A Survey on Bilevel Optimization Under Uncertainty

Yasmine Beck, Ivana Ljubić, and Martin Schmidt

Abstract. Bilevel optimization is a very active field of applied mathematics. The main reason is that bilevel optimization problems can serve as a powerful tool for modeling hierarchical decision making processes. This ability, however, also makes the resulting problems challenging to solve—both in theory and practice. Fortunately, there have been significant algorithmic advances in the field of bilevel optimization so that we can solve much larger and also more complicated problems today compared to what was possible to solve two decades ago. This results in more and more challenging bilevel problems that researchers try to solve today. This survey gives a detailed overview of one of these more challenging classes of bilevel problems: bilevel optimization under uncertainty. We review the classic ways of addressing uncertainties in bilevel optimization using stochastic or robust techniques. Moreover, we highlight that the sources of uncertainty in bilevel optimization are much richer than for usual, i.e., single-level, problems since not only the problem's data can be uncertain but also the (observation of the) decisions of the two players can be subject to uncertainty. We thus also review the field of bilevel optimization under limited observability, the area of problems considering only near-optimal decisions, and discuss intermediate solution concepts between the optimistic and pessimistic cases. Finally, we also review the rich literature on applications studied using uncertain bilevel problems such as in energy, for interdiction games and security applications, in management sciences, and networks.

1. Introduction

Bilevel optimization is a rather young field of research that dates back to the early publications by Bracken and McGill (1973) as well as Candler and Norton (1977), having its game-theoretic foundations dating back to the seminal works by von Stackelberg (1934, 1952). While being very powerful modeling tools that allow to consider hierarchical decision making processes, bilevel optimization models are also very hard to solve—both in theory and practice. For instance, NP-hardness (Jeroslow 1985) and strong NP-hardness (Hansen et al. 1992) have been shown in the 1980s and early 1990s. The intrinsic hardness of bilevel optimization leads to the fact that, on the one hand, the field has been propelled theoretically first (see, e.g., the seminal textbook by Dempe (2002) and the more recent book by Dempe, Kalashnikov, et al. (2015)) but that computational bilevel optimization still has been in its infancy until the late 2000s. Since then, several innovative works pushed the computational study of these problems so that we can solve relevant practical instances of realistic size today; see Kleinert et al. (2021) for a very recent survey on this and related topics as well as the annotated bibliography on bilevel optimization by Dempe (2020).

Rather naturally, the operations research, mathematics, engineering, and economics communities, which all use bilevel optimization to model and solve real-world problems in their respective fields, started to study more and more complicated

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bilevel problems. These more complicated problems are even harder than "usual", e.g., continuous and maybe linear, bilevel problems since they introduce different aspects that make the resulting problems both more challenging in theory and practice—among others: (i) mixed-integer aspects, in particular in the lower-level problem, (ii) nonlinearities and nonconvexities both in the upper- and the lower-level problem, or (iii) large-scale instances that could not have been solved a few decades ago.

Additionally, and this leads us to the main topic of this survey, more and more researchers from all of the above mentioned communities started to study bilevel optimization problems under uncertainty. In classic, i.e., single-level, optimization, there have been mainly two paths to address uncertainty in optimization models: stochastic optimization (Birge and Louveaux 2011; Kall and Wallace 1994) and robust optimization (Ben-Tal, El Ghaoui, et al. 2009; Ben-Tal and Nemirovski 1998; Bertsimas, Brown, et al. 2011; Soyster 1973). The same two paths have been followed as well in bilevel optimization starting from the 1990s on.

However, the sources of uncertainty are much richer in bilevel optimization compared to usual, i.e., single-level, optimization. To make this more concrete, a linear optimization problem $\min\{c^\top x \colon Ax \geq b\}$ can only be subject to uncertainty due to uncertainties in the problem's data c, A, and b. Throughout this survey, we will denote this setting as *data uncertainty*. Moreover, bilevel optimization may be subject to an additional source of uncertainty, which is due to its nature that combines two different decision makers in one model. Hence, there can be further uncertainty involved either if the leader is not sure about the reaction of the follower or if the follower is not certain about the observed leader's decision. We will denote this additional type of uncertainty as *decision uncertainty*. Obviously, decision uncertainty does not play any role in single-level optimization since only one decision maker is involved.

Both data as well as decision uncertainty can—and maybe should—be considered under the wider umbrella of what economists call *bounded rationality*; see, e.g., Rubinstein (1998) and Simon (1972). In economics, *rationality* usually means that the different agents of a system (e.g., the leader and the follower in bilevel optimization or in a Stackelberg game) act as optimizers, meaning that they all implement a fully rational decision process that consists in solving an optimization problem to global optimality while knowing all of the data required to parameterize the given instance of the problem to be solved. Moreover, in a game-theoretical context (as bilevel optimization naturally is), a fully rational decision process also needs to include that the decision of the other players can either be observed or anticipated perfectly.

This point of view leads to a wider scope for bilevel optimization under uncertainty than it can be the case for single-level optimization. Besides the classic topic of data uncertainty, bilevel optimization under uncertainty may cover the following aspects:

(i) The leader may be uncertain about her[1] anticipation of the follower's rational reaction and, thus, may want to hedge against this uncertainty; see, e.g., Besançon et al. (2019).

(ii) As an extreme case of the former aspect it may be the case that the upper-level player knows that the follower will play against her. This is the setting of a pessimistic bilevel optimization problem, which is rather naturally connected to the field of robust optimization; see, e.g., Wiesemann et al. (2013). However, if the level of cooperation or confrontation of the follower

[1] According to the experimental results collected by the male author of this survey while assigning work to co-authors during the writing process, we decided to use "her" for the leader and "his" for the follower throughout the paper.

is not known, this leads to intermediate cases in between the optimistic and the pessimistic case; see, e.g., Aboussoror and Loridan (1995) and Mallozzi and Morgan (1996). Rather obviously, this is another realization of decision uncertainty.

(iii) It can also be the case that the leader can anticipate the rational reaction of the follower but that the follower is not able to perfectly observe the leader's decision. In this case, the follower—if aware of this aspect—usually tries to hedge against this uncertainty (Bagwell 1995; Beck and Schmidt 2021; van Damme and Hurkens 1997).

(iv) Even if all data and the rational reaction of the follower is known and even if the leader can, in principle, fully anticipate the optimal reaction of the follower, it might still be the case that limited intellectual or computational resources make it impossible that globally optimal decisions are taken. In contrast, only approximately optimal or heuristic answers of, e.g., the follower need to be considered, imposing the challenge that the leader does not know which heuristic or which approximation is applied by the follower. As a good primer in this context, we refer to the recent paper by Zare, Prokopyev, et al. (2020).

This list is not comprehensive but should make clear how much more diverse the sources of uncertainty can be in bilevel optimization as compared to single-level optimization.

Throughout this survey, we will highlight different aspects of bounded rationality as it has been roughly discussed above. Most of the papers, but not all of them, that we will review are concerned with the typical setting of data uncertainty. However, the interest of the mathematical optimization as well operations research community in decision uncertainty is growing. The survey thus has two main goals. First, to almost comprehensively describe the state-of-the-art of bilevel optimization under uncertainty. Second, to also view the existing research under a bounded-rationality lens to put the existing literature into a broader (game-theoretic or economic) context as well as to open doors to future research that maybe would stay locked—or not even seen—while using a different lens.

The remainder of this survey is structured as follows. In Section 2, we define the overall problem statement and discuss both data as well as decision uncertainty using illustrating examples. Afterward, in Section 3, we then discuss the existing (and mostly theoretical) literature on bilevel optimization under uncertainty along the lines on how uncertainty is modeled. Thus, we explicitly consider stochastic bilevel problems, bilevel problems with robust modeling of data uncertainty, bilevel problems with near-optimal lower-level decisions, limited observability of decisions of the leader, and the field of intermediate solution concepts between the optimistic and pessimistic cases. In Section 4, we then review papers on bilevel optimization under uncertainty that are devoted to specific applications. Here, we focus on applications from the field of energy, security, management science, and networks. We close the paper with some concluding words in Section 5, where we also mention open questions and other possible directions for future research. Throughout the survey, we assume that the reader is familiar with standard concepts of robust and stochastic optimization.

2. General Problem Statement

We study bilevel problems of the general form

$$\text{“}\min_{x \in X}\text{”} \quad F(x, y) \tag{1a}$$
$$\text{s.t.} \quad G(x, y) \geq 0, \tag{1b}$$
$$y \in S(x), \tag{1c}$$

where $S(x)$ denotes the set of optimal solutions of the x-parameterized problem

$$\min_{y \in Y} \quad f(x, y) \tag{2a}$$
$$\text{s.t.} \quad g(x, y) \geq 0. \tag{2b}$$

We refer to Problem (1) as the upper-level (or the leader's) problem and to Problem (2) as the lower-level (or the follower's) problem. Moreover, the variables $x \in X$ and $y \in Y$ are called the leader's and the follower's variables, respectively. The sets $X \subseteq \mathbb{R}^{n_x}$ and $Y \subseteq \mathbb{R}^{n_y}$ can be used to denote integrality constraints. The objective functions are given by $F, f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ and the constraint functions by $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^m$ as well as $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^\ell$. A summary of important notation used throughout this paper can also be found in Table 1. The quotation marks in (1a) express the ill-posedness of the bilevel problem in the case that the lower-level problem does not have a unique solution. To deal with this ambiguity, it is common to pursue either an optimistic or a pessimistic approach to bilevel optimization; see, e.g., Dempe (2002). For the ease of presentation, we focus on the optimistic setting at this point, i.e., we study

$$\min_{x,y} \quad F(x, y) \quad \text{s.t.} \quad G(x, y) \geq 0,\ x \in X,\ y \in S(x).$$

We consider bilevel problems of the above form, which are, however, affected by various kinds of uncertainty. This setting is relevant for many practical applications since uncertainty is an important aspect of bounded rationality; see, e.g., Simon (1972). In this survey article, we mainly distinguish between two types of uncertainty: data uncertainty and decision uncertainty.

2.1. Data Uncertainty.

Data uncertainty arises if, e.g., the lower-level player only has access to inaccurate or incomplete data. To illustrate this aspect, let us assume that the right-hand sides of the lower-level constraints are uncertain. For a feasible leader's decision x and a specific realization of the uncertainty u, the set of optimal follower's decisions is then given by

$$S(x, u) := \operatorname*{arg\,min}_{y \in Y} \left\{ f(x, y) \colon g(x, y) \geq z(u) \right\},$$

where $z(u) \in \mathbb{R}^\ell$ represents the lower-level right-hand side vector for the given uncertainty realization u. In mathematical optimization, it is common to use one of the following two variants to deal with data uncertainty.

(i) Uncertainties are assumed to take values in a given uncertainty set $\mathcal{U}$. Pursuing a robust approach, we hedge against the worst-case realization of the uncertainties w.r.t. the leader's optimal objective function value. For the ease of presentation, we assume that there are no coupling constraints, i.e., there are no upper-level constraints that explicitly depend on y. To this end, we define $\bar{X} := \{x \in X \colon G(x) \geq 0\}$ with $G : \mathbb{R}^{n_x} \to \mathbb{R}^m$. We then solve

$$\min_{x \in \bar{X}} \max_{u \in \mathcal{U}} \min_{y \in S(x,u)} \quad F(x, y).$$

Here, we consider the robust and optimistic case in which the leader may influence the follower's decision in her favor. However, the consideration of

TABLE 1. Central Notation.

Sets	
$X \subseteq \mathbb{R}^{n_x}$	Set of all admissible upper-level decisions
$\mathcal{X}(x) \subseteq \mathbb{R}^{n_x}$	(Imperfectly) perceived upper-level feasible region
$\bar{X} \subseteq \mathbb{R}^{n_x}$	Upper-level feasible region without coupling constraints
$Y \subseteq \mathbb{R}^{n_y}$	Set of all admissible lower-level decisions
$\mathcal{U}$	Uncertainty set
Ω	Finite set of possible scenarios
Variables	
$x \in X$	Upper-level decision
$\bar{x} \in \mathcal{X}(x)$	(Imperfectly) observed upper-level decision
$u \in \mathcal{U}$, $\omega \in \Omega$, z	Uncertainty realization
$y = y(x) \in Y$	Lower-level response for given x
$y(x, u)$, $y(x, \omega) \in Y$	Lower-level response for given x and uncertainty realization u or ω
Functions	
$F, f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$	Upper and lower-level objective functions
$G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^m$	Upper-level constraints
$g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^\ell$	Lower-level constraints
$\varphi : X \to \mathbb{R}$	Lower-level optimal-value function
Point-to-Set Mappings	
$Y(x)$	Lower-level feasible set for given x
$S(x)$	Set of optimal lower-level solutions for given x
$S(x, u)$, $S(x, z)$	Set of optimal lower-level solutions for given x and uncertainty realization u or z
$S(x, \varepsilon)$	Set of ε-optimal lower-level solutions for given x with $\varepsilon > 0$

other solution concepts is also possible for uncertain bilevel problems. For instance, the most conservative situation in which the leader anticipates a pessimistic follower is given by

$$\min_{x \in \bar{X}} \max_{u \in \mathcal{U}} \max_{y \in S(x,u)} \quad F(x, y).$$

Eventually, the application at hand dictates which model is appropriate.

(ii) Adopting a stochastic approach, it is assumed that the uncertainties can be described by a given probability distribution. Here, we hedge against uncertainties in a probabilistic sense by optimizing, e.g., the expected value. In line with all the existing literature on stochastic bilevel optimization (see also Section 3.1), we again focus on the setting without coupling constraints. Hence, we solve

$$\min_{x \in \bar{X}} \quad \mathbb{E}_u \left[\Phi_u(x)\right] \quad \text{with} \quad \Phi_u(x) := \min_{y \in S(x,u)} F(x, y).$$

In both cases, we consider the timing

$$\text{leader } x \quad \curvearrowright \quad \text{uncertainty } u \quad \curvearrowright \quad \text{follower } y = y(x, u).$$

This means that the leader first takes a here-and-now decision, i.e., without knowing the realization of uncertainty. Then, the uncertainty realizes and, finally, the follower decides in a wait-and-see manner, taking the leader's decision as well as the realization of the uncertainty into account. However, other timings are possible as well, for instance, if we consider problems of the form

$$\begin{aligned}\min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \geq 0,\ x \in X, \\ & y \in \arg\min_{\bar{y} \in Y} \left\{ f(x,\bar{y}) \colon g(x,\bar{y}) \geq z(u) \text{ for all } u \in \mathcal{U} \right\}.\end{aligned}$$

This is another robust bilevel optimization problem but this time, the leader takes a here-and-now decision and the follower also decides before the uncertainty realizes. The decision of the follower is robust in the sense that it is required to remain feasible for all possible realizations of the uncertainty. Hence, one considers the timing

$$\text{leader } x \quad \curvearrowright \quad \text{follower } y = y(x) \quad \curvearrowright \quad \text{uncertainty } u. \tag{3}$$

So far, we have discussed the case of lower-level right-hand side uncertainty. Of course, data uncertainty may also occur at other locations of the problem such as, e.g., in the upper-level problem's data or in the objective function of the follower. The following examples illustrate robust and stochastic approaches to deal with uncertain data at various locations of the lower-level problem.

Example 1. *Let us consider the linear bilevel problem*

$$\min_{x,y \in \mathbb{R}} \quad F(x,y) = x - 4y \tag{4a}$$
$$s.t. \quad x - y \geq -1, \tag{4b}$$
$$3x + y \geq 3, \tag{4c}$$
$$y \in S(x), \tag{4d}$$

where $S(x)$ denotes the set of optimal solutions of the x-parameterized lower-level problem

$$\min_{y \in \mathbb{R}} \quad f(x,y) = -0.1y \tag{5a}$$
$$s.t. \quad -2x + y \geq -7, \tag{5b}$$
$$-3x - 2y \geq -14, \tag{5c}$$
$$0 \leq y \leq 2.5. \tag{5d}$$

The problem is depicted in Figure 1 (left). The upper- and lower-level constraints are represented with dashed and solid lines, respectively. The optimal solution $(x,y) = (1.5, 2.5)$ of the deterministic bilevel problem (4) and (5) is illustrated by the thick dot. Suppose now that the lower-level objective function is uncertain. We follow a robust approach and assume that the follower decides in a here-and-now fashion, i.e., we consider the timing in (3). The uncertain objective function coefficient is assumed to take values in the uncertainty set $\mathcal{U} = \{-0.1 + \zeta \colon |\zeta| \leq 0.5\} = [-0.6, 0.4]$, which leads to a modified gradient of the lower-level objective function. This effect is shown in Figure 1 (right). The optimal solution of the uncertain bilevel problem is represented by the thick square. In particular, we obtain a completely different solution than in the deterministic case if we take data uncertainty into account. Let us further point out that, in this example, we do not have to distinguish between the optimistic and the pessimistic case since the solution of the (robust) lower-level problem is unique for every feasible x.

Example 2. *Consider the linear bilevel problem*

$$\min_{x,y \in \mathbb{R}} \quad F(x,y) = -y \quad s.t. \quad x \geq 0, \quad y \in S(x), \tag{6}$$

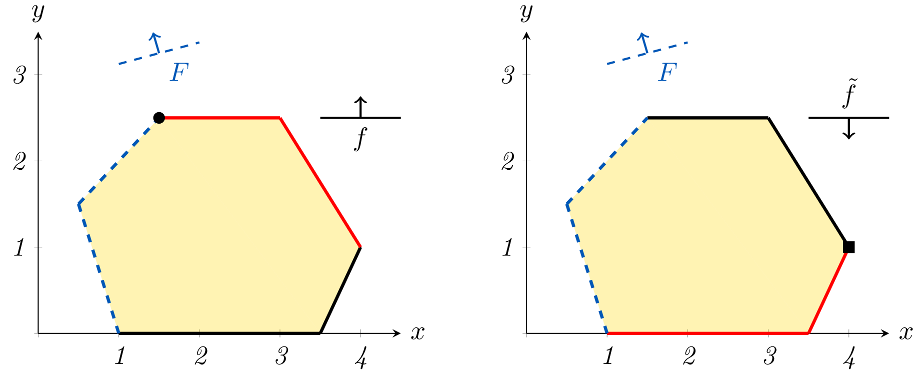

Figure 1. Both figures show the upper-level constraints (dashed blue lines), the lower-level constraints (solid black and red lines), the shared constraint set (yellow area), and the bilevel feasible set (solid red lines) of the bilevel problem (4) and (5). The deterministic variant of the problem is depicted on the left and the variant of the problem that accounts for a robust modeling of an uncertain lower-level objective function $\tilde{f}$ is given on the right.

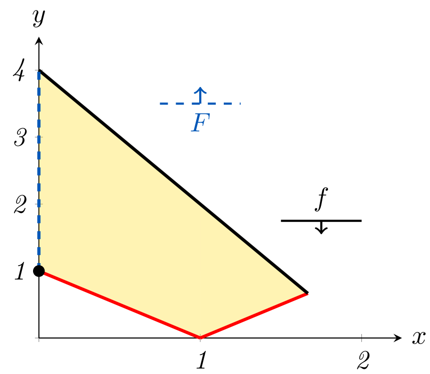

Figure 2. The upper-level constraint (dashed blue line), the lower-level constraints (solid black and red lines), the shared constraint set (yellow area), and the bilevel feasible set (solid red lines) of the bilevel problem (6) and (7).

where $S(x)$ denotes the set of optimal solutions of the x-parameterized lower-level problem

$$\min_{y \geq 0} \quad f(x, y) = y \tag{7a}$$

$$\text{s.t.} \quad x + y \geq 1, \tag{7b}$$

$$-x + y \geq -1, \tag{7c}$$

$$-2x - y \geq -4. \tag{7d}$$

The problem is depicted in Figure 2. The upper- and lower-level constraints are represented with dashed and solid lines, respectively. The unique optimal solution $(x, y) = (0, 1)$ of the deterministic bilevel problem (6) and (7) is illustrated by the thick dot. Suppose now that the right-hand side of Constraint (7c) is uncertain. We pursue a stochastic approach and assume that the right-hand side $b(\omega) \in \mathbb{R}$

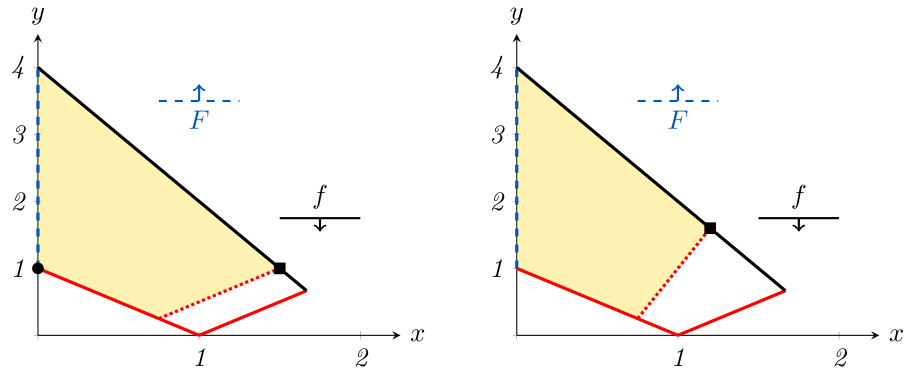

FIGURE 3. Variants of the bilevel problem (6) and (7) with lower-level right-hand side uncertainty (left) and uncertain lower-level constraint coefficients (right).

depends on the scenario $\omega \in \Omega = \{\omega^1, \omega^2\}$ *with* $b(\omega^1) = -1$ *and* $b(\omega^2) = -1/2$. *We further assume that both scenarios have probability* $p^1 = p^2 = 1/2$. *We start by considering each scenario individually. Note that the realization of* ω^1 *corresponds to the deterministic setting. Hence, the unique optimal solution for scenario* ω^1 *is given by* $(x, y^1) = (0, 1)$. *Here and in what follows, we set* $y^i = y(\omega^i)$ *for* $i = 1, 2$. *The realization of scenario* ω^2 *leads to a parallel shift of the uncertain lower-level constraint. This effect is shown in Figure 3 (left). It can also be seen that the solution is not unique anymore if scenario* ω^2 *is considered. Both* $(0, 1)$ *and* $(3/2, 1)$*—which are illustrated by the thick dot and the thick square, respectively—yield an optimal objective function value of* -1. *To hedge against lower-level right-hand side uncertainty, we optimize the expected value of the upper-level objective function, i.e., we solve*

$$\min_{x,y^1,y^2 \in \mathbb{R}} \quad -p^1 y^1 - p^2 y^2 \quad \text{s.t.} \quad x \geq 0,\ y^1 \in S(x, \omega^1),\ y^2 \in S(x, \omega^2). \tag{8}$$

The unique solution for the variant of Problem (6) *and* (7) *with lower-level right-hand side uncertainty is given by* $(x, y^1, y^2) = (0, 1, 1)$. *Despite the consideration of data uncertainty, the overall bilevel solution does not change significantly compared to the deterministic setting. However, the following shows that this may not always be the case.*

To this end, we focus on the stochastic modeling of uncertain constraint coefficients in (7c). *The constraint coefficients* $a(\omega) \in \mathbb{R}^2$ *are assumed to depend on the scenario* $\omega \in \Omega = \{\omega^1, \omega^2\}$ *with* $a(\omega^1) = (-1, 1)$ *and* $a(\omega^2) = (-3/2, 1/2)$. *We further assume that scenario* ω^1 *has probability* $p^1 = 1/3$, *whereas* ω^2 *has probability* $p^2 = 2/3$. *Again, the realization of scenario* ω^1 *corresponds to the deterministic setting. Thus, the unique optimal solution for scenario* ω^1 *is given by* $(x, y^1) = (0, 1)$. *The setting in which scenario* ω^2 *realizes is shown in Figure 3 (right). The unique optimal solution* $(x, y^2) = (6/5, 8/5)$ *is illustrated by the thick square. Hedging against data uncertainty by optimizing over the expected value yields the unique overall stochastic bilevel solution* $(x, y^1, y^2) = (6/5, 1/5, 8/5)$, *which can be obtained by solving the corresponding scenario-expanded formulation* (8). *In particular, the solution is attained at a completely different vertex of the bilevel feasible set than in the deterministic case.*

2.2. **Decision Uncertainty.** Decision uncertainty refers to the case in which the players may face uncertainties regarding the decision of the other player. For instance, the follower may lack the ability or the resources to obtain an optimal solution and, thus, takes a "satisfactory" solution instead of an optimal one. For a given leader's decision x, this can be modeled using the set of ε-optimal reactions of the follower, which is given by

$$S(x, \varepsilon) = \{y \in Y : g(x, y) \geq 0,\ f(x, y) \leq \varphi(x) + \varepsilon\}, \quad \varepsilon > 0,$$

where

$$\varphi(x) := \min_{y \in Y} \{f(x, y) : g(x, y) \geq 0\}$$

denotes the lower-level's optimal-value function. The parameter ε quantifies the follower's willingness to deviate from his optimal objective function value. As a consequence of the follower's ε-optimality, the leader is uncertain about the actual response of the follower. The aim of the leader may thus be to hedge against the worst-case ε-optimal reaction of the follower, i.e., one studies the problem

$$\min_{x \in X} \max_{y \in S(x,\varepsilon)} \quad F(x, y) \quad \text{s.t.} \quad G(x, \hat{y}) \geq 0 \text{ for all } \hat{y} \in S(x, \varepsilon). \tag{9}$$

In particular, Problem (9) can be reformulated as a specific instance of the pessimistic bilevel problem considered by Wiesemann et al. (2013).

Another reason for decision uncertainty may be the follower's limited capability to observe the decision of the leader. A reasonable assumption in this context, however, may be that the follower has an insight into the leader's scope of action. One way to account for this type of decision uncertainty is the following modeling. Let $\mathcal{X}(x) \subset \mathbb{R}^{n_x}$ denote the x-dependent set containing all possible decisions of the leader, which is assumed to be known by the follower. Clearly, the actual decision x of the leader belongs to $\mathcal{X}(x)$. The follower then takes his decision to hedge against all possible leader decisions $\bar{x} \in \mathcal{X}(x)$, e.g., by pursuing a robust approach that leads to the bilevel model

$$\begin{aligned} \min_{x,y} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \geq 0,\ x \in X, \\ & y \in \arg\min_{\bar{y} \in Y} \left\{ \max_{\bar{x} \in \mathcal{X}(x)} f(\bar{x}, \bar{y}) : g(\bar{x}, \bar{y}) \geq 0 \text{ for all } \bar{x} \in \mathcal{X}(x) \right\} \end{aligned}$$

with a robustified follower's objective function and a robustified feasible set of the lower-level problem.

Example 3. *To illustrate some modeling approaches for decision uncertainty discussed so far, we consider the following example taken from Beck and Schmidt (2021) and Besançon et al. (2019):*

$$\begin{aligned} \min_{x,y \in \mathbb{R}} \quad & F(x, y) = x - 10y && \text{(10a)} \\ \textit{s.t.} \quad & x - 4y \geq -11, && \text{(10b)} \\ & -x - 2y \geq -13, && \text{(10c)} \\ & x \geq 0,\ y \in S(x). && \text{(10d)} \end{aligned}$$

Here, $S(x)$ again denotes the set of optimal solutions of the x-parameterized lower-level problem, which is given by

$$\begin{aligned} \min_{y \in \mathbb{R}} \quad & f(x, y) = y && \text{(11a)} \\ \textit{s.t.} \quad & 2x + y \geq 5, && \text{(11b)} \\ & -5x + 4y \geq -30, && \text{(11c)} \\ & y \geq 0. && \text{(11d)} \end{aligned}$$

The deterministic bilevel problem (10) and (11) is depicted in Figure 4. Again, the upper- and lower-level constraints are represented with dashed and solid lines,

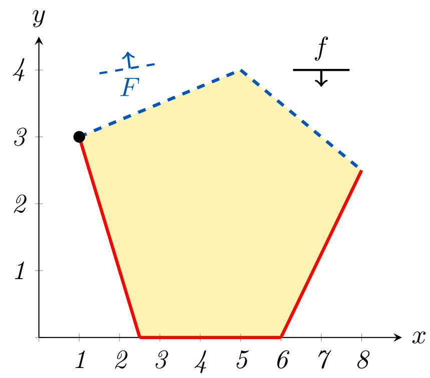

Figure 4. The upper-level constraints (dashed blue lines), the lower-level constraints as well as the bilevel feasible set (solid red lines), and the shared constraint set (yellow area) of the bilevel problem (10) and (11).

respectively, and the optimal solution $(x, y) = (1, 3)$ of the problem is illustrated by the thick dot. Note that, in the deterministic case, the solid lines also correspond to the bilevel feasible set. The modeling of an ε-optimal follower leads to a parallel shift of the upper-level constraints since the leader needs to make sure that her decision remains feasible for every ε-optimal decision of the follower. Hence, the bilevel feasible set is reduced as it can be seen in Figure 5 (left) for $\varepsilon = 0.5$. The optimal solution of the bilevel problem (10) and (11) with an ε-optimal follower is given by the thick dot.

To illustrate the effect of a follower with limited capability to perfectly observe the actual decision x of the leader, let us assume that the perceived leader's decision $\bar{x}$ belongs to the uncertainty set $\mathcal{X}(x) = \{x + \zeta \colon |\zeta| \leq 0.5\}$. Similar to the setting with an ε-optimal follower, pursuing a robust approach to account for limited observability reduces the bilevel feasible set as it can be seen in Figure 5 (right). Here, however, the consideration of limited observability leads to a parallel shift of the lower-level constraints that explicitly depend on the variable of the leader. The reason is that, in this setting, the follower makes a decision, which must be feasible for all possible realizations of the leader's decision. The optimal solution of the bilevel problem (10) and (11) with a follower who cannot perfectly observe the actual decision of the leader is illustrated by the thick square.

To sum up, the consideration of uncertainties in bilevel optimization may impact the solution of the problem significantly. Moreover, the obtained solution depends on the specific modeling of uncertainty that is taken into account. Hence, uncertainties are important to be considered—especially if decision makers are involved who are subject to bounded rationality.

Apart from the aforementioned aspects, there are many other modeling approaches to account for uncertainties in bilevel optimization for which we provide an in-depth discussion in the following sections.

3. Different Approaches to Account for Uncertainty

3.1. **Stochastic Bilevel Problems.** To the best of our knowledge, the first series of papers on stochastic bilevel problems are the ones by Patriksson and Wynter (1999, 1997); see also Christiansen et al. (2001) for a follow-up paper. Before we review the stochastic setups discussed in these papers, let us first state the deterministic

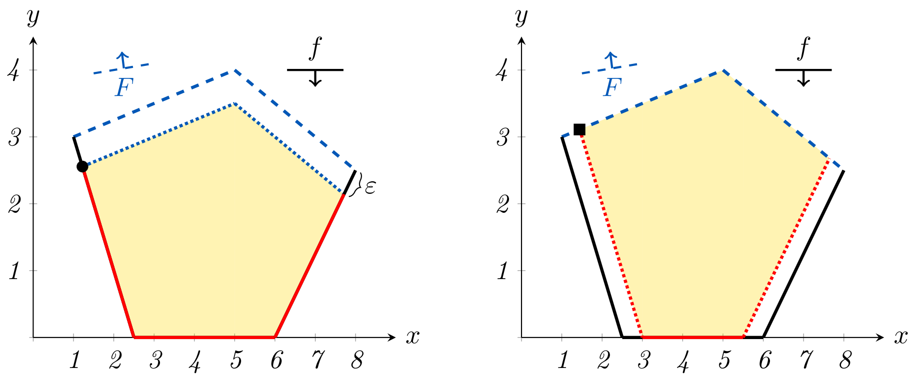

Figure 5. Both figures show the shared constraint set (yellow area) and the bilevel feasible set (red lines) of the linear bilevel problem (10) and (11) with an ε-optimal follower with $\varepsilon = 0.5$ (left) and the variant of the problem with a follower facing limited observability (right).

problem, which the authors call a "generalized" bilevel optimization problem since the lower-level is given by a variational inequality problem. To define this, we consider a bilevel setting in which no coupling constraints are present, i.e., there are no upper-level constraints that explicitly depend on the follower's variables. Further, we consider the function $T : X \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ and the lower-level feasible set $Y(x)$. Then, the variational inequality problem is to find a point $y^* \in Y(x)$ such that

$$T(x, y^*)^\top (y - y^*) \geq 0 \quad \text{for all} \quad y \in Y(x) \tag{12}$$

holds for a given leader's decision $x \in X$. In the cited papers, the authors state this problem in a geometric way by using normal cones. To this end, let $Y(x)$ be convex for all possible leader decisions $x \in X$. We call a vector v a normal of the convex set $Y(x)$ at a point $\bar{y} \in Y(x)$ if

$$v^\top (y - \bar{y}) \leq 0 \quad \text{for all} \quad y \in Y(x)$$

holds. The set of such vectors is then denoted by $N_{Y(x)}(\bar{y})$ and is called the normal cone to $Y(x)$ at $\bar{y}$. Having this notation at hand, we can re-write the variational inequality (12) as

$$-T(x, y^*) \in N_{Y(x)}(y^*). \tag{13}$$

For more background on variational analysis we refer the interested reader to Rockafellar and Wets (1998). Hence, the overall deterministic, generalized bilevel problem is given by

$$\min_{x \in X} \quad F(x, y) \quad \text{s.t.} \quad -T(x, y) \in N_{Y(x)}(y).$$

The classic, i.e., non-generalized, bilevel problem is covered since for continuously differentiable and pseudo-convex lower-level objective functions f as well as non-empty, closed, and convex lower-level feasible sets $Y(x)$, the minimum principle ensures that the parameterized variational inequality (13) characterizes the global optimal solutions of the parameterized lower-level problem

$$\min_{y} \quad f(x, y) \quad \text{s.t.} \quad y \in Y(x)$$

if one identifies $T(x, y) = \nabla_y f(x, y)$ and $Y(x) = \{y \in \mathbb{R}^{n_y} : g(x, y) \geq 0\}$.

3.1.1. *Risk-Neutral Models.* In Patriksson and Wynter (1997), the authors generalize traditional two-stage[2] stochastic problems to generalized bilevel optimization problems under uncertainty. They illustrate this novel class of problems using an example from the field of traffic equilibrium modeling that includes uncertain travel demands as stochastic parameters. As a solution approach, the authors propose a descent method that uses sensitivity analysis to obtain derivatives for setting up search directions, a line-search method (of Armijo-type) to get step sizes, and a projection onto the feasible set of the upper-level player to obtain a new iterate.

In Patriksson and Wynter (1999), the same authors consider stochastic mathematical programs with equilibrium constraints (SMPECs)—a class of problems that comprises stochastic bilevel problems if compact optimality conditions such as the Karush–Kuhn–Tucker (KKT) conditions or the above minimum principle are both necessary and sufficient for lower-level optimality. We refer the reader to Lin and Fukushima (2010) for a survey on SMPECs. Given the deterministic setup above, the stochastic and risk-neutral counterpart reads

$$\min_{x \in X} \quad \mathbb{E}_\omega \left[F(x, y(\omega)) \right], \tag{14}$$

where for all $\omega \in \Omega$,

$$y(\omega) \in \left\{ y \in \mathbb{R}^{n_y} : \, -T(x, \omega, y) \in N_{Y(x,\omega)}(y) \right\} \tag{15}$$

denotes the solutions of the lower-level variational inequality problem, which is parameterized by the upper-level decision x and the random variable ω. We recover the stochastic bilevel problem by identifying $T(x, \omega, y) = \nabla_y f(x, \omega, y)$ and $Y(x, \omega) = \{y \in \mathbb{R}^{n_y} : g(x, \omega, y) \geq 0\}$. As usual, the random variable ω is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The timing covered by the problem given in (14) and (15) is the following. The leader first takes her decision in a here-and-now manner, then the uncertainty realizes and, finally, the follower decides based on the realization of uncertainty and the leader's decision. Hence, we consider the timing

$$\text{leader } x \quad \curvearrowright \quad \text{uncertainty } \omega \quad \curvearrowright \quad \text{follower } y = y(x, \omega) \tag{16}$$

and the considered stochastic setting is risk-neutral since the expected value is minimized in the upper-level objective function. The presented theoretical results in Patriksson and Wynter (1999) are concerned with existence of solutions as well as convexity and directional differentiability of the (implicitly defined) upper-level objective function. Finally, a subgradient descent method is sketched that follows the main ideas already discussed for the special case in Patriksson and Wynter (1997). Going further, Patriksson (2008a,b) again considers SMPECs (in the context of equilibrium problems from structural optimization and traffic assignment problems) and shows that, under some additional assumptions, the SMPEC solutions continuously depend on the probability distribution used to model the uncertainty in the lower-level problem. Note that the author calls this continuity property "robustness", which has nothing to do with robustness in the sense of robust optimization. The papers discussed so far do not contain any numerical results.

In Christiansen et al. (2001), the authors study stochastic bilevel problems for truss topology optimization problems in which the external load applied to the truss is uncertain and the random variables are assumed to be discrete and finite. Hence, a finite set of scenarios is considered. Using classic inf-compactness assumptions, i.e., lower semicontinuity and bounded level sets of the upper-level objective function, existence of solutions is shown. This is extended by an existence result without requiring inf-compactness. However, additional and problem-specific assumptions

[2]Here and in what follows, we try to clearly distinguish between "stages" and "levels", where "stages" always refer to stages as considered in, e.g., two-stage stochastic optimization, whereas "levels" are always used in the sense of multilevel optimization, e.g., as in bilevel optimization.

are used such as that the upper-level objective function is quadratic, which then allows to invoke the classic existence theorem by Frank and Wolfe (1956). As in the other papers discussed so far, the authors derive further results on the directional differentiability of the implicitly given upper-level objective function to design a subgradient method that further exploits parallelization across the scenarios. In contrast to the papers discussed before, Christiansen et al. (2001) also present a small numerical case study for a stochastic truss topology optimization problem.

These early results on stochastic bilevel optimization are wrapped up in the handbook chapter by Wynter (2008). Again, existence of optimal solutions is discussed for the case of a discrete set of scenarios. Moreover, sufficient conditions for the convexity of stochastic bilevel problems are given as well. Convexity, however, is only possible if the upper-level problem does not contain any coupling constraints and if the objective function of the upper level only depends on the optimal value of the lower level and not on the follower's decision itself. As already done in the original works discussed above, the author also collects sufficient conditions that ensure that the upper-level objective function is Lipschitz continuous and directionally differentiable. The handbook chapter closes with a sketch of the subgradient method from the previously discussed original works.

The above mentioned parallelization techniques are a special case of decomposition methods that are very prominent in the literature on stochastic optimization; cf., e.g., the L-shaped and other Benders-inspired decomposition methods for two- or multistage stochastic optimization; see, e.g., Birge and Louveaux (2011), Fischetti, Ljubić, and Sinnl (2016, 2017), Rahmaniani et al. (2018), and Van Slyke and Wets (1969). These decomposition techniques have also been carried over to the case of stochastic MPECs. For this, see, e.g., Shapiro and Xu (2008), where scenario generation techniques are discussed to obtain a finite-dimensional deterministic equivalent (which is a deterministic MPEC or bilevel optimization problem) with separate blocks of follower decisions for each realization of the uncertainty. Such block structures are then exploited by decomposition methods. For an overview of these methods, we refer the reader to the PhD thesis by Henkel (2014), where a broad literature review is given w.r.t. what has been published up to 2014. The PhD thesis itself studies both the classic KKT as well as the optimal-value reformulation for stochastic bilevel problems with discrete and finite probability distributions. Based on the KKT approach, an integer-programming based method is designed and evaluated that also uses problem-tailored decomposition techniques. The numerical results contain instances with up to 20 scenarios and with up to 20 lower-level variables.

Rather recently, Bolusani et al. (2020) consider the similarities between multilevel mixed-integer linear optimization and multistage stochastic mixed-integer linear optimization with recourse. For the bilevel stochastic setting mentioned above (with a discrete set of scenarios, a wait-and-see follower, and a risk-neutral leader), they exploit the block-angular structure of the bilevel deterministic equivalent and propose both a Benders-like decomposition as well as a cutting-plane method. These methods are also implemented in MibS; see Tahernejad (2019).

Note that for a risk-neutral here-and-now follower and with a given set of discrete scenarios, one can easily turn bilevel two-stage stochastic optimization problems into their bilevel deterministic counterparts as it is done in two-stage stochastic optimization. Hence, any general purpose solver for deterministic bilevel problems (see, e.g., Fischetti, Ljubić, Monaci, et al. (2017) or MibS by Tahernejad et al. (2020), see `https://coin-or.github.io/MibS/`) can be used for that purpose.

3.1.2. *Bilevel Models with a Quantile Criterion.* Another branch of research does not consider the optimization of an expected value in the leader's objective function but studies a quantile criterion that ensures that a certain upper-level objective function value is not exceeded with a given probability. In different application contexts, these settings have been considered first in Chen et al. (2007) and Katagiri et al. (2014) and the first theoretical analysis has been carried out, to the best of our knowledge, by Ivanov (2014). The mathematical setup is given as follows. The lower-level solution set is given by

$$S(x, z) = \arg\min_{y} \left\{ d^\top y \colon Dy \geq z - Cx,\ y \geq 0 \right\},$$

where $z \in Z$ is a realization of a random vector. For the ease of presentation, we omit to state the dimensions of all vectors and matrices. With this at hand, the so-called loss function of the leader is defined as

$$\Phi(x, z) = \begin{cases} \min_{y \in S(x,z)} c_y^\top y, & \text{if } S(x, z) \neq \emptyset, \\ +\infty, & \text{if } S(x, z) = \emptyset, \end{cases}$$

and the corresponding α-quantile function reads

$$\Phi_\alpha(x) = \min \left\{ \varphi \colon \mathbb{P}(\Phi(x, Z) \leq \varphi) \geq \alpha \right\},$$

where $\mathbb{P}(\cdot)$ is the probability measure induced by the distribution of the random vector Z. Finally, the bilevel optimization problem with a quantile criterion is given by

$$\min_{x \in X} \quad c_x^\top x + \Phi_\alpha(x).$$

The paper first presents theoretical results regarding Lipschitz continuity of the leader's loss function. After also proving the continuity of the corresponding quantile function, an existence result is obtained. Finally, the paper shows that the studied problem can be reformulated as a single-level mixed-integer linear optimization problem in the case of a discrete and finite distribution of the random variables. To this end, the classic linearization technique by Fortuny-Amat and McCarl (1981) is used, including the choice of sufficiently large big-M values and by introducing additional binary variables to linearize the KKT complementarity conditions. Finally, a numerical case study is presented with 2 upper-level and 3 lower-level variables as well as 4 upper-level and 2 lower-level constraints. The number of considered scenarios is 25.

The paper by Dempe, Ivanov, et al. (2017) builds on Ivanov (2014) and generalizes the setting. Again, a stochastic bilevel problem with a quantile criterion is considered in which the lower-level problem depends on the realization of a random vector and on the leader's decision. In contrast to Ivanov (2014), the authors now consider the so-called "a priori statement" of the problem, meaning that the follower's decision variables are chosen from a set of functions depending on random parameters, whereas the so-called "a posteriori statement" is studied in Ivanov (2014). Hence, Dempe, Ivanov, et al. (2017) study the timing

$$\text{leader } x \quad \curvearrowright \quad \text{follower } y = y(x) \quad \curvearrowright \quad \text{uncertainty } \omega, \tag{17}$$

which differs significantly from the one in (16) since the follower now also takes his decision before the uncertainty is realized. In Dempe, Ivanov, et al. (2017), the leader's problem can be nonlinear and the follower's problem is linear in the follower's variables. The authors establish a mixed-integer nonlinear single-level reformulation for the case that the random vector has a finite set of realizations and propose assumptions under which an optimal solution of the original problem exists. The numerical case study considers a single academic instance of roughly the same size as in Ivanov (2014).

In contrast to Ivanov (2014), where only the lower-level feasible set is affected by randomness, in Ivanov (2018), the setting is studied in which the randomness affects the objective function of the lower-level problem in a way such that its objective function is linear for a given realization of uncertainty and a given leader's decision. The lower level's feasible set, however, is both independent of the leader's decision as well as of the realization of uncertainty. The leader again minimizes a quantile function of her loss function depending on the leader's and the follower's decision. It is shown that the lower-level problem has a unique solution (with probability 1) if the distribution is absolutely continuous and if the lower-level's feasible set is non-empty and bounded. The loss function is proven to be lower semicontinuous, which implies that the overall bilevel problem with a quantile criterion has an optimal solution if the upper-level's feasible set is non-empty and bounded as well. The quantile function itself, however, is not continuous. Finally, sample average approximation is applied and the convergence of its limit points is shown before a small-scale optimal tax rate problem is considered in a case study.

3.1.3. *Convex Risk Measures.* The aforementioned models based on a quantile criterion just discussed can be seen as models that incorporate some kind of risk measure. In the case of quantiles, these risk measures (such as the value-at-risk; VaR) are not convex. The key idea of incorporating risk measures in the upper-level's objective function of stochastic and linear bilevel problems (such as the quantile function in the previous section) is also studied in Burtscheidt, Claus, and Dempe (2020), where the setting of law-invariant, convex, and coherent risk measures is considered for uncertainties in the right-hand side of the follower's problem. More formally, the considered problem is an optimistic parametric bilevel linear problem of the form

$$\min_x \left\{ c_x^\top x + \min_y \left\{ c_y^\top y \colon y \in S(x,z) \right\} : x \in X \right\}$$

with a parameter z and the lower-level solution set mapping

$$S(x,z) = \arg\min_y \left\{ d^\top y \colon Dy \geq z - Cx \right\}.$$

The stochastic bilevel problem is then obtained by assuming that the parameter $z = Z(\omega)$ is the realization of a random vector Z defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define now

$$G = \{(x,z) \colon \exists y \text{ with } Dy \geq z - Cx\}$$

to be the set of all upper-level decisions and parameters so that there is a feasible lower-level solution. Moreover, set

$$F(x,z) = c_x^\top x + \min_y \left\{ c_y^\top y \colon y \in S(x,z) \right\}$$

as an abbreviation for the upper-level objective function in dependence of the leader's decision x and the parameter z. An additional nonanticipativity constraint then leads us to the timing in (16), i.e., the leader takes a here-and-now decision while the follower takes a wait-and-see decision. Further, denote the Borel probability measure induced by the random vector Z by $\mu_Z = \mathbb{P} \circ Z^{-1}$. With this, it is assumed that

$$X \subseteq \{x \colon (x,z) \in G \text{ for all } z \in \operatorname{supp}(\mu_Z)\},$$

i.e., there exists a lower-level feasible point for all possible leader decisions and all realizations of uncertainty. After fixing properly chosen function spaces and a suitable risk measure $\mathcal{R}$, the stochastic bilevel problem is given by

$$\min_x \quad \mathcal{R}[F(x, Z(\cdot))] \quad \text{s.t.} \quad x \in X.$$

The class of considered risk measures includes, among others, the expected value, the conditional value-at-risk (CVaR), or a worst-case risk measure. Note that the latter includes the case of robust and linear bilevel problems. The key results of the paper are about (local) Lipschitz continuity of the upper-level objective function. Moreover, differentiability results are given for those risk measures that are based on expectations. For the case of discrete and finite distributions, equivalent deterministic bilevel problems are derived for the expectation-based risk measures. Finally, the MPCC regularization strategy proposed by Scholtes (2001) is applied to the KKT reformulation of the deterministic equivalent and a convergence result for the limit points of this regularization approach is proven.

Most of the content presented in Burtscheidt, Claus, and Dempe (2020) can also be found in the book chapter by Burtscheidt and Claus (2020), where the same class of problems is considered. The new content given there is about stochastic dominance constraints. Moreover, equivalent deterministic counterparts are proven for some of the cases that have not been included in Burtscheidt, Claus, and Dempe (2020).

In a follow-up paper, Claus (2021a) considers the same risk-neutral setting as in Burtscheidt, Claus, and Dempe (2020) and the first-order necessary optimality conditions for the risk-neutral case from the latter paper are accompanied by second-order sufficient conditions. It was known from Burtscheidt, Claus, and Dempe (2020) that the expectation functional, i.e., the upper-level objective function in the risk-neutral case, is continuously differentiable if the underlying probability measure is absolutely continuous w.r.t. the Lebesgue measure. However, (even local) Lipschitz continuity of the gradient may fail to hold under the assumptions used in Burtscheidt, Claus, and Dempe (2020). The main novel assumptions now are the boundedness of the support and the uniform boundedness of the Lebesgue density of the probability measure, which ensure (as the main result of the paper) the Lipschitz continuity of the gradient.

In the paper by Claus (2021c), the overall timing with a leader taking a here-and-now decision, while the follower reacts in a wait-and-see manner, is kept but the way how randomness enters the lower-level problem is different. Instead of considering a stochastic right-hand side, the lower-level's feasible set is now a fixed polyhedron but the coefficients of the linear objective function of the follower are affected by randomness. This still leads to the case that the upper-level objective function value is a random variable itself that is parameterized by the leader's decision. The main result is the development of sufficient conditions for the existence of optimal solutions for a wide class of risk measures, including the expected value, CVaR, or the worst-case risk measure that leads to a robust and linear bilevel problem. The main stepping stone towards this result is the proof of the continuity of the risk functionals. Surprisingly, these sufficient conditions are the same both for the optimistic and the pessimistic bilevel problem, which is not the case in the deterministic setting.

The, up to now, last paper in this row of research is Claus (2021b) in which the two settings considered so far are combined: randomness in the lower-level's right-hand side as well as in its objective function coefficients. The results of Claus (2021c) cannot be applied anymore since the extreme points of the lower-level's feasible set now depend on the realization of the randomness as well. However, lower semicontinuity can be achieved for a class of functions derived from convex risk measures. Further, a continuity result is derived that can also be applied to the pessimistic case as well. These continuity properties can then, as usual, be used to obtain existence results under further but classic compactness assumptions. As it is the case for all other papers discussed so far in this subsection, the paper does not contain any numerical results.

In Burtscheidt, Claus, Conti, et al. (2021), the same analytical spirit is followed but is now brought to the field of pessimistic and stochastic bilevel optimization.

Again, continuity results are derived in order to finally obtain an existence result. After these theoretical developments, the authors study an applied problem from the field of mechanical shape optimization that is modeled as a pessimistic and stochastic bilevel problem and also present some numerical results.

3.1.4. *Chance Constraints.* Up to now, all papers except for Dempe, Ivanov, et al. (2017) studied the timing in (16). In the case of the other timing (17), fulfilling the random lower-level constraints cannot be guaranteed almost surely in general but only with a certain probability. In a natural way, this leads to the use of probabilistic or chance constraints.

To the best of our knowledge, the first combination of bilevel optimization and chance constraints is studied in Kosuch et al. (2012). The authors study a standard linear bilevel problem that is extended by a probabilistic knapsack constraint in the upper-level problem. The setting is motivated by pricing applications in networks. For finite probability distributions, the authors first derive a deterministic equivalent formulation using additional binary variables and big-M constraints. This all takes place in the upper-level problem and is completely independent of the lower level. Afterward, the authors apply the classic mixed-integer linear reformulation by Fortuny-Amat and McCarl (1981), leading to a mixed-integer linear single-level problem. Based on this reformulation, a so-called min-max scheme is designed with bounds obtained from suitably chosen Lagrangian relaxations. The authors present numerical results for up to 1000 lower- and upper-level variables and for up to 100 scenarios.

From a more application-driven point of view, other chance-constrained bilevel problems have been studied in Pramanik and Banerjee (2012) and Yang, Zhang, et al. (2009). However, the only paper that we are aware of in which a chance constraint is considered in the lower-level problem and in which, thus, the timing in (17) is considered, is Heitsch et al. (2022). There, the chance constraint is shown to be convex and oracles for function as well as gradient evaluations are provided while the chance constraint itself cannot be stated in closed form. This setting is considered as having a convex black-box function in the lower level which is then tackled by an outer-approximation based cutting-plane method. However, since the lower-level chance constraint can thus only by satisfied by a prescribed tolerance $\varepsilon > 0$, it does not seem to be possible to prove anything about the upper-level objective function value obtained by this method—an issue that is also considered in the recent paper by Beck, Schmidt, et al. (2022) on continuous but nonconvex lower-level problems.

3.1.5. *Knapsack Problems.* The first paper on bilevel knapsack problems is Dempe and Richter (2000), where the authors consider the problem in a purely deterministic setting. The first paper combining the bilevel knapsack problem with random data appeared ten years later (Özaltın et al. 2010). The setting is as follows. In the lower-level problem, the follower solves the knapsack problem

$$\max_{y} \quad d^\top y \quad \text{s.t.} \quad a^\top y \leq b(x, \omega), \ y \in \{0,1\}^n, \tag{18}$$

in which the knapsack's capacity depends on the scenario $\omega \in \Omega$ and on the scalar leader's decision x, where b is a non-decreasing function in x. Moreover, the leader solves the problem

$$\begin{aligned} \max_{x \in \mathbb{R}} \quad & \mathbb{E}_\omega \left[c_y^\top y(x, \omega) - c_x x \right] \\ \text{s.t.} \quad & x \in [x^-, x^+] \subset \mathbb{R}, \quad y(x, \omega) \in S(x, \omega) \text{ for all } \omega \in \Omega, \end{aligned}$$

where $S(x, \omega)$ is the set of optimal solutions of the (x, ω)-parameterized problem (18). Thus, the leader maximizes the value of the items in the knapsack (packed by the follower) but has different values c_y for the separate items, compared to the values d

of the follower. Consequently, we are again in the setting of (16), consider the risk-neutral case, and the uncertainty is modeled by a finite set of scenarios. The authors develop necessary and sufficient conditions for the existence of an optimal solution. Under the additional assumption that the leader's decision can only take integer values, the problem is reformulated as a two-stage stochastic program with binary first- and second-stage decisions that uses the optimal-value function of the lower-level problem. For evaluating the subproblem of this two-stage stochastic program, the authors design a so-called branch-and-backtrack algorithm and then, using this sub-routine, develop a branch-and-cut method to solve the overall problem. Computational results are reported on 16 randomly generated test instances with up to 100 items and 200 scenarios.

Very recently, Buchheim, Henke, and Irmai (2022) considered the continuous bilevel knapsack problem. Here, the lower-level problem is the continuous knapsack problem

$$\max_{y} \quad d(\omega)^\top y \quad \text{s.t.} \quad a^\top y \leq x, \; y \in [0,1]^n,$$

which depends on a random variable ω so that the lower-level's objective function is uncertain. The upper-level player then solves

$$\max_{x \in \mathbb{R}} \quad \mathbb{E}_\omega \left[c_y^\top y(x,\omega) - c_x x \right] \quad \text{s.t.} \quad x \in [x^-, x^+],$$

where $y(x,\omega)$ is a solution of the lower-level problem given above, which is parameterized by the uncertainty ω and the upper-level knapsack capacity decision x. Hence, the leader has different values of the items to be packed and the follower's values of the items are uncertain for the leader. This models the risk-neutral case—for a robust consideration of this setup, we refer the reader to Buchheim and Henke (2022). The authors purely focus on complexity questions. The deterministic problem is known to be solvable in polynomial time; see, e.g., Dempe, Kalashnikov, et al. (2015). First, it is shown that the problem stays polynomial-time solvable if the random variable has finite support, which needs to belong to the input of the problem together with the corresponding probabilities. If, however, the random variable has a finite and componentwise uniform distribution, the problem becomes #P-hard, which is shown by a reduction from #Knapsack.[3] The same hardness result holds true for continuous instead of finite, componentwise uniform distributions. Moreover, even the evaluation of the upper-level objective function is #P-hard for these two cases. Since these results all are hardness results in the weak sense, the authors also derive tailored pseudo-polynomial time algorithms based on dynamic programming. Finally, an additive approximation scheme is derived for arbitrary continuous distributions with independent components. Open questions in this field are the consideration of risk measures other than the expected value as well as the study of cases in which the uncertain lower-level coefficients are correlated.

3.1.6. *General Mixed-Integer Stochastic Bilevel Problems.* Except for Özaltın et al. (2010), where the special situation of a bilevel knapsack problem is studied, no other general mixed-integer stochastic bilevel problems have been considered until the paper by Yanıkoğlu and Kuhn (2018) in which a stochastic bilevel problem is studied with the following properties. The leader decides here-and-now and takes a binary decision without knowing the follower's objective function coefficients and right-hand side. Moreover, the objective function coefficients for the follower's variables in the leader's objective are unknown as well. The uncertainty then reveals

[3]The #-symbol indicates that the corresponding counting version of the given decision problem is considered.

and the follower takes his continuous decision having full information. The main focus is on the pessimistic setting, i.e., the bilevel problem at hand is given by

$$\begin{aligned}\min_{x\in X} \quad \sup_{y\in L_n^2} \quad & c_x^\top x + \mathbb{E}_\omega \left[c_y(\omega)^\top y(\omega)\right] \\ \text{s.t.} \quad & y(\omega) \in \operatorname*{arg\,min}_{y'\in\mathbb{R}^n} \left\{d(\omega)^\top y' \colon Ay' \le b(x,\omega)\right\}.\end{aligned}$$

Here, $X \subseteq \{0,1\}^{n_x}$ and L_n^2 is the set of all n-dimensional square-integrable functions of ω. Hence, the risk-neutral setting with the classic timing (16) is considered but in a rather general setting with randomness both in the upper- as well as in the lower-level problem. The key results from this contribution are that the authors develop both primal and dual decision rules to restrict the search space L_n^2 for the lower-level answers from the point of view of the leader. Both bounding problems are equivalent to MILPs that are not significantly harder than the nominal variant of the original problem. The developed techniques are then applied to a facility location problem with up to 35 locations. To the best of our knowledge, this paper is the first one that specifically focuses on stochastic and pessimistic bilevel optimization problems.

Even more recently, Zhang and Özaltın (2021) consider an even more complicated setting w.r.t. the decision variables since they consider stochastic integer bilevel problems, i.e., problems with purely integer variables both at the upper and the lower level. However, they focus on the optimistic setting while claiming that the developed techniques also work for the pessimistic case. Moreover, randomness only appears in the lower-level's right-hand side and has a discrete distribution with finite support. Their solution approach is based on the value-function reformulation and uses an integer complementarity slackness theorem that is extended to bilevel integer programs. Using these approaches, they finally solve the given class of problems using a tailored branch-and-bound method, which is tested on bilevel facility interdiction problems with up to 200 constraints and 400 variables in the lower level and with up to 50 000 scenarios. It turns out that the approach is rather insensitive w.r.t. the number of scenarios but is sensitive w.r.t. the number of random right-hand sides.

3.2. **Robust Approaches to Model Data Uncertainty.** To the best of our knowledge, robust approaches to model uncertainties in the context of bilevel optimization have been much less investigated compared to stochastic approaches. For an overview of robust optimization techniques for single-level optimization, we refer the reader to Ben-Tal, El Ghaoui, et al. (2009), Ben-Tal, Goryashko, et al. (2004), Ben-Tal and Nemirovski (1998), Bertsimas, Brown, et al. (2011), and Soyster (1973). In robust setups, it is assumed that uncertainties take values in a given uncertainty set $\mathcal{U}$. In single-level optimization, uncertainty sets are typically modeled using geometries such as boxes, polyhedra, ellipsoids, or cones. When considered in a bilevel setting, however, the literature so far focuses mainly on either box or polyhedral uncertainty sets.

Chuong and Jeyakumar (2017) study optimistic bilevel problems with linear constraints and a linear lower-level objective function, while the objective of the leader is a polynomial. They allow for interval uncertainties in the upper- and lower-level constraints. Hence, the authors consider the problem

$$\begin{aligned}\min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & \tilde{A}_{i\cdot} x + \tilde{B}_{i\cdot} y \ge \tilde{a}_i \quad \text{for all } \left(\tilde{A}_{i\cdot}, \tilde{B}_{i\cdot}, \tilde{a}_i\right) \in \mathcal{U}_i^{\mathrm{L}},\ i \in [m], \\ & x \in \mathbb{R}^{n_x},\ y \in S(x)\end{aligned}$$

with $[m] := \{1, \dots, m\}$. Here, uncertain parameters are emphasized by a tilde, $\tilde{A}_{i\cdot}$ denotes the ith row of the matrix $\tilde{A}$, and $S(x)$ is the set of optimal solutions of the robust lower-level problem

$$\begin{aligned} \min_y \quad & d^\top y \\ \text{s.t.} \quad & C_{j\cdot} x + \tilde{D}_{j\cdot} y \geq \tilde{b}_j \quad \text{for all } \left(\tilde{D}_{j\cdot}, \tilde{b}_j\right) \in \mathcal{U}_j^{\mathrm{F}},\ j \in [\ell]. \end{aligned}$$

The uncertainty sets on the upper and the lower level are given by $\mathcal{U}_i^{\mathrm{L}}$ and $\mathcal{U}_j^{\mathrm{F}}$, respectively. The matrix C is assumed to be certain. To solve the robust counterpart of the problem, a sequence of single-level nonconvex polynomial relaxations of the uncertain bilevel problem is introduced, which in turn can be reformulated as a sequence of semidefinite linear problems. The authors show that the optimal values of the relaxed problem converge to the robust global optimal value given that the leader's objective function is coercive and that the lower-level problem satisfies Slater's constraint qualification for every feasible decision of the leader and for every possible realization of the uncertainty in the lower-level constraints.

Uncertainties regarding the lower-level objective function are considered in Borrero et al. (2022) in the context of sequential optimistic linear bilevel problems in which the leader and the follower interact over multiple time periods. The follower's time-invariant objective function coefficients are initially unknown to the leader but they are assumed to take values in a given uncertainty set. In each time period, the leader may refine her perception of the uncertainty set by observing the follower's optimal response, which in turn is based on the leader's decision given her current knowledge of the problem data. Various mechanisms to update the uncertainty set are proposed, which differ in the amount of information from the follower's feedback that is taken into account. The updating process may then characterize the leader's policy, which is a sequence of functions that map the information from previous time periods, i.e., previous upper-level decisions and the observed feedback of the follower, to a feasible leader's decision in the current time period. The authors discuss different policies of the leader that can be interpreted as a robust modeling of a follower who is also uncertain about his objective function coefficients. Results on the convergence of the leader's policies to a full-information solution are provided for different update mechanisms. Moreover, an upper bound on the number of time periods necessary for said convergence is established, which is referred to as "time stability". In the worst-case, the upper bound may be exponential but the presented computational results suggest that this bound is rather loose. To illustrate the performance of the proposed policies, the authors provide numerical experiments on 20 randomly generated road network instances with a layered topology with 4 layers and 4 nodes per layer. Further, experiments are performed on 3 variants of the "infiltration network" near the Arizona-Mexico border with a network of 38 nodes and 109 arcs, which has been described in Unsal (2010).

Recently, Zhang, Liu, et al. (2022) are concerned with the existence of solutions of uncertain multi-leader-multi-follower problems that are modeled as Nash–Stackelberg–Nash games. Uncertainties arise in the objective functions of both the leaders' and the followers' problems as well as in the strategy sets of the followers. In the considered setting, the leaders first take a here-and-now decision that particularly influences the description of the uncertainty set. Thus, the authors consider a special type of decision-dependent uncertainty; see also Lappas and Gounaris (2018) and Nohadani and Sharma (2018) for general works on decision-dependent uncertainty. Then, the uncertainty realizes and, finally, the followers decide on their actions taking the leaders' decision as well as the realization of uncertainty into account. To hedge against decision-dependent uncertainties, a worst-case approach is considered.

Buchheim and Henke (2020, 2022) are concerned with complexity questions for the bilevel continuous knapsack problem in which the leader controls the capacity of the knapsack and the follower faces uncertainties regarding the profits of the items. The authors' main focus is on the pessimistic setting and a worst-case oriented approach is pursued to account for data uncertainty. While the deterministic problem can be solved in polynomial time (Dempe, Kalashnikov, et al. 2015), the complexity of the robust variant of the problem strongly depends on the considered type of the uncertainty set. First, the authors show that the robust counterpart remains solvable in polynomial time for discrete uncertainty sets as well as for interval uncertainty under the independence assumption, i.e., if the follower's objective function coefficients independently take values in given intervals. However, the problem is NP-hard if the uncertainty set is the Cartesian product of discrete sets. In particular, this means that replacing the uncertainty set by its convex hull can significantly change the problem in the bilevel context, which is in contrast to the situation in single-level optimization. NP-hardness is also shown for the variants of the problem with polytopal uncertainty sets and uncertainty sets that are defined by a p-norm with $p \in [1, \infty)$. Moreover, even the evaluation of the leader's objective function is NP-hard for the latter three cases.

The complexity of robust bilevel combinatorial problems with a linear follower facing an uncertain objective function is addressed in Buchheim, Henke, and Hommelsheim (2021). The deterministic variant of the problem is known to be NP-easy.[4] It is shown that interval uncertainty renders the bilevel problem significantly harder than the consideration of discrete uncertainty sets. To be more precise, the robust counterpart can be Σ_2^P-hard for interval uncertainty under the independence assumption, whereas it can be NP-hard for uncertainty sets $\mathcal{U}$ with $|\mathcal{U}| = 2$ and strongly NP-hard for general discrete uncertainty sets. In particular, it is shown that replacing the discrete uncertainty set by its convex hull may increase the complexity of the problem at hand, which is in line with the results in Buchheim and Henke (2020, 2022).

All papers discussed so far deal with data uncertainties in the sense of strict robustness. Beck, Ljubić, et al. (2022) propose a Γ-robust approach for mixed-integer linear min-max problems with lower-level data uncertainty. They follow the notion of Γ-robustness introduced by Bertsimas and Sim (2003, 2004) to account for an uncertain follower's objective, i.e., they study the problem

$$
\begin{aligned}
\min_{x} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax \geq a, \; x \in X \subseteq \mathbb{Z}^{n_x}, \\
& y \in \operatorname*{arg\,max}_{\bar{y} \in Y(x)} \left\{ d^\top \bar{y} - \max_{\{S \subseteq [n_y] :\, |S| \leq \Gamma\}} \sum_{i \in S} \Delta d_i \bar{y}_i \right\}
\end{aligned}
$$

with $\Gamma \in [n_y]$ and the lower-level feasible set $Y(x) \subseteq \mathbb{Z}_+^{n_y}$. Furthermore, uncertainties in a single packing-type constraint are considered. Two approaches to model this situation are presented—an extended formulation and a multi-scenario formulation. In particular, the authors establish that the Γ-robust bilevel problem can be interpreted as a single-leader-multi-follower problem with independent followers in the case that all of the follower's variables are binary. A branch-and-cut framework to solve the robustified bilevel problem is proposed. As an application, the authors consider the knapsack interdiction problem for which problem-tailored cuts are provided; see, e.g., Section 4.2. The authors conduct a computational study on

[4] A decision problem is NP-easy if it can be polynomially reduced to an NP-complete decision problem (Buchheim, Henke, and Hommelsheim 2021).

200 robustified knapsack interdiction instances with up to 55 items, which are based on the nominal instances described in Caprara et al. (2016).

3.3. **Lower-Level Near-Optimality.** In the classic setting of bilevel optimization, it is assumed that the leader anticipates an optimal reaction of the follower. In many practical applications, however, exact solutions of the lower-level problem cannot be expected. Possible reasons might be that an exact solution cannot be obtained in a reasonable amount of time or that there is simply no exact solution method to solve the problem at hand. Hence, the follower will take any "satisfactory" decision instead of an optimal one, i.e., one considers the set of ε-optimal follower's decisions

$$S(x,\varepsilon) = \{y \in Y \colon g(x,y) \geq 0,\ f(x,y) \leq \varphi(x) + \varepsilon\}\,, \quad \varepsilon > 0,$$

for a feasible decision x of the leader. Here, the parameter ε specifies the follower's willingness to deviate from his optimal objective function value. Since the leader faces response uncertainty due to the follower's near-optimality, this modeling approach accounts for decision uncertainty w.r.t. the lower-level player. This is in contrast to the concepts presented in the previous sections that cover data uncertainty.

Different approaches to deal with an ε-optimal follower have been studied in the literature. Maybe the first appearance of an ε-optimal follower can be found in Loridan and Morgan (1989), where the ε is used as a regularization parameter in order to prove existence results for the bilevel problem at hand.

Besançon et al. (2019) consider the effect of near-optimality on the upper-level constraints by exploiting a robust approach to hedge against deviations from the optimal reaction of the follower. In this setting, an optimal solution of the overall bilevel problem is required to remain feasible for all ε-optimal follower's decisions. To this end, the leader hedges against the worst-possible reaction of the follower by considering the problem

$$\begin{aligned}
\min_{x,z} \quad & F(x,z) \\
\text{s.t.} \quad & f(x,z) \leq \varphi(x),\ g(x,z) \geq 0,\ x \in X,\ z \in Y, \\
& \min_y \{G(x,y) \colon y \in S(x,\varepsilon)\} \geq 0.
\end{aligned}$$

Here, the leader controls the variables z that model the follower's optimal response, i.e., $z \in S(x,0) \subseteq S(x,\varepsilon)$, and the upper-level constraints are protected against ε-optimal follower's decisions in a robust way. Based on the Karush–Kuhn–Tucker (KKT) conditions of the lower-level problem, a single-level reformulation of the bilevel problem with an ε-optimal follower is provided if the lower level is a convex problem. Finally, the authors propose a solution method for purely linear near-optimal robust bilevel problems. The applicability of the proposed method is assessed in a computational study on 1200 randomly generated linear near-optimal robust bilevel instances with up to 20 variables and 20 constraints each on the upper and the lower level.

The complexity of near-optimal robustness concepts is analyzed in Besançon et al. (2021). The authors not only consider near-optimal robust bilevel problems but also investigate general multilevel optimization problems with an ε-optimal decision maker at an arbitrary lower level of the problem. Under suitable assumptions, they show that the robust modeling of near-optimality at a lower level remains in the same complexity class as the problem without uncertainty.

Motivated by military and law-enforcement applications, Zare, Özaltın, et al. (2018) consider bilevel problems with a follower that willingly deviates from his optimal objective function value to adversely affect the leader. For a feasible decision x of the leader, the authors define the set of near-optimal follower's decisions as

$$S_\alpha(x) = \{y \in Y : f(x,y) \le \alpha\varphi(x) + (1-\alpha)U,\, g(x,y) \ge 0\}.$$

Here, U is an upper bound for the lower-level objective function value and $\alpha \in [0,1]$ denotes the follower's willingness to deviate from his optimal objective function value. Note that for $\alpha = 1$, the set of (exact) optimal decisions of the follower is considered, whereas any feasible follower's decision can be chosen by the follower for $\alpha = 0$. This notion of near-optimality is considered in the context of pessimistic bilevel optimization (see, e.g., Liu, Fan, et al. (2020), Tsoukalas et al. (2009), and Wiesemann et al. (2013)) by introducing the so-called α-pessimistic bilevel problem

$$\min_{x \in X} \max_{y \in S_\alpha(x)} F(x,y).$$

The authors focus on the case in which (i) there are no coupling constraints and (ii) the functions F, f, and g are affine. Furthermore, they allow for integer variables on the upper level. The proposed model accounts for different levels of conservatism regarding the uncertainty of the follower's commitment. In particular, the α-pessimistic bilevel problem includes the standard pessimistic bilevel problem as well as the min-max problem by either setting $\alpha = 1$ or $\alpha = 0$, respectively. As an extension of the proposed model, the authors further embed an α-pessimistic follower into the context of strong-weak bilevel problems; see Section 3.5.

Pita, Jain, Ordóñez, et al. (2009), Pita, Jain, Tambe, et al. (2010), and Pita, Portway, et al. (2008) address ε-optimal follower's decisions in the context of Bayesian Stackelberg games (Conitzer and Sandholm 2006). They consider the setting in which there is only one leader type, e.g., a security entity, and multiple follower types, e.g., multiple attacker types. Each follower type may select an ε-optimal strategy and the actual follower type is unknown to the leader. To hedge against near-optimality of each follower type, the leader pursues a worst-case oriented approach. Given an a priori probability distribution over the follower types, the leader then optimizes over expected values under the worst-case assumption. The authors further combine the modeling of near-optimal follower types with the concept of limited observability regarding the leader's strategy, which is discussed in more detail in the following section.

In many practical applications, the lower-level problem cannot be solved to global optimality either because there is no exact solution method available or due to tractability reasons. Beck, Schmidt, et al. (2022) consider an illustrative example of a bilevel problem with a continuous but nonconvex lower-level problem for which only ε-feasible solutions—at least for the nonlinear constraints of the lower-level problem—can be anticipated. The authors show that such ε-feasible bilevel solutions can be arbitrarily far away from the overall exact optimal solution of the bilevel problem.

Zare, Prokopyev, et al. (2020) study the setting in which the follower can choose any solution method out of a discrete set $\mathcal{H}$ of possible options that may include exact methods, heuristics, or approximation algorithms. The leader knows this set of potential solution methods but she is uncertain regarding the actual choice of the follower. To hedge against sub-optimal follower's decisions that may stem from the follower's use of an inexact solution method, the authors propose three modeling approaches. First, they follow a robust approach by hedging against the worst-possible choice of the follower's solution algorithm. The leader then solves the problem

$$\min_{x \in X} \max_{h \in \mathcal{H}} F(x, y^h),$$

where y^h denotes the "solution" of the lower-level problem using method $h \in \mathcal{H}$. Let us point out that the proposed model does not contain coupling constraints but allows for integer upper- and lower-level variables. The second approach follows to some extent the notion of Γ-robustness, which has been proposed in Bertsimas and Sim (2003) for single-level optimization. Here, the previous model is adapted such that the leader only hedges against the Γth least damaging choices of the solution algorithm for the lower-level problem. The parameter $\Gamma \in \{1, \dots, |\mathcal{H}|\}$ is used to control the leader's level of conservatism. In contrast to the aforementioned modeling approaches, the third model relies on the leader's prior knowledge about the probability p_h that the follower will use method $h \in \mathcal{H}$. Hence, the leader hedges against lower-level algorithmic uncertainty in a probabilistic sense by optimizing over the expected value, i.e.,

$$\min_{x \in X} \ \mathbb{E}_h \left[F(x, y^h)\right] := \sum_{h \in \mathcal{H}} p_h F(x, y^h).$$

The authors provide a detailed discussion of the proposed models for bilevel knapsack problems with a follower that may choose from an exact cutting plane method or several greedy approaches. In their computational study, the authors focus on defender-attacker problems that can be formulated as bilevel knapsack models. Numerical results are presented for 32 randomly generated instances with 15 items in which the follower can choose from an exact method and two greedy heuristics.

Shi et al. (2020) pursue the same idea as Zare, Prokopyev, et al. (2020) and study bilevel problems with an inexact follower. However, instead of specifying the available solution methods, the authors exploit the notion of k-optimality to capture local optimal solutions of the follower. The modeling of a k-optimal follower is then used to derive valid lower and upper bounds for the original mixed-integer linear bilevel problem in which the lower level is solved to global optimality. An extensive computational study on randomly generated instances of the knapsack interdiction problem, the bilevel vertex cover problem, and the bilevel minimum spanning tree problem affirms the quality of the proposed bounds.

3.4. **Limited Observability.** Another approach to account for decision uncertainty in bilevel optimization considered in the literature is known under the notion of limited observability. In this setting, the follower may, e.g., not be able to perfectly observe the leader's decision due to cognitive limitations and the leader thus faces follower's response uncertainty. This aspect plays a significant role for many practical applications—especially in defender-attacker scenarios; see Section 4.

To the best of our knowledge, uncertainties regarding the observability of the upper-level decision are first addressed in Bagwell (1995) and van Damme and Hurkens (1997). The authors consider Stackelberg games involving noise in the follower's observation of the leader's strategy. It is shown that the leader's first-mover advantage is completely lost for pure-strategy equilibria of the "noisy Stackelberg game". Nevertheless, a mixed-strategy equilibrium may exist for which the outcome converges to the outcome of the Stackelberg game under perfect observability.

Following a different approach to account for limited observability of the leader's decision, Pita, Portway, et al. (2008) are concerned with Bayesian Stackelberg games in which the leader's decision $\bar{x}$ that is perceived by the follower may deviate from the actual leader's strategy x by a bounded observation error δ. A discrete set of observation errors—which is known in advance—specifies the strategies of the leader the follower might observe. As the follower assumes that $\bar{x}$ is the strategy the leader actually plays, the follower's response is thus based on $\bar{x}$ instead of x. The leader then hedges against the worst-possible reaction of the follower due to his erroneous observation. In particular, the authors model limited observability for the setting in which there is only one leader type, e.g., a security entity, and multiple follower types, e.g., multiple types of attackers. The leader does not know the actual follower type but has prior knowledge on the probability distribution over the follower types and thus optimizes over expected values. Pita, Jain, Ordóñez, et al. (2009) and Pita, Jain, Tambe, et al. (2010) provide an extension of the previous approach by using anchoring theory (Kelly et al. 2006; Tversky and Koehler 1994) to model the follower's limited capability of observing the actual decision of the leader. Given that—due to limited observability—the follower does not have any information on the actual decision of the leader, the follower tends to assign an equal probability, i.e., a uniform distribution, to each feasible leader's strategy. The more information is revealed to the follower, e.g., via further observations, the more the follower relies on his perception of the leader's strategy. Hence, the perceived decision of the leader $\bar{x}$ is represented as

$$\bar{x}_i = \alpha \frac{1}{N} + (1-\alpha) x_i, \quad i \in [n_x], \tag{19}$$

where N denotes the number of pure strategies available to the leader and $\alpha \in [0,1]$ indicates how much weight the follower leaves on the uniform distribution. To be comprehensive, let us further mention that the discussed articles provide an additional extension by combining the concept of limited observability of the leader's decision with near-optimal decisions of the follower.

A similar setting as in Pita, Jain, Ordóñez, et al. (2009), Pita, Jain, Tambe, et al. (2010), and Pita, Portway, et al. (2008) is considered in Yin and Tambe (2012). In contrast to the literature discussed so far, however, they model the perceived leader's decision $\bar{x}$ as a linear perturbation of the actual leader's decision x, i.e., $\bar{x} = F^\top x + f$, where F and f have appropriate dimensions and are assumed to follow some known continuous distribution. Moreover, the authors allow for data uncertainty and a noisy execution of the leader's strategy. A branch-and-cut method to solve this type of Bayesian Stackelberg game is proposed that incorporates Benders decomposition and heuristic branching rules. The method is assessed using 30 Stackelberg games with randomly generated utility matrices, 5 pure strategies per player, and up to 200 follower types.

Due to the vast amount of publications on limited observability in the context of Bayesian Stackelberg games or, more generally, Stackelberg security games, the literature discussed so far is far from being comprehensive. Hence, we refer the interested reader to Jain et al. (2010), Kar et al. (2018), Kiekintveld et al. (2011), Paruchuri, Pearce, Marecki, et al. (2008), Paruchuri, Pearce, Tambe, et al. (2007), Pita, Jain, Ordóñez, et al. (2009), Pita, Jain, Tambe, et al. (2010), Pita, Portway, et al. (2008), Sinha et al. (2018), Yin, Jain, et al. (2011), and Yin and Tambe (2012), as well as to the references therein.

Beck and Schmidt (2021) consider limited observability regarding the upper-level decision for bilinear bilevel problems. The authors assume that the perceived leader's decision $\bar{x}$ only takes values in a given polyhedral uncertainty set

$$\mathcal{X}(x) = \{x + P\zeta \colon H\zeta \geq h\},$$

where x denotes the actual decision of the leader and P, H, h as well as the perturbation vector ζ have appropriate dimensions. The leader hedges against follower's response uncertainty due to limited observability by pursuing a worst-case oriented approach. A single-level reformulation of the robustified bilevel problem is obtained by exploiting the KKT conditions of the lower-level problem. In particular, it is shown that the robustified bilevel problem remains in the same problem class as the problem without taking limited observability into account. Moreover, an ex-post relation between the modeling of limited observability and robust bilevel problems with an uncertain right-hand side of the lower level is established.

Korzhyk et al. (2011) consider a game in extensive form to model limited observability. To this end, nature is introduced as an additional player in the game. The game then proceeds as follows. Based on a given probability distribution, nature first determines whether the follower can perfectly observe the leader's decision or not. Second, the leader selects a probability distribution over her set of pure strategies without knowing the decision made by nature. Finally, the follower responds optimally after possibly observing the leader's decision depending on nature's choice. The authors propose a method that alternates between solving Nash and Stackelberg games in each iteration. Upper bounds on the number of iterations to obtain an equilibrium are provided.

In Karwowski et al. (2020), limited observability of the leader's strategy is again modeled using anchoring theory. The authors extend the approach proposed in Pita, Jain, Ordóñez, et al. (2009) and Pita, Jain, Tambe, et al. (2010) to account for multiple time steps by considering multi-step extensive-form games. In this setting, a fixed number of leader-follower games is played successively and, in each round of the game, the perceived leader's strategy takes the form in (19). This leads to nonlinearities which cannot be tackled by classic solution methods. Hence, the authors propose to simplify this modeling of limited observability by assuming that only the leader's decision in the last round of the game cannot be perfectly observed such as to avoid nonlinear terms. This modeling of limited observability is embedded in three state-of-the-art MILP methods as well as two other methods (Monte-Carlo sampling and an evolutionary algorithm) for solving sequential Stackelberg games, whose performance is assessed in a computational study on 25 warehouse games with up to 7 time steps.

Lu et al. (2015) introduce a novel notion of robustness for bilevel problems with upper- and lower-level decision uncertainty. In their modeling, the perceived leader's and follower's decisions are assumed to take values in a small neighborhood around the actual strategies. They pursue a robust approach in the sense that an optimal solution of the bilevel problem must be feasible for all possible realizations of the uncertainty. In contrast to traditional robustness concepts (Ben-Tal, El Ghaoui, et al. 2009; Bertsimas and Sim 2003), however, they do not follow a worst-case oriented philosophy. Instead, they hedge against uncertainties by either optimizing mean objective function values or by optimizing the relative deviations from the (exact) optimal objective function values w.r.t. a predefined tolerance.

Finally, Molan and Schmidt (2022) consider one of the extreme cases of limited observability and assume that the leader does not know the optimization problem of the follower at all but tries to learn the best-response function based on past data regarding the outcomes of the same bilevel game. The authors use a neural network to learn the best-response function and apply a tailored Lipschitz optimization approach to solve the resulting optimization problem that contains the input-output mapping of the trained neural network as a constraint.

3.5. **Intermediate Solution Concepts in Between the Optimistic and the Pessimistic Case.** In general, the bilevel problem as defined in (1) and (2) is ill-posed if the lower-level problem does not have a unique solution. To overcome this issue, it is common to pursue either the optimistic or the pessimistic approach to bilevel optimization; see, e.g., Dempe (2002). The optimistic (or strong) approach corresponds to the setting in which the leader and the follower fully cooperate. Thus, the follower chooses his solution such as to favor the leader w.r.t. the leader's objective function value, i.e., the leader considers the problem

$$\min_{x \in \bar{X}} \min_{y \in S(x)} \quad F(x, y)$$

with $\bar{X} := \{x \in X : G(x) \geq 0\}$ and $G : \mathbb{R}^{n_x} \to \mathbb{R}^m$. Here, and in the remainder of this section, we focus on the setting without coupling constraints. In the pessimistic (or weak) setting, the follower aims to harm the leader by selecting the worst-possible reaction in terms of the leader's objective function value. The leader thus considers the problem

$$\min_{x \in \bar{X}} \max_{y \in S(x)} \quad F(x, y).$$

In particular, the pessimistic bilevel problem is a special case of a robust problem in which the set $S(x)$ is interpreted as the uncertainty set. In the literature, the optimistic approach is predominantly used. However, the general pessimistic setting has gained increasing attention recently; see, e.g., Aboussoror and Mansouri (2005), Aussel and Svensson (2019), Tsoukalas et al. (2009), Wiesemann et al. (2013), and Zheng, Wan, Sun, et al. (2013) or the recent surveys in Liu, Fan, et al. (2020, 2018) and the references therein. Nevertheless, the optimistic and the pessimistic approach represent two extreme situations regarding the follower's level of cooperation, which may be inappropriate for certain applications, e.g., if the leader is uncertain regarding the level of cooperation of the follower. The resulting necessity for intermediate solution concepts is, e.g., tackled by so-called strong-weak bilevel problems. This modeling approach allows for a partial cooperation of the follower by considering a convex combination of the leader's objective functions in the optimistic and the pessimistic setting. So far and to the best of our knowledge, intermediate solution concepts have only been considered in the literature for bilevel problems without coupling constraints, i.e., one considers the problem

$$\min_{x \in \bar{X}} \quad \left\{ \beta \min_{y \in S(x)} F(x, y) + (1 - \beta) \max_{y \in S(x)} F(x, y) \right\}.$$

Here, the parameter $\beta \in [0, 1]$ can be interpreted as the follower's probability of cooperation and is used to adjust the leader's level of conservatism. Note that for $\beta = 1$, the optimistic setting is considered, whereas for $\beta = 0$, this modeling corresponds to the standard pessimistic bilevel problem.

Among the first works that address intermediate solution concepts in between the optimistic and the pessimistic case are Aboussoror and Loridan (1995) and Mallozzi and Morgan (1996). In Mallozzi and Morgan (1996) the following two cases are distinguished. First, the authors consider the case in which the set of optimal follower's decisions is discrete, i.e., $S(x) = \{y^1(x), \dots, y^k(x)\}$. It is assumed that the leader has prior knowledge on the likelihood that the follower will choose a certain solution and model this setting in a stochastic sense by optimizing over expected values. Hence, the authors consider the problem

$$\min_{x \in X} \quad \sum_{j=1}^{k} p_j(x) F(x, y^j(x)),$$

where $p_j(x)$ denotes the probability that the follower chooses $y^j(x)$. Second, more general reaction sets $S(x)$ of the follower are considered which are assumed to be Lebesgue measurable with non-zero measure. In this case, the leader assigns a probability measure $\mu_x(y)$ on $S(x)$ for every feasible upper-level decision $x \in X$. The authors then consider the problem

$$\min_{x \in X} \quad \int_{S(x)} F(x, y) \, \mathrm{d}\mu_x(y).$$

For both cases, illustrative examples and comparisons of the obtained results with the pessimistic formulation are provided.

Sufficient conditions for the existence of solutions of strong-weak bilevel problems in finite-dimensional spaces are established in Aboussoror and Loridan (1995).

The authors show that a sequence of ε-optimal solutions converges to an optimal solution of the original problem under sufficient conditions. Similarly, existence results for the variant of the problem in infinite-dimensional spaces are provided in Aboussoror, Adly, et al. (2017). In particular, and in contrast to the techniques used in Aboussoror and Loridan (1995), no convexity assumptions are required to establish these existence results.

It is well known that bilevel problems are intrinsically hard to solve. Even linear (optimistic) bilevel problems are strongly NP-hard; see, e.g., Hansen et al. (1992). Since the consideration of a pessimistic follower adds another level to the problem formulation, pessimistic bilevel optimization—and thus strong-weak bilevel optimization—is expected to be even more challenging. When it comes to developing solution methods, the literature so far focuses on the easiest instantiations of intermediate bilevel formulations, namely linear strong-weak bilevel problems. Cao and Leung (2002) propose a penalty-based approach for this type of problem. They reformulate the strong-weak problem as a classic linear bilevel problem which can be tackled using standard solution approaches, e.g., by solving the KKT reformulation. To obtain an optimal intermediate solution, however, the resulting problem needs to be solved for all possible values of the exogenous parameter β, which models the follower's level of cooperation. This issue is addressed in Zheng, Wan, Jia, et al. (2015). The authors provide a method that avoids the "enumeration" over all $\beta \in [0, 1]$ to determine the critical points of the optimal-value function of the leader.

A relaxation-and-correction scheme to solve the linear strong-weak bilevel problem is presented in Zeng (2020). The author proposes a relaxation of the original problem by including two sets of variables and constraints of the follower—one for the optimistic and one for the pessimistic case—and relaxing the optimality of the pessimistic follower. The original problem is thus reduced to a single-leader-multi-follower problem with two independent followers. Using the KKT conditions of each of the lower-level problems, a single-level reformulation is obtained, which can be solved by state-of-the-art solvers. Afterward, the optimality of the pessimistic follower's solution is ensured via a correction step.

All the mentioned approaches so far take the follower's level of cooperation β as a parameter that is specified by the leader in advance. Jia, Wan, et al. (2011) introduce the concept of considering the follower's level of cooperation as a decision-dependent random function $\beta(x)$ of the leader's variables x. Salas and Svensson (2020) follow this approach in the context of multi-leader-multi-follower problems. In this setting, the classic optimistic and pessimistic approaches are ill-defined since the cooperation of the followers with one of the leaders may result in non-cooperation or partial cooperation with another leader. To overcome this issue, the authors assume that each leader i has a "belief" β^i regarding the followers' choice. The belief assigns a probability measure to an optimal solution of the followers for each feasible leaders' decision $x \in X$. Hence, the followers' response is a random variable that follows the decision-dependent probability distribution $\beta^i(x)$. Each leader then hedges against uncertainties regarding the followers' response by optimizing over expected values. The authors provide results on the existence of optimal solutions for this setting.

Jia and Zheng (2013) consider another intermediate solution concept that allows the leader to make side-payments to the follower. In this setting, the leader willingly gives up a portion of her optimal objective function value to offer an incentive for the follower to cooperate, i.e., one considers the problem

$$\begin{aligned}\max_{x,y,\beta} \quad & \beta F(x,y) \\ \text{s.t.} \quad & f(x,y) + (1-\beta)F(x,y) \geq \bar{\varphi}(x), \\ & x \in \bar{X},\ y \in S(x),\ \beta \in [\alpha, 1]\end{aligned}$$

with the lower-level optimal-value function

$$\bar{\varphi}(x) := \max_{y \in Y} \{f(x,y) \colon g(x,y) \geq 0\}.$$

Here, $\beta \in [\alpha, 1]$ determines how much of the leader's optimal objective function value is given to the follower as a side-payment. The parameter $\alpha \in [0,1]$ can be interpreted as a minimum allocation proportion. In particular, β is understood as a variable of the problem, which is optimized in the proposed procedure. For the linear formulation of the problem, the authors present a solution method that relies on a penalty-based reformulation and that exploits strong duality.

4. Applications

In this section, we review different areas of application in which bilevel optimization under uncertainty is used. We start with energy applications in Section 4.1, discuss the large field of interdiction problems in Section 4.2, and end in Section 4.3 with a discussion of applications from management science.

4.1. **Energy.**

There is a rather large amount of research papers in which uncertain bilevel optimization is applied to the field of energy research. Before we review the separate contributions in detail, let us first briefly summarize the main commonalities of the research in this area.

(i) In the majority of papers, a linear or bilinear setting is considered. If a bilinearity is present, it usually consists of the mix of an upper- and a lower-level variable so that the lower-level problem stays a parameterized linear problem for which compact optimality certificates such as the strong duality or the KKT theorem are available.

(ii) Except for Heitsch et al. (2022), see also Section 3.1.4, all papers consider the timing in (16), i.e., the leader takes her decision here-and-now, then the uncertainty realizes, and the follower afterward takes a wait-and-see decision.

(iii) The vast majority of papers considers a stochastic setup and only a few papers also include robustness aspects. The considered stochastic setting is usually given by a discrete and finite probability distribution that allows for stating the deterministic equivalent.

4.1.1. *Power Markets, Contracting, and Networks.*

An important application of bilevel optimization in the field of electricity is the optimization of retailer problems. Maybe the earliest application of uncertain bilevel optimization in this area is Carrión et al. (2009), where the authors consider stochastic bilevel optimization for determining optimal retailer trading strategies in future markets. Here, the retailer acts as the leader and takes her here-and-now decisions under uncertainty regarding future pool prices, the clients' demands, and the prices of the rivaling retailers. The clients act as the follower in a wait-and-see manner. Risk aversion in the upper-level is modeled using a weighted sum of the expected value and the CVaR. Moreover, stochasticity is covered by using a finite set of scenarios, which can additionally be reduced in size by exploiting further scenario reduction techniques as, e.g., discussed in Growe-Kuska et al. (2003). This allows to consider the deterministic equivalent, which has a linear lower-level problem (in the follower's variables) so that the classic

KKT approach with big-M reformulations à la Fortuny-Amat and McCarl (1981) can be applied. The remaining bilinearities that are all products of prices and quantities are then linearized using duality theory.

In Askeland et al. (2020), the authors also consider a retailer's problem but try to figure out how to use stochastic bilevel optimization to design electricity network tariffs to incentivize flexible end-users (so-called prosumers) to adapt their consumption patterns. To this end, the upper level determines the tariff at the planning stage and the curtailment of load at the operational stage while the lower level models end-users of electricity, which can either be consumers or prosumers. Thus, the lower-level problem models the operational decisions. Uncertainty is again modeled via a finite scenario set and the classic KKT reformulation is used to obtain a single-level optimization problem in which the nonlinear KKT complementarity constraints are tackled via SOS-1 techniques.

A related setting is considered in Fanzeres, Street, et al. (2015), where the authors study a hybrid approach by mixing stochastic and robust optimization to determine optimal energy supply contracting strategies. A single-level reformulation is obtained via strong duality and, as before, stochasticity is modeled using a finite number of scenarios.

Another important application of bilevel optimization in the field of power markets is to determine the optimal bidding strategy of a strategic generator in wholesale electricity markets; see, e.g., Fampa et al. (2008). There, the authors consider the classic setup in the sense that the strategic generator is the upper-level player. However, the lower-level problem does not cover a Nash equilibrium problem to model the rival's behavior at the market but models the (unknown) actions of the competitors using a finite set of scenarios with exogenous probabilities. In this setup, the upper level maximizes the expected profit of the strategic generator while in the lower-level problem, a market clearing is determined that minimizes the overall system costs. After some reformulations, the model at hand has a large similarity to bilevel models considered for optimal taxation of goods and services such as discussed in, e.g., Brotcorne et al. (2000) and Labbé, Marcotte, et al. (1998). A tailored heuristic is proposed that is motivated by the latter papers and to solve the problem to global optimality, the KKT reformulation is used including binary decompositions of integer variables to linearize the occurring nonlinearities.

In Haghighat (2014), the same overall question is tackled but with a transmission-constrained economic dispatch model in the lower level. Again, uncertainty enters the model in order to capture the unknown offers of the rivals and the market demand. In contrast to Fampa et al. (2008), however, uncertainty is handled using the Γ-robust approach (Bertsimas and Sim 2004). After deriving the robust counterpart using duality-based techniques of continuous robust optimization, a single-level reformulation is obtained by using the classic KKT approach and further nonlinearities are handled via big-M constraints and relaxation-linearization techniques (Sherali and Adams 1990). Finally, a hybrid model is presented that also takes stochasticity into account by a finite number of scenarios in order to model uncertain quantity offers of rival generators.

Ambrosius, Egerer, Grimm, and Weijde (2020) and Ambrosius, Egerer, Grimm, and van der Weijde (2022) consider multilevel optimization models for electricity markets that tackle the situation in which investment decisions have to be taken subject to uncertainty w.r.t. the future network congestion management regime such as nodal or zonal pricing.

All papers discussed so far in this section model stochasticity by considering finite scenario sets and by writing down the deterministic equivalent. For large scenario sets, this leads to large-scale single-level reformulations that might be hard to solve even for state-of-the-art commercial solvers. This is especially the case for modeling strategic investment decisions that need to incorporate a large planning horizon and, thus, very large scenario sets. This issue is considered in Kallabis et al. (2020), where the authors tackle this situation using a rolling-horizon approach. Here, the strategic investment by the generator is part of the upper level, whereas the lower level includes the decisions of the market operator that clears the market in order to maximize welfare for given consumer demands, given installed generation capacities, and given price bids of the producers. The investment process is split into multiple stages so that wait-and-see decisions can be modified over time. The rolling-horizon approach is then applied to an MPEC that is obtained by the deterministic equivalent for given scenario trees of electricity demand and by using KKT conditions for the lower level as well as SOS-1 techniques to tackle KKT complementarity conditions.

In another very recent contribution (Zeng, Dong, et al. 2020), the optimal configuration of electricity vehicle charging stations and the corresponding pricing schemes are studied. These two decisions are modeled in the upper level and the lower-level problem comprises the actual charging decisions of plug-in electric vehicle owners. The lower-level objective function is of min-max type and, thus, models a robust setup. The classic KKT reformulation leads to a single-level problem with a max-min-max structure in the objective, which is then solved using the column-and-constraint generation method developed by Zeng and Zhao (2013).

Furthermore, Kovacevic and Pflug (2014) survey bilevel modeling of electricity swing option pricing. The authors carefully develop their model, which leads to a stochastic and dynamic multistage bilevel problem. The survey also discusses solution techniques—in particular for bilinear swing option problems.

4.1.2. *Gas Market and Further Energy Applications.* Besides applications in the power sector, there are also other energy sectors that have been modeled using uncertain bilevel optimization problems. For instance, in U-tapao et al. (2016), the authors set up a bilevel model for optimizing wastewater treatment plans by deciding on the size of compressed natural gas (CNG) filling stations and their locations. The lower-level problem consists of many downstream markets including agriculture, CNG transportation, residential natural gas, and electricity markets. Each downstream market is modeled by its own KKT conditions plus suitably chosen market clearing conditions. Uncertainties stem from, e.g., fuel or electricity prices and are modeled using a finite set of scenarios. It is assumed that there is no correlation between these uncertain aspects—which is, what the authors admit, a simplification. The overall setting leads to an SMPEC, which is reformulated as a mixed-integer linear problem via SOS-1 techniques.

Another branch of research on the European natural gas market started rather recently with the modeling paper by Grimm et al. (2019). There, a four-level model of the so-called European entry-exit gas market system is developed and it is shown that this system can be reduced to a bilevel problem under suitable economic assumptions such as perfect competition. The upper level then consists of the decisions of the transmission system operator (TSO) whereas the lower-level problem models the long- to short-term market behavior of gas buyers and sellers. Due to the EU regulation, the upper-level problem contains a robust constraint so that the overall problem of the TSO can be seen as a special case of an adjustable robust problem (Ben-Tal, El Ghaoui, et al. 2009; Ben-Tal, Goryashko, et al. 2004); see, e.g., Labbé, Plein, Schmidt, and Thürauf (2021) for an in-depth discussion of this relationship. The overall model is rather challenging; see Labbé, Plein, and Schmidt (2020), Labbé, Plein, Schmidt, and Thürauf (2021), Schewe et al. (2020), and Thürauf (2022) for complexity studies and Böttger et al. (2021), Plein et al.

(2021), and Schewe et al. (2022) for solution techniques and numerical results. Let us finally remark that the paper by Heitsch et al. (2022), which we already discussed in Section 3.1.4, also considers this bilevel problem with additional uncertainties in the lower-level problem that are modeled using a chance constraint.

4.2. **Interdiction, Defender-Attacker, and Security Applications.** Interdiction problems are a special class of bilevel optimization problems in which the leader (who acts as a defender) aims at preventing adversary activities of the follower (who acts as an attacker). They are typically used for identifying vulnerabilities of a system when it comes to potential disruptions (being accidental or intentional) to its infrastructure. Interdiction problems follow the common structure of bilevel problems without coupling constraints, namely

$$\max_{x \in \bar{X}} \left\{ \min_{y \in Y} f(x, y) \colon g(x, y) \geq 0 \right\},$$

where the set $\bar{X} = \{x \in X \colon G(x) \geq 0\}$ describes feasible interdiction policies. Let N_y denote the assets that can be interdicted by the leader. Decisions of the leader and the follower are linked through constraints $g(x, y) \geq 0$, which are typically given as $y_i \leq U_i(1 - x_i)$ for all $i \in N_y$. Here, U_i is the default available capacity of asset i (if not interdicted). Interdiction actions (modeled by variables x_i) can be discrete (in which case the assets are made unavailable for the follower) or continuous (if the capacity of the asset is modified, based on the intensity of interdiction). If the interdiction affects only the objective function of the follower, then their nominal objective function value determined as $\sum_{i \in N_y} d_i y_i$ is modified by adding a bilinear term $\sum_{i \in N_y} \delta_i y_i x_i$ to it, where δ_i represents the cost increase for each asset interdicted by the leader. For a comprehensive survey on interdiction problems, we refer to Smith and Song (2020) and Section 6 in Kleinert et al. (2021). Interdiction problems on networks are the most frequently studied problem variants in which the leader controls the network resources (nodes, edges) by eliminating them, reducing their capacities, or increasing the costs of their usage.

In more realistic settings of defender-attacker games, either party may not have complete information about their opponent's strategy or about the underlying conditions such as the network topology, arc or node costs, or their capacities. Hence, interdiction problems under uncertainty are gaining increasing attention of the bilevel optimization community. Also, for these problems, we can distinguish between wait-and-see followers (who observe the leader's decision and the realization of random variables), which is determined by the timing as in (16), and here-and-now followers (who—as customary—observe the leader's decision but need to deal with parameter uncertainty in the second stage of the lower-level problem), which is the timing given in (17).

Early examples of interdiction problems under uncertainty include the stochastic shortest path interdiction introduced by Israeli (1999) or the stochastic maximum-flow interdiction studied by Cormican et al. (1998).

4.2.1. *Stochastic Interdiction Problems (SIPs).* In SIPs, the uncertainty can be in costs or capacities as well as in the effect of interdiction, i.e., an interdicted resource can be partially or completely destroyed only with a certain probability. It is commonly assumed that the underlying probability distribution of the random variables is known to the leader, who is a risk-neutral decision maker and is thus optimizing the expected value of the opposite of the follower's objective function.

For the example given above, the SIP variant is given by

$$\max_{x \in \bar{X}} \ \mathbb{E}_\omega \left[\min_{y \in Y} \sum_{i \in N_y} (d_i + \delta_i(\omega) x_i) y_i \right], \tag{20}$$

where the actual value of $\delta_i(\omega)$ is revealed to the follower before he makes his decision. If the uncertainty is in the asset cost after the interdiction, then $\delta_i(\omega)$ typically represents the increase of cost in scenario ω. On the other hand, if the uncertainty is in the effect of interdiction, a binary random vector $\tilde{s}_i$ is associated to each asset $i \in N_y$. The value of $\tilde{s}_i$ is equal to one with probability p_i (indicating that the interdiction attempt of this asset is successful) or zero with probability $1 - p_i$ (otherwise). It is typically assumed that separate interdiction attempts are independent and that each asset can be interdicted at most once. In this case, for a given δ_i representing the increase of the asset cost after interdiction, we have $\delta_i(\omega) = \delta_i \tilde{s}_i(\omega)$, where ω represents a possible scenario realization.

Two main approaches to model the uncertainty are adopted in the literature: (i) sample average approximation (SAA) or (ii) sequential approximation (SA). SAA allows to transform the original two-stage (or multi-stage) stochastic problem into its deterministic equivalent while guaranteeing a certain quality of the obtained solution for a sufficiently large number of scenarios; see Kleywegt et al. (2002) for further details. On the other hand, SA starts with a small set of discrete scenarios for which valid lower and upper bounds for the original problem are derived. These scenarios are then iteratively refined by partitioning the uncertainty space until a sufficiently small gap between lower and upper bounds is obtained.

Both SAA and SA allow to transform the original problem into its deterministic interdiction counterpart with a potentially large number of discrete scenarios $\omega \in \Omega$. For problem (20), its deterministic equivalent is given as

$$\max_{x \in \bar{X}} \sum_{\omega \in \Omega} p_\omega \left\{ \min_{y^\omega \in Y} \sum_{i \in N_y} (d_i + \delta_i(\omega) x_i) y_i^\omega \right\}, \tag{21}$$

where y^ω refers to lower-level variables representing the optimal response of the follower in scenario ω. The latter bilevel problem can then be reformulated as a single-level MI(N)LP using common techniques for interdiction problems such as dualization or a strong-duality-based reformulation (if the lower level problem is convex for a given choice of x and ω) as well as penalization (otherwise). More details on these reformulation techniques can be found in recent surveys by Kleinert et al. (2021) and Smith and Song (2020). Due to the large number of scenarios involved, the employment of sophisticated decomposition techniques is indispensable in order to develop computationally effective methods. Bailey et al. (2006) consider a generalization of SIPs in which the leader acts as an interdictor, and the follower's decision making process is modeled using a Markov decision process (MDP). As in SIPs, we are given a finite set of discrete scenarios, however, for each (discrete) choice of the leader and for each scenario realization, the follower is solving an MDP. The authors propose a Benders decomposition approach under the assumption that transition probabilities in the MDP are not affected by the decisions of the leader.

In the following, we review some of the most studied applications of stochastic interdiction problems in networks.

Stochastic Shortest Path Interdiction: In this setting, the leader wishes to interdict arcs of a given network by increasing their cost within a limited interdiction budget so that the shortest path between two distinct nodes s and t in the resulting network is maximized. In his PhD thesis, Israeli (1999) assumes that the success of an interdiction attempt is uncertain and, hence, the original deterministic max-min objective function is replaced by maximizing the expected length of the shortest path over all possible interdiction decisions. This corresponds to Model (21) in which binary interdiction variables are associated to arcs of the network, the set $\bar{X}$ models all feasible interdictions under a given knapsack-like budget constraint, and the set Y models all s-t paths in the given network. The author points out that single-level reformulations can be derived following reformulation methods proposed by Israeli and Wood (2002) for the deterministic shortest path interdiction. However, the major difficulty arises from the exponential number of possible scenarios and the author proposes several SA-based methods to deal with them.

Nguyen and Smith (2022) study a variant of this problem in which the base arc cost $d_i \geq 0$ introduced above is not known to the leader, whereas the cost increase $\delta_i \geq 0$ caused by interdiction is certain. The leader assumes that the base cost values are uniformly distributed within given (arc-specific) intervals, whereas the follower acts in a wait-and-see manner. As customary, the leader maximizes the expected shortest path cost attainable by the follower. Nguyen and Smith (2022) develop an SA approach inspired by the work of Cormican et al. (1998) with bounds derived using Jensen's inequality. The authors also provide several algorithmic strategies for accelerating the convergence of their exact approach. Computational results are provided for randomly generated networks with up to 20 nodes.

Held et al. (2005) consider a class of network interdiction problems introduced by Hemmecke et al. (2003) in which the network topology is uncertain and probabilities of each possible network configuration scenario $\omega \in \Omega$ are provided as data: If an arc is not available in a given scenario, its cost is given by $d_i(\omega) = \infty$ and, as customary, $\delta_i(\omega) \geq 0$ denotes the cost increase in case of arc interdiction in scenario ω. In this setting, the leader wishes to maximize the probability of *sufficient disruption*, e.g., to maximize the probability that the shortest path length in the interdicted network is above a given threshold value $1 > \varphi > 0$. The problem is stated as

$$\max_{x \in \bar{X}} \mathbb{P}\left(\min_{y \in Y(\omega)} \sum_{i \in N_y} (d_i(\omega) + \delta_i(\omega) x_i) y(\omega)_i \geq \varphi \right).$$

The authors show that the latter problem can be decomposed by scenarios and solved in a cutting-plane fashion using the method of Riis and Schultz (2003). A computational study is conducted on networks with up to 110 nodes and considering up to 100 scenarios. In a similar setting studied by Song and Shen (2016), dubed *risk-averse* shortest path interdiction, the sufficient disruption is imposed in form of a chance constraint over a discrete set of scenarios. The leader controls interdiction variables x and minimizes the interdiction cost subject to $\sum_{\omega \in \Omega} p_\omega z_\omega \geq 1 - \varepsilon$. Here, the binary variable z_ω is set to one if and only if the follower's shortest path in scenario ω is above the threshold φ, i.e.,

$$\min_{y \in Y(\omega)} \sum_{i \in N_y} (d_i(\omega) + \delta_i(\omega) x_i) y(\omega)_i \geq \varphi z_\omega.$$

The authors propose several families of valid inequalities and develop a branch-and-cut algorithm based on scenario decomposition. In this work, computational results are provided for small grid graphs with up to 64 nodes and for two transportation networks with up to 44 nodes. In both cases, up to 1000 scenarios are considered.

Stochastic Maximum Flow Interdiction: Cormican et al. (1998) investigate the case of minimizing the expected maximum s-t flow in a given network by removing some arcs (or reducing their capacities), assuming that the effect of interdiction is uncertain. More precisely, the interdiction success of each arc is assumed to be an independent binary random variable such that a successful interdiction (which can be performed with some known probability) leaves the arc with no capacity. They also introduce other variants of the problem in which both interdiction success and arc capacities are random or in which multiple interdictions per arc are allowed. The authors apply SA based on classic bounding techniques in stochastic optimization and gradually refine the set of discrete scenarios until obtaining a sufficiently small optimality gap. Janjarassuk and Linderoth (2008) apply a duality-based MILP reformulation, combined with SAA and Benders decomposition, and implement a distributed algorithm. The authors obtain significant speed-ups compared to previous techniques, which is due to a successful combination of decomposition, sampling, parallel computing, and heuristics. They conduct a computational study on grid graphs with up to 400 nodes and consider between 50 and 5000 scenarios.

Lei et al. (2018) consider a variant of the problem with uncertain interdiction effects in which the follower, after observing the leader's interdiction action and before the uncertainty is revealed, can add additional arc capacities to mitigate flow losses. The authors study risk-neutral and risk-averse approaches to model the leader's behavior, i.e., they incorporate the expectation, left-tail, and right-tail CVaR for evaluating maximum flows under uncertainty in the leader's and follower's objectives. The resulting bilevel or trilevel mathematical models are reformulated into single-level MILP formulations and, using an SAA approach, are applied to real-world network instances. Their instances are derived from SNDlib (see `sndlib.zib.de`) and contain 100 scenarios. Finally, Atamtürk et al. (2020) assume that arc capacities are uncertain and that their mean values along with the covariance matrix are known to the leader. The leader removes a subset of arcs from the network while minimizing the VaR of the maximum flow on the resulting network for a given confidence level. The authors propose a heuristic procedure based on successive quadratic optimization embedded in a bisection search. Computational results are reported for a set of grid networks with $q \times q$ nodes (created in a similar way as in Janjarassuk and Linderoth (2008)) with q ranging between 20 and 100.

Maximum Reliability Path Interdiction: In this family of problems, the origin and the destination node chosen by the follower may be uncertain. The leader installs sensors at some arcs of the network and the follower seeks to find a path in the resulting network that maximizes the probability of remaining undetected. The probabilities of being detected with and without sensors installed are known to both players. The problem is, e.g., relevant for preventing nuclear smuggling activities between two countries by placing sensors at their borders (Morton et al. 2007; Pan, Charlton, et al. 2003; Pan and Morton 2008). The leader installs sensors within a limited budget so as to minimize the expected value of the maximum reliability path over all possible source-target choices of the follower. When the probabilities of traversing interdicted arcs are all strictly positive and the follower's source-target choice is known, one can obtain a deterministic shortest-path interdiction problem with a logarithmic transformation; see Morton (2011) and Towle and Luedtke (2018). Morton et al. (2007) and Pan and Morton (2008) propose step inequalities to exploit the relationships between evasion probabilities associated with different paths. Sullivan et al. (2014) use the problem transformation in a bipartite network, provide polyhedral results for a single-scenario, i.e., the deterministic, case, and show how these can be exploited in a multi-scenario setting. The problem can also be reformulated as a two-stage stochastic problem and Bodur et al. (2017) use this reformulation for assessing novel generic ideas to strengthen Benders cuts by exploiting integrality of the first-stage variables. Towle and Luedtke (2018) provide an alternative path-based reformulation and develop a branch-and-cut algorithm that is based on supermodular cuts from Nemhauser et al. (1978) and Ahmed and Atamtürk (2011). The latter approach is shown to be the new state-of-the-art w.r.t. the set of instances considered in the previous literature. These instances are built from a network with 783 nodes as well as 2586 arcs and 456 scenarios are considered. A generalization of this problem with applications in cyber security is studied by Ertem and Bier (2021). Lunday and Sherali (2012) study the problem setting in which the source-target pair is known to the leader, however, there are multiple interdiction resources that can be employed and the impact of their combination can be nonlinear.

Michalopoulos et al. (2015) assume that, in addition to the above uncertainties, the leader is uncertain about the interdiction budget as well. Hence, a three-stage stochastic interdiction approach is proposed in which the leader starts by forming a priority list and assigning an appropriate number of interdiction locations to each priority level. After the budget is revealed, the leader installs sensors at the highest priority locations until the budget is exhausted. Finally, in the third stage, the follower solves the maximum-reliability path problem. A tabu search heuristic is used to solve this challenging problem.

Contrary to the above assumption that the follower's path is deterministic, once the source-target pair and the interdicted arcs are revealed, Collado et al. (2017) assume that the path chosen by the follower remains uncertain to the leader. This setting corresponds to a decision uncertainty caused, e.g., by the uncertainty regarding the follower's criterion for choosing his path through the network or by random influences encountered by the follower while traversing the network. Hence, the follower's path choice is known to the leader only in terms of a probability distribution reflecting her beliefs. The authors propose a risk-neutral and a risk-averse modeling approach while assuming that the leader deploys her resources (modeled using continuous interdiction variables) before and after discovering the source chosen by the follower. The authors employ a mean-upper semideviation risk measure for the risk-aversion approach. The risk-averse problem is approximated and reformulated as a single-level LP model.

4.2.2. *Robust Interdiction Problems.* There is much less literature available for interdiction problems in which there is no assumption on the distribution of the uncertain parameters. Chauhan (2020) studies maximum-flow interdiction with interval uncertainty w.r.t. arc capacities and consumption of resources required to interdict an arc. Following the Γ-robust modeling approach (Bertsimas and Sim 2004), the leader assumes that only a limited number of arcs can be subject to uncertainty, both w.r.t. interdiction budget and arc capacities. It is a wait-and-see setting in which the follower observes arcs removed by the leader and a realization of the uncertain arc capacities for the remaining ones. The leader seeks for an interdiction strategy that protects her from the worst-case outcome in which the capacity of Γ arcs is the largest possible. An MILP formulation is proposed along with three heuristics based on Lagrangian relaxation, Benders decomposition, and a combination of the two.

Nikoofal and Zhuang (2011) consider defender-attacker problems with applications in counter-terrorism. The leader and the follower have different perceptions regarding the valuation of the damage caused by attacking given targets. The leader searches to minimize the damage caused by the follower, however, the follower's valuation of targets is unknown and it is assumed to belong to bounded intervals. Using the Γ-robust approach, the authors assume that the total scaled deviation of the uncertainty parameters cannot exceed a given threshold Γ and the corresponding robustness constraint is added at the upper level. In a follow-up article, Nikoofal and Zhuang (2015) study the trade-off between disclosure of the defense strategy by the leader (which corresponds to a Stackelberg game) versus secrecy (which corresponds to a simultaneous game), assuming in both cases that the leader only knows intervals to which the attacker's valuation of targets belongs. Their results show that the leader's benefit by making the first move, i.e., by playing the Stackelberg game, is only considerable if the follower and the leader share a similar valuation of the targets. Thus, the optimal defense allocation in a simultaneous game provides a better protection against uncertainty in the follower's valuation of targets.

Gillen et al. (2021) study the spread of cascading behavior in a social network using a linear threshold (LT) model with a given set of activated nodes. In the cascading LT model, new nodes are getting activated if the sum of arc weights of their already activated neighbors is above a given threshold value. In the defender-attacker problem studied by Gillen et al. (2021), the leader tries to fortify some nodes by increasing their influence threshold within a limited budget in order to reduce the total number of activated nodes at the end of the propagation process. The follower's problem models the propagation process and the authors assume that the arc weights (modeling the influence of a node to its neighbors) are subject to interval uncertainty. The authors use a Γ-robust approach to deal with uncertainty and develop an iterative procedure in which the problem is solved on a subgraph using a cutting-plane procedure. The subgraph is then expanded and the procedure is repeated until the convergence criteria are met. A computational study is conducted on a selection of real-world networks from the SNAP library (see https://snap.stanford.edu) with some of them containing more than 200 000 nodes.

Beck, Ljubić, et al. (2022) propose a generic branch-and-cut framework for solving min-max mixed-integer optimization problems with a Γ-robust uncertainty modeling in the lower level. The follower takes decisions after observing the action of the leader and before facing the uncertainty, i.e., the timing (17) is assumed. The follower aims to hedge against a subset of deviations of uncertain parameters and the lower-level problem contains discrete variables. Two cases of interval uncertainty are studied: in the coefficients of the lower-level's objective function and in the coefficients of a single packing-type constraint. Two generic reformulations as a single-level MILP are proposed and problem-tailored cuts have been derived for the Γ-robust variants of the knapsack interdiction problem. These cuts assume that Γ-robust follower sub-problems satisfy a downward monotonicity property, which arises in many packing-type applications. In this context, these cuts are a generalization of what has been proposed in Fischetti, Ljubić, Monaci, et al. (2019) for the deterministic knapsack interdiction problem.

4.2.3. *Other Interdiction Problems with Incomplete or Asymmetric Information.* Bayrak and Bailey (2008) study a shortest path network interdiction problem with asymmetric information in which the follower has inaccurate or incomplete information while the leader has complete knowledge of the network. Hence, the perceived arc costs and their increase caused by interdiction are different for the two players. Therefore, the objective functions of the leader and the follower do not coincide, and the problem assumes a structure of a more general bilevel optimization problem with a convex lower-level model. The authors propose a reformulation as a single-level MILP and demonstrate in their computational study that asymmetric information allows to obtain improved interdiction policies compared to those using symmetric information.

Pay et al. (2018) consider a stochastic shortest path interdiction problem with incomplete risk preferences of the leader. As customary, there is uncertainty in the arc costs and in the interdiction effect on each arc and, after the interdiction, the follower reacts in a wait-and-see fashion. A finite set of scenarios is used to model uncertainty realizations. Contrary to previous studies, the authors assume that the leader is risk-averse and that there exists a utility function that summarizes her risk preferences but that her knowledge about this function is incomplete. To this end, the authors propose two ways to deal with this ambiguity: (i) to use historical data and pairwise comparison of lotteries to fit a piecewise concave utility function and run the stochastic interdiction model afterward, or (ii) to integrate utility estimation within the optimization model. The latter leads to a robust approach (originally proposed in Armbruster and Delage (2015) and Hu and Mehrotra (2015)) in which the leader searches for an interdiction strategy that maximizes her utility in a robust fashion, i.e., by considering an infimum over the function space of all possible utility functions $u \in U$ consistent with leader's preferences:

$$\max_{x \in X} \inf_{u \in U} \mathbb{E}\left[u\left(\min_{y \in Y} \sum_{\omega \in \Omega} p_\omega \sum_{i \in N_y} (d_i + \delta_i(\omega) x_i) y_i\right)\right].$$

A single-level MILP reformulation for this problem is proposed and solved using a branch-and-cut procedure. Computational results are reported for small grid networks with up to 49 nodes and up to 1000 scenarios.

Sequential shortest path interdiction games with asymmetric information have been studied in a series of papers by Borrero et al. (2019, 2016) and Yang, Borrero, et al. (2021). In the sequential decision making setting proposed by Borrero et al. (2016), there is a repeated interaction between the leader and the follower. At each period, the leader attacks the follower by interdicting a subset of assets with the goal of maximizing the cumulative follower's costs over the given time period. The leader has incomplete knowledge concerning the structure and arc costs in the network, with only lower and upper bounds on the arc costs being available. The leader learns about the network and arc costs trough sequential interdiction actions (thanks to the optimal responses of the follower) and dynamically adapts her interdiction strategy. The leader observes the chosen path along with its cost in order to possibly update her strategy in the following iteration. The authors introduce the concept of "policy time-stability", representing the number of learning iterations needed for the leader to reach the same interdiction strategy as if she would have complete information. Two policies (a greedy and a pessimistic one) are proposed and studied from the theoretical and computational perspective. A computational study is carried out on randomly generated uniform graphs with 40 and 50 nodes. The results have been later generalized in Borrero et al. (2019), where general interdiction problems are considered such that the leader does not know all of the follower's resources and constraints and the follower's cost coefficients are assumed to belong to a polyhedral uncertainty set. Three types of follower's feedback are studied: standard (the leader observes the value of the follower's objective only), value-perfect (cost coefficients of the follower's activity are revealed, too), and response-perfect (the full decision vector of the follower is revealed to the leader). For their computational study, the authors use knapsack interdiction instances with up to 15 items.

In a follow-up article, Yang, Borrero, et al. (2021) also study the sequential shortest path interdiction problem with incomplete information. Contrary to Borrero et al. (2016), where the feedback includes the shortest path chosen by the follower, in Yang, Borrero, et al. (2021), this feedback is limited to the length of the chosen path. In terms of policy time stability, the authors show that, in the worst case, the number of iterations of the proposed greedy interdiction policies is exponential and that convergence in polynomial time is possible if more information is provided through the feedback. Computational experiments are conducted on layered graph networks with between 3 and 10 layers as well as 7 nodes in each layer. Finally, more general two-player sequential games are studied in Borrero et al. (2022), see also Section 3.2.

4.3. **Management Science.** In this last part of this section, we review applications of uncertain bilevel optimization in the fields of networks, supply chains, facility location, and finance.

Networks: Toll pricing in networks is a bilevel optimization problem in which the leader sets the tolls for road segments of a transportation network so that the revenue raised from tariffs is maximized. The leader anticipates that user flows are assigned to cheapest paths in the resulting network; see Brotcorne et al. (2000), Labbé, Marcotte, et al. (1998), and Labbé and Violin (2016). Alizadeh et al. (2013) consider a two-stage stochastic toll pricing problem in which the leader faces uncertainties regarding travel demand and travel costs. These uncertainties are modeled through a discrete set of scenarios. The first-stage decision of the leader is to set the tariffs while maximizing the expected revenues. The tariffs set in the first stage can be modified within a pre-defined interval in the second stage once the uncertainties are revealed. The authors show how to reduce the problem to its deterministic bilevel equivalent and conduct a sensitivity analysis w.r.t. the constraints linking the tariffs at the first and the second stage. Gilbert et al. (2015) consider another stochastic variant in which the users choose their paths according to a discrete choice model. The followers namely choose a path that minimizes their disutility, which, besides the arc costs and tolls, contains an additional additive component unknown to the leader. This unobserved term is assumed to have a logistic distribution. The authors provide two heuristics derived from approximations of the logit revenue function. In her PhD thesis, Violin (2014) studies the toll pricing problem with interval uncertainty considering travel demand or the cost of toll-free paths. She applies Γ-robust models to some cases whose deterministic counterparts can be solved in polynomial time, e.g., pricing on a single arc, the single commodity case, or the application of a unit toll. Dokka et al. (2016), see also Dokka et al. (2017), study the toll pricing problem under uncertainty on non-toll costs. For the leader, the distribution of non-toll costs is unknown but, based on historical information, she assumes that the distribution is fixed and belongs to a set of non-negative distributions. The followers observe the toll rates and use the full knowledge of non-toll data (in a wait-and-see fashion) to choose the shortest paths in the resulting network. The authors consider a single follower case and model the uncertainty using the concept of distributional robustness; see, e.g., Goh and Sim (2010). For this highly complex problem, they provide mathematical formulations and heuristics for networks with multiple parallel source-target arcs and networks with a polynomial number of source-target paths.

In the literature, robust optimization is frequently applied to the hazmat network design problem. Given an existing road network, the deterministic hazmat network design problem asks for selecting the road segments that should be closed for hazmat transport so as to minimize the total risk. Here, each commodity has its own arc risk, which is taken into consideration if the arc is used in the resulting transportation network. Hence, the problem can be seen as a bilevel problem in which the leader selects the road segments to be closed while the followers, i.e., the hazmat carriers, solve shortest path problems with different source-target pairs. Moreover, it is assumed that there are no congestion effects in the resulting network. Longsheng et al. (2017) propose a robust approach to model *generalized bounded rationality* in route choice behavior modeling. They test their concept on the robust hazmat network design problem in which robust optimization models the uncertainty in the cost of shortest paths of the lower-level problems. The uncertainty models the bounded rationality of lower-level decision makers caused by their perception error. Among others, polyhedral and ellipsoidal uncertainty sets are considered and an exact method based on cutting planes is proposed. A similar problem is studied by

Xin et al. (2015), however, it is assumed that the arc risks are subject to interval uncertainty instead, whereas the arc lengths are deterministic and known to both players. The leader searches for a subset of road segments to block so that the maximum regret w.r.t. path risks over all commodities is minimized. The authors propose a heuristic approach and test it using a case study of the road network of the Guangdong province in China. Arguing that the minimax regret approach of Xin et al. (2015) is too conservative, Sun et al. (2015) use Γ-robustness to deal with arc risk uncertainties. Two cases are considered in which Γ corresponds to (i) the overall number of arcs that are subject to uncertainty or (ii) the number of arcs which are subject to uncertainty for all shipments. The authors provide single-level reformulations and a Lagrangian heuristic. In related works, Berglund and Kwon (2014) and Liu and Kwon (2020) consider a combined facility location and hazmat network design problem. They assume interval uncertainty w.r.t. transportation demands and arc risks and adopt the Γ-robust approach. The leader minimizes a linear combination of fixed facility opening cost and the worst-case arc risk exposure. The followers choose the routes that minimize the transportation cost to the nearest hazmat facility. A single-level MILP formulation and a genetic algorithm are proposed in Berglund and Kwon (2014). Liu and Kwon (2020) develop an exact method based on cutting planes combined with Benders decomposition. In this latest computational study, the authors consider a road network of the city of Ravenna with 110 nodes and 143 arcs.

Attack graphs, represented by trees or directed acyclic graphs, are frequently used to model vulnerabilities of a system. In such a network, nodes represent attack states and arcs correspond to the transition of states fulfilled by attack activities; see, e.g., Bhuiyan (2018) and further references therein. Zheng and Albert (2019) use an attack graph with completion times on its arcs to model applications for infrastructure protection planning, e.g., to mitigate cyber-security or supply chain attacks. The leader, who acts as the defender, implements policies and invests in cyber-infrastructure security, whereas multiple adversaries try to exploit vulnerabilities of the system to carry out attacks as soon as possible. An attack corresponds to a "project" whose fastest completion time is associated with a critical path in the attack graph. In their setting, arc delay times (imposed by interdiction activities) are uncertain, the followers act in a wait-and-see manner, and the leader deploys limited resources to impose delays on the arcs so that the total weighted expected completion time of all adversarial attacks is maximized. After considering a finite set of discrete scenarios and dualizing the lower-level problem, the resulting max-max problem is decomposed using a Lagrangian heuristic approach based on subgradient optimization. Another more general Stackelberg game on attack graphs is studied by Letchford and Vorobeychik (2013) and a risk-averse approach based on a CVaR model is studied by Bhuiyan (2018).

In telecommunication industry, bilevel optimization can be used to model hierarchical decision making between a network operator (the leader) and virtual operators (the followers). Audestad et al. (2006) study the problem in which the network operator solves the pricing problem so as to maximize her profit or the market share. The leader decides on the capacity leased to the virtual operators and sets the prices for these capacities as well as for the service to the customers. The followers maximize their own profit function after observing the decisions of the leader. The authors describe a two-stage stochastic bilevel optimization model in which the uncertainty of customer demands is modeled with a discrete set of scenarios. Once the customer demand is revealed, the leader has a possibility to extend the service and to change the pricing decisions. Additional results, including a Lagrangian optimization method for finding locally optimal solutions, are provided in the PhD thesis of Werner (2004). DeMiguel and Xu (2009) consider stochastic multi-leader-follower games with demand uncertainty and use it to model competition in the telecommunication industry. The leaders only know the distribution of the demand, whereas the followers have information on the exact realization of uncertainty. The leaders compete in a Cournot setting and each leader searches for a Stackelberg solution w.r.t. her followers. The followers compete in a Cournot fashion with all leaders and the other followers. The authors show the existence and uniqueness of an equilibrium for the considered stochastic model and propose a solution approach based on sample average approximation. We refer to Hu and Fukushima (2015) for further studies on multi-leader-follower Stackelberg games.

Supply Chains and Facility Location: Multi-period facility location interdiction with stochastic resource constraints is studied by Zhang and Özaltın (2021). The problem is modeled as a stochastic bilevel problem with integer variables at both levels and a branch-and-bound procedure is proposed, see Section 3.1.6 for further details. Ryu et al. (2004) consider supply chain optimization in which the upper-level problem is a plant planning problem and the lower-level problem is a distribution network problem with stochastic demands. A similar problem but in a multi-objective setting is investigated by Roghanian et al. (2007). Yeh et al. (2015) use bilevel optimization to model timber supply chains in which the harvesters decide first on the quantity to be harvested and the manufacturers decide later on how much to utilize. The authors study a problem of investing in biofuel production under uncertainty and model it as a two-stage stochastic problem in which, for each realization of uncertain parameters, the second stage is the aforementioned bilevel problem. In Su and Geunes (2013), a Stackelberg game is considered between a single-supplier and a multi-retailer supply chain under asymmetric demand information. The supplier sets price discounts first (in a multi-period setting) under uncertainty concerning the retail-store demand. After the actual demand of each store is revealed, individual store order quantities are determined in each period. The goal of the supplier is to determine the pricing strategy that maximizes her expected payoff while anticipating the store order quantities.

Finance: Yan et al. (2014) consider a stochastic approach to supply chain financing under demand uncertainty. Fanzeres, Ahmed, et al. (2019) investigate a revenue-maximizing strategic bidding problem with uncertainty concerning the competitors' bidding strategy. A two-stage robust optimization model with equilibrium constraints is proposed and reformulated as a bilevel problem with equilibrium constraints. After deriving a single-level reformulation, the authors implement a column-and-constraint generation approach.

5. Possible Directions for Future Research

Although the study of bilevel optimization problems under uncertainty started rather recently in the 1990s, there already has been substantial work in this field; see, e.g., Figure 6, which shows the number of papers per year cited in this survey.

Nevertheless, there are very many topics still open for future research. In this last section of the survey, we thus sketch a few of these potential future research topics.

(i) For the stochastic approach to address data uncertainty, only a few works exist (see, e.g., Carrión et al. (2009) and Kovacevic and Pflug (2014)) that go beyond the risk-neutral case for nonlinear bilevel models. This leaves a wide open space for future research combining risk-averse modeling (such as using the CVaR as in the papers mentioned above) for nonlinear models. Except for some interdiction problems, nonlinear bilevel models

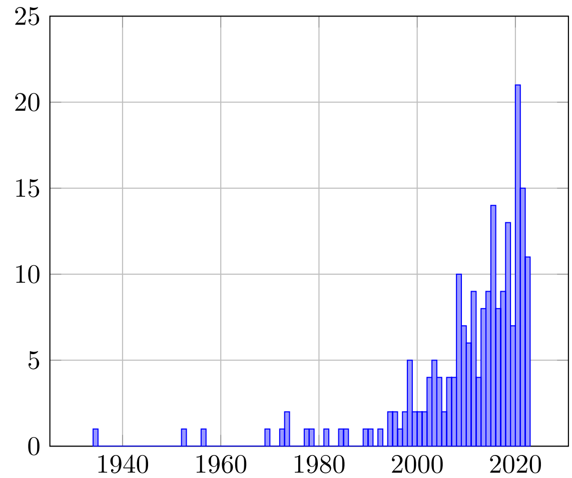

FIGURE 6. Papers per year as cited in this survey

under uncertainty are only rarely discussed in the literature and most of the research on stochastic bilevel optimization so far focuses on the linear case. The reason, most likely, is that even nonlinear bilevel optimization is an extremely challenging field and the combination of such nonlinearities with further uncertainty considerations makes such problems even more hard to study and solve.

(ii) For stochastic setups, there have been some works that algorithmically exploit (quasi-)block structures. However, apart from Held et al. (2005) and Song and Shen (2016) who study special interdiction problems, most of these works consider the risk-neutral case. Hence, the field of structure-exploiting algorithms for risk-averse bilevel models contains many possible directions for future research as well.

(iii) It seems that there is only little research (see, e.g., Adasme et al. (2013) and Dokka et al. (2016)) that combines distributional robustness (Goh and Sim 2010) and bilevel optimization although both fields, standalone, are very active fields of research today. However, since both fields on their own are already very difficult, their combination will be even more challenging.

(iv) The literature on stochastic bilevel optimization is rather theoretical. There is (again, to the best of our knowledge) no general computational paper in this field besides those that consider specific applications and that almost all use the deterministic equivalent as the main algorithmic workhorse.

(v) Robust approaches to account for uncertainties in the context of bilevel optimization are still in their infancy. So far, most works focus on the strictly or Γ-robust case. Hence, the consideration of other robustness concepts such as light robustness (Fischetti and Monaci 2009) or adjustable robustness (Ben-Tal, Goryashko, et al. 2004) may be reasonable directions for future research.

(vi) In robust setups, uncertainties are predominantly modeled using either interval or polyhedral uncertainty sets. This leaves room to study other uncertainty set geometries and, in particular, models with discrete uncertainty sets.

(vii) We are not aware of any work that considers intermediate solution concepts between the optimistic and the pessimistic approach for bilevel problems with coupling constraints. Moreover, and to the best of our knowledge, solution methods for strong-weak bilevel problems have only been considered for the linear case.

(viii) Almost all papers on uncertain bilevel optimization consider the timing in which the uncertainty realizes after the decision of the leader has been taken and before the follower decides. Although some first works (Beck, Ljubić, et al. 2022; Heitsch et al. 2022) have been published on the alternative timing given in (17), this setup still is a rather open field—in particular in the case of chance constraints as part of the lower-level problem.

(ix) There are only a few papers (Salas and Svensson 2020; Zhang, Liu, et al. 2022) that started to pave the way for the study of decision-dependent uncertainties. This is, however, another completely open field of research.

(x) Finally, there are no well-curated collections of instances in the field of bilevel optimization under uncertainty, which would, of course, help the community a lot when it comes to developing and testing novel algorithmic ideas.

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