On the weakest constraint qualification for sharp local minimizers

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Abstract
The sharp local minimality of feasible points of nonlinear optimization problems is known to possess a characterization by a strengthened version of the Karush-Kuhn-Tucker conditions, as long as the Mangasarian-Fromovitz constraint qualification holds. This strengthened condition is not easy to check algorithmically since it involves the topological interior of some set. In this paper we derive an algorithmically tractable version of this condition, called strong Karush-Kuhn-Tucker condition, and we show that the weakest condition under which a feasible point is a strong Karush-Kuhn-Tucker point for every at this point continuously differentiable objective function possessing the point as a sharp local minimizer, is the Guignard constraint qualification. As an application, our results yield an algebraic characterization of strict local minimizers of linear programs with cardinality constraints.

KEYWORDS
Sharp local minimizer, strict local minimizer of order one, strongly unique local minimizer, Guignard constraint qualification, weakest constraint qualification, cardinality constraint

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1. Introduction

We consider nonlinear optimization problems of the form

$$P: \quad \min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0,\ h(x) = 0$$

with defining functions $f \in C^1(\mathbb{R}^n, \mathbb{R})$, $g \in C^1(\mathbb{R}^n, \mathbb{R}^p)$ and $h \in C^1(\mathbb{R}^n, \mathbb{R}^q)$. The feasible set of P will be denoted by X, $\nabla f(x)$ stands for the gradient of f at x, and $\nabla g(x)$, $\nabla h(x)$ are the (transposed) Jacobians of g and h, respectively, at x. With the active index set $A(\bar{x}) = \{i \in \{1, \ldots, p\} \mid g_i(\bar{x}) = 0\}$ of $\bar{x}$ the matrix $\nabla g_A(\bar{x})$ possesses the columns $\nabla g_i(\bar{x})$, $i \in A(\bar{x})$.

We will be interested in necessary and sufficient optimality conditions for sharp local minimizers of P, that is, points $\bar{x} \in X$ for which a neighborhood U and some $\alpha > 0$ exist with

$$\forall\, x \in X \cap U: \quad f(x) \geq f(\bar{x}) + \alpha \|x - \bar{x}\|.$$

Example 1.1. *For $n = 2$ let $f(x) = x_1 + x_2$, $g_1(x) = -x_1$, $g_2(x) = -x_2$ and $g_3(x) = x_1 x_2$. It is not hard to see that $\bar{x} = 0$ is a sharp local minimizer with respect to the Euclidean norm, where one may choose $U = \mathbb{R}^2$ (i.e., $\bar{x}$ is even a sharp global minimizer) and $\alpha = 1$.*

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Sharp local minimizers are also called strong [1], strongly unique [2], or strict of order one [3]. Since sharp local minimizers $\bar{x}$ of P are local minimizers, under some constraint qualification they are necessarily Karush-Kuhn-Tucker (KKT) points, that is, there exist $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^q$ with

$$\nabla f(\bar{x}) + \nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu = 0, \ \lambda \geq 0. \tag{1}$$

Moreover, since sharp local minimizers are special local minimizers, one may expect that they also satisfy a strengthened version of the KKT conditions, and that this condition may even be sufficient for sharp local minimality. Such a condition is given in [1, Th. 3.6] under the Mangasarian-Fromovitz constraint qualification (MFCQ) at $\bar{x}$, which assumes $\operatorname{rank} \nabla h(\bar{x}) = q$ and the existence of some vector $d \in \mathbb{R}^n$ with $\nabla g_A(\bar{x})^\intercal d < 0$ and $\nabla h(\bar{x})^\intercal d = 0$. Note that (1) may be rewritten as

$$-\nabla f(\bar{x}) \in \{\nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu \mid \lambda \in \mathbb{R}^p, \ \mu \in \mathbb{R}^q, \ \lambda \geq 0\}.$$

Theorem 1.2 ([1]). *Let the MFCQ hold at $\bar{x} \in X$. Then $\bar{x}$ is a sharp local minimizer of P if and only if*

$$-\nabla f(\bar{x}) \in \operatorname{int}\{\nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu \mid \lambda \in \mathbb{R}^p, \ \mu \in \mathbb{R}^q, \ \lambda \geq 0\} \tag{2}$$

holds.

We remark that condition (2) is also used in [4] for the investigation of unique minimal points of problems P with convex feasible sets. In the present paper, however, we do not impose any convexity assumption on P.

The aim of this paper is twofold. Firstly, we will derive an algorithmically more tractable version of the condition (2) and, secondly, we wish to identify a weakest constraint qualification under which sharp local minimality is characterized by (2). Both is possible by using techniques which were introduced for the characterization of sharp minimizers of linear semi-infinite problems in [5] and, independently, of strict local Pareto optimal points of order one in multicriteria optimization [6,7].

As, to the best of the authors' knowledge, the corresponding results in single objective nonlinear optimization have not been formulated explicitly so far, the present paper first closes this gap by introducing the concept of strong Karush-Kuhn-Tucker points in Section 2 as well as the related stationarity condition and constraint qualifications in Section 3. In Section 4 we show that the Guignard constraint qualification is weakest possible for the characterization of sharp local minimality by the strong KKT property, and in Section 5 we apply our results to strict local minimizers of cardinality-constrained linear optimization problems. Section 6 closes this paper with some final remarks.

2. Strong Karush-Kuhn-Tucker points

In the sequel the following notion will be useful.

Definition 2.1. *We call $\bar{x} \in X$ a* strong Karush-Kuhn-Tucker point *if*

$$\operatorname{rank}(\nabla g_A(\bar{x}), \nabla h(\bar{x})) = n \tag{3}$$

holds and if there exist $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^q$ with

$$\nabla f(\bar{x}) + \nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu = 0, \ \lambda > 0. \tag{4}$$

We remark that (3) is not a constraint qualification (cf. Ex. 4.3 below). The linear independence constraint qualification (LICQ) $\mathrm{rank}(\nabla g_A(\bar{x}), \nabla h(\bar{x})) = |A(\bar{x})| + q$, the identity $|A(\bar{x})| + q = n$ and the strict complementary slackness condition $\lambda > 0$ are sufficient for $\bar{x} \in X$ to be a strong KKT point. These conditions are, however, not necessary since at strong KKT points $|A(\bar{x})| + q > n$ may hold, and the multipliers λ and μ do not need to be unique.

Example 2.2. *In Example 1.1 we have* $\nabla f(\bar{x}) = (1,1)^\intercal$, $A(\bar{x}) = \{1,2,3\}$ *and* $\nabla g_A(\bar{x}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ *so that LICQ is violated at* $\bar{x}$ *while* (3) *is satisfied. Indeed,* $\bar{x}$ *is even a strong KKT point with the (nonunique) multiplier* $\lambda = (1,1,1)^\intercal > 0$.

We will characterize strong KKT points by means of Tucker's theorem of the alternative.

Lemma 2.3 ([8])**.** *For matrices A and B, with A being nonvacuous, exactly one of the following alternatives hold:*

a) $Ax \le 0$, $Ax \ne 0$, $Bx = 0$ *possesses a solution* x.
b) $A^\intercal y + B^\intercal z = 0$, $y > 0$ *possesses a solution* (y, z).

The proof of the following characterization is almost identical to the one for the multicriteria case from [6, Th. 3.4] where, however, a weaker assertion is stated as the result of the proof. A proof of the same result from the point of view of linear semi-infinite programming is given in [5, Th. 3.1] and used in, e.g., [9,10]. For completeness we repeat the arguments here. We define the set $C_\le(f,\bar{x}) = \{d \in \mathbb{R}^n \mid \nabla f(\bar{x})^\intercal d \le 0\}$ of (potential) descent directions for f at $\bar{x} \in X$ and the linearization cone $L(g,h,\bar{x}) = \{d \in \mathbb{R}^n \mid \nabla g_A(\bar{x})^\intercal d \le 0,\ \nabla h(\bar{x})^\intercal d = 0\}$ to X at $\bar{x}$.

Lemma 2.4. *A point $\bar{x}$ is a strong KKT point of P if and only if $\bar{x} \in X$ and $C_\le(f,\bar{x}) \cap L(g,h,\bar{x}) = \{0\}$ hold.*

Proof. With $A^\intercal = (\nabla f(\bar{x}), \nabla g_A(\bar{x}))$ and $B^\intercal = \nabla h(\bar{x})$ we have $C_\le(f,\bar{x}) \cap L(g,h,\bar{x}) = \{0\}$ if and only if the system $Ad \le 0$, $Bd = 0$ possesses only the trivial solution $d = 0$. The latter is equivalent to the fact that, both, the system $Ad = 0$, $Bd = 0$, $d \ne 0$ is unsolvable, and the system $Ad \le 0$, $Ad \ne 0$, $Bd = 0$ is unsolvable. The unsolvability of the first system is equivalent to the linear independence of the n rows of the matrix $(A^\intercal, B^\intercal)$, that is, to

$$\mathrm{rank}(\nabla f(\bar{x}), \nabla g_A(\bar{x}), \nabla h(\bar{x})) = n. \tag{5}$$

Moreover, by Lemma 2.3 the second system is unsolvable if and only if there exist $\kappa > 0$, $\lambda > 0$ with $\kappa \nabla f(\bar{x}) + \nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu = 0$. After division of this equation by κ (and renaming λ) this is condition (4). Finally, under (4), (5) is equivalent to (3). $\square$

The following reformulation of Lemma 2.4 will be useful, where $A^\circ = \{v \in \mathbb{R}^n \mid v^\intercal d \le 0\, \forall d \in A\}$ denotes the polar cone of a cone $A \subseteq \mathbb{R}^n$.

Lemma 2.5. *A point $\bar{x}$ is a strong KKT point of P if and only if $\bar{x} \in X$ and*

$$-\nabla f(\bar{x}) \in \mathrm{int}\, L^\circ(g,h,\bar{x}) \tag{6}$$

hold.

Proof. It is well-known that the Farkas lemma yields

$$L^\circ(g,h,\bar{x}) \;=\; \{\nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu \mid \lambda \ge 0,\ \mu \in \mathbb{R}^q\}. \tag{7}$$

Since $L(g,h,\bar{x})$ is a closed cone, [11, Ex. 6.22] implies

$$\mathrm{int}\, L^\circ(g,h,\bar{x}) = \{v \in \mathbb{R}^n \mid v^\intercal d < 0\, \forall d \in L(g,h,\bar{x}) \setminus \{0\}\}. \tag{8}$$

Therefore (6) is equivalent to $C_{\leq}(f,\bar{x}) \cap L(g,h,\bar{x}) = \{0\}$, and the assertion follows from Lemma 2.4. $\square$

As a direct consequence of Lemma 2.5 and (7) we obtain that condition (2) in Theorem 1.2 is equivalent to $\bar{x} \in X$ being a strong KKT point, where the strong KKT property is easier to check algorithmically than the topological condition (2).

Theorem 2.6. *A point $\bar{x}$ is a strong KKT point of P if and only if $\bar{x} \in X$ and condition (2) hold.*

Theorem 2.6 allows us to reformulate Theorem 1.2 as the statement that, under the MFCQ at $\bar{x} \in X$, this point is a sharp local minimizer of P if and only if it is a strong KKT point.

Remark 2.7. *The strong KKT property at $\bar{x} \in X$ admits the following geometric interpretation which is, however, less suitable for a transparent discussion of constraint qualifications. Indeed, by Lemma 2.5 we have $v \in \operatorname{int} L^{\circ}(g,h,\bar{x})$ if and only if $\bar{x}$ is a strong KKT point for the minimization of $f_v(x) := -v^{\intercal}x$ over X. The latter is equivalent to the rank condition (3) and $v \in \{\nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu \mid \lambda > 0,\ \mu \in \mathbb{R}^q\}$. Therefore $\operatorname{int} L^{\circ}(g,h,\bar{x}) \neq \emptyset$ is characterized by (3), and under (3) one has*

$$\operatorname{int} L^{\circ}(g,h,\bar{x}) = \{\nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu \mid \lambda > 0,\ \mu \in \mathbb{R}^q\}. \tag{9}$$

On the other hand, (4) may hold while (3) is violated, $-\nabla f(\bar{x}) \in \operatorname{int} L^{\circ}(g,h,\bar{x})$ can then not be concluded from (4), and $\bar{x}$ is not a strong KKT point, but only a strict complementary KKT point (see Example 4.2 below).

Remark 2.8. *The last step in the proof of Lemma 2.4, which removes the gradient of the objective function from (5) and thus establishes (3) as a condition purely on the feasible set, is not possible in the multicriteria setting. There only one of the finitely many gradients of objective functions could be removed from the condition, so that the rank condition (5), with the (transposed) Jacobian $\nabla f(\bar{x})$ of the vector-valued objective function f, is usually not stated in such a reduced from. As a consequence, $\operatorname{int} L^{\circ}(g,h,\bar{x})$ may be empty at strong KKT points and, in particular, strong KKT points even exist for unconstrained multicriteria problems [12].*

3. A stationarity condition and constraint qualifications

In Example 1.1 the point $\bar{x} = 0$ is, both, a sharp local minimizer and a strong KKT point, while the MFCQ is violated there. This raises the question whether in Theorem 1.2 the MFCQ can be replaced by some weaker constraint qualification.

In the following we use the (Bouligand) tangent cone

$$T(X,\bar{x}) = \{d \in \mathbb{R}^n \mid \exists\, t^k \searrow 0,\ (x^k) \subseteq X : \lim_k (x^k - \bar{x})/t^k = d\}$$

to X at $\bar{x}$. The proof of the following result employs similar ideas as the ones of [7, Th. 4.1] and [1, Th. 3.2].

Lemma 3.1. *A point $\bar{x}$ is a sharp local minimizer of P if and only if $\bar{x} \in X$ and $C_{\leq}(f,\bar{x}) \cap T(X,\bar{x}) = \{0\}$ hold.*

Proof. Let $\bar{x} \in X$ not be a sharp local minimizer of P. Then for each $k \in \mathbb{N}$ there exists some $x^k \in X$ with $\|x^k - \bar{x}\| \leq 1/k$ and $f(x^k) < f(\bar{x}) + (1/k)\|x^k - \bar{x}\|$. The sequence (t^k) with $t^k = \|x^k - \bar{x}\|$ satisfies $t^k \searrow 0$ and, by the compactness of the unit sphere, without loss of generality the sequence of directions $d^k = (x^k - \bar{x})/t^k$ converges to some $d \in T(X,\bar{x})$ with $\|d\| = 1$. Moreover, the differentiability of f yields $\nabla f(\bar{x})^{\intercal} d = \lim_k (f(x^k) - f(\bar{x}))/t^k$ with

$(f(x^k) - f(\bar{x}))/t^k < 1/k$ for all k and, thus, $\nabla f(\bar{x})^\intercal d \leq 0$. This means that $d \neq 0$ lies in $C_{\leq}(f,\bar{x}) \cap T(X,\bar{x})$ and therefore $C_{\leq}(f,\bar{x}) \cap T(X,\bar{x}) \supsetneq \{0\}$ holds.

On the other hand, let $C_{\leq}(f,\bar{x}) \cap T(X,\bar{x}) \supsetneq \{0\}$ for $\bar{x} \in X$ and choose some $d \neq 0$ from $C_{\leq}(f,\bar{x}) \cap T(X,\bar{x})$. Then there exist some $t^k \searrow 0$ and $(x^k) \subseteq X$ with $d^k = (x^k - \bar{x})/t^k \to d$. Assume that $\bar{x}$ is a sharp local minimizer. Then, with some $\alpha > 0$, for all sufficiently large k we have $(f(x^k) - f(\bar{x}))/t^k \geq (\alpha \|x^k - \bar{x}\|)/t^k = \alpha \|d^k\|$. This yields the contradiction $0 \geq \nabla f(\bar{x})^\intercal d = \lim_k (f(x^k) - f(\bar{x}))/t^k \geq \alpha \|d\| > 0$. □

Example 3.2. *In Example 1.1 we have* $C_{\leq}(f,0) = \{d \in \mathbb{R}^2 \mid d_1 + d_2 \leq 0\}$ *and* $T(X,0) = X = (\mathbb{R}_{\geq} \times \{0\}) \cup (\{0\} \times \mathbb{R}_{\geq})$. *Therefore* $C_{\leq}(f,0) \cap T(X,0) = \{0\}$ *holds, and Lemma 3.1 shows that* $\bar{x} = 0$ *is a sharp local minimizer, without the need to specify* U *and* α.

As the relation $T(X,\bar{x}) \subseteq L(g,h,\bar{x})$ is true without further assumptions, the combination of Lemma 2.4 and Lemma 3.1 yields that being a strong KKT point is sufficient for $\bar{x} \in X$ to be a sharp local minimizer. If additionally the Abadie constraint qualification (ACQ) $L(g,h,\bar{x}) \subseteq T(X,\bar{x})$ holds at $\bar{x}$, then the same combination implies that being a strong KKT point is necessary for $\bar{x} \in X$ to be a sharp local minimizer. Therefore, in Theorem 1.2 the MFCQ can be replaced by the weaker ACQ.

However, in Example 1.1 also the ACQ is violated at $\bar{x}$ while this point is, both, a sharp local minimizer and a strong KKT point. To formulate an even weaker constraint qualifcation, we state a reformulation of Lemma 3.1 which is analogous to the reformulation of Lemma 2.4 by Lemma 2.5. In fact, the proof runs along the same lines, using the closedness of the cone $T(X,\bar{x})$.

Lemma 3.3. *A point* $\bar{x}$ *is a sharp local minimizer of* P *if and only if* $\bar{x} \in X$ *and* $-\nabla f(\bar{x}) \in \operatorname{int} T^\circ(X,\bar{x})$ *hold.*

Lemma 2.5 and Lemma 3.3 imply that also under the condition

$$\operatorname{int} T^\circ(X,\bar{x}) \subseteq \operatorname{int} L^\circ(g,h,\bar{x}) \tag{10}$$

being a strong KKT point is necessary for $\bar{x} \in X$ to be a sharp local minimizer. Condition (10) is a consequence of the Guignard constraint qualification (GCQ) $T^\circ(X,\bar{x}) \subseteq L^\circ(g,h,\bar{x})$ which, in turn, follows from the ACQ. Hence, in Theorem 1.2 the MFCQ may even be replaced by the GCQ. The GCQ does hold at $\bar{x} = 0$ in Example 1.1, since the sets $T^\circ(X,\bar{x})$ and $L^\circ(g,\bar{x})$ both coincide with $\{v \in \mathbb{R}^2 | \, v \leq 0\}$.

We have thus shown the following result.

Theorem 3.4. *Let the GCQ hold at* $\bar{x} \in X$. *Then* $\bar{x}$ *is a sharp local minimizer of* P *if and only if it is a strong KKT point.*

As (10) is a consequence of the GCQ, one may ask why in Theorem 3.4 the GCQ is not replaced by this potentially weaker condition. Indeed, in the subsequent section we will show that the GCQ is the weakest condition under which sharp local minimality can be characterized by the strong KKT property. Therefore the condition (10) cannot be strictly weaker than the GCQ.

The following explicit proof of this result sheds some more light on the underlying reason. Observe that $T^\circ(X,\bar{x})$ coincides with the regular normal cone $\widehat{N}(X,\bar{x})$ to X at $\bar{x}$ [11], so that the GCQ at $\bar{x}$ may be rewritten as $\widehat{N}(X,\bar{x}) \subseteq L^\circ(g,h,\bar{x})$, and (10) as $\operatorname{int} \widehat{N}(X,\bar{x}) \subseteq \operatorname{int} L^\circ(g,h,\bar{x})$.

In this notation Lemma 3.3 states that $\bar{x}$ is a sharp local minimizer if and only if $\bar{x} \in X$ and $-\nabla f(\bar{x}) \in \operatorname{int} \widehat{N}(X,\bar{x})$ hold. Thus we are only interested in the case $\operatorname{int} \widehat{N}(X,\bar{x}) \neq \emptyset$.

Proposition 3.5. *For* $\operatorname{int} \widehat{N}(X,\bar{x}) \neq \emptyset$ *the condition* (10) *implies the GCQ at* $\bar{x}$.

Proof. For $\operatorname{int} \widehat{N}(X,\bar{x}) \neq \emptyset$ the convex set $\widehat{N}(X,\bar{x})$ and, under (10), also $L^\circ(g,h,\bar{x})$ are full dimensional, so that their relative interiors coincide with their interiors. Therefore [13, Th. 6.3] implies $\operatorname{cl}\operatorname{int}\widehat{N}(X,\bar{x}) = \operatorname{cl}\widehat{N}(X,\bar{x}) = \widehat{N}(X,\bar{x})$ as well as $\operatorname{cl}\operatorname{int} L^\circ(g,h,\bar{x}) = \operatorname{cl} L^\circ(g,h,\bar{x}) = L^\circ(g,h,\bar{x})$, where the respective second identities follow from the closedness of polar cones. Since (10) yields $\operatorname{cl}\operatorname{int}\widehat{N}(X,\bar{x}) \subseteq \operatorname{cl}\operatorname{int} L^\circ(g,h,\bar{x})$, the GCQ follows. □

We remark that (10) is trivially fulfilled at any $\bar{x} \in X$ with $\operatorname{int}\widehat{N}(X,\bar{x}) = \emptyset$. In this case the GCQ may be violated at $\bar{x}$, as the example $X = \{x \in \mathbb{R}^2 \mid x_1^3 \le 0\}$ with $\bar{x} = 0$ shows.

4. The weakest constraint qualification

Recall from [14] that the GCQ is the weakest condition under which a point $\bar{x} \in X$ is a KKT point for every $f \in \mathcal{F}(X,\bar{x})$, where $\mathcal{F}(X,\bar{x})$ denotes the set of at $\bar{x}$ continuously differentiable functions possessing $\bar{x}$ as a local minimizer on X.

In view of Theorem 3.4 and Proposition 3.5 the GCQ at $\bar{x}$ may also be the weakest condition under which a point $\bar{x} \in X$ is a strong KKT point for every $f \in \mathcal{F}_s(X,\bar{x})$, where $\mathcal{F}_s(X,\bar{x})$ denotes the set of at $\bar{x}$ continuously differentiable functions possessing $\bar{x}$ as a *sharp* local minimizer on X. The following result verifies this.

Theorem 4.1. *The weakest condition under which a point $\bar{x} \in X$ is a strong KKT point for every $f \in \mathcal{F}_s(X,\bar{x})$ is the GCQ at $\bar{x}$.*

Proof. By Theorem 3.4 the GCQ at $\bar{x}$ is some condition under which $\bar{x}$ is a strong KKT point for every $f \in \mathcal{F}_s(X,\bar{x})$. On the other hand, let $\bar{x}$ be a strong KKT point for every $f \in \mathcal{F}_s(X,\bar{x})$. We will show that then the GCQ $\widehat{N}(X,\bar{x}) \subseteq L^\circ(g,h,\bar{x})$ necessarily holds at $\bar{x}$.

In a first step we show that under the current assumption the GCQ trivially holds at $\bar{x}$ in the case $\operatorname{int}\widehat{N}(X,\bar{x}) = \emptyset$, since then no continuously differentiable function f can possess $\bar{x}$ as a sharp local minimizer. Indeed, Lemma 3.3 would then imply $-\nabla f(\bar{x}) \in \operatorname{int}\widehat{N}(X,\bar{x})$, in contradiction to $\operatorname{int}\widehat{N}(X,\bar{x}) = \emptyset$. This implies $\mathcal{F}_s(X,\bar{x}) = \emptyset$ and, therefore, the trivial correctness of the assertion.

In the remainder of the proof let $\operatorname{int}\widehat{N}(X,\bar{x}) \neq \emptyset$. Then by Proposition 3.5 it is sufficient to show (10) for the proof of the GCQ at $\bar{x}$. Indeed, choose $v \in \operatorname{int}\widehat{N}(X,\bar{x})$. By Lemma 3.3 the linear function $f_v(x) = -v^\intercal x$ possesses $\bar{x}$ as a sharp local minimizer on X, implying $f_v \in \mathcal{F}_s(X,\bar{x})$. By assumption $\bar{x}$ is then also a strong KKT point of f_v on X, that is, $\operatorname{rank}(\nabla g_A(\bar{x}), \nabla h(\bar{x})) = n$ holds and there exist $\lambda > 0$ and μ with

$$v = \nabla g_A(\bar{x})\lambda + \nabla h(\bar{x})\mu.$$

For all $d \in L(g,h,\bar{x}) \setminus \{0\}$ this implies

$$v^\intercal d = \lambda^\intercal \nabla g_A(\bar{x})^\intercal d + \mu^\intercal \nabla h(\bar{x})^\intercal d \le 0.$$

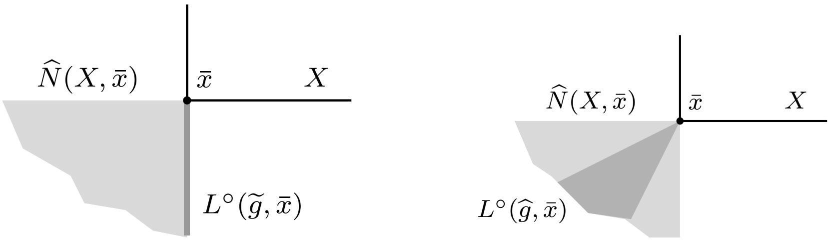
Moreover, in the case $v^\intercal d = 0$ we would obtain

$$0 = \lambda^\intercal \nabla g_A(\bar{x})^\intercal d + \mu^\intercal \nabla h(\bar{x})^\intercal d$$

which, in view of $\lambda > 0$, is only possible for $\nabla g_A(\bar{x})^\intercal d = 0$. Therefore we arrive at $d^\intercal(\nabla g_A(\bar{x}), \nabla h(\bar{x})) = 0$. The rank condition (3) implies $d = 0$, which contradicts the choice $d \neq 0$. We have thus shown $v^\intercal d < 0$ for all $d \in L(g,h,\bar{x}) \setminus \{0\}$. By (8) this means $v \in \operatorname{int} L^\circ(g,h,\bar{x})$, so that (10) is shown, and the proof is complete. □

The following examples illustrate two situations of a sharp local minimizer at which the GCQ is violated.

Example 4.2. *In Example 1.1 let us replace the function $g_1(x) = -x_1$ by $\widetilde{g}_1(x) = -x_1^3$ and set $\widetilde{g}_2 := g_2$, $\widetilde{g}_3 := g_3$. Then the set X remains unchanged, so that $\bar{x} = 0$ is still a sharp local*



(a) Example 4.2 (b) Example 4.3

Figure 1.: Normal cones and polars of linearization cones

minimizer, and we still have $\widehat{N}(X,\bar{x}) = \{v \in \mathbb{R}^2 |\, v \le 0\}$. *On the other hand, the linearization cone becomes* $L(\widetilde{g},\bar{x}) = \{d \in \mathbb{R}^2 |\, d_2 \ge 0\}$ *with the polar cone* $L^\circ(\widetilde{g},\bar{x}) = \{v \in \mathbb{R}^2 |\, v_1 = 0,\ v_2 \le 0\}$ *(Fig. 1a). Hence the GCQ is violated at* $\bar{x}$.

By Theorem 4.1 there exists at least one function $f \in \mathcal{F}_s(X,\bar{x})$ *for which* $\bar{x}$ *is not a strong KKT point. Indeed the function* $f(x) = x_1 + x_2$ *from Example 1.1 serves this purpose, where neither condition* (3) *nor* (4) *can be fulfilled (*$\bar{x}$ *is only a Fritz-John point).*

In fact, by Lemma 3.3 the set $\mathcal{F}_s(X,\bar{x})$ *contains exactly the at* $\bar{x}$ *continuously differentiable functions* f *with* $\nabla f(\bar{x}) > 0$, *but due to* $\operatorname{int} L^\circ(\widetilde{g},\bar{x}) = \emptyset$ *and Lemma 2.5,* $\bar{x}$ *is not a strong KKT point for any of them. For the latter conclusion one may also argue that* (3) *is violated independently of the choice of* f, *so that* $\operatorname{int} L^\circ(\widetilde{g},\bar{x})$ *is empty and* $\bar{x}$ *can thus not be a strong KKT point for any* f. *With respect to Remark 2.7 note, however, that* $\bar{x}$ *is a strict complementary KKT point for* $f(x) = x_2$ *on* X.

The next example verifies that the violation of GCQ at $\bar{x}$ does not force the violation of (3) (as in Example 4.2), so that $\operatorname{int} L^\circ(g,\bar{x})$ is then nonempty and $\bar{x}$ is a strong KKT point at least for some $f \in \mathcal{F}_s(X,\bar{x})$.

Example 4.3. *In Example 1.1 we replace the functions* g_1 *and* g_2 *by* $\widehat{g}_1(x) = -x_1^3$, $\widehat{g}_2(x) = -x_2^3$, *set* $\widehat{g}_3(x) := g_3(x) = x_1 x_2$, *and add the constraints* $\widehat{g}_4(x) = -2x_1 - x_2 \le 0$ *and* $\widehat{g}_5(x) = -x_1 - 2x_2 \le 0$. *The set* X *then still remains the same,* $\bar{x} = 0$ *is a sharp local minimizer, and* $\widehat{N}(X,\bar{x}) = \{v \in \mathbb{R}^2 |\, v \le 0\}$ *holds. The linearization cone, however, becomes* $L(\widehat{g},\bar{x}) = \{d \in \mathbb{R}^2 |\, \widehat{g}_4(d) \le 0,\ \widehat{g}_5(d) \le 0\}$ *with the polar cone* $L^\circ(\widehat{g},\bar{x}) = \{v \in \mathbb{R}^2 |\, 2v_1 \le v_2 \le v_1/2\}$ *(Fig. 1b). Hence again the GCQ is violated at* $\bar{x}$.

As in Example 4.2, by Lemma 3.3 the set $\mathcal{F}_s(X,\bar{x})$ *contains exactly the at* $\bar{x}$ *continuously differentiable functions* f *with* $\nabla f(\bar{x}) > 0$. *Moreover, by Lemma 2.5* $\bar{x}$ *is a strong KKT point if and only if* $-\nabla f(\bar{x}) \in \operatorname{int} L^\circ(\widetilde{g},\bar{x}) = \{v \in \mathbb{R}^2 |\, 2v_1 < v_2 < v_1/2\}$ *holds. Thus, for* $f(x) = x_1 + x_2$ *from Example 1.1* $\bar{x}$ *is a strong KKT point, while for* $\widehat{f}(x) = 3x_1 + x_2$ *with* $\widehat{f} \in \mathcal{F}_s(X,\bar{x})$ *it is not.*

5. Application to cardinality-constrained linear programs

In linear programming every unique minimizer is a vertex of the feasible set and, hence, satisfies the rank condition (3). Moreover, since the Goldman-Tucker theorem [15] states that every solvable linear program possesses a strictly complementary optimal point, any unique minimizer is strictly complementary and, thus, satisfies (4). Therefore, in linear programming every unique minimizer is a strong KKT point. An alternative line of arguments for this result uses that unique minimizers of linear programs are necessarily sharp. Since the ACQ and, thus, the GCQ are satisfied everywhere in a polyhedral set, by Theorem 3.4 the unique minimizers of linear programs are even characterized by the strong KKT property. The latter also implies that a basic optimal point is strictly complementary if and only if it is the only optimal point [16], since this may be rephrased as the fact that minimizers with (3) satisfy (4) if and only if they are unique, and since minimizers of linear programs are characterized as KKT points.

The above characterization of unique (global) minimizers as strong KKT points in the case of LPs also holds for more general problem classes in a local version, as formulated in the following result. Recall that a local minimizer $\bar{x}$ is called strict if $f(x) > f(\bar{x})$ holds for all $x \in X \cap U$ for some neighborhood U of $\bar{x}$. A global minimizer is strict if and only if it is unique. Every sharp local minimizer is strict, while a strict local minimizer need not be sharp.

The following theorem is an immediate consequence of Theorem 3.4.

Theorem 5.1. *Consider a nonlinear optimization problem P satisfying the following properties:*

a) The GCQ holds everywhere in the feasible set X.
b) Every strict local minimizer of P is sharp.

Then $\bar{x}$ is a strict local minimizer of P if and only if it is a strong KKT point.

From Example 1.1 one may expect that assumptions a and b from Theorem 5.1 hold for all linear programs with complementarity constraints (LPCCs [17,18]). The following example shows that this is not the case.

Example 5.2. *[19, Ex. 3] Let*

$$X = \{x \in \mathbb{R}^3 |\ x_1, x_2 \geq 0,\ x_1 x_2 \leq 0,\ -4x_1 + x_3 \leq 0,\ -4x_2 + x_3 \leq 0\}$$

and $\bar{x} = 0$. Then $T(X, \bar{x}) = X$ and

$$L(g, \bar{x}) = \{d \in \mathbb{R}^3 |\ d_1, d_2 \geq 0,\ -4d_1 + d_3 \leq 0,\ -4d_2 + d_3 \leq 0\}$$

hold. Since the vector $(-1, -1, 1)^\intercal$ lies in $T^\circ(X, \bar{x})$, but not in $L^\circ(g, \bar{x})$ (due to $(1, 1, 3)^\intercal \in L(g, \bar{x})$), the GCQ is violated at $\bar{x}$. In particular, the minimization of $f(x) = x_1 + x_2 - x_3$ over X constitutes an LPCC at whose unique global minimizer $\bar{x}$ the GCQ is violated. In fact, $\bar{x}$ is not even a KKT point, let alone a strong one.

It turns out, however, that an application relevant subclass of LPCCs can be treated by Theorem 5.1, namely cardinality-constrained linear programs (CCLPs). They possess the form

$$CCLP: \quad \min_{x \in \mathbb{R}^n} c^\intercal x \quad \text{s.t.} \quad Ax \leq \alpha,\ Bx = \beta,\ \|x\|_0 \leq \kappa,$$

where the number $\|x\|_0$ of nonzero entries of the vector x is bounded above by some $\kappa \in \{1, \ldots, n-1\}$. In [20] the continuous relaxation

$$\begin{aligned} RCCLP: \quad \min_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} c^\intercal x \quad \text{s.t.} \quad & Ax \leq \alpha,\ Bx = \beta, \\ & e^\intercal y \geq n - \kappa, \\ & 0 \leq y \leq e, \\ & x_i y_i = 0,\ i = 1, \ldots, n \end{aligned}$$

of a mixed-integer reformulation of $CCLP$ is studied, where e denotes the all-ones vector in $\mathbb{R}^n$. Note that $RCCLP$ is an LPCC if the system $Ax \leq \alpha$ includes nonnegativity constraints on x. By [20, Th. 3.2] a point $\bar{x}$ is a global minimizer of $CCLP$ if and only if there exists some $\bar{y} \in \mathbb{R}^n$ such that $(\bar{x}, \bar{y})$ is a global minimizer of $RCCLP$.

By [20, Cor. 4.5] the GCQ holds everywhere in the feasible set of $RCCLP$, so that we have the following result.

Lemma 5.3. *Every problem RCCLP satisfies condition a from Theorem 5.1.*

For the proof of condition b in Theorem 5.1 we recall the local patch structure of the feasible set Z of *RCCLP* presented in [20]. For $\bar z := (\bar x, \bar y) \in Z$ we define the active index sets

$$\begin{aligned}
I_{\pm 0}(\bar z) &= \{i \in \{1,\ldots,n\} |\ \bar x_i \neq 0,\ \bar y_i = 0\},\\
I_{00}(\bar z) &= \{i \in \{1,\ldots,n\} |\ \bar x_i = 0,\ \bar y_i = 0\},\\
I_{0+}(\bar z) &= \{i \in \{1,\ldots,n\} |\ \bar x_i = 0,\ \bar y_i \in (0,1)\},\\
I_{01}(\bar z) &= \{i \in \{1,\ldots,n\} |\ \bar x_i = 0,\ \bar y_i = 1\},
\end{aligned}$$

and for each $I \subseteq I_{00}(\bar z)$ the local patch

$$\begin{aligned}
Z_I(\bar z) \ = \ \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n |\ & Ax \le \alpha,\ Bx = \beta,\ e^\intercal y \ge n - \kappa,\\
& x_i = 0,\ y_i \in [0,1],\ i \in I_{0+}(\bar z) \cup I_{01}(\bar z) \cup I,\\
& y_i = 0,\ i \in I_{\pm 0}(\bar z) \cup (I_{00}(\bar z) \setminus I)\}.
\end{aligned}$$

Lemma 5.4. *[20, Prop. 4.1, Lem. 4.2] Every $\bar z \in Z$ possesses some neighborhood U with*

$$U \cap Z \ = \ U \cap \bigcup_{I \subseteq I_{00}(\bar z)} Z_I(\bar z), \tag{11}$$

and the tangent cone to Z at $\bar z$ satisfies

$$T(Z, \bar z) \ = \ \bigcup_{I \subseteq I_{00}(\bar z)} T(Z_I(\bar z), \bar z). \tag{12}$$

Lemma 5.5. *Every problem RCCLP satisfies condition b from Theorem 5.1.*

Proof. Let $\bar z = (\bar x, \bar y)$ be a strict local minimizer of *RCCLP*. Then the standard stationarity condition for local minimizers of nonlinear optimization problems (i.e., B-stationarity in the terminology of MPCCs) yields $-c \in \widehat{N}(Z, \bar z)$. For $-c \in \operatorname{int} \widehat{N}(Z, \bar z)$ Lemma 3.3 implies that $\bar z$ is a sharp local minimizer. Thus it remains to show that $-c \in \operatorname{bd} \widehat{N}(Z, \bar z)$ results in a contradiction to $\bar z$ being a strict local minimizer.

Indeed, by

$$\operatorname{int} \widehat{N}(Z, \bar z) = \operatorname{int} T^\circ(Z, \bar z) = \{v \in \mathbb{R}^n \mid v^\intercal d < 0\, \forall d \in T(Z, \bar z) \setminus \{0\}\}$$

and the closedness of $\widehat{N}(Z, \bar z)$, from $-c \in \operatorname{bd} \widehat{N}(Z, \bar z)$ we obtain the existence of some $d \in T(Z, \bar z) \setminus \{0\}$ with $c^\intercal d = 0$. In view of (12) there exists some $I \subseteq I_{00}(\bar z)$ with $d \in T(Z_I(\bar z), \bar z)$. Since the patch $Z_I(\bar z)$ is a polyhedral set, together with (11) we obtain $\bar z + td \in Z_I(\bar z) \subseteq Z$ for all sufficiently small $t > 0$. Due to $c^\intercal(\bar z + td) = c^\intercal \bar z$ this rules out that $\bar z$ is a strict local minimizer. □

Lemma 5.3 and Lemma 5.5 together with Theorem 5.1 yield the main result of this section.

Theorem 5.6. *In any problem RCCLP the set of strict local minimizers coincides with the set of strong KKT points.*

From an application point of view a result would be even more interesting which states that the set of strict local minimizers of any problem *CCLP* coincides with the set of strong KKT points of *RCCLP*. However, this cannot be expected, since [20, Ex. 2, Ex. 3] show that *RCCLP* may possess spurious local minimizers, that is, local minimizers $(\bar x, \bar y)$ for which $\bar x$ is not a local minimizer of *CCLP*. While these examples employ a nonlinear objective function, the following example shows by a similar construction that this effect persists for CCLPs, and also that a sharp global minimizer of *CCLP* does not need to correspond to a sharp global minimizer of *RCCLP*.

Example 5.7. *The problem*

$$CCLP: \qquad \min_{x\in\mathbb{R}^3} x_1 \quad s.t. \quad \|x\|_1 \le 1,\ \|x\|_0 \le 2$$

possesses the sharp global minimizer $\bar{x} = (-1,0,0)^\intercal$. *Any corresponding* $\bar{y} \in \mathbb{R}^3$ *such that* $(\bar{x},\bar{y})$ *is a global minimizer of*

$$RCCLP: \qquad \min_{(x,y)\in\mathbb{R}^3\times\mathbb{R}^3} x_1 \quad s.t. \quad \|x\|_1 \le 1,\ e^\intercal y \ge 1,\ 0 \le y \le e,\ x_i y_i = 0,\ i=1,\ldots,n,$$

needs to satisfy $\bar{y}_1 = 0$, $\bar{y}_2, \bar{y}_3 \in [0,1]$ *and* $\bar{y}_2 + \bar{y}_3 \ge 1$. *Since the choices of* $\bar{y}_2$ *and* $\bar{y}_3$ *are nonunique,* $(\bar{x},\bar{y})$ *is a nonstrict global minimizer of RCCLP and therefore not a sharp global minimizer. More explicitly, for no* $\bar{y} \in \mathbb{R}^3$ *such that* $(\bar{x},\bar{y})$ *is a global minimizer of RCCLP there exist a neighborhood* U *of* $(\bar{x},\bar{y})$ *and some* $\alpha > 0$ *such that* $f(x) \ge f(\bar{x}) + \alpha\|(x,y)-(\bar{x},\bar{y})\|$ *holds for all* $(x,y) \in Z \cap U$.

Moreover, the point $(\widetilde{x},\widetilde{y})$ *with* $\widetilde{x} = (0,0,0)^\intercal$ *and* $\widetilde{y} = (1,0,0)^\intercal$ *is feasible for RCCLP with objective value* $f(\widetilde{x}) = 0$, *and for every feasible point* (x,y) *from a sufficiently small neighborhood of* $(\widetilde{x},\widetilde{y})$ *the condition* $y_1 \ne 0$ *enforces* $x_1 = 0$ *and, therefore* $f(x) = 0 = f(\widetilde{x})$. *Hence* $(\widetilde{x},\widetilde{y})$ *is a local minimizer of RCCLP, while it is easily seen that* $\widetilde{x}$ *is not a local minimizer of CCLP.*

In Example 5.7 the cardinality constraint $\|x\|_0 \le 2$ is inactive at $\bar{x}$. As observed in [20, Prop. 3.5], better results can be formulated for feasible points $x \in X$ of *CCLP* with $\|x\|_0 = \kappa$. This is due to the fact that, with $S = \{i \in I|\, x_i \ne 0\}$ denoting the support of x, for $(x,y) \in Z$ the constraints of *RCCLP* imply $y_S = 0$, $0 \le y_{S^c} \le e$ and $e^\intercal y_{S^c} \ge n-\kappa$, where the value $e^\intercal y_{S^c}$ ranges in the interval $[0,|S^c|] = [0, n-\|x\|_0]$. Therefore, in the case $\|x\|_0 = \kappa$ the point y is uniquely determined to $y(x)$ with $y_S(x) = 0$ and $y_{S^c}(x) = e$. On the other hand, for $\|x\|_0 < \kappa$ the condition $(x,y) \in Z$ possesses more than one solution y.

By [20, Th. 3.4] for every local minimizer x of *CCLP* there exists some $y \in \mathbb{R}^n$ such that (x,y) is a local minimizer of *RCCLP*. In the case $\|x\|_0 = \kappa$ this means that $(x,y(x))$ is a local minimizer of *RCCLP*. Moreover, by [20, Th. 3.6] for any local minimizer (x,y) with $\|x\|_0 = \kappa$ the point x is a local minimizer of *CCLP*. Therefore, a point x with $\|x\|_0 = \kappa$ is a local minimizer of *CCLP* if and only if $(x,y(x))$ is a local minimizer of *RCCLP*. Strengthening the latter statement to strict local minimizers yields the following result.

Corollary 5.8. *For any problem CCLP the set of strict local minimizers* $\bar{x}$ *with* $\|\bar{x}\|_0 = \kappa$ *coincides with the set of strong KKT points* $(\bar{x},\bar{y})$ *of RCCLP with* $\|\bar{x}\|_0 = \kappa$.

Proof. In view of Theorem 5.6 the set of strong KKT points $(\bar{x},\bar{y})$ of *RCCLP* with $\|\bar{x}\|_0 = \kappa$ coincides with the set of strict local minimizers $(\bar{x},\bar{y})$ of *RCCLP* with $\|\bar{x}\|_0 = \kappa$. Hence it remains to show the mentioned correspondence between strict local minimizers of *CCLP* and *RCCLP*. The proof of this part uses similar arguments as the ones presented in [20] .

Indeed, let $\bar{x}$ be a strict local minimizer of *CCLP* with $\|\bar{x}\|_0 = \kappa$. Then there exists a neighborhood U of $\bar{x}$ such that $f(x) > f(\bar{x})$ holds for all $x \in U \cap P$ with $\|x\|_0 \le \kappa$, where we put $P = \{x \in \mathbb{R}^n \mid Ax \le \alpha,\ Bx = \beta\}$. As seen above, the point $(\bar{x}, y(\bar{x}))$ is feasible for *RCCLP*. With the neighborhood $V := \{y \in \mathbb{R}^n \mid \|y - y(\bar{x})\|_\infty < 1/2\}$ of $y(\bar{x})$ all points $(x,y) \in (U \times V) \cap Z$ satisfy $x \in U \cap P$ and $y_i \ne 0$ for all $i \in S^c$. The latter implies $x_{S^c} = 0$ and therefore $\|x\|_0 \le \kappa$. By assumption we have $f(x) > f(\bar{x})$, so that $(\bar{x}, y(\bar{x}))$ is a strict local minimizer of *RCCLP* with $\|\bar{x}\|_0 = \kappa$.

On the other hand, let $(\bar{x},\bar{y})$ be a strict local minimizer of *RCCLP* with $\|\bar{x}\|_0 = \kappa$. This implies $\bar{y} = y(\bar{x})$, and there exist neighborhoods U of $\bar{x}$ and V of $y(\bar{x})$ with $f(x) > f(\bar{x})$ for all $(x,y) \in (U \times V) \cap Z$. We will show that $f(x) > f(\bar{x})$ holds for all $x \in U \cap P$ with $\|x\|_0 \le \kappa$, which completes the proof. In fact, for sufficiently small U all $x \in U$ fulfill $x_i \ne 0$, $i \in S$. This yields $\|x\|_0 = \kappa$ for all $x \in U$ with $\|x\|_0 \le \kappa$, and the supports of x and $\bar{x}$ need to coincide. As a consequence, also $y(x)$ and $y(\bar{x})$ coincide, so that all $x \in U \cap P$ with $\|x\|_0 \le \kappa$ fulfill $(x, y(x)) = (x, y(\bar{x})) \in (U \times V) \cap Z$, and $f(x) > f(\bar{x})$ follows. □

We mention that generically the cardinality constraint is active at minimizers of cardinality-constrained nonlinear optimization problems with twice continuously differentiable defining functions [21, Th. 4]. While this does not necessarily imply the same result for CCLPs, it suggests that also the cardinality assumption of Corollary 5.8 may be satisfied for CCLPs with defining functions in general position and can, thus, be considered a weak assumption.

6. Final remarks

The nonuniqueness of the minimal point set of *RCCLP* in Example 5.7 suggests to study extensions of the present investigation to the concept of weak sharp minima [22], which takes nonuniqueness into account. A different route to the study of sharp minimizers of CCLPs via RCCLPs would be the introduction of partial sharpness for problems which depend on two groups of decision variables, x and y, where the first order growth condition is only measured with respect to x. We leave such extensions to future research.

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