Solving the $n_1 \times n_2 \times n_3$ Points Problem for $n_3 < 6$

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Abstract: In this paper, we show enhanced upper bounds of the nontrivial $n_1 \times n_2 \times n_3$ points problem for every $n_1 \leq n_2 \leq n_3 < 6$. We present new patterns that drastically improve the previously known algorithms for finding minimum-link covering paths.
Keywords: Graph theory, Three-dimensional, Link-length, Connectivity, Outside the box, Game, Upper bound, Point, Covering path.
2020 Mathematics Subject Classification: 05C57.

1 Introduction

The $n_1 \times n_2 \times n_3$ points problem [10] is a three-dimensional extension of the classic nine-dot problem appeared in Samuel Loyd's Cyclopedia of Puzzles (see [8], p. 301), and it is related to the well-known NP-hard traveling salesman problem (TSP), minimizing the number of turns in the tour instead of the total distance traveled [1, 13].

Given $n_1 \cdot n_2 \cdot n_3$ points in $\mathbb{R}^3$, our goal is to visit all of them (at least once) with a polygonal path that has the minimum number of line segments connected at their endpoints (links or generically lines), the so-called Minimum-link Covering Path [2–4, 7]. In particular, we are interested in the best solutions to the nontrivial $n_1 \times n_2 \times n_3$ dots problem, where (by definition) $1 \leq n_1 \leq n_2 \leq n_3$ and $n_3 < 6$.

Let $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3) \leq h_u(n_1, n_2, n_3)$ be the length of the covering path with the minimum number of links for the $n_1 \times n_2 \times n_3$ points problem, we define the best known upper bound as $h_u(n_1, n_2, n_3) \geq h(n_1, n_2, n_3)$, and we denote as $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$ the proved lower bound. For the simplest cases, the same problem has already been solved [2].

In details, if $n_1 = 1$ and $n_2 < n_3$, then $h(n_1, n_2, n_3) = h(n_2) = 2 \cdot n_2 - 1$, while $h(n_1 = 1, n_2 = n_3 \geq 3) = 2 \cdot n_2 - 2$ [5].

Hence, for $n_1 = 2$, it can be easily proved that

$$h(2, n_2, n_3) = 2 \cdot h(1, n_2, n_3) + 1 = \begin{cases} 4 \cdot n_2 - 1 & \text{iff} \quad n_2 < n_3 \\ 4 \cdot n_2 - 3 & \text{iff} \quad n_2 = n_3 \end{cases} . \tag{1}$$

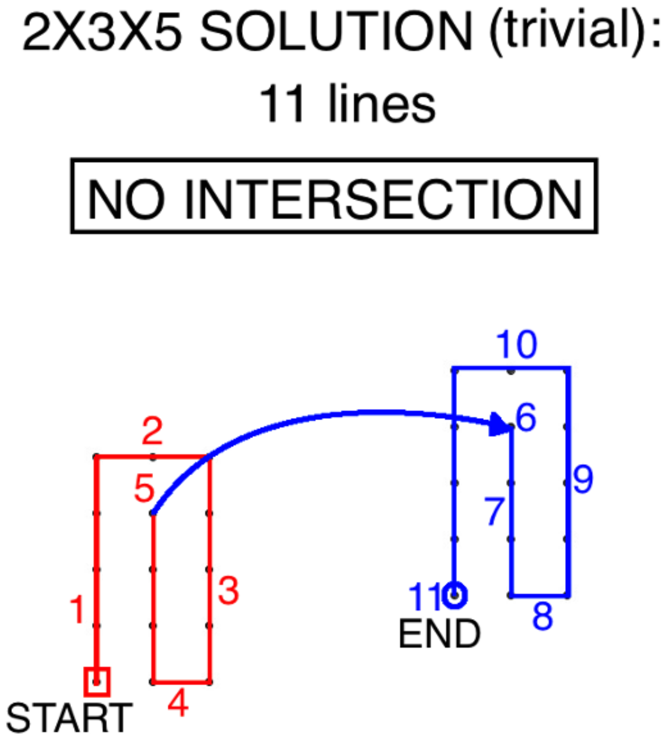

Figure 1: A trivial Hamiltonian path completely solves the $2 \times 3 \times 5$ points puzzle (avoiding self-intersections).

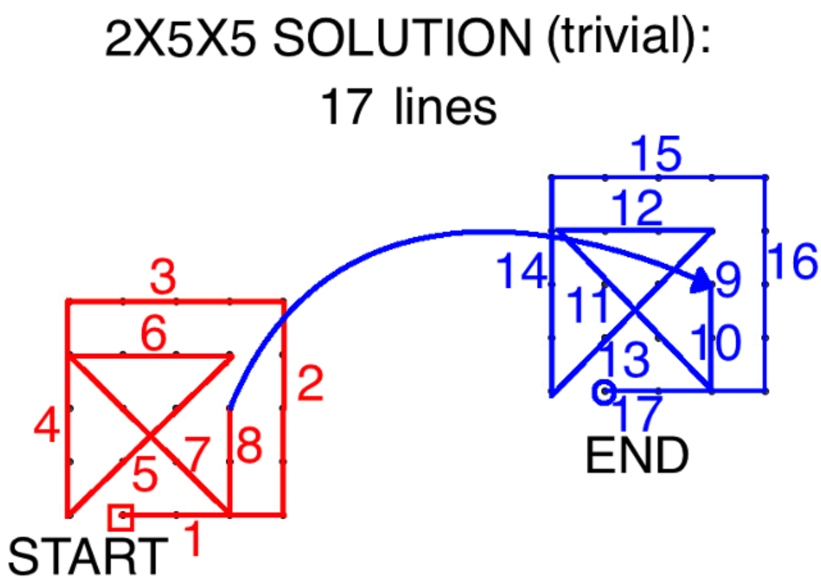

Figure 2: Another example of a trivial pattern, solving the $2 \times 5 \times 5$ points puzzle.

Therefore, the aim of the present paper is to solve the ten aforementioned nontrivial cases where the current upper bound does not match the proved lower bound.

2 Improving the solution of the $n_1 \times n_2 \times n_3$ points problem for $n_3 < 6$

In this complex brain challenge we need to stretch our pattern recognition [6, 9] in order to find a plastic strategy that improves the known upper bounds [2, 10] for the most interesting cases (and the $3 \times 3 \times 3$ problem, which is the three-dimensional extension of the immortal nine-dot puzzle, is by far the most valuable one [11]), avoiding those standardized methods which are based on fixed patterns that lead to suboptimal covering paths, as the approache presented in [7, 10].

Theorem 1. If $2 < n_1 \leq n_2 \leq n_3$, then a lower bound of the general $n_1 \times n_2 \times n_3$ problem is given by

$$h_l(n_1, n_2, n_3) = \left\lceil \frac{3 \cdot (n_3 \cdot n_2 \cdot n_1 - n_1)}{2 \cdot n_3 + n_2 - 3} \right\rceil + 1, \tag{2}$$

where the ceiling $\lceil q \rceil$ denotes the function which takes the rational number q as input and returns as output the least integer greater than or equal to q.

Proof. Let $\{0, 1, ..., n_1-2, n_1-1\} \times \{0, 1, ..., n_2-2, n_2-1\} \times \cdots \times \{0, 1, ..., n_k-2, n_k-1\}$ be a set of $\prod_{i=1}^{k} n_i$ points, in the Euclidean vector space $\mathbb{R}^3$, such that $3 \leq n_1 \leq n_2 \leq ... \leq n_k$. Although we could argue that it is not possible to intersect more than $(n_k - 1) + (n_{k-1} - 1) + n_k - 1 = 2 \cdot n_k + n_{k-1} - 3$ unvisited points with three straight lines connected at their endpoints (i.e., using three consecutive edges), we observe that there is one exception (which, for simplicity, we may assume as in the case of the first line drawn). In this circumstance, it is possible to fit n_k points with the first line, $n_{k-1} - 1$ points using the second line, $n_k - 1$ points with the next one, and so forth. In general, the third and the last line of the aforementioned group will join (at most) $n_k - 1$ unvisited points each.

In order to complete the covering path, reaching every edge of our hyper-parallelepiped, we need at least one more link for any of the remaining n_i, and this implies that $k - 2$ lines cannot join a total of more than $n_{k-2} - 1 + n_{k-3} - 1 + ... + n_1 - 1 = \sum_{i=1}^{k-2} n_i - k + 2$ unvisited points.

Thus, the considered lower bound $h_l(n_1, n_2, ..., n_k)$ satisfies the relation

$$\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 2 - 1 \leq (2 \cdot n_k + n_{k-1} - 3) \cdot \left(\frac{h_l(n_1, n_2, ..., n_k)}{3} - k + 2 \right). \tag{3}$$

Hence,

$$h_l(n_1, n_2, ..., n_k) = \left\lceil 3 \cdot \frac{\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 3}{2 \cdot n_k + n_{k-1} - 3} \right\rceil + k - 2. \tag{4}$$

Substituting $k = 3$ into (4), we get the statement of Theorem 1. □

The current best results are listed in Table 1, and a direct proof follows for each nontrivial upper bound shown below.

n_1	n_2	n_3	**Best Lower bound h_l**	**Best Upper bound h_u**	**Discovered by**	**Gap $(h_u - h_l)$**
2	2	2	6	6	Koki Goma, proved in Aug. 2021 (see [12])	0
2	2	3	7	7	trivial	0
2	3	3	9	9	trivial	0
3	3	3	13	13	Marco Ripà, proved in June 2020 (see [11])	0
2	2	3	7	7	trivial	0
2	3	4	11	11	trivial	0
2	4	4	13	13	trivial	0
3	3	4	14	15	Marco Ripà, June 2019	1
3	4	4	16	19	Marco Ripà, June 2019	3
4	4	4	21	23	Marco Ripà, 2019 (see NNTDM, 25(2), p. 70, Fig. 1)	2
2	2	5	7	7	trivial	0
2	3	5	11	11	trivial	0
2	4	5	15	15	trivial	0
2	5	5	17	17	trivial	0
3	3	5	14	16	Marco Ripà, June 2019	2
3	4	5	17	20	Marco Ripà, June 2019	3
3	5	5	19	24	Marco Ripà, June 2019	5
4	4	5	22	26	Marco Ripà, June 2019	4
4	5	5	25	31	Marco Ripà, June 2019	6
5	5	5	31	36	Marco Ripà, July 2019	5

Table 1: Current solutions to the $n_1 \times n_2 \times n_3$ points problem, where $n_1 \leq n_2 \leq n_3 < 6$.

Figures 3 to 12 show the patterns used to solve the $n_1 \times n_2 \times n_3$ puzzle (case by case). In particular, by combining (2) with the original results shown in Figures 3&4, we obtain a formal proof for the major $3 \times 3 \times 3$ points problem, plus very tight bounds for the $3 \times 3 \times 4$ case.

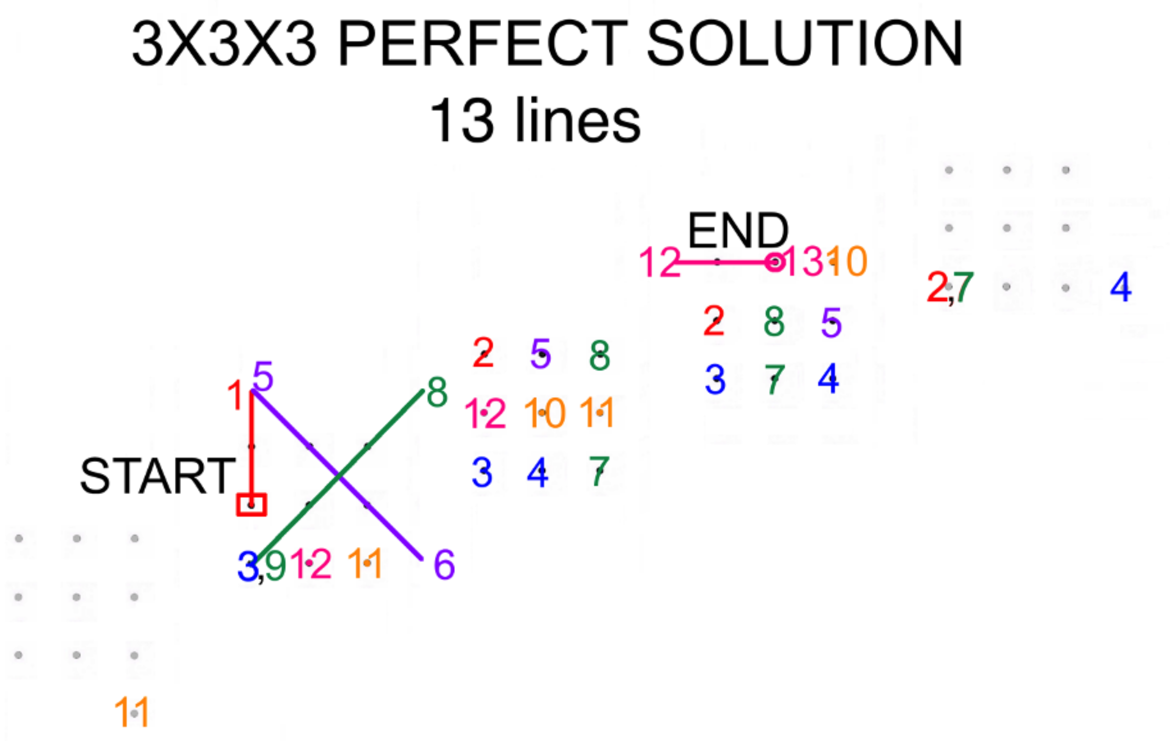

Figure 3: The k-dimensional $3 \times 3 \times \cdots \times 3$ puzzle has constructively been solved for any $k \in \mathbb{Z}^+$ (since $h_u(3, 3, ..., 3) = h_l(3, 3, ..., 3) = 0.5 \cdot (3^k - 1)$, see [11]). In particular, the above solution for the three-dimensional case was provided by Ripà on June 19, 2020, and is optimal by Corollary 1.

Corollary 1.

$$h_l(3, 3, 3) = h_u(3, 3, 3) = 13. \tag{5}$$

Proof. The covering path for the $3 \times 3 \times 3$ case shown in Figure 3 consists of 13 straight lines connected at their endpoints, and Eq. (2) gives $h_l(3, 3, 3) = \lceil 12 \rceil + 1 = 13$. □

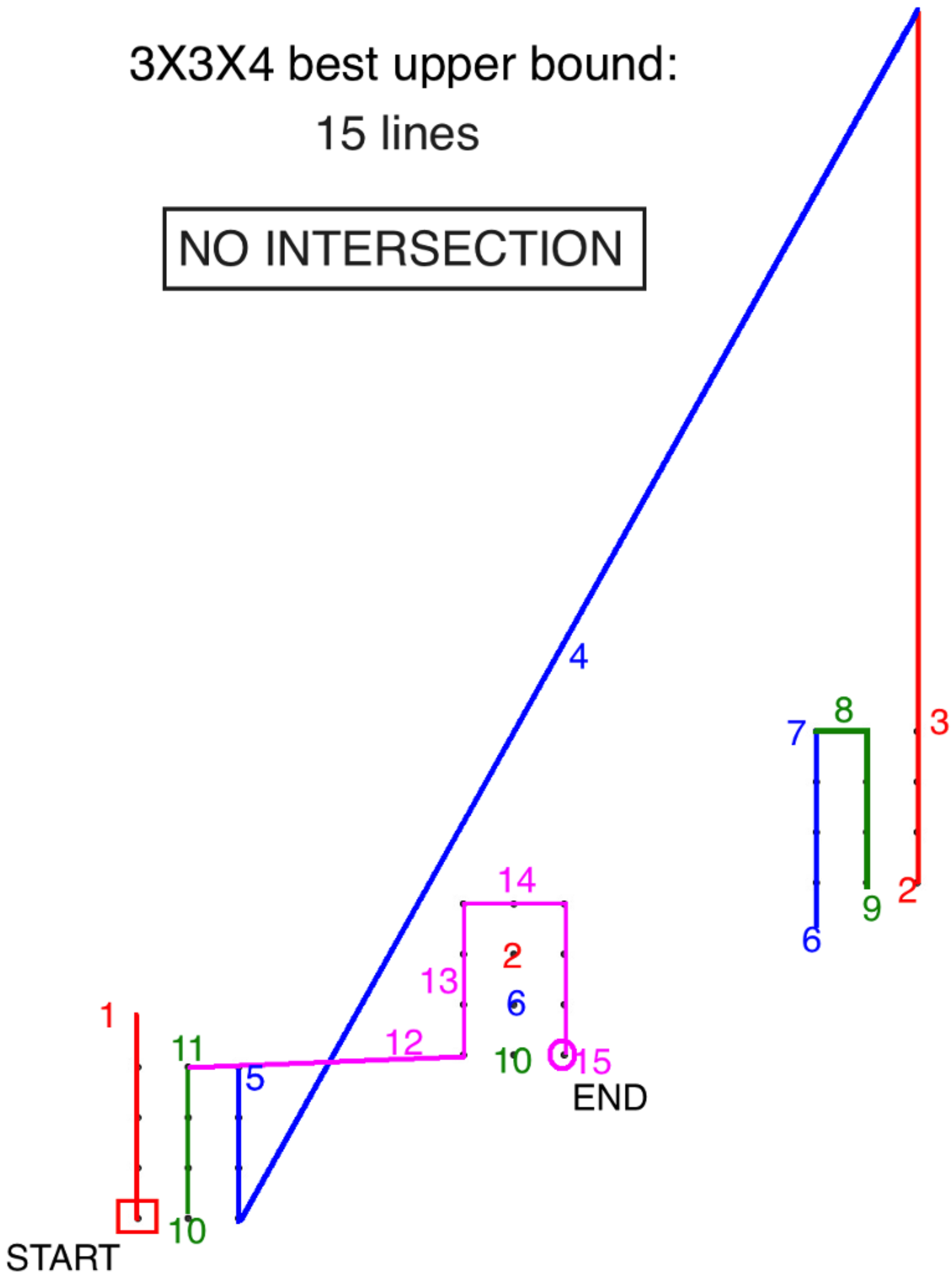

Figure 4: Best known (non-crossing) covering path for the $3 \times 3 \times 4$ puzzle. $15 = h_u = h_l + 1$.

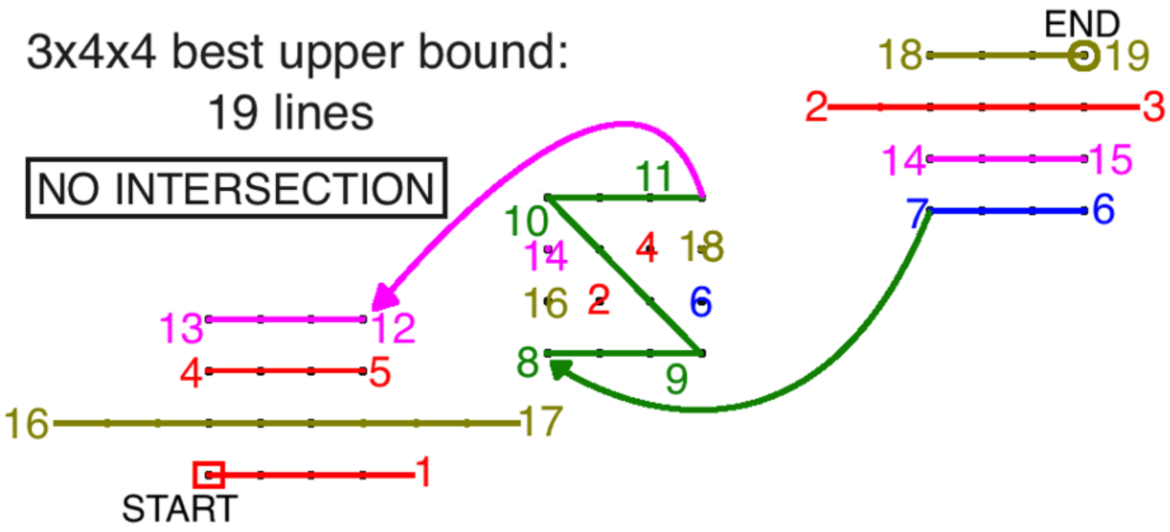

Figure 5: Best known (non-crossing) covering path for the $3 \times 4 \times 4$ puzzle. $19 = h_u = h_l + 3$.

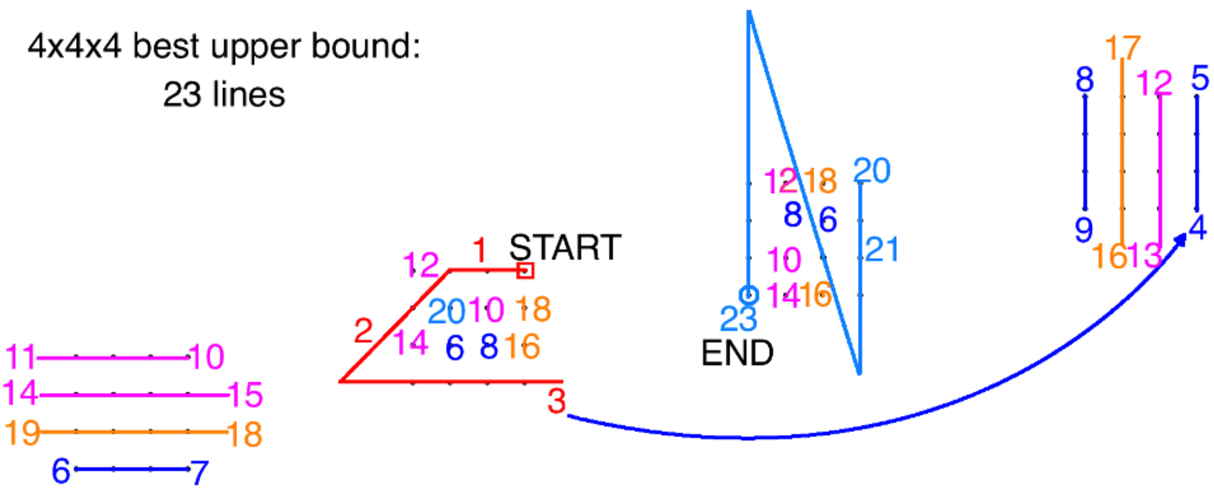

Figure 6: An original covering path for the $4 \times 4 \times 4$ puzzle. $23 = h_u = h_l + 2$.

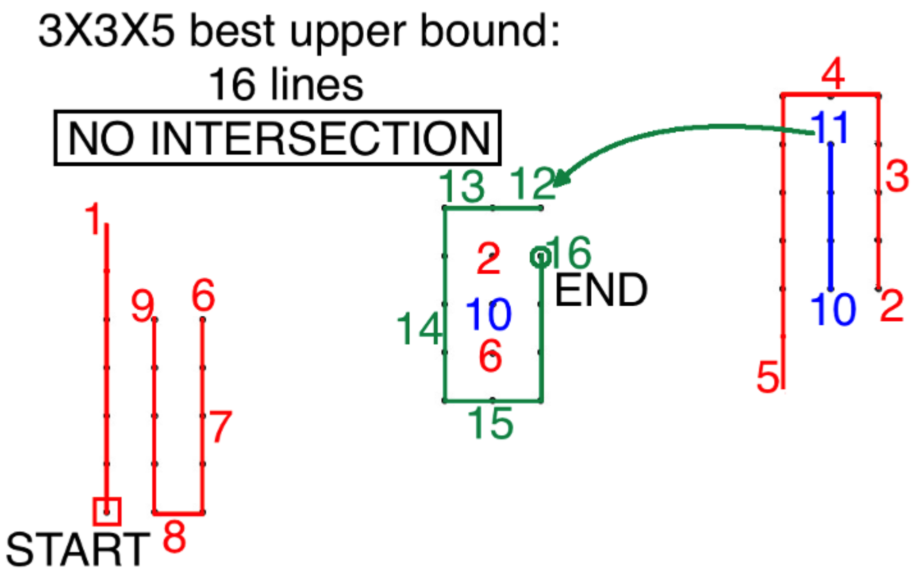

Figure 7: Best known (non-crossing) covering path for the $3 \times 3 \times 5$ puzzle. $16 = h_u = h_l + 2$.

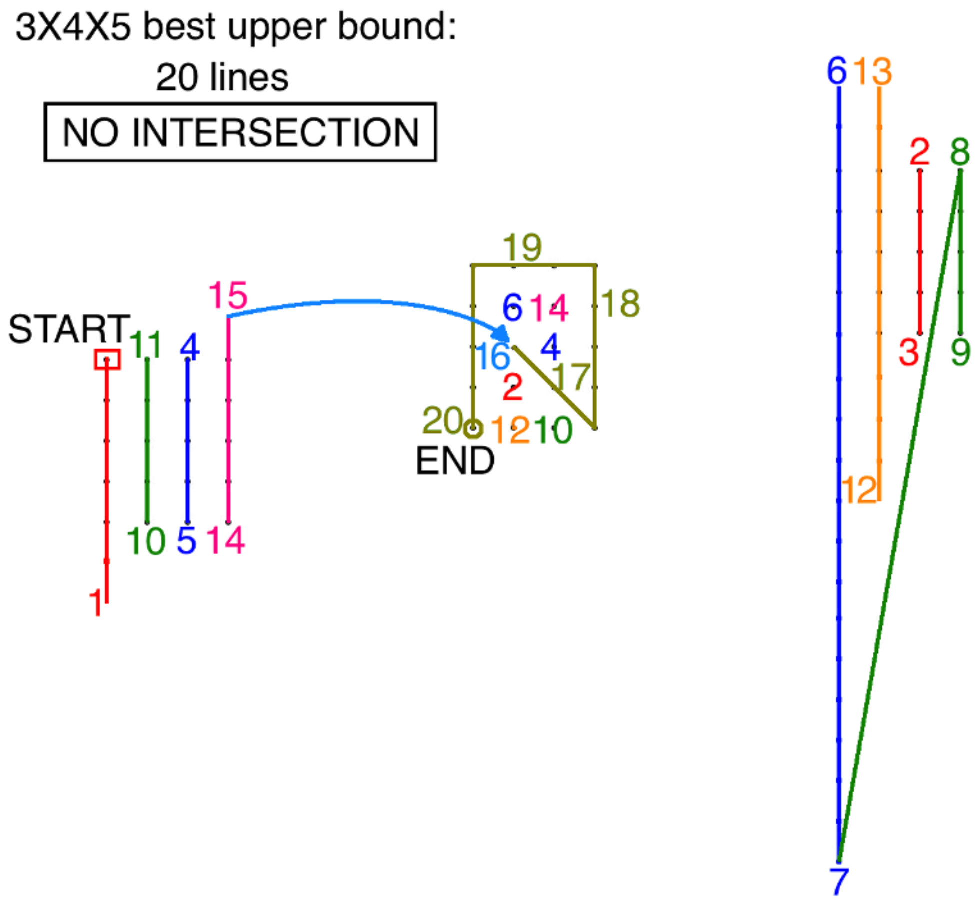

Figure 8: Best known (non-crossing) covering path for the $3 \times 4 \times 5$ puzzle, consisting of $20 = h_u = h_l + 3$ lines.

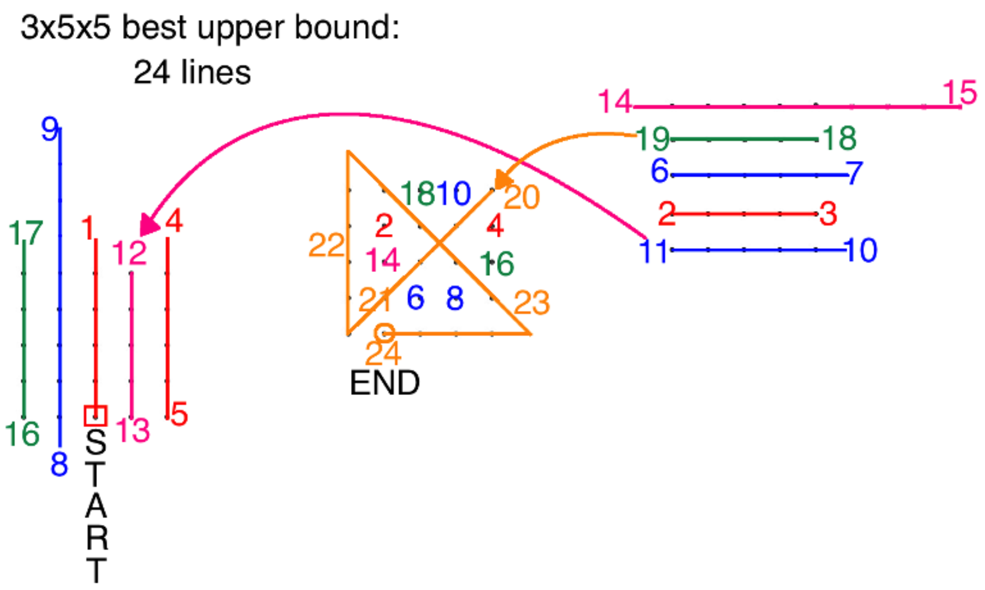

Figure 9: Best known covering path for the $3 \times 5 \times 5$ puzzle. $24 = h_u = h_l + 5$.

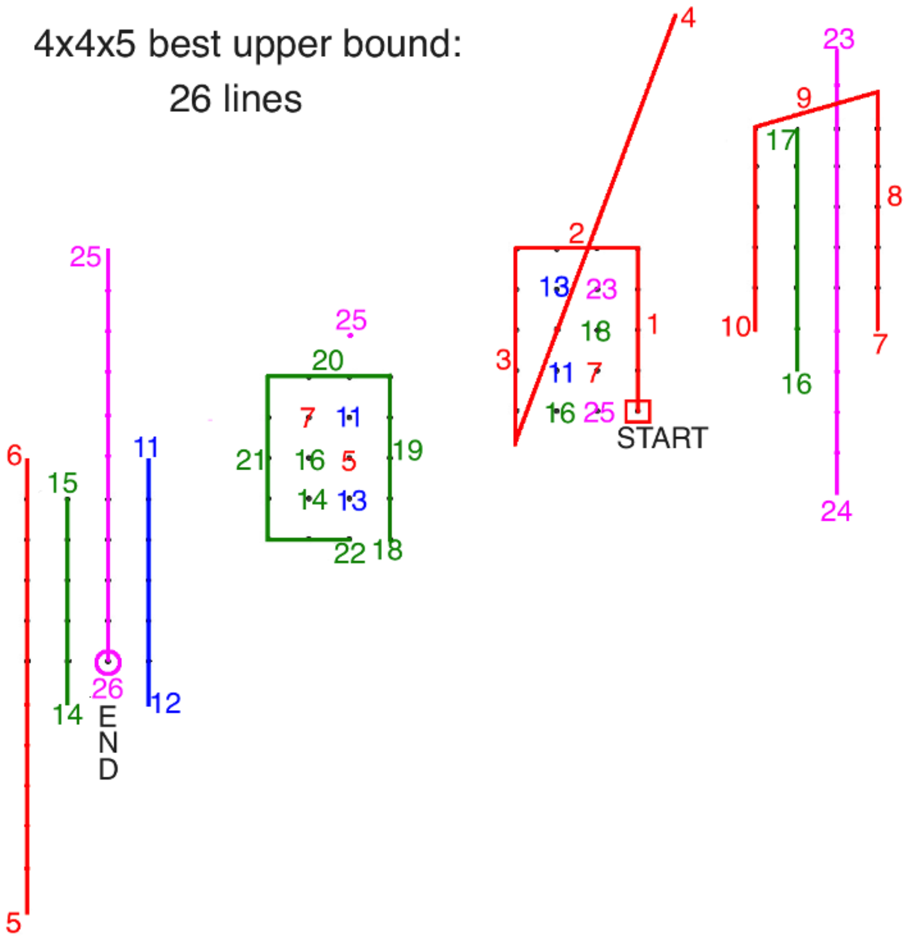

Figure 10: Best known covering path for the $4 \times 4 \times 5$ puzzle. $26 = h_u = h_l + 4$.

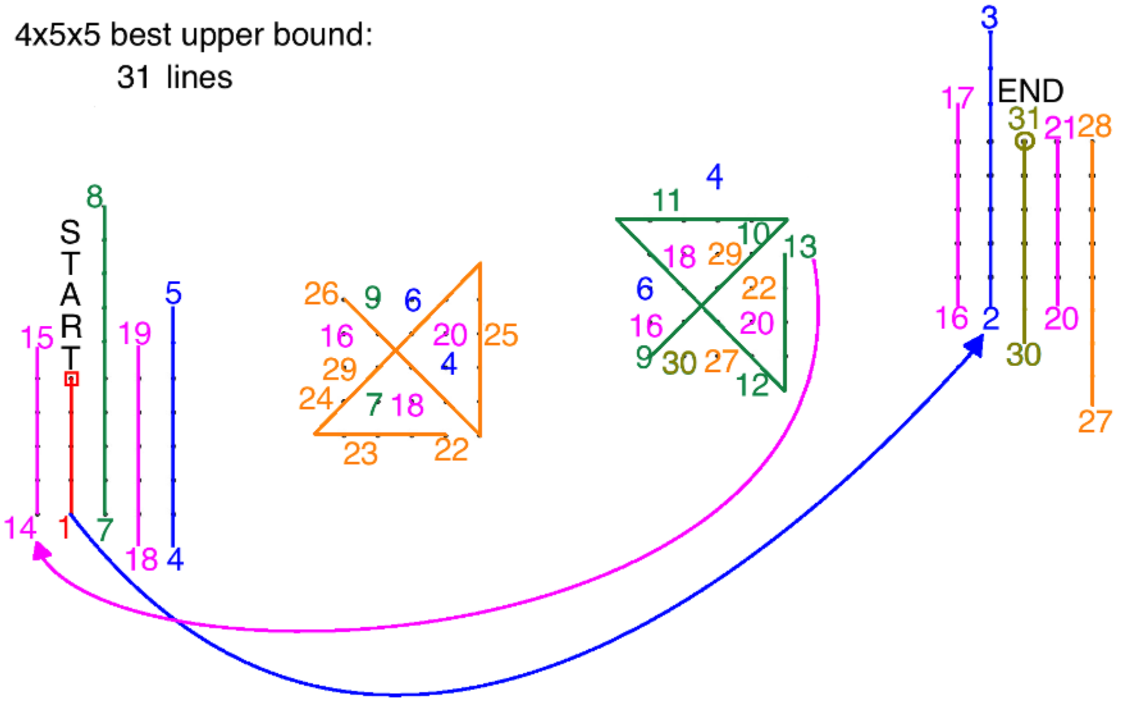

Figure 11: Best known covering path for the $4 \times 5 \times 5$ puzzle. $31 = h_u = h_l + 6$.

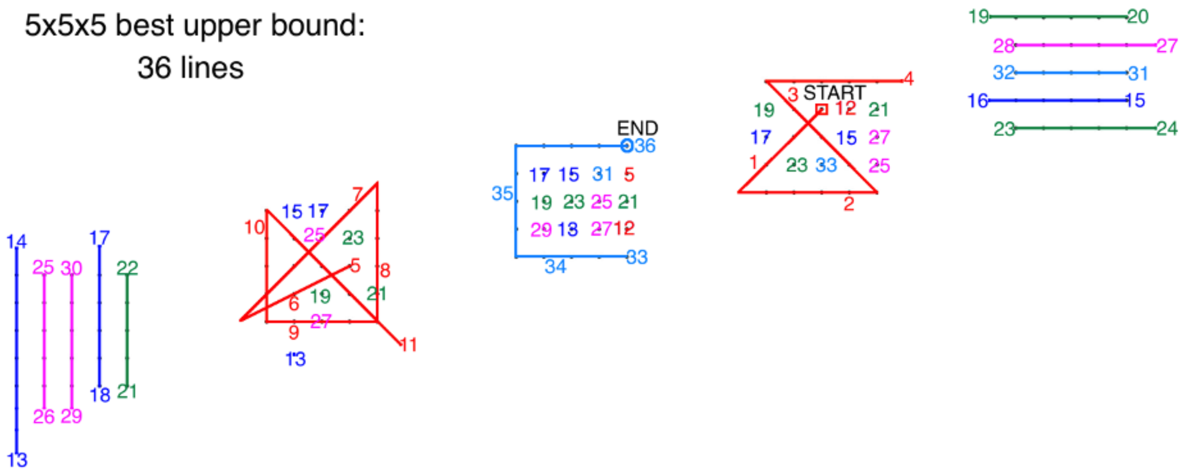

Figure 12: Best known upper bound of the $5 \times 5 \times 5$ puzzle. $36 = h_u = h_l + 5$.

Finally, it is interesting to note that the improved $h_u(n_1, n_2, n_3)$ can lower down also the upper bound of the generalized k-dimensional puzzle. As an example, we can apply the aforementioned 3D patterns to the generalized $n_1 \times n_2 \times \cdots \times n_k$ points problem using the simple method described in [10].

Let $k \geq 4$, given $n_k \leq n_{k-1} \leq ... \leq n_4 \leq n_1 \leq n_2 \leq n_3$, we can conclude that

$$h_u(n_1, n_2, n_3, ..., n_k) = (h_u(n_1, n_2, n_3) + 1) \cdot \prod_{j=4}^{k} n_j - 1. \tag{6}$$

3 Conclusion

In the present paper, we have drastically reduced the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$ for every previously unsolved puzzle such that $n_3 < 6$.

We do not know if any of the patterns shown in Figures 4 to 12 represent optimal solutions, since (by definition) $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$. Therefore, some open questions about the NP-complete [2] $n_1 \times n_2 \times n_3$ points problem remain to be answered, and the research in order to cancel the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$, at least for every $n_3 \leq 5$, is not over yet.

Moreover, since $h(2, 2, 2) = 6$, it automatically follows that $h_u(2, 2, 2, 2) = 6 \cdot 2 + 1$ [12]. On the other hand, $h(2, 2, 2, 2) \geq 11$ by the methods shown in the proof of Theorem 1. Finding the exact value of $h(2, 2, 2, 2)$ is an intriguing open problem in $\mathbb{R}^4$.

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