A UNIFIED SCHEME FOR SCALARIZATION IN SET OPTIMIZATION

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ABSTRACT. In this work, we propose a new scheme for scalarization in set optimization studied with the Kuroiwa set appoach. First, we define an abstract scalarizing function possessing properties such as global Lipschizity, sublinearity, cone monotonicity, cone representation property, cone interior representation property and uniform positivity. Next, we use this function to define the so called signed Hausdorff-type half-distances and Hausdorff-type distances. As the first applications, we obtain characterizations of Kuroiwa's set order relations and some optimal solutions of a set optimization problem. This scheme provides a unified approach to scalarization involving different scalarizing functions such as the Gerstewizt (Tammer) function, the Hiriart-Urruty signed distance and the function proposed by Kasimbeyli.

Key Words: Set optimization, scalarization, set order relation, optimality condition

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1. Introduction

Optimization problems with objective set-valued maps, in brief (SOP), have been recently attracted more attention due to their extensive real-world applications in finance (set-valued risk measures), statistics, game theory, and multicriteria decision making, see for instance [10, 18, 20, 24, 34]. There are several approaches to defining optimal solutions for (SOP) but in this paper, we will restrict ourselves to the Kuroiwa set approach [40] in which one compares sets with respect to set order relations.

Scalarization is an important technique in vector and set optimization, which means that a scalar problem is solved in order to obtain the solutions of the original vector or set optimization problem, see [7, 10, 23, 24, 44, 45, 46]. Well-known scalarization functions are the Gerstewitz (Tammer) function [6], the Hiriart-Urruty signed distance function [21] and the function proposed by Kasimbeyli [31]. There is a rich literature devoted to properties, applications, set extensions of these functions, see for instance [1, 2, 4, 5, 9, 10, 13, 14, 20, 27, 28, 29, 30, 32, 46, 48] and the references therein, and relations among them, see [3, 43].

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An axiomatic approach to scalarization in vector optimization was introduced by Wierzbicki in [47] and developed further in [3, 45], see also [34]. These axioms are those of monotonicity and order (or cone) representability which were shown to be necessary and sufficient for characterizing some solution sets of a vector optimization problem. Scalarization scheme has also been introduced for binary relations (in particular, for quasi-orders) with applications in the study of nondominated solutions of (SOP) [10, 13, 26]. The properties such as order representing property and order preserving property considered in [10, 13, 36, 45] allowed to obtain the necessary and/or sufficient conditions of these solutions.

Following Wierzbicki's axiomatic approach and aiming at unifying different scalarizing functions, we propose a new scheme for scalarization in set optimization. First, we define an abstract scalarizing function satisfying some properties, such as global Lipschizity, sublinearity, cone monotonicity, cone representation property, cone interior representation property, uniform positivity. It turns out that the mentioned above three scalarizing functions are examples of the abstract one. Next, in a similar way as in [14, 15, 28, 30], we use this abstract scalarizing function to define functions of sup-inf and inf-sup types which we call signed Hausdorff-type half-distances and Hausdorff-type distances. The latters allow us to obtain characterizations of the six set order relations defined by Kuroiwa in [40] and some optimal solutions of (SOP). Similar to the works [14, 15], one can use them to define slope and directional derivatives of set-valued maps, which are useful tools in formulating optimality conditions and error bounds for (SOP).

The paper is organized as follows. In the next section, we recall some notations from vector and set optimization and provide some auxiliary results. Section 3 is devoted to properties of an abstract scalarizing function in vector case. In Section 4, we define signed Hausdorff-type half-distances and study their properties. In Section 5, we characterize set order relations and some efficient solutions of (SOP). In Section 6, we define Hausdorff-type distances. Some conclusions are given in the last section.

2. Some notions from vector optimization and set optimization

In this section, we recall some concepts of cone boundedness, Pareto efficient points and set order relations that will be used throughout the paper.

Let Y be a Banach space with the dual Y^* and the pairing $\langle \cdot, \cdot \rangle$. The family of all nonempty subsets of Y is denoted by 2^Y. By $\mathbb{B}$ we denote the closed unit ball in a normed space. For nonempty subsets A and B of Y and a scalar t, we define the algebraic sum (also called Hausdorff sum or Minkowski addition) $A + B$, the algebraic difference $A - B$ and the set tA as follows: $A + B := \{a + b \mid a \in A,\ b \in B\}$, $A - B := \{a - b \mid a \in A,\ b \in B\}$ and $tA := \{ta \mid a \in A\}$. The distance from a point u to a nonempty set U of Y is denoted by $d_U(u)$.

Throughout the paper, unless otherwise specified, let $K \subset Y$ be a closed convex cone and $K^* := \{y^* \in Y^* \mid \langle y^*, k\rangle \geq 0,\ \forall\ k \in K\}$. We say that K is *pointed* if $K \cap (-K) = \{0\}$ and K is *solid* if K has a nonempty interior. The cone K induces a partial order in Y: for any $y_1, y_2 \in Y$ we write $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$. When K is solid, it induces a strict order on Y: for any $y_1, y_2 \in Y$, we write $y_1 <_K y_2$ if $y_2 - y_1 \in \text{int}K$. When no confusion occurs, we will omit the subscript K and write simply $\leq$ and $<$.

In this paper, $\mathbb{R}^n$ and $\mathbb{R}^n_+$ denote the n-dimensional space and its nonnegative orthant (the norm in this space is euclidean unless otherwise specified).

We recall some concepts related to a nonempty subset A of Y. We say that A is K-*bounded* if there exists a bounded nonempty set $M \subset Y$ such that $A \subset M + K$, A is K-*closed* if $A + K$ is closed and A is K-*compact* if any its cover of the form $\{U_\alpha + K \colon \alpha \in I,\ U_\alpha \text{ are open}\}$ admits a finite subcover, see [44]. It is well-known that if A is K-compact, then it is K-closed and K-bounded.

Let us pay more attention on the K-properness, a useful cone boundedness concept introduced in [20, Definition 2.15] and consequently used in [10, 11, 27, 28, 29, 30]. Recall that a nonempty subset A of Y is said to be K-*proper* if $A + K \neq Y$. It has been shown that the K-properness is weaker than other usual cone boundedness assumptions, in particular, the K-boundedness implies the K-properness [12, Theorem 3.1]. A necessary and sufficient condition for a set to be K-proper when K is solid has been established in [27, Lemma 5.4]. Below we characterize the solidness of the cone K in terms of the K-properness.

Lemma 2.1. *The following statements are true.*

(a) *A is K-proper iff there exists $y \in Y$ such that $A \subseteq \{y\} + (Y \setminus (-K))$.*
(b) *K is solid iff there exists $\rho > 0$ such that $\rho\mathbb{B} + (Y \setminus (-K)) \cup \{0\}$ is K-proper. Equivalently, K is nonsolid iff $\rho\mathbb{B} + (Y \setminus (-K) \cup \{0\}) = Y$ for all $\rho > 0$.*

Proof. It is easy to check that

$$Y \setminus (-K) \cup \{0\} + K = Y \setminus (-K) \cup \{0\}.$$

The assertions are immediate from the following implications

$$\begin{aligned} A + K \neq Y \quad &\Longleftrightarrow \exists y \in Y \text{ such that } y \notin A + K \\ &\Longleftrightarrow \exists y \in Y \text{ such that } a - y \in Y \setminus (-K)\ \forall a \in A \\ &\Longleftrightarrow \exists y \in Y \text{ such that } A \subseteq \{y\} + (Y \setminus (-K)) \end{aligned}$$

and

$$\begin{aligned} K \text{ is solid} \quad &\Longleftrightarrow \exists y \in Y\ \exists \rho > 0 \text{ such that } y + \rho\mathbb{B} \subset -\text{int}K \\ &\Longleftrightarrow \exists y \in Y\ \exists \rho > 0 \text{ such that } y \notin \rho\mathbb{B} + Y \setminus (-K) \cup \{0\} \\ &\Longleftrightarrow \exists \rho > 0 \text{ such that } \rho\mathbb{B} + (Y \setminus (-K) \cup \{0\}) \neq Y \\ &\Longleftrightarrow \exists \rho > 0 \text{ such that } \rho\mathbb{B} + (Y \setminus (-K) \cup \{0\}) + K \neq Y. \end{aligned}$$

□

In what follows, the expression "A is $(-K)$-..." means "$-A$ is K-...". For instance "A is $(-K)$-bounded" means "$-A$ is K-bounded".

Definition 2.1 ([24, 44])**.** Let $A \subset Y$ be a nonempty set and $a \in A$. We say that a is an *efficient point* or a *Pareto minimal point* of A (denoted by $a \in \mathrm{Min}(A)$) if $a' \leq a$ for some $a' \in A$, then $a \leq a'$ and a is an *ideal efficient point* or an *ideal minimal point* of A (denoted by $a \in \mathrm{IMin}(A)$) if $a \leq a'$ for all $a' \in A$.

If $\mathrm{IMin}(A)$ is nonempty, then $\mathrm{Min}(A) = \mathrm{IMin}(A)$ and if in addition K is pointed, then $\mathrm{IMin}(A)$ is a singleton. Recall [44] that a nonempty subset A of Y has the *domination property* if $\mathrm{Min}(A)$ is nonempty and $A \subseteq \mathrm{Min}(A) + K$. Let us collect some facts that will be used later.

Proposition 2.1. *Let A and B be a nonempty subsets of Y.*

(i) *A has the domination property iff* $\mathrm{Min}(A)$ *is nonempty and $A+K = \mathrm{Min}(A)+K$. If* $\mathrm{IMin}(A)$ *is nonempty, then A has the domination property and $A+K = \mathrm{IMin}(A)+K$.*

(ii) *If A is K-compact, then* $\mathrm{Min}(A) \neq \emptyset$ *and A has the domination property.*

(iii) *Assume that $A + K = B + K$. If A is K-closed (K-bounded, K-compact), then so is B. If A and B are K-compact and K is pointed, then* $\mathrm{Min}(A) = \mathrm{Min}(B)$.

Proof. The assertions (i) and (iii) can be easily proved and the assertion (ii) has been established in [44]. □

Finally, we recall some *set order relations.*

Definition 2.2. Let A and B be nonempty subsets of Y.

(i) The *l-type less order relation* $\preceq_l$ is defined by

$$A \preceq_l B \quad :\Longleftrightarrow \quad \big(\forall b \in B \; \exists \; a \in A : a \leq b\big) \Longleftrightarrow B \subseteq A + K.$$

(ii) The *u-type less order relation* $\preceq_u$ is defined by

$$A \preceq_u B \quad :\Longleftrightarrow \quad \big(\forall a \in A \; \exists \; b \in B : a \leq b\big) \Longleftrightarrow A \subseteq B - K.$$

(iii) The *possibly less order relation* $\preceq_p$ is defined by

$$A \preceq_p B \quad :\Longleftrightarrow \quad \big(\exists a \in A \; \exists \; b \in B : a \leq b\big) \Longleftrightarrow (A - B) \cap (-K) \neq \emptyset.$$

(iv) The *certainly less order relation* $\preceq_c$ is defined by

$$A \preceq_c B \quad :\Longleftrightarrow \quad (\forall a \in A \; \forall b \in B : a \leq b) \Longleftrightarrow \; A - B \subseteq -K.$$

(v) The *possibly-certainly less order relation* $\preceq_{pc}$ is defined by

$$A \preceq_{pc} B \quad :\Longleftrightarrow \quad \big(\exists a \in A \; \forall b \in B : a \leq b\big) \Longleftrightarrow \exists a \in A \text{ such that } \; B \subseteq a + K.$$

(vi) The *certainly-possibly less order relation* $\preceq_{cp}$ is defined by

$$A \preceq_{cp} B \quad :\Longleftrightarrow \quad \big(\exists b \in B \; \forall a \in A : a \leq b\big) \Longleftrightarrow \exists b \in B \text{ such that } A \subseteq b - K.$$

Alongside with the set order relations, we will consider *strict set order relations* in the case K has a nonempty interior.

Definition 2.3. Assume that K is solid. Let A and B be nonempty subsets of Y.

(i) $A \prec_l B \quad :\Longleftrightarrow \quad \big(\forall b \in B \; \exists \; a \in A : a < b\big) \Longleftrightarrow B \subseteq A + \text{int}K.$
(ii) $A \prec_u B \quad :\Longleftrightarrow \quad \big(\forall a \in A \; \exists \; b \in B : a < b\big) \Longleftrightarrow A \subseteq B - \text{int}K.$
(iii) $A \prec_p B \quad :\Longleftrightarrow \quad \big(\exists a \in A \; \exists \; b \in B : a < b\big) \Longleftrightarrow (A - B) \cap (-\text{int}K) \neq \emptyset.$
(iv) $A \prec_c B \quad :\Longleftrightarrow \quad (\forall a \in A \; \forall b \in B : a < b) \Longleftrightarrow A - B \subseteq -\text{int}K.$
(v) $A \prec_{pc} B \quad :\Longleftrightarrow \quad \big(\exists a \in A \; \forall b \in B : a < b\big) \Longleftrightarrow \exists a \in A \text{ such that } \; B \subseteq a + \text{int}K.$
(vi) $A \prec_{cp} B \quad :\Longleftrightarrow \quad \big(\exists b \in B \; \forall a \in A : a < b\big) \Longleftrightarrow \exists b \in B \text{ such that } \; A \subseteq b - \text{int}K.$

The order relations in Definition 2.2 have been introduced in [40], see also [41]. In the literature, they are denoted in different ways, see [42, Definition 2.1] and [30, Definition 2.1]. For the names of the set order relations $\preceq_p$ and $\preceq_c$, see [26] (note that the relation $A \preceq_c B$ has been defined in [26] as $A = B$ or $A - B \subseteq -K$). The prefixes "possibly-certainly" and "certainly-possibly" in the order relations $A \preceq_{pc} B$ and $A \preceq_{cp} B$ are motivated by the fact that the relation $a \leq b$ holds for at least one point a in A and for all points b in B in the first case and for all points in A and some point b in B in the second case.

Let us state some properties of the order relations which are immediate from the definition.

Lemma 2.2. *(compare* [30, Lemma 2.5]*) Let A and B be nonempty subsets of Y. Then*

(a) $A \preceq^u B$ *iff* $(-B) \preceq^l (-A)$.
(b) $A \preceq^{cp} B$ *iff* $(-B) \preceq^{pc} (-A)$.
(c) $A \preceq_c B \implies A \preceq_l B \implies A \preceq_p B$.

In view of Lemma 2.2, results for $\preceq^u$ and $\preceq^{cp}$ can be deduced from the ones obtained for $\preceq^l$ and $\preceq^{pc}$, respectively. Motivated by this fact, we will **restrict ourselves to $\preceq_r$ with $r \in \{l, p, c, pc\}$ and write simply $\preceq_r$ when there is no need to specify which r is under consideration**.

Lemma 2.3. *(compare* [26, 42]*)The following assertions are true.*

(a) *The order relation $\preceq_r$ is transitive for $r \in \{l, c, pc\}$.*
(b) *The order relation $\preceq_r$ is reflexive for $r \in \{l, p\}$. The order relations $\preceq_{pc}$ is reflexive provided considered sets possess ideal minimal points.*

Let A and B be nonempty subsets of Y. We define a relation $\sim_r$ as follows

$$A \sim_r B \; :\Longleftrightarrow \; A \preceq_r B \text{ and } B \preceq_r A.$$

Lemma 2.3 implies that the relations $\sim_l$ and $\sim_{pc}$ are reflexive and hence, are quasi-orders (in the case $\sim_{pc}$, the sets under consideration are assumed to possess ideal minimal points).

Lemma 2.4. *Let A and B be nonempty subsets of Y. Then*

(a) *$A \sim_l B$ iff $A + K = B + K$. Assume that A and B are K-compact. Then $A \sim_l B$ iff $\mathrm{Min}(A) + K = \mathrm{Min}(B) + K$ and if K is pointed, then $A \sim_l B$ iff the sets $\mathrm{Min}(A)$ and $\mathrm{Min}(B)$ coincide.*

(b) *$A \sim_c B$ iff $A - B \subseteq K \cap (-K)$. If K is pointed, then $A \sim_c B$ iff A and B are singletons and coincide.*

(c) *$A \sim_{pc} B$ iff the sets $\mathrm{IMin}(A)$, $\mathrm{IMin}(B)$ are nonempty and $\mathrm{IMin}(A) + K = \mathrm{IMin}(B) + K$. If K is pointed, then $A \sim_{pc} B$ iff $\mathrm{IMin}(A)$ and $\mathrm{IMin}(B)$ are singletons and coincide.*

Proof. (a) The facts that $A \sim_l B$ iff $A + K = B + K$ is well-known. The next two other cases can be checked using Proposition 2.1 and a simple argument.

(b) It is obvious.

(c) The "only if" part. As $A \sim_{pc} B$, there exist $\bar{a} \in A$ and $\bar{b} \in B$ such that $\bar{a} \leq b$ for all $b \in B$ and $\bar{b} \leq a$ for all $a \in A$. Then $\bar{a} \leq \bar{b} \leq a$ for all $a \in A$ and $\bar{b} \leq \bar{a} \leq b$ for all $b \in B$. Hence, $\bar{a} \in \mathrm{IMin}(A)$ and $\bar{b} \in \mathrm{IMin}(B)$. Since $a \in \bar{b} + K$ for all $a \in A$, it follows that $A \subset \mathrm{IMin}(B) + K$ and $\mathrm{IMin}(A) + K \subseteq \mathrm{IMin}(B) + K$. Similarly, we have $\mathrm{IMin}(B) + K \subseteq \mathrm{IMin}(A) + K$. Therefore, $\mathrm{IMin}(A) + K = \mathrm{IMin}(B) + K$. If K is pointed, then $\mathrm{IMin}(A) = \{\bar{a}\}$ and $\mathrm{IMin}(B) = \{\bar{b}\}$ and it follows from $\bar{a} \leq \bar{b} \leq \bar{a}$ that $\bar{a} = \bar{b}$.

The "if" part. Assume that $\mathrm{IMin}(A)+K = \mathrm{IMin}(B)+K$. Since $\mathrm{IMin}(B) \subseteq \mathrm{IMin}(B)+K = \mathrm{IMin}(A) + K$, there exist $\bar{a} \in \mathrm{IMin}(A)$ and $\bar{b} \in \mathrm{IMin}(B)$ such that $\bar{a} \leq \bar{b}$ and since $\bar{b} \leq b$ for all $b \in B$ we get $\bar{a} \leq b$ for all $b \in B$, which means that $A \preceq_{pc} B$. The inverse relation $B \preceq_{pc} A$ can be proved similarly. □

3. An abstract scalarizing function defined on Y: properties and examples

In this section, we define an abstract scalarizing function $\theta \colon Y \to \mathbb{R}$ satisfying a set of properties and show that the known scalarizing functions involving Gerstewizt's function, Hiriart-Urruty signed distance and the function proposed by Kasimbeyli are examples of such a function.

3.1. **Desired properties for an abstract scalarizing function defined on Y.** We will consider the following properties.

(P1) *Global Lipschizity property*: θ is global Lipschitz with some Lipschitz constant $\mathcal{L}$.

(P2) *Sublinearity*: $\theta(ty) = t\theta(y)$ for any $y \in Y$ and scalar $t \geq 0$ (positive homogenuity) and $\theta(y_1 + y_2) \leq \theta(y_1) + \theta(y_2)$ for any $y_1, y_2 \in Y$ (subadditivity).

(P3) *K-monotonicity*: $\theta(y_1) \leq \theta(y_2)$ for any $y_1, y_2 \in Y,\ y_1 \leq y_2$.

(P4) *Cone representation property*:

$$-K = \{y \in Y \colon \theta(y) \leq 0\}.$$

(P5) *Cone interior representation property*

$$-\text{int}K = \{y \in Y \colon \theta(y) < 0\}.$$

(P6) *Uniform positivity*: For any $\alpha > 0$, there exists a scalar $\beta > 0$ such that

$$y + \alpha\mathbb{B} \subset Y \setminus (-K) \Longrightarrow \theta(y) \geq \beta.$$

Definition 3.1. We say that θ is an *abstract scalarizing function* if it satisfies Properties (P1)-(P4).

In what follows, by saying "θ satisfies Property (P5)", we implicitly mean that K is solid. Property (P5) can be expressed in term of the so called *Slater condition*, which means that there exists $y \in Y$ such that $\theta(y) < 0$. Moreover, we have.

Lemma 3.1. *Assume that θ is continuous and satisfies Property (P4). Then*

(a) *If θ satisfies the Slater condition, then K is solid. Moreover, θ possesses Property (P5) iff θ satisfies the Slater condition.*

(b) *Properties (P2) and (P5) implies Property (P6).*

Proof. (a) It is obvious.

(b) Let $\alpha > 0$ be given. Take $\bar{y} \in -\text{int}K$ with $\|\bar{y}\| = 1$ and denote $\beta := -\alpha\theta(\bar{y})$. Clearly, $\beta > 0$. Let $y \in Y$ be such that $y + \alpha\mathbb{B} \subset Y \setminus (-K)$. Since $y + \alpha\bar{y} \in y + \alpha\mathbb{B} \subset Y \setminus (-K)$, Property (P4) implies $\theta(y+\alpha\bar{y}) > 0$ and by Property (P2), we have $\theta(y) \geq \theta(y+\alpha\bar{y}) - \theta(\alpha\bar{y})$. Therefore, $\theta(y) \geq \theta(y + \alpha\bar{y}) - \theta(\alpha\bar{y}) > -\alpha\theta(\bar{y}) = \beta$. □

Remark 3.1. Let us discuss Properties (P1)-(P6).

(i) We may assume simply that θ is continuous, but we consider the global Lipschitzity property because this property is important for the finiteness of scalarizing functions of "sup-inf"- and "inf-sup"-types appeared in set optimization.

(ii) Properties (P2) and (P3) imply a part of Property (P4), namely if θ is positively homogeneous and K-monotone, then $-K \subset \{y \in Y \colon \theta(y) \leq 0\}$ (because $\theta(0) = 0$ and $-k \leq 0$ for all $k \in K$).

(iii) It will be shown in (forthcoming) Example 3.1 that an abstract scalarizing function may satisfy Property (P6) without satisfying Property (P5).

(iv) Assume that K is a solid pointed closed convex cone in a Banach space. In terms of [3, Definition 3], a continuous function θ satisfying (P4) and (P5) gives a *robust* *representation* of the cone $-K$. Recall ([3, Definition 7]) that a continuous function θ satisfies the *order representability property* if

(a) $\{y \in Y\colon \theta(y) \leq 0\} \subseteq -K$;

(b) $\{y \in Y\colon \theta(y) < 0\} \subseteq -\text{int}K$.

As it is mentioned in (ii), if in addition θ is K-monotone and positively homogenous, then instead of (a), we have the equality $-K = \{y \in Y\colon \theta(y) \leq 0\}$.

Next, we provide an estimation of the subdifferential of the function θ that could be used for formulating optimality conditions in set optimization involving some tools of variational analysis such as the coderivative, slope and directional derivative. We also prove a characterization of (P5) in terms of the subdifferential of θ at zero.

Lemma 3.2. *Assume that θ possesses Properties (P1)-(P4). Then*

(a) *θ is convex and for all $y \in Y$ we have $|\theta(y)| \leq \mathcal{L}\|y\|$ and $\partial\theta(y) \subset K^* \cap \mathcal{L}\mathbb{B}^*$.*

(b) *(K is assumed to be solid) θ satisfies (P5) iff there exists $\alpha > 0$ such that*

$$\|y^*\| \geq \alpha, \quad \forall y^* \in \partial\theta(0).$$

Proof. (a) The fact that θ is convex is obvious. The inequality $|\theta(y)| \leq \mathcal{L}\|y\|$ follows from the Lipschizity of θ and the equality $\theta(0) = 0$. The inclusion $\partial\theta(y) \subset K^* \cap \mathcal{L}\mathbb{B}^*$ can be proved using the K-monotonicity and the Lipschitzity of θ.

(b) Assume that θ does not satisfy (P5). Then $\theta(y) \geq 0$ for all $y \in Y$ and $\theta(0) = 0$. Hence, $0 \in \partial\theta(0)$ and the mentioned scalar α does not exist. Next assume that θ satisfies (P5). Then there exists $\bar{k} \in \text{int}K$ such that $\|\bar{k}\| = 1$ and $\alpha := -\theta(-\bar{k}) > 0$. For any $y^* \in \partial\theta(0)$, we have $\langle y^*, -\bar{k}\rangle \leq \theta(-\bar{k}) - \theta(0) = \theta(-\bar{k}) = -\alpha$. It follows that $\alpha \leq \langle y^*, \bar{k}\rangle \leq \|y^*\|\|\bar{k}\| = \|y^*\|$. □

We recall a result that will be used in our study. In what follows, we say that a function $g : Y \to \mathbb{R}$ is K-monotone (resp., $(-K)$-monotone) if $g(y_1) \leq g(y_2)$ (resp., $g(y_1) \geq g(y_2)$) for any $y_1, y_2 \in Y$ satisfying $y_1 \leq y_2$.

Lemma 3.3. [29, Lemma 4.9] *Assume $g : Y \to \mathbb{R}$ be a function and that A is a K-compact subset of Y. If g is continuous and K-monotone (resp., $(-K)$-monotone), then g achieves its minimum (resp., maximum) on A.*

3.2. **Examples of abstract scalarizing functions.** We recall three scalarizing functions which are important tools in vector and set optimization. These functions are defined in more general settings, but we restrict ourselves to a simple form of definition.

1. *The scalarizing function proposed by Gerstewizt (Tammer)* [7]. Let K be a solid pointed closed convex cone and $k_0 \in \text{int}K$ be a fixed vector. Define a function $\varphi_{k_0}\colon Y \to \mathbb{R}$ by

$$\varphi_{k_0}(y) := \inf\{t \in \mathbb{R}\colon y \in tk_0 - K\} = \inf\{t \in \mathbb{R}\colon y \leq tk_0\}.$$

Note that this function is a special case of the function considered in [19, Definition 4.1]. We show that φ_{k_0} is well-defined by providing estimations for its values. Let

$$\rho := \sup\{r \in \mathbb{R} \colon \mathbb{B}(k_0, r) \subset K\},$$

where $\mathbb{B}(k_0, r)$ is the closed ball centered at k_0 with radius ρ. Since $k_0 \in \text{int}K$ and K is pointed, one can prove that $0 < \rho < \infty$. Let $y \in Y$. It follows from the relations $k_0 \pm \frac{\rho'}{\|y\|} y \in \mathbb{B}(k_0, \rho') \subset K$ and $k_0 + \frac{\rho'}{\|y\|} y \notin -K$ for any $\rho' \in]0, \rho[$ that

$$-\frac{\|y\|}{\rho} k_0 \leq y \leq \frac{\|y\|}{\rho} k_0 \quad \text{and} \quad -\frac{\|y\|}{\rho} \leq \varphi_{k_0}(y) \leq \frac{\|y\|}{\rho}. \tag{1}$$

Proposition 3.1. *The function φ_{k_0} has Properties (P1)-(P6) and is global Lipschitz with the constant $1/\rho$.*

Proof. For Properties (P1)-(P4), see [19, Lemma 4.2]. Further, the Slater condition is satisfied because $\varphi_{k_0}(-k_0) = -1$ and Lemma 3.1 implies that θ has Properties (P5)-(P6). Finally, the subadditivity and the second inequality in (1) imply $\varphi_{k_0}(u) - \varphi_{k_0}(v) \leq \varphi_{k_0}(u - v) \leq \|u - v\|/\rho$ and $\varphi_{k_0}(v) - \varphi_{k_0}(u) \leq \varphi_{k_0}(v - u) \leq \|u - v\|/\rho$. Hence, φ_{k_0} is global Lipschitz with the constant $1/\rho$. □

Remark 3.2. Given $k_0 \in \text{int}K$, Krasnoselski introduced the so called k_0-norm in [39] as follows: for any $y \in Y$,

$$\|y\|_{k_0} := \inf\{t \in \mathbb{R}_+ \colon -tk_0 \leq y \leq tk_0\}.$$

Then the first inequality in (1) implies that k_0-norm is well-defined. Note that in contrast to the k_0-norm, φ_{k_0} may have both negative and positive values.

2. *The Hiriart-Urruty signed distance* Δ_{-K} associated to the cone K is defined by

$$\Delta_{-K}(y) := d_{-K}(y) - d_{Y\setminus(-K)}(y) = \begin{cases} -d_{Y\setminus(-K)}(y) & \text{if } y \in -K \\ d_{-K}(y) & \text{otherwise.} \end{cases}$$

We refer the reader to [21] for the definition of this function in the general case.

Proposition 3.2. *The function Δ_{-K} has Properties (P1)-(P6) and is global Lipschitz with the constant 1.*

Proof. Properties (P1)-(P4) and the constant of the Lipschizity of the function Δ_{-K} are known, see [21, Propositions 2 and 5] and [49]. Next, assume that K is solid. Let $k_0 \in \text{int}K$. Then $-k_0 + \rho\mathbb{B} \subset -K$ for some $\rho > 0$ and $\theta(-k_0) = \Delta_{-K}(-k_0) = -d_{Y\setminus(-K)}(-k_0) < -\rho < 0$. Hence θ satisfies the cone interior representation property (P5). Finally, Lemma 3.1 implies that θ satisfies Property (P6): for any $\alpha > 0$ we can take $\beta = \alpha$ because for $y \in Y$ satisfying $y + \alpha\mathbb{B} \subset Y \setminus (-K)$, one has $\theta(y) = \Delta_{-K}(y) = d_{-K}(y) \geq \alpha$. □

3. *The scalarizing function used by Kasimbeyli* in the conic scalarization method (see [31, 32, 33] for the history, applications of this method and comparison to other scalarization methods). Here, we recall this function in the case $K = C(\ell)$, where $C(\ell)$ is a Bishop-Phelps cone defined by a continuous linear functional $\ell \in Y^*$ with $\|\ell\| \geq 1$ as follows

$$C(\ell) := \{y \in Y : \ell(y) \geq \|y\|\}.$$

Clearly, K is a closed pointed convex cone. When $\|\ell\| = 1$, $C(\ell)$ has the form

$$C(\ell) = \{y \in Y : \ell(y) = \|y\|\}$$

[16] and it is then called a Bishop-Phelps cone given by an equation in [17]. The scalarizing function $\xi_\ell \colon Y \to \mathbb{R}$ proposed by Kasimbeyli is defined as follows: for every $y \in Y$

$$\xi_\ell(y) := \ell(y) + \|y\|.$$

Proposition 3.3. *The function ξ_ℓ has Properties (P1)-(P4) and is globally Lipschitz with the constant $\|\ell\| + 1$. It also has Properties (P5)-(P6) in the case $\|\theta\| > 1$.*

Proof. Note that Properties (P1) -(P2) are obvious and Property (P3) has been established in [31, Theorem 3.5]. Property (P4) is satisfied because

$$y \in -K \Longleftrightarrow -y \in C(\ell) \Longleftrightarrow \ell(-y) \geq \| - y\| \Longleftrightarrow \xi_\ell(y) = \ell(y) + \|y\| \leq 0.$$

Further, for any $y_1, y_2 \in Y$ we have

$$\xi_\ell(y_1) - \xi_\ell(y_2)| = \ell(y_1 - y_2) + \|y_1\| - \|y_2\| \leq \ell(y_1 - y_2) + \|y_1 - y_2\| \leq (\|\ell\| + 1)\|y_1 - y_2\|$$

and, similarly, $\xi_\ell(y_2) - \xi_\ell(y_1)| \leq (\|\ell\| + 1)\|y_2 - y_1\|$. Hence, ξ_ℓ is globally Lipschitz with the constant $\|\ell\| + 1$. Finally, assume that $\|\ell\| > 1$. Then $C(\ell)$ is solid and $\text{int}C(\ell) = \{x \in X : \ell(x) > \|x\|\}$ [25]. It is easy to see that $-\text{int}K = \{y \in Y : \theta_\ell(y) < 0\}$, which means that Property (P5) is satisfied. Finally, Lemma 3.1 implies that θ satisfies Property (P6). □

Example 3.1. It is of interest to know about properties of $\theta = \xi_\ell$ in the case $\|\ell\| = 1$, i.e. K is a Bishop-Phelps cone defined by an equation. In this case, it is clear that θ does not satisfied (P5), however it still may satisfy (P6). To see this, let us first provide examples of Bishop-Phelps cone defined by an equation. Let $L^1_{[0,1]}$ be the Banach space of integrable functions defined on $[0, 1]$ and l_p $(1 \leq p < \infty)$ be the Banach space of p-power summable sequences. It has been shown in [17] that nonnegative orthants in $\mathbb{R}^n$, l_1 and $L^1_{[0,1]}$, the Lorentz cones (also called second-order cones) in $\mathbb{R}^n$ and the extended Lorentz cones in l_p $(1 \leq p < \infty)$ are Bishop-Phelps cones given by an equation. In the finite dimensional space setting, let consider the cone K being the nonnegative orthant in $\mathbb{R}^n$ with the l_1 norm. Then

$K = C(\ell)$, where $\ell = (1, ..., 1)$ with $\|\ell\| = 1$ and $\ell(y) = \sum_{j=1}^{n} y_j$ for any $y = (y_1, ..., y_n)$. The function ξ_ℓ has the form

$$\xi_\ell(y) = \sum_{j=1}^{n} y_j + \sum_{j=1}^{n} |y_j|$$

and $-K = \{y \colon \xi_\ell(y) = 0\}$. The cone K is solid but ξ_ℓ does not satisfy (P5). We show that, nevertheless, ξ_ℓ satisfies (P6). Given $\alpha > 0$ we show that (P6) is satisfied with $\beta = 2\alpha/n$. Let $y \in \mathbb{R}^n$ be such that $y + \alpha\mathbb{B} \subset \mathbb{R}^n \setminus (-\mathbb{R}^n_+)$. If $y_j < \alpha/n$ for all $j = 1, ..., n$, then $y + \alpha b \in -\mathbb{R}^n_+$, where $b = (-1/n, ..., -1/n) \in \mathbb{B}$, a contradiction to $y + \alpha\mathbb{B} \subset \mathbb{R}^n \setminus (-\mathbb{R}^n_+)$. Hence, there is an index j such that $y_j \geq \alpha/n$ and we get $\xi_\ell(y) \geq 2\alpha/n = \beta$.

Remark 3.3. We would like to stay more in the work [3], in which a unified characterization of nonlinear scalarizing functionals in optimization has been presented. Recall that if θ is a continuous sublinear K-monotone function possessing the representability property, then there exists a weak*-compact convex generator G of K^* (this means $0 \notin G$ and $K^* = \text{cone}G = \{\lambda g \colon \lambda \geq 0\}$) such that $\theta(y) = \sigma_G(y) := \sup_{y^* \in G} y^{\cdot}(y)$ for all $y \in Y$ [3, Theorem 3]. The function σ_G has been used by Drummond-Svaiter as a scalarizing function [8]. The authors established several relationships in the sense of inclusion between three mayor classes of scalarizing functions, namely that of the Gerstewitz, that of Hiriart-Urruty and that of Drummond-Svaiter and introduced new scalarizing functions that are not necessarily convex, but quasidifferentiable and positively homogeneous.

3.3. **The finiteness of θ on cone boundedness set.** From now on, we make a convention that θ is an abstract scalarizing function, i.e.,

θ satisfies Properties (P1)-(P4).

In this subsection, we study conditions ensuring the finiteness of θ on some subset A of Y. Similar to the work [27], in which θ is the Hiriart-Urruty signed distance, sometimes we need the assumption that either the cone K is nonsolid or A is K-proper. To combine these two situations, we will use the expression *"A is W-bounded"* in the sense that $A \subseteq M + W$ for some bounded set M, where

$$W := (Y \setminus (-K)) \cup \{0\}.$$

Note that in view of Lemma 2.1, if K is nonsolid, then every nonempty subset of Y is W-bounded and if K is solid, then a set A is K-proper iff A is W-bounded. Observe that $W + K = W$ and that A is W-bounded iff so is the set $A + K$. In what follows, we say that A is $(-W)$-bounded if $(-A)$ is W-bounded.

Proposition 3.4. *Let A be a nonempty subset of Y. Then*

(a) $\sup_{a \in A} \theta(a) < \infty$ *iff A is $(-K)$-bounded (the "only if" part is true provided θ has Property (P5)).*

(b) $\inf_{a\in A}\theta(a) > -\infty$ *iff* A *is* W*-bounded (the "only if" part is true provided either* K *is nonsolid or* θ *has Property (P5)).*

Proof. (a) The " if" part: Let M be a bounded set such that $A \subseteq M - K$. For an arbitrary $a \in A$, there exist $m \in M$ such that $a \leq m$. Since θ is K-monotone and Lipschitz, we get $\theta(a) \leq \theta(m) \leq \mathcal{L}\sup_{m\in M}\|m\|$. Therefore, $\sup_{a\in A}\theta(a) < \infty$.

The "only if" part: Let $e \in \mathbb{B}$ such that $\theta(e) < 0$. Suppose that A is not $(-K)$-bounded. Then $A \not\subseteq j\mathbb{B} - K$ for all $j = 1, 2, ...$ For each j there exists $a_j \in A$ such that $a_j \notin j\mathbb{B} - K$ and therefore, $a_j + je \notin -K$ and $\theta(a_j + je) > 0$. Since $\theta(a_j) + \theta(je) \geq \theta(a_j + je) > 0$, we get $\sup_{a\in A}\theta(a) \geq \theta(a_j) > -j\theta(e)$ for all $j = 1, 2, ...$ Hence, we obtain $\sup_{a\in A}\theta(a) = \infty$, a contradiction.

(b) The " if" part: Let M be a bounded set such that $A \subseteq M + W$. For any $a \in A$ there exist $m \in M$ and $w \in W$ such that $a = m + w$. Property (P4) implies that $\theta(w) \geq 0$. Since θ is sublinear and Lipschitz, we get

$$\theta(a) = \theta(m + w) \geq \theta(w) - \theta(-m) \geq -\theta(-m) \geq -\mathcal{L}\sup_{m\in M}\|m\| > -\infty.$$

The "only if" part: Lemma 2.1 implies that every subset A of Y is W-bounded when K is non-solid. It remains to consider the case K is solid. By the assumption, one can find a nonzero vector $\bar{y} \in \mathbb{B}$ such that $\theta(\bar{y}) < 0$. Let A be a nonempty set of Y such that $\inf_{a\in A}\theta(a) > -\infty$. Suppose to the contrary that A is not W-bounded. Then for any $j = 1, 2, ...$ we have $A \not\subseteq j\mathbb{B}+W$ and we can find $a_j \in A$ such that $a_j \notin j\mathbb{B}+W$. Then $a_j - j\bar{y} \notin W$ and $\theta(a_j - j\bar{y}) \leq 0$ for all j. It follows that $\inf_{a\in A}\theta(a) \leq \theta(a_j) \leq \theta(a_j - j\bar{y}) + \theta(j\bar{y}) \leq j\theta(\bar{y})$ for all $j = 1, 2, ...$, a contradiction to $\inf_{a\in A}\theta(a) > -\infty$. □

Remark 3.4. (a) Let us stay more on the first assertion of Proposition 3.4.

Note that the $(-K)$-boundedness assumption in the "if " part cannot be relaxed to $(-W)$-boundedness. Indeed, let $Y = L_{[0,1]}$ and K be its nonnegative orthant, which is known to be a Bishop-Phelps cone given by an equation $C(\ell)$ with $\ell(y) = \int_0^1 y(t)dt$, namely, $K = \{y \in L_{[0,1]} : \ell(y) = \|y\|\}$. The scalarizing function θ is given by $\theta(y) = \int_0^1 y(t)dt + \int_0^1 |y(t)|dt$. Let $A \subset L_{[0,1]}$ be the set consisting of functions $y_i(t)$, $i = 1, 2, ...$ of the form $y_i(t) = i$ for $t \in [0, 0.5]$ and $y_i(t) = -1$ for $t \in [0.5, 1]$. Since $A \subset Y \setminus K$, this set is $(-W)$-bounded. One can check that $\sup_{y\in A}\theta(y) = \infty$. In view of Proposition 3.4, this set is not $(-K)$-bounded.

(b) The second assertion of Proposition 3.4 is inspired by the characterization of the K-properness of a set via the finiteness of a scalarizing function involving the Hiriart-Urruty signed distance [27, Proposition 5.5]. Note that if θ does not satisfy the Slater condition, then $\inf_{a\in A}\theta(a) \geq 0$ for any nonempty set A.

Proposition 3.5. *Let* A *be a nonempty subset of* Y*. Then*

(a) *Assume that* $\inf_{a\in A}\theta(a) > -\infty$. *Then the function* $g : Y \to \mathbb{R}$ *defined by* $g(y) = \inf_{a\in A}\theta(a-y)$ *is globally Lipschitz and* $(-K)$*-monotone.*

(b) *Assume that* $\sup_{a\in A}\theta(a) < \infty$. *Then the function* $v : Y \to \mathbb{R}$ *defined by* $v(y) = \sup_{a\in A}\theta(a+y)$ *is globally Lipschitz and* K*-monotone.*

Proof. We prove the assertion (a) and omit a similar proof for the assertion (b). Let $y_1, y_2 \in Y$ be arbitrary vectors. For any $\epsilon > 0$, there exist $\bar{a}_1 \in A$ and $\bar{a}_2 \in A$ such that $\theta(\bar{a}_1 - y_1) \leq \inf_{a\in A}\theta(a-y_1)+\epsilon$ and $\theta(\bar{a}_2-y_2) \leq \inf_{a\in A}\theta(a-y_2)+\epsilon$. Since $\theta(\bar{a}_2-y_1) \leq \theta(\bar{a}_2-y_2)+\mathcal{L}\|y_1-y_2\|$, we get

$$\inf_{a\in A}\theta(a-y_1) \leq \theta(\bar{a}_2-y_1) \leq \theta(\bar{a}_2-y_2)+\mathcal{L}\|y_1-y_2\| \leq \inf_{a\in A}\theta(a-y_2)+\epsilon+\mathcal{L}\|y_1-y_2\|.$$

Similarly, we have $\inf_{a\in A}\theta(a-y_2) \leq \inf_{a\in A}\theta(a-y_1)+\epsilon+\mathcal{L}\|y_1-y_2\|$. Since ϵ is an arebitrary positive scalar, it follows that the function g is globally Lipschitz.

Assume in addition that $y_1 \geq y_2$. For any $u \in A$ we have $\theta(u-y_1) \leq \theta(u-y_2)$ and $\inf_{a\in A}\theta(a-y_1) \leq \theta(u-y_1) \leq \theta(u-y_2)$. Hence, $\inf_{a\in A}\theta(a-y_1) \leq \inf_{a\in A}\theta(a-y_2)$, which means that the function g is $(-K)$-monotone. □

4. Signed Hausdorff-type half-distances: Definition and properties

We assume as before that $\theta : Y \to \mathbb{R}$ is an abstract scalarizing function with Properties (P1)-(P4). With the help of the function θ, we define scalars $h^r(A,B)$ associated to any two nonempty subsets A and B of Y and use them to characterize set order relations $A \preceq_r B$. We call these scalars signed Hausdorff-type half-distances between two sets and the motivation of this name will be explained later.

Note that similar scalars have been defined with different notations for θ being the Hiriart-Urruty function Δ_{-K} in [14] ($r \in \{l,u\}$) and in [27] ($r \in \{l,u,p,c,pc,cp\}$), and for θ being the Gerstewitz function in [37] ($r \in \{l,u\}$), see also [2, 22, 28, 29, 30]. Results of this section are inspired mainly by the works [14, 27, 30].

Let A and B be nonempty subsets of Y. We define the following quantities

$$\begin{aligned}
h^l_\theta(A,B) &:= \sup_{b\in B}\inf_{a\in A}\theta(a-b),\\
h^u_\theta(A,B) &:= \sup_{a\in A}\inf_{b\in B}\theta(a-b),\\
h^p_\theta(A,B) &:= \inf_{a\in A}\inf_{b\in B}\theta(a-b),\\
h^c_\theta(A,B) &:= \sup_{a\in A}\sup_{b\in B}\theta(a-b),\\
h^{pc}_\theta(A,B) &:= \inf_{a\in A}\sup_{b\in B}\theta(a-b),\\
h^{cp}_\theta(A,B) &:= \inf_{b\in B}\sup_{a\in A}\theta(a-b).
\end{aligned}$$

Let us note a correspondence between the definition of the quantity h^r_θ and the order relation $\preceq_r$: the "sup" or "inf" in the definition of h^r_θ corresponds to "$\forall$" or "$\exists$" in the definition of $\preceq_r$, respectively. For instance, $A \preceq_l B$ means $\forall b \in B$, $\exists a \in A$ such that $a \leq b$ and $h^l_\theta(A, B) = \sup_{b\in B} \inf_{a\in A} \theta(a - b)$. It is easy to see that

$$h^p_\theta(A, B) = \inf_{b\in B} \inf_{a\in A} \theta(a - b) \text{ and } h^c_\theta(A, B) = \sup_{b\in B} \sup_{a\in A} \theta(a - b).$$

Some of $h^r_\theta(A, B)$ are closely related to the classical Hausdorff half-distance $H(A, B)$ defined by

$$H(A, B) := \sup_{b\in B} \inf_{a\in A} \|a - b\|.$$

Namely, in the special case $\theta = \Delta_{-K}$ and $K = \{0\}$, we have $\theta(y) = \Delta_{\{0\}}(y) = \|y\|$ and

$$\begin{array}{rcl} h^l_\theta(A, B) & = & H(A, B) \\ h^u_\theta(A, B) & = & H(B, A) \\ h^p_\theta(A, B) & = & H(A - B, \{0\}) \\ h^c_\theta(A, B) & = & H(\{0\}, A - B). \end{array}$$

Due to the first equality, $h^l_\theta(A, B)$ with $\theta = \Delta_{-K}$ (K is an arbitrary pointed closed convex cone) was called a Hausdorff-type half-distance between A and B in [14]. Note that some scalars $h^r_\theta(A, B)$ can be negative when θ satisfies the Slater condition. Motivates by these facts, we call $h^r_\theta(A, B)$ ($r \in \{l, u, p, c, pc, cp\}$) *signed Hausdorff-type half-distances* between the sets A and B.

In what follows, we drop the subscript θ in the notation h^r_θ for $r \in \{l, u, p, c, pc, cp\}$. Some elementary properties of h^r are formulated in the following.

Lemma 4.1. *Let A and B be nonempty subsets of Y. Then*

(a) $h^u(A, B) = h^l(-B, -A)$.
(b) $h^{cp}(A, B) = h^{pc}(-B, -A)$.
(c) $h^p(A, B) = h^p(-B, -A)$.
(d) $h^c(A, B) = h^c(-B, -A)$.
(e) $h^p(A, B) \leq h^l(A, B) \leq h^{pc}(A, B) \leq h^c(A, B)$.
(f) $h^p(A, B) \leq h^u(A, B) \leq h^{cp}(A, B) \leq h^c(A, B)$.

Proof. These relations are immediate from the definitions and elementary properties of the operations "inf" and "sup". □

Remark 4.1. The assertions (a),(b),(e) and (f) in Lemma 4.1 with θ being the Hiriart-Urruty signed distance have been established in [30, Lemmas 3.2 and 3.3].

In what follows, **we will restrict ourselves to $r \in \{l, p, c, pc\}$ and write simply r when there is no need to specify which $r \in \{l, p, c, pc\}$ is involved**. Results for the cases

$r = u$ and $r = cp$ can be deduced from the ones established for the cases $r = l$ and $r = pc$, respectively, thanks to the equalities $h^u(A, B) = h^l(-B, -A)$ and $h^{cp}(A, B) = h^{pc}(-B, -A)$.

The following proposition establishes conditions for the finiteness of $h^r(A, B)$.

Proposition 4.1. *Let A and B be nonempty subsets of Y. Then*

(a) *$h^l(A, B) > -\infty$ iff A is W-bounded (the "only if" part is true provided either K is nonsolid or θ satisfies Property (P5)) and $h^l(A, B) < \infty$ iff B is K-bounded (the "only if" part is true provided A is K-bounded and θ satisfies Property (P5)).*

(b) *$h^p(A, B) > -\infty$ if either A is W-bounded and $-B$ is K-bounded or A is K-bounded and $-B$ is W-bounded. Conversely, if θ satisfies Property (P5), then $h^p(A, B) > -\infty$ only if A and $-B$ are K-proper.*

(c) *$h^c(A, B) < \infty$ iff $-A$ and B are K-bounded (the "only if" part is true provided θ satisfies Property (P5)).*

(d) *$h^{pc}(A, B) > -\infty$ if A is W-bounded and $h^{pc}(A, B) < \infty$ if B is K-bounded.*

Proof. (a) and (d) We show that for $r \in \{l, pc\}$, one has $h^r(A, B) > -\infty$ if A is W-bounded and $h^r(A, B) < \infty$ if B is K-bounded . First, taking Proposition 3.4 into account, we get

$$\begin{aligned} h^l(A, B) > -\infty \quad &\Longleftrightarrow \exists \bar{b} \in B \text{ such that } \inf_{a\in A} \theta(a - \bar{b}) > -\infty \\ &\Longleftrightarrow A - \bar{b} \text{ is } W - \text{bounded} \Longleftrightarrow A \text{ is } W - \text{bounded}. \end{aligned}$$

Thus, if A is W-bounded then $h^l(A, B) > -\infty$ and since $h^l(A, B) \leq h^{pc}(A, B)$ (see Lemma 4.1), we also have $h^{pc}(A, B) > -\infty$. Further, if B is K-bounded, Proposition 3.4 implies that for each $a \in A$, $\sup_{b\in B} \theta(a - b) = \sup_{b\in -B} \theta(a + b) < \infty$. Then $h^{pc}(A, B) < \infty$ and, since $h^l(A, B) \leq h^{pc}(A, B)$, we also have $h^l(A, B) < \infty$. It remains to show that if θ satisfies Property (P5) and A is K-bounded, then $h^l(A, B) < \infty$ implies B is K-bounded. Suppose to the contrary that $B \not\subseteq j\mathbb{B} + K$ for every $j = 1, 2, ...$ Let $b_j \in B$ be such that $b_j \notin j\mathbb{B} + K$. Choose $e \in \mathbb{B}$ such that $\theta(e) < 0$. Then $je - b_j \notin -K$ and $\theta(e - b_j) \geq 0$. Let M be a bounded subset of Y such that $A \subseteq M + K$. For any $a \in A$, there exists $m \in M$ such that $m \leq a$ and hence,

$$\theta(a - b_j) \geq \theta(m - b_j) \geq \theta(je - b_j) - \theta(-m) - \theta(je) \geq -\theta(-m) - j\theta(e).$$

Since the set M is bounded and $\theta(e) < 0$, we get $h^l(A, B) = \infty$, a contradiction.

(b) Consider first the case when A is W-bounded and B is $(-K)$-bounded. Proposition 3.4 implies $\inf_{a\in A} \theta(a) > -\infty$ and $\sup_{b\in B} \theta(b) < \infty$. Since $\theta(a - b) \geq \theta(a) - \theta(b)$, we get $\theta(a - b) \geq \inf_{a\in A} \theta(a) - \sup_{b\in B} \theta(b) > -\infty$ and hence, $h^p(A, B) > -\infty$. The case A is K-bounded and B is $(-W)$-bounded follows from the previous one and the equality $h^p(A, B) = h^p(-B, -A)$.

Next, assume that θ satisfies Property (P5) and $h^p(A,B) > -\infty$. One can find $\overline{a} \in A$ and $\overline{b} \in B$ such that $\inf_{b\in B} \theta(\overline{a}-b) > -\infty$ and $\inf_{a\in A} \theta(a-\overline{b}) > -\infty$. Proposition 3.4 implies that the sets $-B-\overline{a}$ and $A-\overline{b}$ are W-bounded. Hence, A and $-B$ are W-bounded.

(c) If $-A$ and B are K-bounded, then Proposition 3.4 implies $\sup_{a\in A} \theta(a) < \infty$ and $\sup_{b\in B} \theta(-b) = \sup_{b\in -B} \theta(b) < \infty$. Since $\theta(a-b) \le \theta(a) + \theta(-b)$, we get $\theta(a-b) \le \sup_{a\in A} \theta(a) + \sup_{b\in B} \theta(-b) < \infty$ and hence, $h^c(A,B) < \infty$. The case A is K-bounded and B is $(-W)$-bounded follows from the previous one and the equality $h^c(A,B) = h^c(-B,-A)$.

Finally, assume that θ satisfies Property (P5) and $h^c(A,B) < \infty$. One can find $\overline{a} \in A$ and $\overline{b} \in B$ such that $\sup_{b\in B} \theta(\overline{a}-b) < \infty$ and $\sup_{a\in A} \theta(a-\overline{b}) < \infty$. Proposition 3.4 implies that the sets $-B+\overline{a}$ and $A-\overline{b}$ are $(-K)$-bounded. Hence, $-A$ and B are K-bounded. □

Remark 4.2. Let us recall some results similar to the ones stated in Proposition 4.1 but for the case $\theta = \Delta_{-K}$. The condition for $h^l(A,B) > -\infty$ is K-boundedness of A [14] or the weaker one - its K-properness [27, 30]. Moreover, it has been established that if K is solid, then $h^l(A,B) > -\infty$ iff A is K-proper [27, Proposition 5.5] and if A is K-bounded, then $h^l(A,B) < \infty$ iff A is K-bounded [27, Proposition 5.8]. Sufficient conditions for h^r to be finite are stated in [30, Theorem 3.13]. Similar results to [27, Proposition 5.8] are [48, Theorem 3.1] and (for θ being the Gerstewitz's scalarizing function) [20, Theorem 3.6].

In what follows unless otherwise specified, we make *a convention* that

$h^r(.,.)$ **is finite for sets under consideration**

Next, we formulate conditions for "sup" and/or "inf" in the definition of h^r to be replaced by "max" and/or "min". In the case $\theta = \Delta_{-K}$, we refer the interested reader to [15, Proposition 2] for $r = l$ and [30, Corollary 3.18] for $r \in \{l,u,p,c,pc,cp\}$.

Proposition 4.2. *Let A and B be nonempty subsets of Y. Then*

(a) $h^l(A,B) = \max_{b\in B} \min_{a\in A} \theta(a-b)$ *if A and B are K-compact.*
(b) $h^p(A,B) = \min_{a\in A} \min_{b\in B} \theta(a-b)$ *if A and $-B$ are K-compact.*
(c) $h^c(A,B) = \max_{a\in A} \max_{b\in B} \theta(a-b)$ *if $-A$ and B are K-compact.*
(d) $h^{pc}(A,B) = \min_{a\in A} \max_{b\in B} \theta(a-b)$ *if A and B are K-compact.*

Proof. The proof is based on Proposition 3.5 and Lemma 3.3. We provide a detailed proof for the assertion (a) and omit similar proofs of the other assertions. First, observe that due Lemma 3.3, for each $b \in B$, $\inf_{a\in A} \theta(a-b) = \min_{a\in A} \theta(a-b)$. Next, Proposition 3.5 implies that $g(y) = \min_{a\in A} \theta(a-y)$ is globally Lipschitz and $(-K)$-monotone. Lemma 3.3 implies that the function g attains its maximum on the K-compact set B and we get $h^l(A,B) = \sup_{b\in B} g(b) = \max_{b\in B} \min_{a\in A} \theta(a-b)$. □

Proposition 4.3. *Let A and B be nonempty subsets of Y. Then*

(a) $h^l(A,B) = h^l(A+K, B+K)$.
(b) $h^p(A,B) = h^p(A+K, B-K)$.
(c) $h^c(A,B) = h^c(A-K, B+K)$.
(d) $h^{pc}(A,B) = h^{pc}(A+K, B+K)$.

Proof. (a) Let $b \in B$ be an arbitrary vector. Since θ is K-monotone, we have $\inf_{a\in A}\theta(a-b) \leq \inf_{a\in A+K}\theta(a-b)$ and since $A \subset A+K$, we have $\inf_{a\in A+K}\theta(a-b) \leq \inf_{a\in A}\theta(a-b)$. Therefore, $\inf_{a\in A}\theta(a-b) = \inf_{a\in A+K}\theta(a-b)$. Next, since the function $\inf_{a\in A}\theta(a-.)$ is $(-K)$-monotone due to Proposition 3.5, we have $\sup_{b\in B+K}\inf_{a\in A}\theta(a-b) \leq \sup_{b\in B}\inf_{a\in A}\theta(a-b)$. On the other hand, since $B \subset B+K$, we have $\sup_{b\in B}\inf_{a\in A}\theta(a-b) \leq \sup_{b\in B+K}\inf_{a\in A}\theta(a-b)$. Therefore, we have $\sup_{b\in B}\inf_{a\in A}\theta(a-b) = \sup_{b\in B+K}\inf_{a\in A}\theta(a-b)$ and

$$\sup_{b\in B}\inf_{a\in A}\theta(a-b) = \sup_{b\in B+K}\inf_{a\in A}\theta(a-b) = \sup_{b\in B+K}\inf_{a\in A+K}\theta(a-b).$$

(b) Since $A \subseteq A+K$ and $B \subseteq B-K$, we have

$$\inf_{a\in A}\inf_{b\in B}\theta(a-b) \geq \inf_{a\in A+K}\inf_{b\in B-K}\theta(a-b).$$

On the other hand, the K-monotonicity of θ implies that for all $u \in A$, $v \in B$, $k_1, k_2 \in K$ one has

$$\theta(u+k_1-v+k_2) \geq \theta(u-v) \geq \inf_{a\in A}\inf_{b\in B}\theta(a-b).$$

Hence $\inf_{a\in A+K}\inf_{b\in B-K}\theta(a-b) \geq \inf_{a\in A}\inf_{b\in B}\theta(a-b)$.

(c)-(d) We omit similar proofs. □

Results similar to the ones stated in Proposition 4.3 in the case $\theta = \Delta_{-K}$ have been obtained in [15, Proposition 1 (i)] for $r = l$ and [30, Theorem 3.20] for $r \in \{l, u, p, c, pc, cp\}$.

Corollary 4.1. *Let A and B be nonempty subsets of Y. If A and B have the domination property, then*

$$h^r(A,B) = h^r(\mathrm{Min}(A), \mathrm{Min}(B)) \quad \text{for } r \in \{l, pc\}.$$

Proof. Applying Propositions 2.1 and 4.3, we obtain

$$h^r(A,B) = h^r(A+K, B+K) = h^r(\mathrm{Min}(A)+K, \mathrm{Min}(B)+K) = h^r(\mathrm{Min}(A), \mathrm{Min}(B)).$$

□

Proposition 4.4. *Let $r \in \{l, pc\}$ and A_1, A_2, B_1, B_2 be nonempty subsets of Y. Then*

$$A_1 \sim_r A_2 \text{ and } B_1 \sim_r B_2 \implies h^r(A_1,B_1) = h^r(A_2,B_2).$$

Proof. We have $A_1+K = A_2+K$ and $B_1+K = B_2+K$ due to Lemma 2.4 in the case $r = l$ and due to Lemma 2.4 and Proposition 2.1 in the case $r = pc$. Proposition 4.3 implies

$$h^r(A_1,B_1) = h^r(A_1+K, B_1+K) = h^r(A_2+K, B_2+K) = h^r(A_2,B_2).$$

□

Proposition 4.5. *Assume that A, B and C are nonempty subsets of Y. Then*

(i) *(Positive homogenuity) $h^r(\lambda A, \lambda B) = \lambda h^r(A, B)$ for any $\lambda \geq 0$.*
(ii) *(Triangle inequality) $h^r(A, B) \leq h^r(A, C) + h^r(C, B)$.*

Proof. (i) The assertion is immediate from the definition of h^r and Property (P2).

(ii) Recall that the subadditivity of the function θ implies

$$\theta(a-b) \leq \theta(a-c) + \theta(c-b) \quad \forall a \in A, \forall b \in B, \forall c \in C. \tag{2}$$

First, one can see that the inequality (2) implies

$$\inf_{a\in A} \theta(a-b) \leq \inf_{a\in A} \theta(a-c) + \theta(c-b) \leq \sup_{c\in C} \inf_{a\in A} \theta(a-c) + \theta(c-b) \quad \forall b \in B, \forall c \in C.$$

Hence, we have

$$\inf_{a\in A} \theta(a-b) \leq \sup_{c\in C} \inf_{a\in A} \theta(a-c) + \inf_{c\in C} \theta(c-b) \quad \forall b \in B,$$

and

$$\sup_{b\in B} \inf_{a\in A} \theta(a-b) \leq \sup_{c\in C} \inf_{a\in A} \theta(a-c) + \sup_{b\in B} \inf_{c\in C} \theta(c-b).$$

Thus, the desired triangle inequality holds for h^l. The inequality (2) also gives

$$\sup_{b\in B} \theta(a-b) \leq \sup_{c\in C} \theta(a-c) + \sup_{b\in B} \theta(c-b) \quad \forall a \in A, \forall c \in C.$$

Hence, we have

$$\inf_{a\in A} \sup_{b\in B} \theta(a-b) \leq \inf_{a\in A} \sup_{c\in C} \theta(a-c) + \sup_{b\in B} \theta(c-b) \quad \forall c \in C,$$

and

$$\inf_{a\in A} \sup_{b\in B} \theta(a-b) \leq \inf_{a\in A} \sup_{c\in C} \theta(a-c) + \inf_{c\in C} \sup_{b\in B} \theta(c-b),$$

which mean that the desired triangle inequality holds for h^{pc}. The cases h^p and h^c can be proved by similar arguments. □

We study conditions for the equality $h^r(A, A) = 0$.

Proposition 4.6. *Assume that A is a nonempty subset of Y. Then*

(a) *$h^r(A, A) \leq 0$ for $r \in \{l, p\}$ and $h^r(A, A) \geq 0$ for $r \in \{c, pc\}$.*
(b) *$h^l(A, A) = 0$ iff A is W-bounded (the "only if" part is true provided either K is nonsolid or θ satisfies Property (P5)).*
(c) *$h^p(A, A) = 0$ if $A = \mathrm{Min}(A)$.*
(d) *$h^c(A, A) = 0$ iff $A - A \subseteq K \cap (-K)$.*
(e) *$h^{pc}(A, A) = 0$ iff $\mathrm{IMin}(A)$ is nonempty (the "only if" part is true provided A is K-compact).*

Proof. (a) The assertion can be easily derived from the definition.

(b) Let us start with the proof of the "if" part. By Proposition 4.1, $h^l(A,A) > -\infty$. Suppose to the contrary that $h^l(A,A) < 0$. For any $a' \in A$, we have $\inf_{a\in A}\theta(a-a') \leq h^l(A,A)$ and one can find $a \in A$ such that $\theta(a-a') < \frac{1}{2}h^l(A,A)$. Take $a_1 \in A$. For each $i = 1, 2, ...$ one can find $a_{i+1} \in A$ such that $\theta(a_{i+1}-a_i) < \frac{1}{2}h^l(A,A)$. Since $\theta(a_{i+1}-a_i) \geq \theta(a_{i+1})-\theta(a_i)$, we get $\theta(a_{i+1})-\theta(a_i) < \frac{1}{2}h^l(A,A)$. Summarizing these inequalities, we obtain $\theta(a_{i+1}) - \theta(a_1) < \frac{1}{2}ih^l(A,A)$. Then $\inf_{a\in A}\theta(a) = -\infty$, a contradiction to Proposition 3.4. The "only if" part follows from Proposition 4.1.

(c) Since $A = \mathrm{Min}(A)$, for all $a, a' \in A$, $a \neq a'$ we have either $a - a' \in K \cap (-K)$ or $a - a' \notin -K$ and hence $\theta(a-a') \geq 0$. On the other hand, $\theta(a-a) = 0$. Therefore, $h^p(A,A) = 0$.

(d) The "if" part follows from the fact that $\theta(v) = 0$ for all $v \in K \cap (-K)$. Let us prove the "only if " part. Since $h^c(A,A) = 0$, we have $\theta(a-a') \leq 0$ or $a - a' \in -K$ for all $a, a' \in A$. Then we have $A - A \subseteq -K$ and by symmetry, we also have $A - A \subseteq K$. Hence, $A - A \subseteq K \cap (-K)$.

(e) To prove the "if" part, assume that $\mathrm{IMin}(A)$ is nonempty. If $a \in \mathrm{IMin}(A)$, then $a \leq a'$ and $\theta(a-a') \leq 0$ for all $a' \in A$. Therefore, $h^{pc}(A,A) \leq \inf_{a\in \mathrm{IMin}(A)}\sup_{a'\in A}\theta(a-a') \leq 0$. On the other hand, $h^{pc}(A,A) \geq 0$. Hence, $h^{pc}(A,A) = 0$. Next, we prove the "only if" part. Assume that A is K-compact and $h^{pc}(A,A) = 0$. Proposition 4.2 implies the existence of $a \in A$ such that $h^{pc}(A,A) = \max_{a'\in A}\theta(a-a')$. Then $\max_{a'\in A}\theta(a-a') = 0$. It follows that $a - a' \in -K$ and we get $a \in \mathrm{IMin}(A)$. □

Remark 4.3. Note that a necessary and sufficient condition for $h^l(A,A) = 0$ has been stated in [27, Proposition 5.7] for the case $\theta = \Delta_{-K}$ and K is nonsolid. We provide some examples to illustrate Proposition 4.6 in the cases $r \in \{l, p, pc\}$.

(a) Case $r = l$. Let Y be the space $\mathbb{R}^2$ equipped with the l_1-norm. The cone $K = \mathbb{R}^2_+$ is the Bishop-Phelps cone $C(\ell)$ with $\ell(y_1, y_2) = y_1 + y_2$. The scalarizing function θ is of the form $\theta(.) = \ell(.) + \|.\|$. Let $A = \{(t, -1/t) \colon t > 0\}$. The set A is W-bounded (in fact, $A \in \{0\} + W$). Proposition 4.6 implies $h^l(A,A) = 0$.

(b) Case $r = p$. We show that when $A \neq \mathrm{Min}(A)$, both the cases $h^p(A,A) = 0$ and $h^p(A,A) < 0$ are possible. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A_1 = \{(0,0),(1,0)\}$, $A_2 = \{(0,0),(1,1)\}$. It is clear that $A_1 \neq \mathrm{Min}(A_1)$ and $A_2 \neq \mathrm{Min}(A_2)$. If we take $\theta(y) = \varphi_{k_0}(y)$, where $k_0 = (1,1)$, then $h^p(A_1,A_1) = 0$ and $h^p(A_2,A_2) = -1$.

(c) Case $r = pc$. We show that when $\mathrm{IMin}(A)$ is empty, both the cases $h^{pc}(A,A) = 0$ and $h^{pc}(A,A) > 0$ are possible. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $k_0 = (1,1)$ and $\theta(y) = \varphi_{k_0}(y)$. Let $A_1 = \{(1,0),(0,1)\}$ and $A_2 = \{tk_0 \colon t > 0\}$. One has $\mathrm{IMin}(A_1) = \emptyset$, $\mathrm{IMin}(A_2) = \emptyset$, $h^{pc}(A_1,A_1) = 1$ and $h^{pc}(A_2,A_2) = 0$.

Finally, we study the monotonicity of the functions $h^r(.,B) : 2^Y \to \mathbb{R}$ and $h^r(A,.) : 2^Y \to \mathbb{R}$, where A and B are nonempty subsets of Y.

Proposition 4.7. *Let A, A_1, A_2, B, B_1 and B_2 be nonempty subsets of Y and $r \in \{l, c, pc\}$.*

(a) *The function $h^r(.,B)$ is increasing w.r.t. the order relation $\preceq_r$, i.e.,*

$$A_1 \preceq_r A_2 \Longrightarrow h^r(A_1,B) \le h^r(A_2,B).$$

(b) *The function $h^r(A,.)$ is decreasing w.r.t. the order relation $\preceq_r$, i.e.,*

$$B_1 \preceq_r B_2 \Longrightarrow h^r(A,B_1) \ge h^r(A,B_2).$$

Proof. We prove the assertion (a) for the case $r = l$ and omit similar proofs for the other cases as well as for the assertion (b). Since $A_1 \preceq_l A_2$, we have $A_2 \subseteq A_1 + K$. For any $a_2 \in A_2$, there exists $a_1 \in A_1$ such that $a_1 \le a_2$. Hence, for any $b \in B$ we have $\inf_{a\in A_1} \theta(a-b) \le \theta(a_1 - b) \le \theta(a_2 - b)$. Then for any $b \in B$ we have $\inf_{a\in A_1} \theta(a-b) \le \inf_{a\in A_2} \theta(a-b)$, which implies $\sup_{b\in B} \inf_{a\in A_1} \theta(a-b) \le \sup_{b\in B} \inf_{a\in A_2} \theta(a-b)$, i.e, $h^l(A_1,B) \le h^l(A_2,B)$. □

Remark 4.4. In contrast, the assertions of Proposition 4.7 are not true for $r = p$. For instance, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $k_0 = (1,1)$ and $\theta = \varphi_{k_0}$. Let $B = \{(0,0)\}$, $A_1 = \{(0,0),(1,1)\}$ and $A_2 = \{(0,0),(-1,-1)\}$. One has $A_1 \preceq_p A_2$ but $h^p(A_1,B) = 0$ and $h^p(A_2,B) = -1$.

5. Characterizations of set order relations and efficient solutions of (SOP)

This section is devoted to the first applications of the signed Hausdorff-type half-distances. We obtain characterizations of set order relations and efficient solutions of (SOP).

5.1. **Characterizations of set order relations.**

Theorem 5.1. *Let A and B be nonempty subsets of Y. Then the implications*

$$A \preceq_r B \iff h^r(A,B) \le 0$$

are true, where the implication $\Longleftarrow$ is true provided the following conditions are satisfied.

(a) *Case $r = l$: A is K-closed and θ satisfies Property (P6).*
(b) *Case $r = p$: A and $-B$ are K-compact.*
(c) *Case $r = c$: No additional conditions.*
(d) *Case $r = pc$: A is K-compact and B is K-bounded.*

Proof. Case $r = l$. First suppose that $A \preceq_l B$. Since for any $b \in B$ there exists $a \in A$ such that $a \le b$, we get $\inf_{a\in A} \theta(a-b) \le 0$ and hence $\sup_{b\in B} \inf_{a\in A} \theta(a-b) \le 0$. Next, suppose that $A \not\preceq_l B$. There exists $b \in B$ such that $b \in Y \setminus (A+K)$. As A is K-closed, the set $Y \setminus (A+K)$ is open. Let $\alpha > 0$ be a scalar such that $b + \alpha\mathbb{B} \subset Y \setminus (A+K)$. We claim that for all $a \in A$ one has $(a-b)+\alpha\mathbb{B} \subset Y \setminus (-K)$. Indeed, suppose to the contrary that there exist $u \in \mathbb{B}$ such that $(a-b)-\alpha u \in -K$. Then $b+\alpha u \in a+K \subset A+K$, a contradiction to $b+\alpha\mathbb{B} \subset Y \setminus (A+K)$. Property (P6) implies that for the given α there exists $\beta > 0$ such that $\theta(a-b) \geq \beta$ for all $a \in A$. Hence $\inf_{a\in A} \theta(a-b) \geq \beta > 0$ and $h^l(A,B) \geq \beta > 0$.

Case $r = p$. First suppose that $A \preceq_p B$. Then there exist $a \in A$ and $b \in B$ such that $a \leq b$. Hence, $\inf_{a\in A} \inf_{b\in B} \theta(a-b) \leq 0$. Next, assume that $A \not\preceq_p B$. Then for any pair $a \in A$ and $b \in B$, we have $a \not\leq b$ and $\theta(a-b) > 0$. Since A and $-B$ are K-compact, Proposition 4.2 implies that $h^p(A,B) = \min_{a\in A} \min_{b\in B} \theta(a-b) > 0$.

Case $r = c$. We have

$$\begin{aligned} h^c(A,B) \leq 0 \quad &\Longleftrightarrow \sup_{a\in A} \sup_{b\in B} \theta(a-b) \leq 0 \Longleftrightarrow \theta(a-b) \leq 0, \forall a \in A, \forall b \in B \\ &\Longleftrightarrow a-b \in -K, \forall a \in A, \forall b \in B \Longleftrightarrow A \preceq_c B. \end{aligned}$$

Case $r = pc$. First suppose that $A \preceq_{pc} B$. There exists $a \in A$ such that $a \leq b$ for all $b \in B$. Hence, $\sup_{b\in B} \theta(a-b) \leq 0$ and $\inf_{a\in A} \sup_{b\in B} \theta(a-b) \leq 0$. Next, assume that $A \not\preceq_{pc} B$. Since B is K-bounded, Proposition 3.4 implies that any $a \in A$, $\sup_{b\in B} \theta(a-b) < \infty$. Further, $A \not\preceq_{pc} B$ implies that for any $a \in A$ there exists $b \in B$ such that $a \not\leq b$ and $\theta(a-b) > 0$ and hence, $0 < \sup_{b\in B} \theta(a-b) < \infty$. Observe that the function $\sup_{b\in B} \theta(.-b) : Y \to \mathbb{R}$ is globally Lipschitz and K-monotone and since A is K-compact, Lemma 3.3 implies that this function attains its minimum. Hence, $h^{pc}(A,b) = \min_{a\in A} \sup_{b\in B} \theta(a-b) > 0$. □

Remark 5.1. The reader is referred to [30, Theorem 4.1] for the case $\theta = \Delta_{-K}$ and [37, 38] for the case $\theta = \varphi_{k_0}$.

Proposition 5.1. *Assume that A and B are nonempty subsets of Y. Then the implication*

$$A \preceq_r B \Longrightarrow h^r(B,A) \geq 0$$

is true provided that the following conditions are satisfied.

(a) *Case $r = l$: A is W-bounded.*
(b) *Case $r = p$: $A = \mathrm{Min}(A)$.*
(c) *Case $r \in \{c, pc\}$: No additional conditions.*

Proof. We have $h^r(A,A) = 0$ in the cases $r \in \{l,p\}$ and $h^r(A,A) \geq 0$ in the cases $r \in \{c,pc\}$ by Proposition 4.6. Proposition 4.5 implies $0 \leq h^r(A,A) \leq h^r(A,B) + h^r(B,A)$. Since $h^r(A,B) \leq 0$ by Theorem 5.1, it follows that $h^r(B,A) \geq 0$. □

Our next concern is a relation between $\sim_r$ and h^r.

Proposition 5.2. *Let A and B be nonempty subsets of Y. Consider the implications*

$$A \sim_r B \Longleftrightarrow h^r(A,B) = 0 \text{ and } h^r(B,A) = 0.$$

Then

(a) *Case $r = l$: The implication $\Longrightarrow$ holds if either A or B is W-bounded and the implication $\Longleftarrow$ holds if A and B are K-closed and θ possesses Property (P6).*
(b) *Case $r = p$: The implication $\Longrightarrow$ holds if either $A = \mathrm{Min}(A)$ or $B = \mathrm{Min}(B)$ and the implication $\Longleftarrow$ holds if A and B are compact.*
(c) *Case $r = c$: The implications hold without additional assumptions.*
(d) *Case $r = pc$: The implication $\Longrightarrow$ holds without additional assumptions and the implication $\Longleftarrow$ holds if A and B are K-compact.*

Proof. The implication $\Longrightarrow$. Assume that $A \sim_r B$. Theorem 5.1 implies $h^r(A, B) \leq 0$ and $h^r(B, A) \leq 0$. Proposition 4.6 implies $h^r(A, A) = 0$ or $h^r(B, B) = 0$ for $r \in \{l, p\}$ and $h^r(A, A) \geq 0$ for $r \in \{c, pc\}$. Since $0 \leq \max\{h^r(A, A), h^r(B, B)\} \leq h^r(A, B) + h^r(B, A)$, we get $h^r(A, B) = h^r(B, A) = 0$.

The implication $\Longleftarrow$ follows from Theorem 5.1. □

We characterize strict set order relations $\prec_r$ in term of the function h^r.

Theorem 5.2. *Assume that K is solid and θ additionally satisfies Property (P5). Let A and B be nonempty subsets of Y. Then the implications*

$$A \prec_r B \Longleftrightarrow h^r(A, B) < 0,$$

are true, where the implication $\Longrightarrow$ is true provided the following conditions are satisfied.

(a) *Case $r = l$: B is K-compact.*
(b) *Case $r = p$: No additional condition is required.*
(c) *Case $r = c$: $-A$ and B are K-compact.*
(d) *Case $r = pc$: B is K-compact.*

Proof. (a) Case $r = l$. First, assume that $A \prec_l B$. For each $b \in B$, there exists $a \in A$ such that $a - b \in -\mathrm{int}K$ and therefore, $\inf_{a\in A} \theta(a - b) < 0$. Proposition 3.5 implies that the function $\inf_{a\in A} \theta(a - .) : Y \to \mathbb{R}$ is globally Lipschitz and $(-K)$-monotone. Lemma 3.3 implies that this function attains its maximum on B, i.e., $\sup_{b\in B} \inf_{a\in A} \theta(a - b) = \max_{b\in B} \inf_{a\in A} \theta(a-b) < 0$. Thus, we have $h^l(A, B) < 0$. Next, assume that $h^l(A, B) < 0$ but $A \not\prec_l B$. Then there exists $b \in B$ such that for all $a \in A$ one has $a-b \notin -\mathrm{int}K$, $\theta(a-b) \geq 0$ and hence, $\inf_{a\in A} \theta(a - b) \geq 0$. Hence, we get $h^l(A, B) = \sup_{b\in B} \inf_{a\in A} \theta(a - b) \geq 0$, a contradiction.

(b) Case $r = p$. We have

$$\begin{aligned} A \prec_p B &\Longleftrightarrow \exists a \in A\ \exists b \in B \text{ such that } a - b \in -\mathrm{int}K \\ &\Longleftrightarrow \exists a \in A\ \exists b \in B \text{ such that } \theta(a - b) < 0 \\ &\Longleftrightarrow h^p(A, B) = \inf_{a\in A} \inf_{b\in B} \theta(a - b) < 0. \end{aligned}$$

(c) Case $r = c$. First, assume that $A \prec_c B$. For all $a \in A$ and $b \in B$, we have $a-b \in -\mathrm{int}K$ and $\theta(a-b) < 0$. Proposition 4.2 implies that $h^c(A,B) = \max_{a\in A}\max_{b\in B}\theta(a-b) < 0$. Next, assume that $h^c(A,B) < 0$ but $A \not\prec_c B$. Then there exist $a \in A$ and $b \in B$ such that $a-b \notin -\mathrm{int}K$ and $\theta(a-b) \geq 0$. Then we get $h^c(A,B) = \sup_{a\in A}\sup_{b\in B}\theta(a-b) \geq 0$, a contradiction.

(d) Case $r = pc$. First, assume that $A \prec_{pc} B$. There exists $a \in A$ such that $a-b \in -\mathrm{int}K$, $\theta(a-b) < 0$ for all $b \in B$. Since the function $\theta(a-.) : Y \to \mathbb{R}$ is globally Lipschitz and $(-K)$-monotone, Lemma 3.3 implies that this function attains its maximum on B, i.e., $\sup_{b\in B}\theta(a-b) = \max_{b\in B}\theta(a-b) < 0$. Hence, $h^{pc}(A,B) = \inf_{a\in A}\max_{b\in B}\theta(a-b) < 0$. Next, assume that $h^{pc}(A,B) < 0$ but $A \not\prec_{pc} B$. For any $a \in A$ there exists $b \in B$ such that $a-b \notin -\mathrm{int}K$, $\theta(a-b) \geq 0$ and hence $\sup_{b\in B}\theta(a-b) \geq 0$. Therefore, we get $h^{pc}(A,B) = \inf_{a\in A}\sup_{b\in B}\theta(a-b) \geq 0$, a contradiction. □

Proposition 5.3. *Assume that K is solid and θ additionally satisfies Property (P5). Let A and B be nonempty subsets of Y. Then*

$$A \prec_r B \Longrightarrow h^r(B,A) > 0$$

provided the following conditions are satisfied.

(a) *Case $r = l$: A is K-proper and B is K-compact.*
(b) *Case $r = p$: $A = \mathrm{Min}(A)$.*
(c) *Case $r = c$: $-A$ and B are K-compact.*
(d) *Case $r = pc$: B is K-compact.*

Proof. We have $h^r(A,A) = 0$ in the cases $r \in \{l,p\}$ and $h^r(A,A) \geq 0$ in the cases $r \in \{c,pc\}$ by Proposition 4.6. The triangle property stated in Proposition 4.5 implies

$$0 \leq h^r(A,A) \leq h^r(A,B) + h^r(B,A).$$

Since $h^r(A,B) < 0$ by Theorem 5.2, it follows that $h^r(B,A) > 0$. □

Remark 5.2. The assertions of Proposition 5.3 may not be true if the mentioned conditions are not satisfied. For instance, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $k_0(1,1)$ and $\theta = \varphi_{k_0}$. Let $A = \{(1,2),(3,2)\}$ and $B = \{(2,3),(2,1)\}$. One can check that $A \prec_p B$, $\mathrm{Min}(A)$ is nonempty, $A \neq \mathrm{Min}(A)$ and $h^p(B,A) = -1$. Next, let $A = \{(0,0)\}$ and $B = \{(t,t) \colon t > 0\}$. One can check that $A \prec_l B$, $A \prec_{pc} B$, A is K-proper, B is not K-compact and $h^l(B,A) = h^{pc}(B,A) = 0$.

5.2. **Characterizations of efficient solutions to (SOP).** In this section, we use the signed Hausdorff-type half-distances h^r to characterize efficient solutions of (SOP). As before, we assume that θ satisfies (P1)-(P4) and $h^r(.,.)$ is finite for sets under consideration. The reader is referred to [11, 15, 20, 28, 30, 35, 42, 48] for characterizations of efficient solutions in set optimization by scalarizing functions involving Δ_{-K} or φ_{k_0}.

Let X be a nonempty set and $F : X \rightrightarrows Y$ be a set-valued map with nonempty values. We consider global solutions to a minimizing set-valued optimization problem (SOP) of the form

$$\mathrm{Min}_{x\in X} F(x).$$

As before, we restrict ourselves to $\preceq_r$ and h^r with $r \in \{l, p, c, pc\}$. The order set relations $\preceq_u$ and $\preceq_{cp}$ could be used to obtain similar results for a maximizing (SOP).

Definition 5.1. Let $\bar{x} \in X$. We say that

(i) $\bar{x}$ is a $\preceq_r$-*efficient solution* of (SOP) if

$$F(x) \preceq_r F(\bar{x}) \text{ for some } x \in X \text{ implies } F(\bar{x}) \preceq_r F(x).$$

(ii) (K is assumed to be solid) $\bar{x}$ is a *weak* $\preceq_r$-*efficient solution* of (SOP) if

$$F(x) \not\prec_r F(\bar{x}) \;\; \forall x \in X,\; x \neq \bar{x}.$$

(iii) $\bar{x}$ is an *ideal* $\preceq_r$-*efficient solution* of (SOP) if

$$F(\bar{x}) \preceq_r F(x) \;\; \forall x \in X,\; x \neq \bar{x}.$$

Remark 5.3. It is easy to see that $\bar{x}$ is a $\preceq_r$-efficient solution of (SOP) iff for all $x \in X$, $x \neq \bar{x}$ one has

$$\text{either } F(x) \sim_r F(\bar{x}) \text{ or } F(x) \not\preceq_r F(\bar{x}).$$

In the case $r = pc$, if $\mathrm{IMin}(F(\bar{x}))$ does not exist, then $F(\bar{x}) \not\prec_{pc} F(x)$ for all $x \neq \bar{x}$ and hence, for such a $\preceq_{pc}$-efficient solution $\bar{x}$, we have $F(x) \not\preceq_{pc} F(\bar{x})$ for all $x \neq \bar{x}$.

We will characterize solutions of (SOP) as solutions of a scalar optimization problem (OP)

$$\mathrm{Min}_{x\in X} g(x)$$

with an objective function $g : X \to \mathbb{R}$. Recall that $\bar{x} \in X$ is a solution of (OP) if $g(\bar{x}) \leq g(x)$ for all $x \in X$, $x \neq \bar{x}$. In what follows, we say that F is N-valued if for every $x \in X$, $F(x)$ is N, where N is compact, K-bounded, K-closed or bounded.

Proposition 5.4. *$\bar{x} \in X$ is a $\preceq_r$-efficient solution of (SOP) only if it is a solution to* (OP) *with the objective function $g : X \to \mathbb{R}$ given by*

$$g(x) = h^r(F(x), F(\bar{x}))$$

provided the following conditions are satisfied.

(a) *Case $r = l$: θ satisfies Property (P6) and F is K-bounded- K-closed-valued.*

(b) *Case $r = p$: F is compact-valued and $F(\bar{x}) = \mathrm{Min}F(\bar{x})$.*

(c) *Case* $r = c$*: F is bounded-valued and* $F(\bar{x}) - F(\bar{x}) \subset K \cap (-K)$.

(d) *Case* $r = pc$*: F is K-compact-valued and* $\mathrm{IMin} F(\bar{x})$ *is nonempty.*

Proof. First, observe that the function g has finite values due to Proposition 4.1. Next, since $\bar{x}$ is a $\preceq_r$-efficient solution of (SOP), for any $x \in X$, $x \neq \bar{x}$, we have either $F(x) \sim_r F(\bar{x})$ or $F(x) \not\preceq_r F(\bar{x})$. Proposition 5.2 and Theorem 5.1 imply that we have either $h^r(F(x), F(\bar{x})) = 0$ or $h^r(F(x), F(\bar{x})) > 0$. Finally, Proposition 4.6 yields that $h^r(F(\bar{x}), F(\bar{x})) \leq 0$ for $r \in \{l, p\}$ and $h^r(F(\bar{x}), F(\bar{x})) = 0$ for $r \in \{c, pc\}$. Hence $g(\bar{x}) \leq g(x)$ for all $x \in X$ and $\bar{x}$ is a solution to (OP). □

Proposition 5.5. *Assume that K is solid and θ satisfies (P5). Then $\bar{x} \in X$ is a weak $\preceq_r$-efficient solution of (SOP) if and only if it is a solution to* (OP) *with the objective function* $g : X \to \mathbb{R}$ *given by*

$$g(x) = h^r(F(x), F(\bar{x}))$$

provided the following conditions are satisfied.

(a) *Case* $r = l$*: F is K-compact-valued.*

(b) *Case* $r = p$*: F is bounded-valued.*

(c) *Case* $r = c.$*: F is compact-valued and* $F(\bar{x}) - F(\bar{x}) \subset K \cap (-K)$.

(d) *Case* $r = pc$*: F is K-compact-valued and* $\mathrm{IMin} F(\bar{x})$ *is nonempty.*

Proof. First, observe that the function g has finite values due to Proposition 4.1 and that $h^r(F(\bar{x}), F(\bar{x})) \leq 0$ for $r \in \{l, p\}$ and $h^r(F(\bar{x}), F(\bar{x})) = 0$ for $r \in \{c, pc\}$ due to Proposition 4.6. Thus, we have $g(\bar{x}) \leq 0$. By the definition, $\bar{x}$ is a weak $\preceq_r$-efficient solution of (SOP) iff $F(x) \not\prec_r F(\bar{x})$ for all $x \in X$, $x \neq \bar{x}$. Therefore, $\bar{x}$ is a weak $\preceq_r$-efficient solution of (SOP) iff $g(x) = h^r(F(x), F(\bar{x})) \geq 0$ for all $x \in X$ due to Theorem 5.2. The assertion follows. □

Proposition 5.6. *$\bar{x} \in X$ is an ideal $\preceq_r$-efficient solution of (SOP) only if it is solution to* (OP) *with the objective function* $g : X \to \mathbb{R}$ *given by*

$$g(x) = h^r(F(x), F(\bar{x}))$$

provided the following conditions are satisfied.

(a) *F is K-bounded-valued in case* $r \in \{l, pc\}$.

(b) *F is bounded-valued in case* $r \in \{p, c\}$.

Proof. First, observe that the function g has finite values due to Proposition 4.1. Next, since $\bar{x}$ is an ideal $\preceq_r$-efficient solution of (SOP), for any $x \in X$, $x \neq \bar{x}$, we have $F(\bar{x}) \preceq_r F(x)$. Proposition 4.7 implies that $h^r(F(\bar{x}), F(\bar{x})) \leq h^r(F(x), F(\bar{x}))$. The assertion follows. □

6. Hausdorff-type distances

In this section, we use the signed Hausdorff-type half-distances to define some distance d^r_θ that we call the Hausdorff-type distances. We follow the scheme of [14], in which d^l_θ with $\theta = \Delta_{-K}$ has played an important role in defining the directional derivative of set-valued maps and characterizing optimality conditions for (SOP).

Let A and B be two nonempty subsets of Y. Recall that the classical Hausdorff distance $D(A, B)$ between A and B is defined by

$$D(A, B) := \max\{H(A, B), H(B, A)\}.$$

For $r \in \{l, u, p, c, pc, cp\}$, we define scalars $d^r_\theta(A, B)$ as follows

$$d^r_\theta(A, B) := \max\{h^r_\theta(A, B), h^r_\theta(B, A)\}.$$

In the special case $\theta = \Delta_{-K}$ and $K = \{0\}$, we have

$$\begin{aligned} d^l_\theta(A, B) &= D(A, B) \\ d^u_\theta(A, B) &= D(B, A) = D(A, B) \\ d^p_\theta(A, B) &= D(A - B, \{0\}) \\ d^c_\theta(A, B) &= D(\{0\}, A - B) = D(A - B, \{0\}). \end{aligned}$$

Due to the first equality, $d^l_\theta(A, B)$ with $\theta = \Delta_{-K}$ (K is an arbitrary pointed closed convex cone) was called a Hausdorff-type distance between A and B in [14]. Motivated by this fact, we use this name for $d^r_\theta(A, B)$ with θ being any abstract scalarizing function defined on Y (i.e. θ satisfies Properties (P1)-(P4)).

In what follows, we restrict ourselves to $r \in \{l, p, c, pc\}$ and, when no confusion occurs, we *drop the subscript* θ in $d^r_\theta(A, B)$ and $h^r_\theta(A, B)$. We also make *a convention* that

$$d^r(.,.) \textbf{ is finite for sets under consideration}$$

Due to Proposition 4.1, $d^l(A, B)$, $d^{pc}(A, B)$ are finite if the sets A and B are K-bounded and $d^p(A, B)$, $d^c(A, B)$ are finite if A and B are bounded. Moreover, Propositions 4.5 and 4.6 imply the following properties of d^r.

Proposition 6.1. *Assume that A, B and C are nonempty subsets of Y. Then*

(i) *(Symmetry)* $d^r(A, B) = d^r(B, A)$.

(ii) *(Triangle inequality)* $d^r(A, B) \le d^r(A, C) + d^r(C, B)$.

Proposition 6.2. *Assume that A is a nonempty subset of Y. Then*

(a) $d^l(A, A) = 0$ *if A is W-bounded.*

(b) $d^p(A, A) = 0$ *if* $A = \mathrm{Min}(A)$.

(c) $d^c(A, A) = 0$ *if* $A - A \subseteq K \cap (-K)$.

(d) $d^{pc}(A, A) = 0$ *if* $\mathrm{IMin}(A)$ *is nonempty.*

Remark 6.1. It may happen that $d^p(A, A) < 0$ if $\mathrm{Min}(A)$ is nonempty but $A \neq \mathrm{Min}(A)$. For instance, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and $\theta = \Delta_{-K}$. Let $A = \{(1, 2), (3, 2)(2, 3), (2, 1)\}$. One can check that $\mathrm{Min}(A) = \{(1, 2), (2, 1)\}$ and $h^p(A, A) = -\sqrt{2}$.

Next, we state conditions for d^r to be nonnegative.

Proposition 6.3. *Let A and B be nonempty subsets of Y. Then*

$$d^r(A, B) \geq 0$$

provided the following conditions are satisfied:

(a) *Case $r = l$: Either A or B is W-bounded.*
(b) *Case $r = p$: Either $A = \mathrm{Min}(A)$ or $B = \mathrm{Min}(B)$.*
(c) *Case $r \in \{c, pc\}$: No additional conditions.*

Proof. Proposition 4.6 implies that either $h^r(A, A) = 0$ or $h^r(B, B) = 0$ in the cases $r \in \{l, p\}$. Assume for instance that $h^r(A, A) = 0$. The triangle inequality implies $0 = h^r(A, A) \leq h^r(A, B) + h^r(B, A)$ and since $d^r(A, B) = \max\{h^r(A, B), h^r(B, A)\}$, we get $d^r(A, B) \geq 0$. In the cases $r \in \{c, pc\}$, we already have $h^r(A, B) \geq 0$ and $h^r(B, A) \geq 0$. □

Remark 6.2. It may happen that $d^p(A, B) < 0$ if the stated condition is not satisfied. For instance, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $k_0(1, 1)$ and $\theta = \varphi_{k_0}$. Let $A = \{(1, 2), (3, 2)\}$ and $B = \{(2, 3), (2, 1)\}$. One can check that $\mathrm{Min}(A)$ and $\mathrm{Min}(B)$ are nonempty, $A \neq \mathrm{Min}(A)$, $B \neq \mathrm{Min}(B)$, $h^p(A, B) = h^p(B, A) = -1$ and therefore, $d^p(A, B) = -1$.

Proposition 6.4. *Let $r \in \{l, pc\}$. Let A and B be nonempty subsets of Y. Consider the implications*

$$A \sim_r B \Longleftrightarrow d^r(A, B) = 0.$$

(a) *The implication $\Longrightarrow$ is true if either A or B is W-bounded in the case $r = l$ and without additional conditions in the case $r = pc$.*
(b) *The implication $\Longleftarrow$ is true if θ possesses Property (P6) and A, B are K-closed in the case $r = l$ and if A, B are K-compact in the case $r = pc$.*

Proof. The implication $\Longrightarrow$ follows from Proposition 5.2 and the reciprocal one follows from Theorem 5.1. □

Proposition 6.5. *Assume that A and B are nonempty subsets of Y. Then*

(a) $d^l(A, B) = d^l(A + K, B + K)$.
(b) $d^p(A, B) = d^p(A + K, B - K)$.
(c) $d^c(A, B) = d^c(A - K, B + K)$.
(d) $d^{pc}(A, B) = d^{pc}(A + K, B + K)$.

Proof. It follows from Proposition 4.3. □

Proposition 6.6. *Let $r \in \{l, pc\}$ and A_1, A_2, B_1 and B_2 be nonempty subsets of Y. Then*

$$A_1 \sim_r A_2 \text{ and } B_1 \sim_r B_2 \implies d^r(A_1, B_1) = d^r(A_2, B_2).$$

Proof. It follows from Proposition 4.4. □

Our next concern is a limit operation. Let T be an index set in a metric space and A_t ($t \in T$) be a parametrized family of nonempty subsets of Y.

Definition 6.1. We say that a nonempty set $A \subset Y$ is a d^r-limit of A_t when $t \to \bar{t}$ if

$$\lim_{t \to \bar{t}} d^r(A_t, A) = 0.$$

We show the "uniqueness" in some sense of a limit set.

Proposition 6.7. *Let A and B be nonempty subsets of Y and $r \in \{l, pc\}$. Assume that A is a d^r-limit of A_t when $t \to \bar{t}$. Consider the implications*

$$B \text{ is a } d^r\text{−limit of } A_t \iff A \sim_r B.$$

Then

(a) *The implication $\implies$ is true if θ possesses Property (P6) and A, B are K-closed in the case $r = l$ and if A, B are K-compact in the case $r = pc$.*
(b) *The implication $\impliedby$ is true if B is W-bounded in the case $r = l$ and without additional conditions in the case $r = pc$.*

Proof. (a) Assume that B also is a d^r-limit of A_t. Since $\lim_{t\to\bar{t}} d^r(A_t, A) = 0$, $\lim_{t\to\bar{t}} d^r(A_t, B)$ $= 0$ and $d^r(A, B) \leq d^r(A, A_t) + d^r(A_t, B)$, we get $d^r(A, B) \leq 0$. Hence, $h^r(A, B) \leq 0$ and $h^r(B, A) \leq 0$. Theorem 5.1 gives $A \preceq_r B$ and $B \preceq_r A$, which imply $A \sim_r B$.

(b) Assume that $A \sim_r B$. We claim that $d^r(A_t, B) \geq 0$ and $d^r(A, B) = 0$. Indeed, the inequality follows from Proposition 6.3 and the equality follows from Proposition 6.4. Since $0 \leq d^r(A_t, B) \leq d^r(A_t, A) + d^r(A, B)$, we get $\lim_{t\to\bar{t}} d^r(A_t, B) = 0$. □

Next, we show that the limit operation reserves the order relations. This property has been shown very useful for establishing optimality conditions expressed in terms of directional derivative of a set-valued objective map [14].

Proposition 6.8. *Let A and B be nonempty subsets of Y. Assume that A is a d^r-limit of A_t when $t \to \bar{t}$. The assertions*

(i) *$A_t \preceq_r B$ for all t implies $A \preceq_r B$ and*
(ii) *$B \preceq_r A_t$ for all t implies $B \preceq_r A$.*

are true provided the following conditions are satisfied.

(a) *Case $r = l$: A is K-closed and θ possesses Property (P6) for the assertion (i) and B is K-closed and θ possesses Property (P6) for the assertion (ii).*
(b) *Case $r = p$: A and $-B$ are K-compact for the assertion (i) and B and $-A$ are K-compact for the assertion (ii).*
(c) *Case $r = c$: no condition is required.*
(d) *Case $r = pc$: A is K-compact and B is K-bounded for the assertion (i) and B is K-compact and A is K-bounded for the assertion (ii).*

Proof. It is easy to see that

$$h^r(A,B) \leq h^r(A,A_t) + h^r(A_t,B) \leq d^r(A,A_t) + h^r(A_t,B)$$

and

$$h^r(B,A) \leq h^r(B,A_t) + h^r(A_t,A) \leq h^r(B,A_t) + d^r(A_t,A).$$

If $A_t \preceq_r B$ ($B \preceq_r A_t$), Theorem 5.1 yields $h^r(A_t,B) \leq 0$ (resp., $h^r(B,A_t) \leq 0$), which together with $\lim_{t\to 0} d^r(A_t,A) = 0$ imply $h^r(A,B) \leq 0$ (resp., $h^r(B,A) \leq 0$). The assertions now follow from Theorem 5.1. □

It turns out that d^r may induce metric in some families of nonempty subsets of Y. Denote

$$\begin{aligned} \mathcal{Y}_l &:= \{A \in 2^Y : A \text{ is } K-\text{compact}\} \\ \mathcal{Y}_{pc} &:= \{A \in 2^Y : A \text{ is compact and IMin}(A) \text{ is nonempty}\}. \end{aligned}$$

For each $A \in \mathcal{Y}_r$, denote

$$[A]^r := \{A' \in \mathcal{Y}_r \ : \ A' \sim_r A\}$$

and define

$$\mathcal{V}^r := \{[A]^r \ : \ A \in \mathcal{Y}_r\}.$$

Observe that each families $\mathcal{V}^r$ is a semi-linear space with the addition and multiplication operations given by $[A]^r + [B]^r := [A+B]^r$ and $t[A]^r := [tA]^r$ for any pair $[A]^r, [B]^r \in \mathcal{V}^r$ and any nonnegative scalar t. We define a function $d^r_{\mathcal{V}^r} : \mathcal{V}^r \times \mathcal{V}^r \to \mathbb{R}_+$ by

$$d^r_{\mathcal{V}^r}([A]^r, [B]^r) := d^r(A,B).$$

Assume that θ satisfies (P6) in the case $r = l$. Propositions 6.1-6.4 and 6.6 imply that $d^r_{\mathcal{V}^r}$ is well-defined and it induces a metric on $\mathcal{V}^r$. For $[A_i]^r \in \mathcal{V}^r$ ($i = 1,2,...$) and $[A]^r \in \mathcal{V}^r$, we define

$$[A]^r := \lim_{i\to+\infty} [A_i]^r \text{ iff } \lim_{i\to+\infty} d^r_{\mathcal{V}^r}([A_i]^r, [A]^r) = 0.$$

Proposition 6.7 shows that the above limit operation in $\mathcal{V}^r$ is well-defined. The metric space $\mathcal{V}^r$ can be useful in the study of directional derivative of set-valued maps, see [14] for the case $r = l$ and $\theta = \Delta_{-K}$.

7. Conclusions

In this work, we propose a new scheme for scalarization in set optimization studied with the Kuroiwa set approach. Namely, we define an abstract scalarizing function on a Banach space and use it to define the so called signed Hausdorff-type half-distances and Hausdorff-type distances. With the help of the latters, we obtain characterizations of set order relations and some optimal solutions.

This scheme provides a unified approach to scalarization involving different scalarizing functions such as the Gerstewizt (Tammer) function, the Hiriart-Urruty signed distance and the function proposed by Kasimbeyli.

For the future research, one could exploit these half-distances and distances to define slopes and directional derivatives for a set-valued map and to study optimality conditions and error bounds in set optimization.

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