possible future studies include extensions to the constrained problems and problems with unknown structures.

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