Minimum-Link Covering Trails for any Hypercubic Lattice

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Abstract: In 1994, Kranakis et al. published a conjecture about the minimum link-length of every rectilinear covering path for the k-dimensional grid $P(n,k) \coloneqq \{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-1\} \times \cdots \times \{0,1,\ldots,n-1\}$. In this paper we consider the general, NP-complete, Line-Cover problem, where the edges are not required to be axis-parallel, showing that the original Theorem 1 by Kranakis et al. no longer holds when the aforementioned constraint is disregarded. Furthermore, for any given n above two, as k approaches infinity, the link-length of any minimal (non-rectilinear) polygonal chain does not exceed Kranakis' conjectured value of $\frac{k}{k-1} \cdot n^{k-1} + O(n^{k-2})$ only if we introduce a multiplicative constant $c \geq 1.5$ for the lower order terms (e.g., if we select $n = 3$ and assume that $c < 1.5$, starting from a sufficiently large k, it is not possible to visit all the nodes of $P(n,k)$ with a trail of link-length $\frac{k}{k-1} \cdot n^{k-1} + c \cdot n^{k-2}$).
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1 Introduction

We take into account a multidimensional extension of a well-known puzzle involving lateral thinking [5,7,8] and examine a generalization of an intriguing conjecture by Kranakis, Krizanc, and Meertens, published in 1994 on Ars Combinatoria, Vol. 38, p. 191 [6].

To compactly describe the related open problems and the main results of the present paper, let us give a few basic definitions first.

Definition 1.1. *Let n and k be two strictly positive integers. $P(n,k) \coloneqq \{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-1\} \times \cdots \times \{0,1,\ldots,n-1\}$ is a finite set of n^k points in $\mathbb{R}^k$.*

Definition 1.2. *We define as $h(n,k)$ the number of edges of the minimum-link covering trail for $P(n,k)$ (i.e., the link-length of any polygonal chain that visits each node of $P(n,k)$ at least once and that is characterized by the minimum number of edges). Furthermore, let us call $h_l(n,k)$ and $h_u(n,k)$ the proven lower and upper bound for $h(n,k)$ (respectively), so $h_l(n,k) \leq h(n,k) \leq h_u(n,k)$ holds for every $P(n,k) \subset \mathbb{R}^k$.*

Definition 1.3. *Let $k, n_1, n_2, \ldots, n_k$ be strictly positive integers such that $n_1 \leq n_2 \leq \cdots \leq n_k$. $G(n_1,n_2,\ldots,n_k) \coloneqq \{0,1,\ldots,n_1-1\} \times \{0,1,\ldots,n_2-1\} \times \cdots \times \{0,1,\ldots,n_k-1\}$ is a finite set of $\prod_{i=1}^{k} n_i$ points in $\mathbb{R}^k$.*

Definition 1.4. *We define as $h(n_1, n_2, \dots, n_k)$ the number of edges of the minimum-link covering trail for $G(n_1, n_2, \dots, n_k)$. Furthermore, let us call $h_l(n_1, n_2, \dots, n_k)$ a proven lower bound for $h(n_1, n_2, \dots, n_k)$ and, similarly, let us indicate as $h_u(n_1, n_2, \dots, n_k)$ a valid upper bound for the same set of $n_1 \cdot n_2 \cdots n_k$ points.*

The NP-complete problem (see [1], p. 514) that we aim to solve asks to visit, at least once, all the nodes of $P(n, k)$ with a polygonal chain that is characterized by the minimum number of edges [3,6].

In 1994, Evangelos Kranakis et al. [6] conjectured that $h(n, k) \leq \frac{k}{k-1} \cdot n^{k-1} + O(n^{k-2})$ for each $k \in \mathbb{N} - \{0, 1, 2\}$, under two additional conditions: all line-segments are axis-parallel [2], and every point of the k-dimensional grid should not be visited twice.

Kranakis' expected value of the link-length of the minimal rectilinear paths cannot be smaller than any proven lower bound for the same optimization problem, and this proposition would automatically be confirmed if we remove the aforementioned additional constraints (e.g., for the unconstrained case, it has been constructively proved that $h(4, 3) \leq 23 < \frac{k}{k-1} \cdot n^{k-1}$ [10] and also $h(5, 3) \leq 36 < \frac{k}{k-1} \cdot n^{k-1}$ [12]). Thus, we will show that, for any $k \geq 8$ and $n = 3$, $h(n, k)$ falls between Kranakis' conjectured bound of $\frac{k}{k-1} \cdot n^{k-1} + n^{k-2}$ and Bereg's proved upper bound of $\frac{k}{k-1} \cdot n^{k-1} + n^{k-\frac{3}{2}}$ [1].

2 Current upper bound vs. best theoretical solution

Let $k > 2$ and assume also $n \geq 3$. From [9], Eq. (9) (given the fact that $\frac{n_k + n_{k-1}}{2} \in \mathbb{N}$ for every k since $n_k = n_{k-1} = \cdots = n_2 = n_1$ by hypothesis), follows the trivial lower bound

$$h_l(n, k) = \left\lceil \frac{n^k - (k-2) \cdot n^2 + (k-2) \cdot n - n + n \cdot ((k-2) \cdot n - k + 2)}{n - 1} \right\rceil + 1 = \frac{n^k - 1}{n - 1}. \quad (1)$$

Generally speaking, as shown by Theorem 2.1 below, if $3 \leq n_1 \leq n_2 \leq \cdots \leq n_{k:k>2}$, then it is not possible to visit all the nodes of $G(n_1, n_2, \dots, n_k)$ (see Definition 1.3) with a covering trail whose link-length is smaller than $\left\lceil 3 \cdot \frac{\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 3}{2 \cdot n_k + n_{k-1} - 3} \right\rceil + k - 2$.

Theorem 2.1. *Let $k \in \mathbb{N} - \{0, 1, 2\}$. If $3 \leq n_1$, then*

$$h_l(n_1, n_2, \dots, n_k) = \left\lceil 3 \cdot \frac{\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 3}{2 \cdot n_k + n_{k-1} - 3} \right\rceil + k - 2. \quad (2)$$

Proof. Let the k integers $n_1 \leq n_2 \leq \cdots \leq n_k$ be such that $n_1 > 2$. We immediately notice that it is not usually possible to intersect more than $(n_k - 1) + (n_{k-1} - 1) + (n_k - 1) = 2 \cdot n_k + n_{k-1} - 3$ points using three (consecutive) straight lines connected at their endpoints; however, there is one exception (which, for sake of simplicity, we may assume as in the case of the first line drawn). In this circumstance, it is possible to fit n_k points with the first line segment, then $n_{k-1} - 1$ points using the second line and $n_k - 1$ points with the third one (or, alternatively, we could join $n_{k-1} - 2$ points using the second line and n_k points with the next one), and so forth.

In general, it does not exist any polygonal chain consisting of exactly three links that visits more than $2 \cdot n_k + n_{k-1} - 2$ nodes of $G(n_1, n_2, \dots, n_k)$ (i.e., we cannot improve in any way the previous result whether we visit $n_k - 1$ nodes with the first link of a covering trail, n_{k-1} with the second one, and then we spend the third link to join $n_k - 1$ unvisited nodes, or by joining n_k nodes with the first and third link and $n_{k-1} - 2$ nodes thanks to the second one, since $\forall t \in \{1, 2, \dots, h(n_1, n_2, \dots, n_k)\}, n_k + \sum_{i=1}^{t} \left\lfloor \frac{t-1}{2} \right\rfloor \cdot ((n_{k-1} - 2) + n_k) + \frac{(-1)^t+1}{2} \cdot (n_{k-1} - 2)$ $\leq n_k + \sum_{i=1}^{t} \left\lfloor \frac{t-1}{2} \right\rfloor \cdot ((n_{k-1} - 1) + (n_k - 1)) + \frac{(-1)^t+1}{2} \cdot (n_{k-1} - 1)$, while it is also trivial to point out that $(n_k - 1) + n_{k-1} + (n_k - 1) + (n_{k-2} - 1) + \dots \leq n_k + (n_{k-1} - 1) + (n_k - 1) + (n_{k-1} - 1) + \dots$).

In order to complete the covering trail, reaching every node of $G(n_1, n_2, \dots, n_k)$, we need at least one more link for any of the elements of the set $\{n_1, n_2, \dots, n_{k-2}\}$, and consequently $k - 2$ additional lines cannot join a total of more than $\sum_{i=1}^{k-2} n_i - k + 2$ (i.e., $n_{k-2} - 1 + n_{k-3} - 1 + \dots + n_1 - 1$) unvisited points.

Thus, $h(n_1, n_2, \dots, n_k)$ satisfies the relation

$$\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 2 \leq (2 \cdot n_k + n_{k-1} - 3) \cdot \left(\frac{h(n_1, n_2, \dots, n_k)}{3} - k + 2 \right) + 1. \tag{3}$$

Hence,

$$h(n_1, n_2, \dots, n_k) \geq h_l(n_1, n_2, \dots, n_k) := \left\lceil 3 \cdot \frac{\prod_{i=1}^{k} n_i - \sum_{i=1}^{k-2} n_i + k - 3}{2 \cdot n_k + n_{k-1} - 3} \right\rceil + k - 2, \tag{4}$$

and this concludes the proof of Theorem 2.1. □

Now, let $n := n_1 = n_2 = \dots = n_k$ belong to the set $\mathbb{N} - \{0, 1, 2\}$, as usual.

Since $h(n, 3) \leq \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 2$ by Reference [13], it follows that

$$h(n, k) \leq h_u(n, k) := \left(\left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 1 \right) \cdot n^{k-3} - 1 \tag{5}$$

holds for every $k \in \mathbb{N} - \{0, 1, 2\}$.

Thus,

$$\begin{aligned} \lim_{k \to \infty} \frac{n^k}{h_u(n, k)} &= \lim_{k \to \infty} \frac{n^k}{\left(\left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 1 \right) \cdot n^{k-3} - 1} \\ &\geq \lim_{k \to \infty} \frac{n^k}{\left(\frac{3 \cdot n^2}{2} - \frac{n-1}{4} + \frac{n+1}{4} - \frac{n+2}{4} + \frac{n}{4} + n \right) \cdot n^{k-3}} = \lim_{k \to \infty} \frac{2 \cdot n^k}{n^{k-3} \cdot (3 \cdot n^2 + 2 \cdot n)} = \frac{2 \cdot n^2}{3 \cdot n + 2}, \end{aligned} \tag{6}$$

whereas

$$\lim_{k \to \infty} \frac{n^k}{h_l(n, k)} = \lim_{k \to \infty} \frac{n^k}{\frac{n^k - 1}{n - 1}} = n - 1. \tag{7}$$

On the other hand, it is easy to check that, for any $n \geq 3$,

$$\lim_{k \to \infty} \frac{n^k}{h_u(n,k)} = \lim_{k \to \infty} \frac{n^k}{\left(\left\lfloor \frac{3 \cdot n^2}{2} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 1 \right) \cdot n^{k-3} - 1} \Rightarrow \lim_{k \to \infty} \frac{n^k}{h_u(n,k)} < n - 1.$$

It follows that, on average, as k approaches infinity, the efficiency loss for each link is equal to $L(n)$ unvisited points, where $L(n) := \lim_{k \to \infty} \frac{n^k}{h_l(n,k)} - \lim_{k \to \infty} \frac{n^k}{h_u(n,k)}$.

Hence,

$$L(n) \le \lim_{k\to\infty}\left(n-1+\frac{1}{h_l(n,k)}\right)-\frac{2\cdot n^2}{3\cdot n+2}=\frac{n^2-n-2}{3\cdot n+2}. \tag{8}$$

This is reasonable because, by Theorem 2.1, for any given n above two, we know that $n-1+\frac{1}{h_l(n,k)}$ is the maximum average number of "new" nodes of $P(n,k)$ visited by each edge of a trail since the highest theoretical number of unvisited nodes covered by t consecutive edges of a trail is $t\cdot(n-1)+1$ (for any n as above).

In particular, if $n=3$, then we can improve (1) as [10]

$$h_l(3,k)=h(3,k)=\frac{3^k-1}{2}. \tag{9}$$

Now, we observe that

$$h(3,k)<3^{k-1}+\frac{3}{2}\cdot 3^{k-2} \tag{10}$$

holds for every $k\in\mathbb{Z}^+$, and Kranakis' conjectured upper bound $\tilde{h}_u(n:n\ge 3,\ k:k\ge 4)=\frac{k}{k-1}\cdot n^{k-1}+O(n^{k-2})$, concerning every minimal rectilinear walk covering $P(n,k)$, implies the existence of a constant $c\ge\frac{3}{2}$ such that $\frac{k}{k-1}\cdot n^{k-1}+c\cdot n^{k-2}\ge h(n,k)$.

Moreover, we point out that

$$\lim_{k\to\infty}\frac{3^k}{h(3,k)}=\lim_{k\to\infty}\frac{3^k}{\frac{3^k-1}{2}}=2, \tag{11}$$

with zero efficiency loss for each link, since $L(3)=\lim\limits_{k\to\infty}\frac{3^k}{h_l(3,k)}-\lim\limits_{k\to\infty}\frac{3^k}{h_u(3,k)}=2-2$ unvisited points (instead of $\frac{8}{31}$ as given by (8)).

Conjecture 2.1 For any given $P(n,k)$, we conjecture that the edges of each minimal covering trail visit (on average) less than $n-1$ new nodes if and only if $n\in\mathbb{N}-\{0,1,2,3\}\ \wedge\ k\in\mathbb{N}-\{0,1\}$.

Remark 2.1 If Conjecture 2.1 holds, then $\nexists n\in\{1,2,3\}:h(n,k)<\frac{n^k}{n-1}$, and consequently $\frac{n^k}{h(n,k)}\ge n-1$ implies $n\le 3$. This is certainly true for $(n,k)=(2,3)$, since $(1-\sqrt{2},-\sqrt{2},0)$-$(\sqrt{2},\sqrt{2},0)$-$(\frac{1}{2},\frac{1}{2},2\cdot\sqrt{3}-\sqrt{\frac{3}{2}})$-$(\sqrt{2},1-\sqrt{2},0)$-$(1-\sqrt{2},\sqrt{2},0)$-$(\frac{1}{2},\frac{1}{2},2\cdot\sqrt{3}-\sqrt{\frac{3}{2}})$-$(1-\sqrt{2},1-\sqrt{2},0)$ is a covering cycle for $P(2,3)$ (see [11], pp. 162-163) and, by definition, it follows that $h_u(2,3)=6\Rightarrow\frac{2^3}{h_u(2,3)}\le\frac{2^3}{h(2,3)}\Rightarrow\frac{8}{h(2,3)}>2-1$. Likewise, Eq. (9) underlines that Kranakis' Theorem 1 of Reference [6] is not valid if we disregard the rectilinear walk constraint and allow covering paths as the above.

3 Conclusion

We have shown that Kranakis' conjectured upper bound of $\frac{k}{k-1}\cdot n^{k-1}+O(n^{k-2})$, for minimal rectilinear walks covering $P(n,k)$, can be rewritten, taking into account the general covering trails considered in References [10, 13], as $\frac{k}{k-1}\cdot n^{k-1}+c\cdot n^{k-2}$, where $c\ge\frac{3}{2}$. On the contrary, the upper bound proved by Bereg et al. [1] of $\frac{k}{k-1}\cdot 3^{k-1}+3^{k-\frac{3}{2}}$ definitely holds for every finite set of 3^k points in $\mathbb{R}^k$.

Lastly, for any nontrivial value of k, we see that $\frac{2^k}{h(2,k)} > \frac{3^k}{2 \cdot h(3,k)} > \frac{4^k}{3 \cdot h(4,k)}$ and Conjecture 2.1 leads to a more general research question, which can be formulated as follows:

"Does the relation $\frac{n^k}{(n-1) \cdot h(n,k)} > \frac{(n+1)^k}{n \cdot h(n+1,k)}$ hold for any $n, k \in \mathbb{N} - \{0, 1\}$?"

(e.g., the answer is affirmative if we select $k = 2$, since $n > 2 \Rightarrow h(n, 2) = 2 \cdot n - 2$ by [4], and so $\frac{n^2}{(n-1) \cdot h(n,2)} = \frac{1}{2 \cdot (n-1)^2} + \frac{1}{n-1} + \frac{1}{2}$ proves that $\frac{n^2}{(n-1) \cdot h(n,2)} > \frac{(n+1)^2}{n \cdot h(n+1,2)}$ holds for every n).

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