# Solving the $n_1 \times n_2 \times n_3$ Points Problem for $n_3 < 6$

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**Abstract:** In this paper, we show enhanced upper bounds of the nontrivial $n_1 \times n_2 \times n_3$ points problem for every $n_1 \leq n_2 \leq n_3 < 6$. We present new patterns that significantly improve the previously known algorithms for finding minimum-link covering paths and trails.
**Keywords:** Connectivity, Covering trail, Game, Graph theory, Link-length, Outside the box, Point, Three-dimensional, Upper bound.
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## 1 Introduction

The $n_1 \times n_2 \times n_3$ points problem [10] is a three-dimensional extension of the classic nine-dot problem that appeared in Sam Loyd's Cyclopedia of Puzzles (see [8], p. 301). It is related to the well-known NP-hard traveling salesman problem (TSP), which minimizes the number of turns in the tour rather than the total distance traveled [1, 13].

Given $n_1 \cdot n_2 \cdot n_3$ points in $\mathbb{R}^3$, our goal is to visit all of them (at least once) with a polygonal chain that has the minimum number of line segments connected at their endpoints (links or, more generally, lines), the so-called *minimum-link covering trail* [2–4, 7]. In particular, we are interested in the best solutions to the nontrivial $n_1 \times n_2 \times n_3$ points problem, where (by definition) $1 \leq n_1 \leq n_2 \leq n_3$ and $n_3 < 6$.

Let $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3) \leq h_u(n_1, n_2, n_3)$ be the length of the covering trail with the minimum number of links for the $n_1 \times n_2 \times n_3$ points problem, we define the best known upper bound as $h_u(n_1, n_2, n_3) \geq h(n_1, n_2, n_3)$, and we denote as $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$ the proved lower bound.

For simple configurations, this problem has already been solved [2]. In detail, if $n_1 = 1$ and $n_2 < n_3$, then $h(n_1, n_2, n_3) = 2 \cdot n_2 - 1$, while $h(n_1, n_2, n_3) = 2 \cdot n_2 - 2$ when $n_1 = 1$, $n_2 \geq 3$, and $n_3 = n_2$ [5].

Hence, assuming $n_1 = 2$ and $n_3 > 2$, it can be easily proved that

$$h(2, n_2, n_3) = 2 \cdot h(1, n_2, n_3) + 1 = \begin{cases} 4 \cdot n_2 - 1 & \text{iff} \quad n_2 < n_3 \\ 4 \cdot n_2 - 3 & \text{iff} \quad n_2 = n_3 \end{cases} . \quad (1)$$

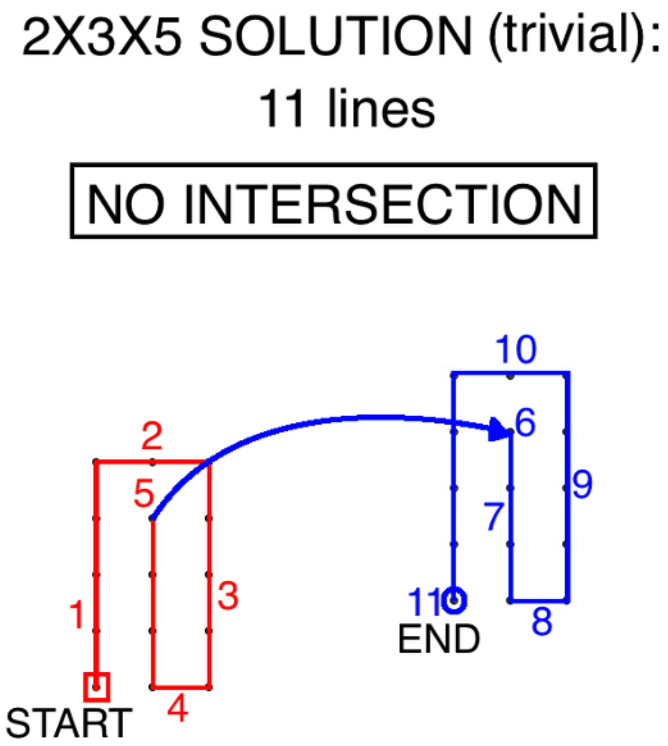


Figure 1: A trivial Hamiltonian path that completely solves the $2 \times 3 \times 5$ points puzzle (without self-intersections).

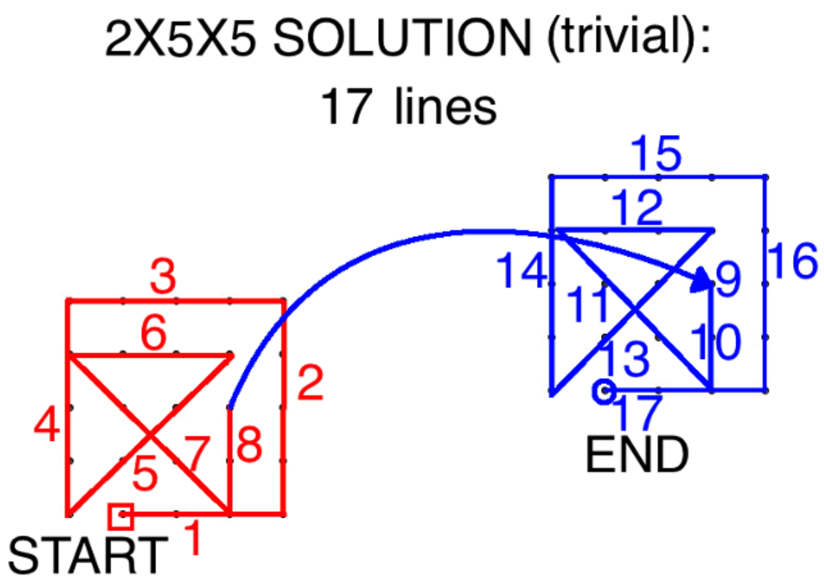


Figure 2: Another example of a trivial pattern: solving the $2 \times 5 \times 5$ points puzzle.

Therefore, the present paper aims to solve the ten above-mentioned nontrivial cases where the current upper bound does not match the proved lower bound.

## 2 Improving the solution of the $n_1 \times n_2 \times n_3$ points problem for $n_3 < 6$

In this complex brain challenge, we need to extend our pattern recognition capabilities [6, 9] in order to find a plastic strategy that improves the known upper bounds [2, 10] for the most interesting cases (and the $3 \times 3 \times 3$ problem, which is the three-dimensional extension of the immortal nine-dot puzzle, is by far the most valuable one [11]), avoiding standardized methods based on fixed patterns, which lead to suboptimal covering paths, such as the approaches presented in [7, 10].

**Theorem 1.** Let $(n_1, n_2, n_3)$ be a triple of integers satisfying $3 \leq n_1 \leq n_2 \leq n_3$. Then, a lower bound for the $n_1 \times n_2 \times n_3$ problem is given by

$$h_l(n_1, n_2, n_3) = \left\lceil \frac{2 \cdot (n_1 \cdot n_2 \cdot n_3 - n_3)}{n_2 + n_3 - 2} \right\rceil + 1. \tag{2}$$

*Proof.* Let $\{0, 1, \ldots, n_1 - 1\} \times \{0, 1, \ldots, n_2 - 1\} \times \{0, 1, \ldots, n_3 - 1\}$ be a set of $n_1 \cdot n_2 \cdot n_3$ points, in the Euclidean vector space $\mathbb{R}^3$, such that $3 \leq n_1 \leq n_2 \leq n_3$.

We immediately notice that, for any given positive integer $t$, we have $\frac{(n_2-1)+(n_3-1)}{2} \geq \frac{\left\lceil \frac{t}{2} \right\rceil \cdot (n_2-1) + \left\lfloor \frac{t}{2} \right\rfloor \cdot (n_3-1)}{t}$, and consequently there does not exist any polygonal chain of $1 + t$ links that visits more than $n_3 + \frac{(n_2-1)+(n_3-1)}{2} \cdot t$ points of the given $n_1 \times n_2 \times n_3$ regular grid.

Thus,

$$n_1 \cdot n_2 \cdot n_3 \leq n_3 + \frac{n_2 + n_3 - 2}{2} \cdot (h(n_1, n_2, n_3) - 1). \tag{3}$$

Hence,

$$h(n_1, n_2, n_3) - 1 \geq \frac{2 \cdot (n_1 \cdot n_2 \cdot n_3 - n_3)}{n_2 + n_3 - 2}.$$

Since $h(n_1, n_2, n_3)$ is a natural number (and given the fact that $h(n_1, n_2, n_3) \geq h_l(n_1, n_2, n_3)$ must hold by definition), we can finally set

$$h_l(n_1, n_2, n_3) := \left\lceil \frac{2 \cdot (n_1 \cdot n_2 \cdot n_3 - n_3)}{n_2 + n_3 - 2} \right\rceil + 1, \tag{4}$$

and this concludes the proof of the theorem. $\square$

Table 1 lists the best results known at the present date, and a direct proof follows for each stated nontrivial upper bound.

| $n_1$ | $n_2$ | $n_3$ | **Best Lower bound $h_l$** | **Best Upper bound $h_u$** | **Discovered by** | **Gap ($h_u - h_l$)** |
|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 6 | 6 | Koki Goma, proved in Aug. 2021 (see [12]) | 0 |
| 2 | 2 | 3 | 7 | 7 | trivial | 0 |
| 2 | 3 | 3 | 9 | 9 | trivial | 0 |
| 3 | 3 | 3 | 13 | 13 | Marco Ripà, proved in June 2020 (see [11]) | 0 |
| 2 | 2 | 3 | 7 | 7 | trivial | 0 |
| 2 | 3 | 4 | 11 | 11 | trivial | 0 |
| 2 | 4 | 4 | 13 | 13 | trivial | 0 |
| 3 | 3 | 4 | 14 | 15 | Marco Ripà, June 2019 | 1 |
| 3 | 4 | 4 | 16 | 19 | Marco Ripà, June 2019 | 3 |
| 4 | 4 | 4 | 21 | 23 | Marco Ripà, 2019 (see NNTDM, 25(2), p. 70, Fig. 1) | 2 |
| 2 | 2 | 5 | 7 | 7 | trivial | 0 |
| 2 | 3 | 5 | 11 | 11 | trivial | 0 |
| 2 | 4 | 5 | 15 | 15 | trivial | 0 |
| 2 | 5 | 5 | 17 | 17 | trivial | 0 |
| 3 | 3 | 5 | 15 | 16 | Marco Ripà, June 2019 | 1 |
| 3 | 4 | 5 | 17 | 20 | Marco Ripà, June 2019 | 3 |
| 3 | 5 | 5 | 19 | 24 | Marco Ripà, June 2019 | 5 |
| 4 | 4 | 5 | 23 | 26 | Marco Ripà, June 2019 | 3 |
| 4 | 5 | 5 | 25 | 31 | Marco Ripà, June 2019 | 6 |
| 5 | 5 | 5 | 31 | 36 | Marco Ripà, July 2019 | 5 |

Table 1: Current solutions to the $n_1 \times n_2 \times n_3$ points problem, where $n_1 \leq n_2 \leq n_3 < 6$.

Figures 3 to 12 show the patterns used to solve the $n_1 \times n_2 \times n_3$ puzzle (case-by-case).

In particular, by combining (2) with the original results shown in Figures 3, 4, and 7, we obtain a formal proof for the crucial $3 \times 3 \times 3$ points problem, as well as very tight bounds for the $3 \times 3 \times 4$ and $3 \times 3 \times 5$ cases.

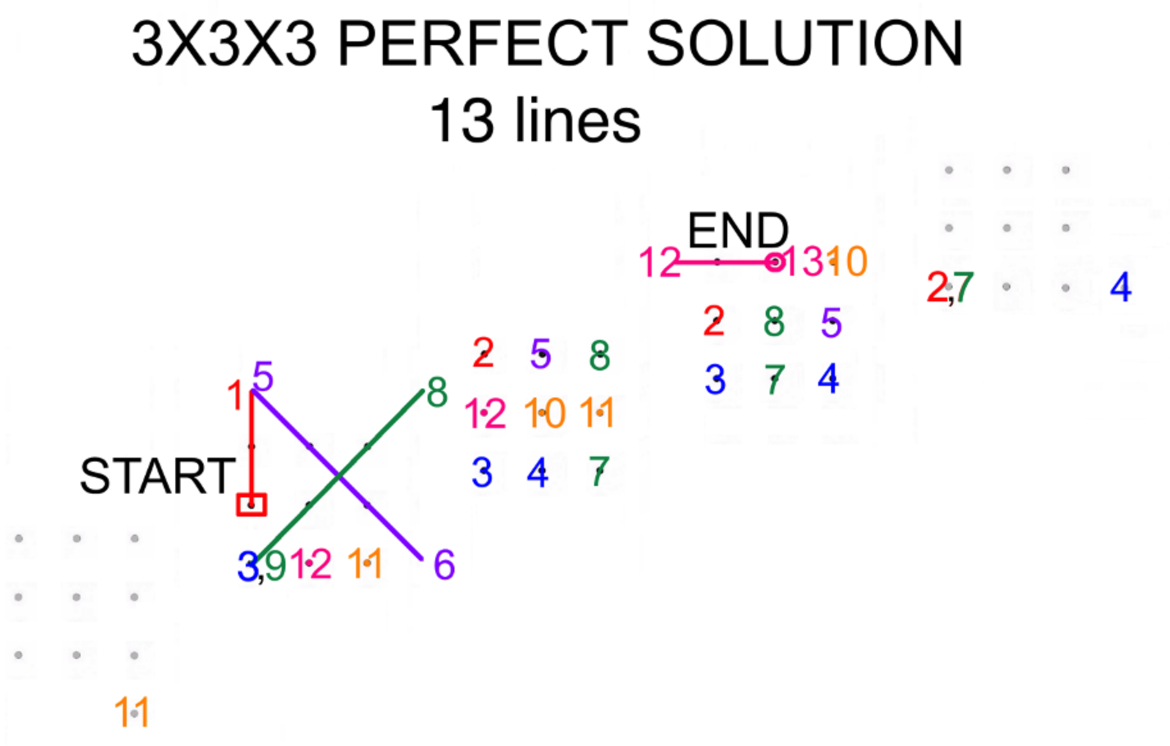


Figure 3: The $k$-dimensional $3 \times 3 \times \cdots \times 3$ puzzle has been explicitly solved for every $k \in \mathbb{Z}^{+}$ (since $h_u(3, 3, \ldots, 3) = h_l(3, 3, \ldots, 3) = \frac{3^k - 1}{2}$, see [11]). In particular, Ripà provided the above solution for the three-dimensional case on June 19, 2020, and it is optimal by Corollary 1.

**Corollary 1.** With regard to the $3 \times 3 \times 3$ points problem, the lower bound and the upper bound satisfy

$$h_l(3, 3, 3) = h_u(3, 3, 3) = 13. \tag{5}$$

*Proof.* The covering trail for the $3 \times 3 \times 3$ case shown in Figure 3 consists of $13$ straight lines connected at their endpoints, and from (2) we obtain $h_l(3, 3, 3) = \frac{3^3 - 1}{3 - 1} = 13$ (in this case the ceiling function is not needed since $\frac{3^k - 1}{2}$ is an integer for any positive integer $k$). $\square$

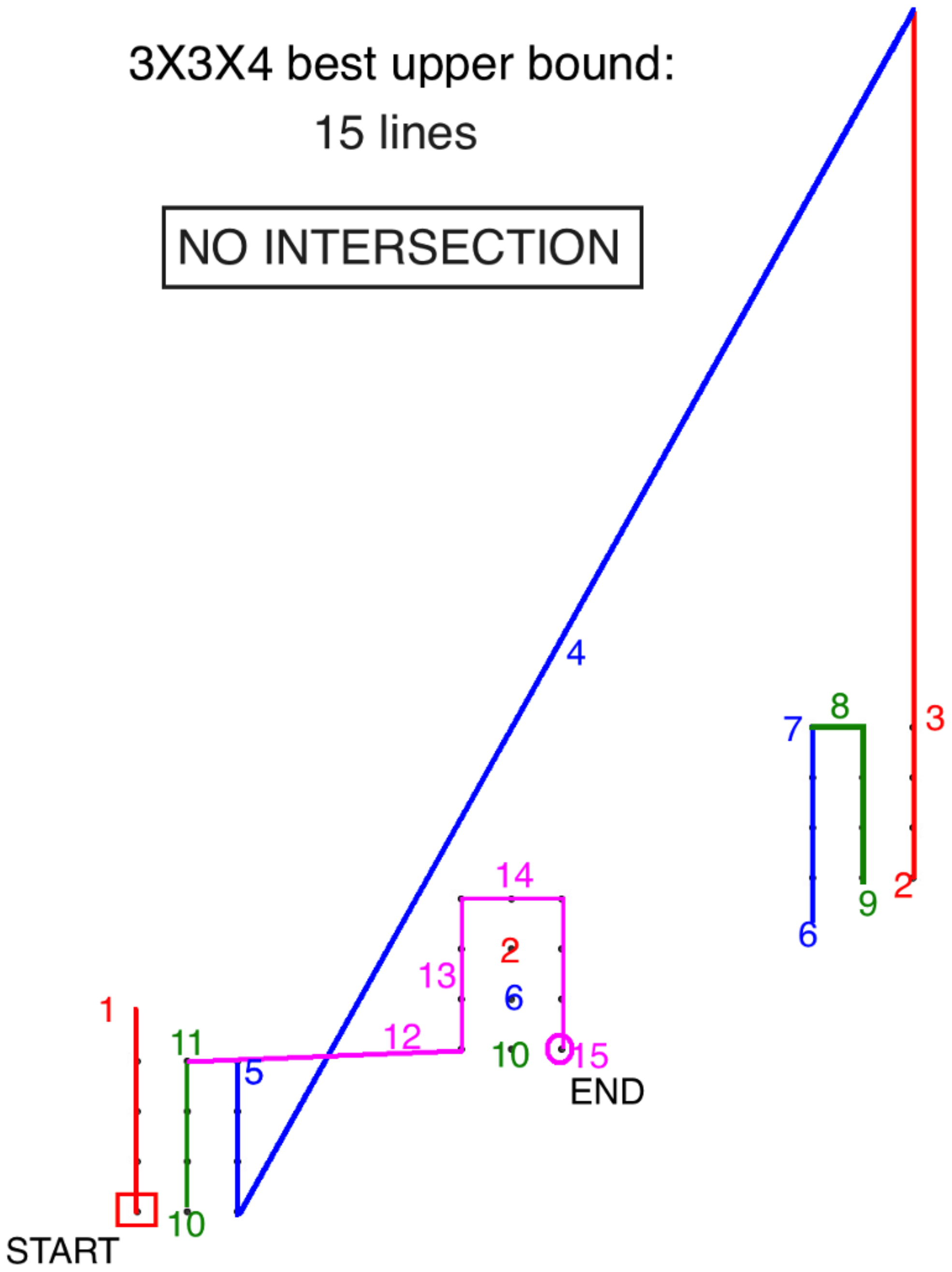


Figure 4: Best known (non-crossing) covering path for the $3 \times 3 \times 4$ puzzle. $15 = h_u = h_l + 1$.

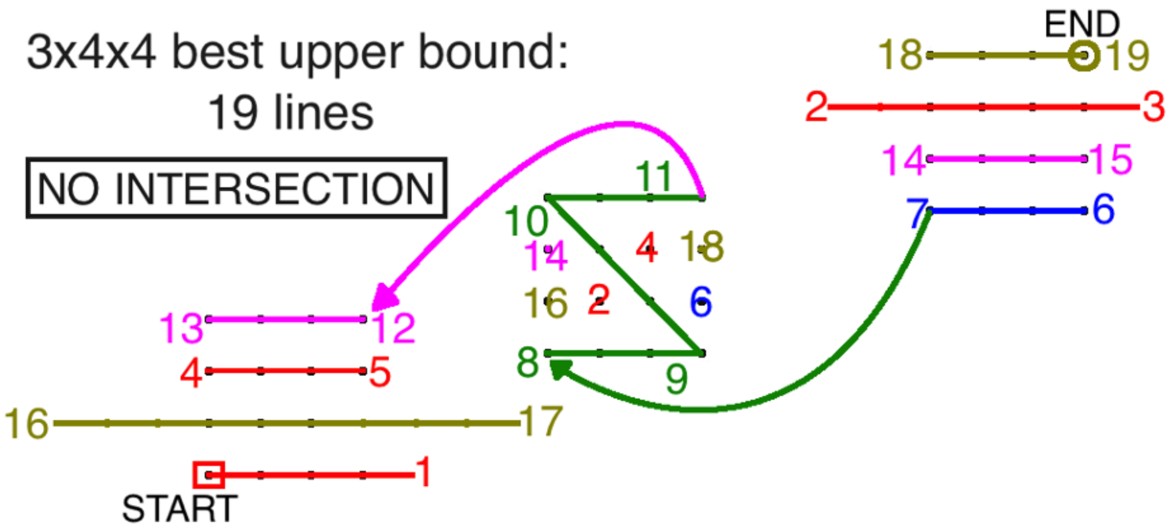


Figure 5: Best known (non-crossing) covering path for the $3 \times 4 \times 4$ puzzle. $19 = h_u = h_l + 3$.

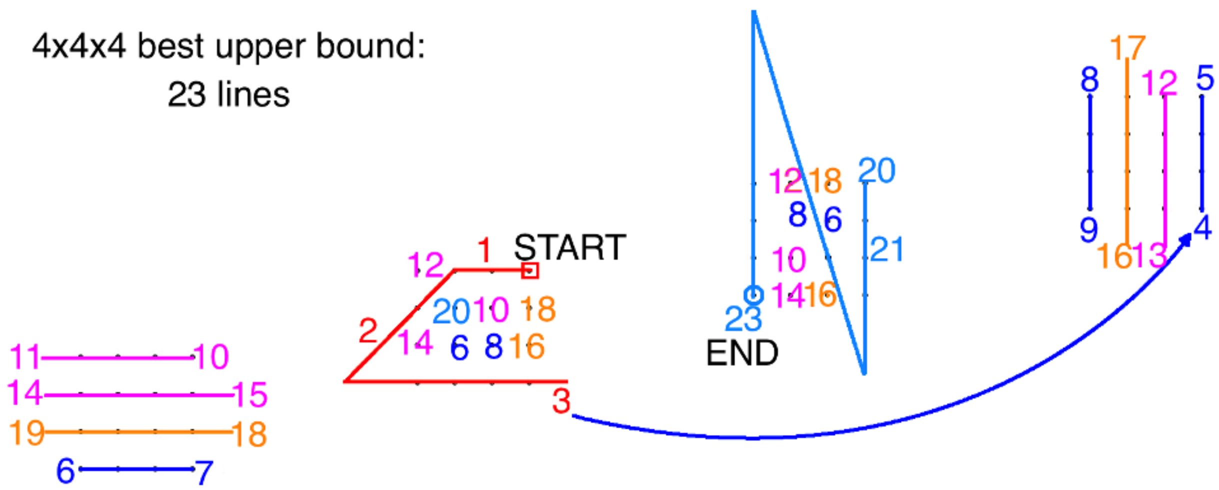


Figure 6: An original covering path for the $4 \times 4 \times 4$ puzzle. $23 = h_u = h_l + 2$.

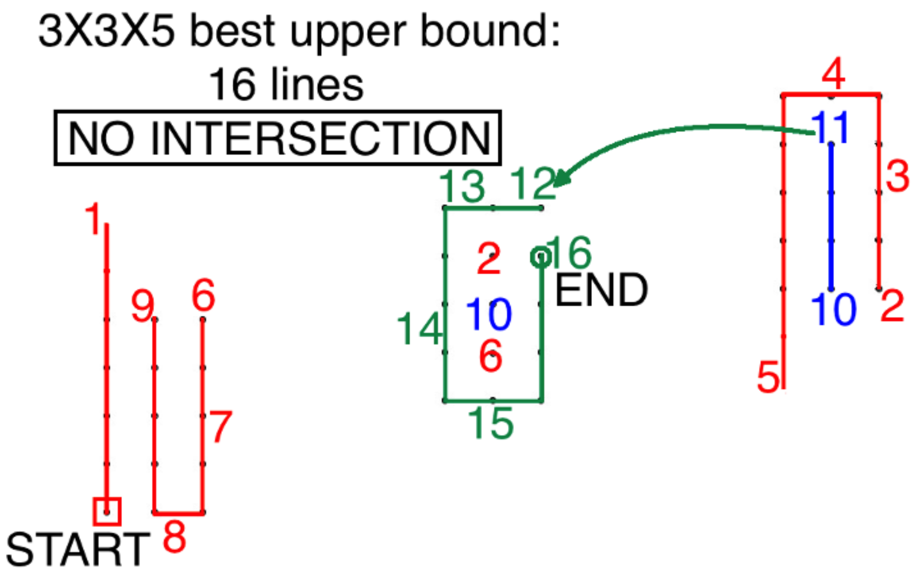


Figure 7: Best known (non-crossing) covering path for the $3 \times 3 \times 5$ puzzle. $16 = h_u = h_l + 1$.

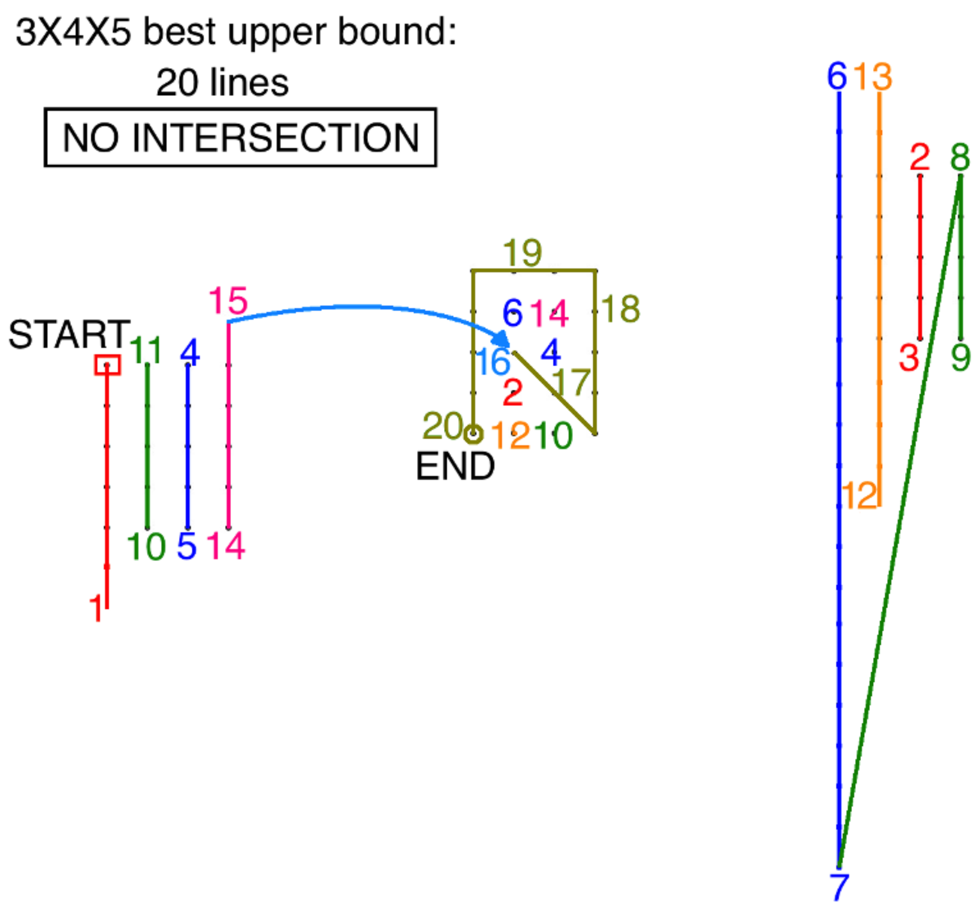


Figure 8: Best known (non-crossing) covering path for the $3 \times 4 \times 5$ puzzle, consisting of $20 = h_u = h_l + 3$ lines.

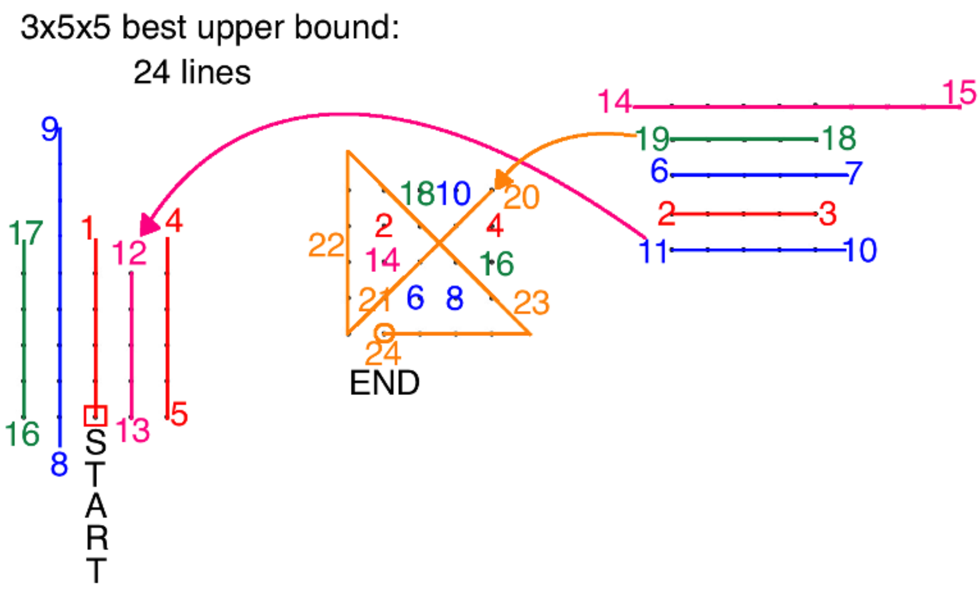


Figure 9: Best known covering path for the $3 \times 5 \times 5$ puzzle. $24 = h_u = h_l + 5$.

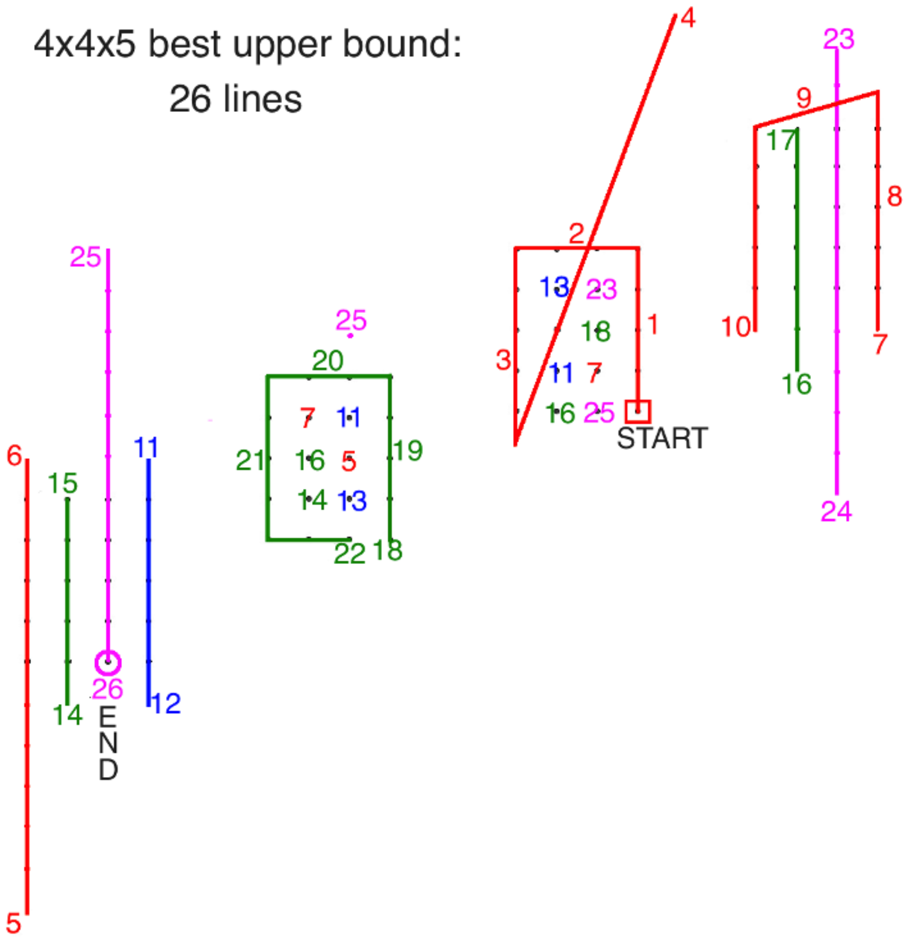


Figure 10: Best known covering path for the $4 \times 4 \times 5$ puzzle. $26 = h_u = h_l + 3$.

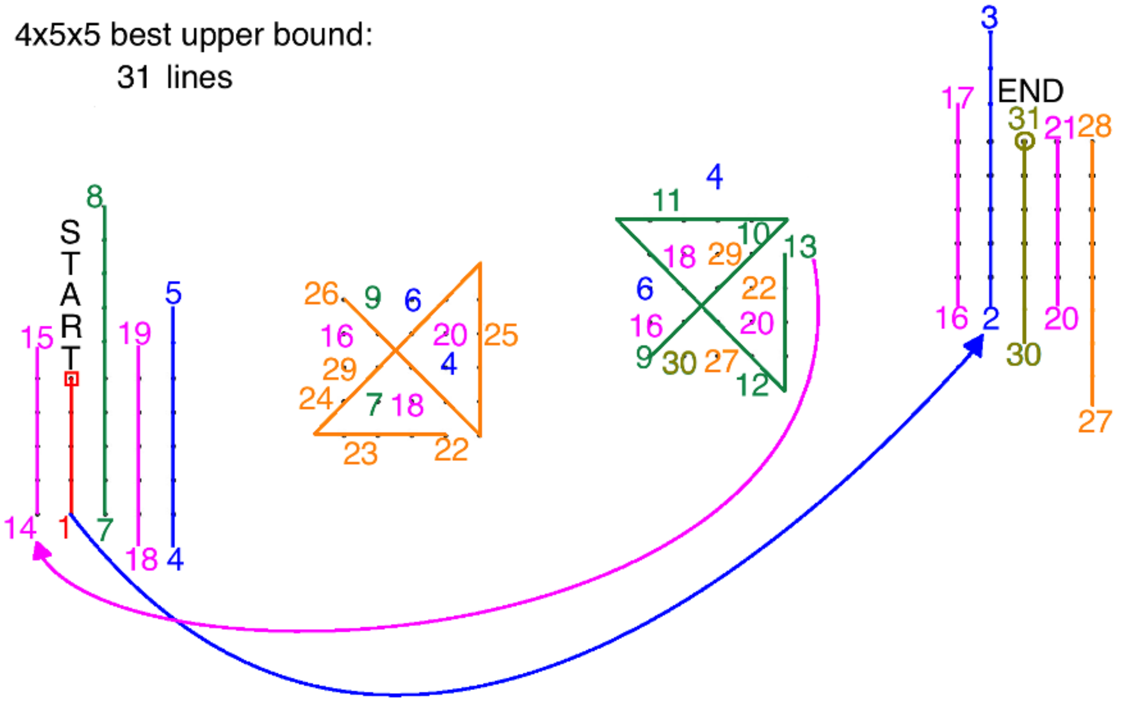


Figure 11: Best known upper bound for the $4 \times 5 \times 5$ puzzle. $31 = h_u = h_l + 6$.

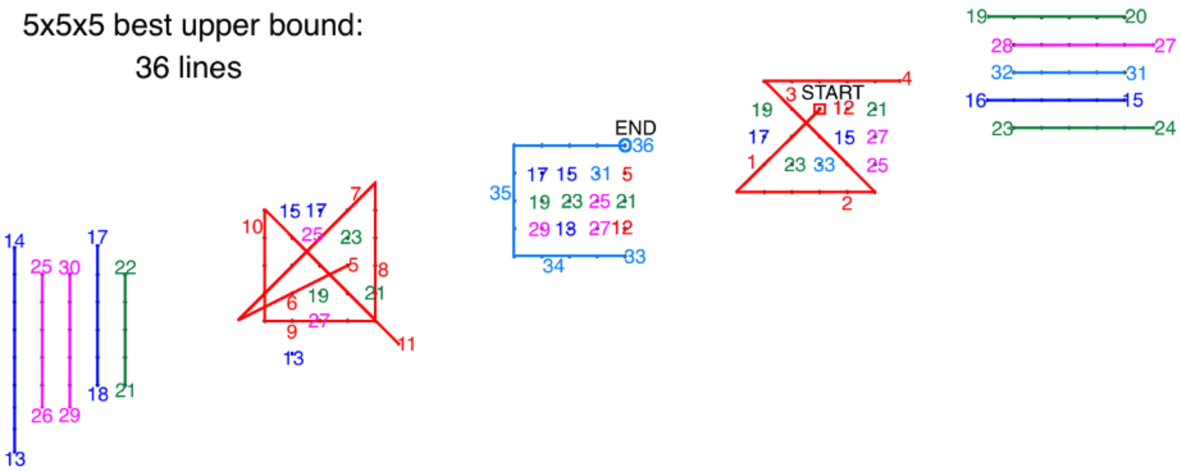


Figure 12: Best known upper bound for the $5 \times 5 \times 5$ puzzle. $36 = h_u = h_l + 5$.

Lastly, it is interesting to note that the reduced value of $h_u(n_1, n_2, n_3)$ also improves the upper bound for the generalized $k$-dimensional puzzle. For example, we can apply the aforementioned 3D patterns to the generalized $n_1 \times n_2 \times \cdots \times n_k$ points problem using the simple method described in [10].

For any given $k \geq 4$, assuming $n_k \leq n_{k-1} \leq \cdots \leq n_4 \leq n_1 \leq n_2 \leq n_3$, we can conclude that

$$h(n_1, n_2, n_3, \ldots, n_k) \leq (h_u(n_1, n_2, n_3) + 1) \cdot \prod_{j=4}^{k} n_j - 1. \qquad (6)$$

# 3 Conclusion

In the present paper, we have significantly reduced the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$ for every previously unsolved puzzle with $n_3 < 6$.

We do not know whether any of the patterns shown in Figures 4 to 12 represent optimal solutions since (by definition) $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$. Therefore, some open questions about the NP-complete [2] $n_1 \times n_2 \times n_3$ points problem remain open, and the research aimed at closing the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$, at least for every $n_3 \leq 5$, is not over yet.

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