Target-Oriented Regret Minimization for Satisficing Monopolists

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We study a robust monopoly pricing problem where a seller aspires to sell an item to a buyer. We assume that the seller, unaware of the buyer's willingness to pay, ambitiously optimizes over a space of all individual rational and incentive compatible mechanisms with a regret-type objective criterion. Using robust optimization, Koçyiğit et al. (2021) analytically derived a mechanism that minimizes the worst-case regret. In this paper, we alternatively adopt robust satisficing which minimizes the excess regret that is above the predetermined target level. We analytically show that the optimal mechanism involves the seller offering a menu of lotteries that charges a buyer-dependent participation fee and allocate the item with a buyer-dependent probability. Then, we consider two additional variants of the problem where the seller restricts her attention to a class of only deterministic posted price mechanisms and where the seller is relieved from specifying the target regret in advance. Finally, we determine a randomized posted price mechanism that is readily implementable and equivalent to the optimal mechanism, compute its statistics, and quantify the strength of the entailed randomization. Besides, we compare the proposed mechanism with a robust benchmark and numerically find that the former is predominantly superior to the latter in terms of the expected regret and the expected revenue when the coefficient of variation of the buyer's value is under a hundred percent.

Key words: Mechanism design; monopoly pricing; regret minimization; robust optimization; satisficing

1. Introduction

We study a variant of the monopoly pricing problem where the seller (*'she'*) offers an item to gain maximum benefit. The buyer (*'he'*) attaches a private value ν to the item offered, and to him this value is a personal monetary equivalent of the item. It indicates the maximum amount of money that he is willing to pay. Consequently, if this value was known to the seller, then she could have simply offered the item at a fixed price ν and earned maximally.

However, a prudent buyer would always keep his value strictly private before the seller making her first move in order to not lose his bargaining power. It is therefore impossible for the seller to earn exactly an amount of ν from the sale transaction. The seller designs and broadcasts a mechanism that sets out the instructions on how the buyer can communicate with her. A mechanism consists of three parts: the set of messages for the buyer to choose from, the allocation rule, and the payment rule. Once the buyer selects a message that to him is the most favourable, he will pay an amount indicated by the payment rule to the seller and be allotted the item with a certain probability that is specified by the allocation rule. Modern studies of mechanism design largely simplify due to the influential Revelation Principle (Myerson 1981) which allows the seller to restrict her attention to a subclass of direct mechanisms, where the message set is simply chosen as the set of all possible values of the buyer's private value ν, without any loss of generality and revenue. In the remainder of the paper, we will henceforth use the terms mechanism and direct mechanism interchangeably.

As ν is unknown to the seller, it may be perceived as a random variable that crisply follows a known probability distribution. If that is the case, then Myerson (1981) and Riley and Zeckhauser (1983) showed that the seller can attain a maximum expected revenue by posting a deterministic price for the item. Although without a doubt this is the most popular variant of the mechanisms currently seen in practice, especially at retail stores, several studies have outlined the advantages of the seller offering the same item at different price points. Wilson (1988) demonstrated that a price dispersion can increase the seller's revenue when the marginal cost of producing the item is not a constant, whereas Salop (1977) showed that the seller may benefit from the consumer heterogeneity in search costs by offering the same item at different prices in different locations.

The assumption that the seller knows the distribution of the buyer's private value ν itself is not innocuous, and a question concerning the robustness of the so-called optimal mechanism has come under the spotlight (Bergemann and Morris 2005). A seller with limited information about ν may consider stepping away from maximizing the expected revenue and instead leverage robust optimization (see *e.g.,* Ben-Tal et al. 2009 and Bertsimas et al. 2011) with an aim to maximize the worst-case revenue. While this approach is methodologically sound, it may recommend a trivial and unrealistically conservative mechanism where the seller keeps the item to herself provided that the uncertainty set for ν contains zero. We remark that if multiple bidders competing to acquire the same product are involved, then the robust mechanism design problem involves a multidimensional uncertainty set, and it is possible to escape such extreme pessimism by using a budgeted uncertainty set; see Bandi and Bertsimas (2014). The complexity of multidimensional mechanism design means however that a specialized algorithm is needed. A distributionally robust version of the problem was also recently considered by Koçyiğit et al. (2020).

To address having limited information and also to avoid being superfluously conservative, Savage (1951) introduced a minimax regret as a decision criterion; see also Poursoltani and Delage (2021) and Perakis and Roels (2008) for further justifications. Specifically in a sale transaction, the seller's regret is defined as the difference between the hypothetical revenue that the seller could have earned if she exactly knew ν and the actual revenue which is specified by the payment rule. Koçyiğit et al. (2021) demonstrated that the mechanism that attains the smallest worst-case regret is non-trivial, and it involves a piece-wise logarithmic allocation rule and a piece-wise linear payment rule even if ν can take a value of zero. The minimax regret criterion has also been adopted in several other papers, for example, Bergemann and Schlag (2008) and Bergemann and Schlag (2011) where their aim was to determine the optimal distribution of the randomized price. In addition, Bergemann and Schlag (2011) aptly noted that "a deterministic pricing policy exposes the seller to substantial regret, and that the seller can decrease her exposure by offering many prices," which underscores the importance of a randomized pricing strategy when a regret-type objective is used by the seller. Slightly differently, Chen et al. (2022) and Wang et al. (2020) accounted for a few summary statistics of ν besides its support and adopted a worst-case relative regret criterion, also known as a competitive ratio, and again the proven benefit of a randomized strategy remains.

The principle of worst-case regret minimization resonates well with that of robust optimization. A duality technique that is frequently leveraged in the robust optimization literature is indispensable for Koçyiğit et al. (2021) to analytically derive the optimal mechanism that minimizes the seller's worst-case regret. We refer our readers to Vohra (2011) for a broader discussion on the link between mechanism design and linear programming as well as the corresponding duality theory.

For the comprehensiveness of our literature review, we remark that a risk-neutral, ambiguity-averse seller may instead use distributionally robust optimization (*e.g.,* Delage and Ye 2010, Wiesemann et al. 2014 and Mohajerin Esfahani and Kuhn 2018) to maximize her worst-case expected revenue when facing the inadequacy of robust optimization in mechanism design. The key component of such a framework is how the worst case should be defined, and to answer this the seller needs to construct a set of all probability distributions of ν that are consistent with her prior information. For instance, Bergemann and Schlag (2011) considered a neighbourhood of a reference distribution of ν with respect to the Prohorov metric and showed that there exists a deterministic posted price mechanism that is optimal. When the neighbourhood is instead based on the Wasserstein metric, Li et al. (2019) proved that the optimal mechanism involves a randomized allocation and payment. Pınar and Kızılkale (2017) argued that when the seller is only informed about the mean valuation of the buyer, the benefit of price randomization manifests. Later, Carrasco et al. (2018) provided an important generalization and considered a seller who has access to the higher-order moments of ν in addition to its mean. Recently, Chen et al. (2021) provided a unified framework that can be used to characterize robustly optimal mechanisms under different ambiguity sets of distributions including some of those previously considered in the above literature, and through duality they established a connection between the distributionally robust mechanism design problem and personalized pricing (see *e.g.* Elmachtoub et al. 2021).

Recently, Long et al. (2022) provided a follow-up on an earlier work on globalized robust optimization due to Ben-Tal et al. (2017) and introduced an alternative to robust optimization known as *'robust satisficing.'* When the objective function representing cost (or regret) is uncertain, robust optimization computes a solution that minimizes the worst-case cost. On the contrary, robust satisficing determines a solution that, in the nominal scenario, has a cost under the predetermined target and, in all other scenarios, a cost that only proportionately deviates from the same target. Abiding by this new principle, the decision maker needs to specify the nominal scenario and to supply the target value. In practice, a robust optimizer typically controls the level of conservativeness by differently sizing the uncertainty set (and oftentimes the uncertainty set has to be chosen uncharacteristically small so that the solution obtained remains useful) and a robust satisficer by calibrating the target for the nominal objective value, which in this paper is the seller's regret. In our context of monopoly pricing, we impose no statistical requirement on the nominal ν. Its value can thus be estimated or acquired with relative ease, and having this additional piece of information could result in the seller's substantial gain (Eren and Maglaras 2010). In this paper, we revisit the monopoly pricing problem that was studied by Koçyiğit et al. (2021), but we formulate the problem using the robust satisficing ideology instead. Computationally, depending on the convexity and linearity properties of the problem (or the lack thereof), robust satisficing solutions could be exactly determined by Fenchel duality (Ben-Tal et al. 2017) or approximately attained by using either a primal decision rule (Long et al. 2022) or a dual linear decision rule (Ramachandra et al. 2022). To our knowledge, we are the first to exactly solve a robust satisficing problem of this level of complexity, that is, the infinite dimensionality of the mechanism design problem and the interaction between the two agents (*i.e.,* the buyer and the seller).

Finally, we list a few references where the optimality of a single-item sale mechanism with respect to both the worst-case expected revenue and the worst-case regret objective can be extended to a multi-item and/or multi-bidder environment: Carroll (2017), Gravin and Lu (2018), Koçyiğit et al. (2021), Koçyiğit et al. (2022) and Zhang (2022). Even though the proofs of such results are often convoluted, we are encouraged by them and hope to consider a similar generalization in the future.

We summarize the main contributions of the paper as follows.

- We use robust satisficing to formulate a mechanism design problem for a monopolist who has an item to liquidate and a regret minimization objective. The problem takes as input parameters the support of the buyer's value for the item and its nominal value as well as the target regret level that should not be exceeded under the nominal scenario.

- We characterize the condition on the problem's input that is equivalent to the problem's feasibility. Whenever it is feasible, we *analytically* propose a candidate solution of the mechanism design problem. We establish that the proposed mechanisms are *optimal* with respect to the problem's given input by developing tight lower bounds of the problem. We argue that each of these lower bounds has an intimate relationship with the problem's dual; see Appendix A.
- We study two additional variants of the mechanism design problem that are similarly analytically solvable. The first extension assumes that the seller restricts her attention to a class of deterministic posted price mechanisms only, whereas the second relaxes the requirement of the target regret needing specifying.
- To increase the acceptance and the relevance of the derived optimal mechanism, we interpret it as a randomized posted price mechanism, compute its statistics and compare it with the optimal deterministic posted price mechanism. We also compare our target-free mechanism with one that minimizes the worst-case regret of the seller; see Koçyiğit et al. (2021). We numerically show the dominance of our mechanism in terms of the seller's expected regret and expected revenue, especially when the coefficient of variation of the buyer's value falls below 100%.

The rest of the paper is structured as follows. Section 2 discusses how regret could be leveraged as a decision criterion in both robust optimization and robust satisficing settings. Section 3 derives the optimal mechanism for different ranges of the input parameters. Section 4 considers a restricted problem where only deterministic posted mechanisms are considered, and Section 5 studies another variation of the problem where the seller is relieved from choosing the target regret. Finally, Section 6 interprets the obtained optimal mechanism as a randomized posted price mechanism and compares it with the optimal deterministic posted price counterpart from Section 4 as well as with a benchmark from the literature.

Notation: For a logical expression $\mathcal{E}$, we define $\mathbb{1}(\mathcal{E}) = 1$ if $\mathcal{E}$ is true; $= 0$ otherwise, and for any real number x, we denote by x^+ its positive part, that is, $x^+ = \max\{x, 0\}$. We adopt the convention for division by zero that $a/0 = \infty$ if $a > 0$; $= 0$ otherwise. All logarithms have a natural base of e. Besides, the set of all (bounded) Borel-measurable functions from $\mathcal{D}$ to $\mathcal{R}$ is denoted by $\mathcal{L}(\mathcal{D}, \mathcal{R})$, and finally the cone of all non-negative Borel measures supported on $\mathcal{A}$ is denoted by $\mathcal{M}_+(\mathcal{A})$.

2. Regret satisficing in robust mechanism design

We consider a prototypical monopoly pricing problem where the seller aims to sell a single product to a buyer. She perceives the value ν that the buyer privately assigns to the item as a stochastic-free uncertain variable which could take any value in the interval $[0, \overline{\nu}]$. The vanishing lower bound is justified by the observation that no matter what the product offered is, it can be inconsequential to some people, and the upper bound $\overline{\nu}$ could perhaps be estimated from the price of a similar yet superior substitute that is currently available in the market. If nothing else about ν is known, a direct mechanism that attains the minimum worst-case regret can be found by solving

$$\begin{array}{lll} \text{minimize} & \sup_{\nu \in [0,\overline{\nu}]} \nu - m(\nu) & \\ \text{subject to} & q \in \mathcal{L}([0,\overline{\nu}],[0,1]),\ m \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}) & \\ & q(\nu)\nu - m(\nu) \geq 0 & \forall \nu \in [0,\overline{\nu}] \\ & q(\nu)\nu - m(\nu) \geq q(\omega)\nu - m(\omega) & \forall \nu, \omega \in [0,\overline{\nu}], \end{array} \tag{1}$$

where q and m denote the allocation and the payment rule, respectively. In particular, $q(\nu)$ represents the probability that the buyer with value ν will obtain the item after he makes a payment of amount $m(\nu)$ to the seller. Besides, Problem (1) assumes that the buyer is risk-neutral and that his expected utility coincides with $q(\nu)\nu - m(\nu)$. The two inequalities are known as the *'individual rationality'* and the *'incentive compatibility'* constraints, respectively. A mechanism is said to be individually rational if it ensures that the buyer's expected utility is always non-negative, and it is said to be incentive compatible if the buyer can maximize his expected utility by truthfully reporting his true value ν, which is unknown to the seller when the mechanism is designed. Both of these constraints must hold for a buyer of any value. Finally, the objective function of Problem (1) contains $\nu - m(\nu)$ which we refer to as a *'regret'* of the seller since it characterizes the difference between the hypothetical revenue that the seller could have earned if she precisely knew ν and the actual revenue that is generated by the mechanism (q, m).

As an allocation of the item is probabilistic, the seller announcing a mechanism is similar to the seller offering a lottery: requesting an upfront payment from the buyer without making a definite promise to deliver the winning prize. Though, a mechanism can be as simple as offering a product at a certain price, say $p \in \mathbb{R}$. In this case, we can write down the mechanism as

$$q(\nu) = \mathbb{1}(\nu \geq p) \quad \text{and} \quad m(\nu) = p\mathbb{1}(\nu \geq p),$$

and we shall refer to it as a *'deterministic posted price mechanism.'* One can readily verify that such a mechanism is always incentive compatible and individually rational. Note that due to the linearity of the constraints involved, even if the price p is chosen at random, $\tilde{p} \sim \mathbb{P}$, a *'randomized posted price mechanism'* with

$$q(\nu) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}(\nu \geq \tilde{p})\right) \quad \text{and} \quad m(\nu) = \mathbb{E}_{\mathbb{P}}\left(\tilde{p}\mathbb{1}(\nu \geq \tilde{p})\right)$$

is too incentive compatible and individually rational. Carrasco et al. (2018) amongst others argued that every incentive compatible and individually rational mechanism with a right-continuous allocation rule can be interpreted as a randomized posted price mechanism.

Bergemann and Schlag (2008) essentially solved the above worst-case regret minimization problem, and subsequently Koçyiğit et al. (2021) provided an extension to this problem in which the seller has multiple items to sell simultaneously. Similarly, both papers showed that the optimal mechanism consists of a piece-wise logarithm allocation rule and a piece-wise linear payment rule.

In this paper, we are going to take a similar yet fundamentally different approach to the seller's worst-case regret minimization problem. Instead of using robust optimization, we adopt a recently proposed *'robust satisficing'* approach and consider the following mechanism design problem.

$$\begin{array}{lll}
\text{minimize} & k & \\
\text{subject to} & q \in \mathcal{L}([0,\overline{\nu}],[0,1]),\ m \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}),\ k \in \mathbb{R}_+ & \\
& q(\nu)\nu - m(\nu) \geq 0 & \forall \nu \in [0,\overline{\nu}] \\
& q(\nu)\nu - m(\nu) \geq q(\omega)\nu - m(\omega) & \forall \nu, \omega \in [0,\overline{\nu}], \\
& \nu - m(\nu) \leq \tau + k|\nu - \hat{\nu}| & \forall \nu \in [0,\overline{\nu}].
\end{array} \tag{$\mathcal{P}$}$$

In this formulation, we have additional parameters $\hat{\nu} \in (0, \overline{\nu})$ and $\tau \in \mathbb{R}$. They represent the nominal value of ν and the target regret (see Long et al. 2022). The target τ represents an admissible upper bound of the seller's regret under the nominal scenario $\nu = \hat{\nu}$. The additional decision variable k characterizes the maximum level of constraint violation of all other scenarios $\nu \neq \hat{\nu}$ in relation to their deviation from the nominal counterpart. Problem ($\mathcal{P}$) seeks the smallest upper bound on the regret $\nu - m(\nu)$ of the form $\tau + k|\nu - \hat{\nu}|$, parameterized by k. This upper bound takes a minimum value when $\nu = \hat{\nu}$ and is increasing when the buyer's value ν further deviates from the nominal value $\hat{\nu}$ in either direction. The inputs $\hat{\nu}$ and $\overline{\nu}$ are to be obtained or statistically estimated from the available data, whereas the target τ is to be specified by the seller to reflect her risk tolerance level. Throughout, we will refer to the last constraint of Problem ($\mathcal{P}$) as the *'satisficing'* constraint.

When a mechanism (q, m) is feasible in Problem ($\mathcal{P}$), the payment rule m can be expressed in terms of the allocation rule q thanks to the following known result, which we will utilize heavily.

PROPOSITION 1 **(Myerson and Satterthwaite 1983)**. *A mechanism (q, m) is incentive compatible if and only if*

(i) $q(\nu)$ *is non-decreasing in* $\nu \in [0, \overline{\nu}]$,

(ii) $m(\nu) = m(0) + q(\nu)\nu - \int_0^\nu q(x)\ \mathrm{d}x \quad \forall \nu \in [0, \overline{\nu}]$.

Our first result is to analyze the feasibility of Problem ($\mathcal{P}$).

THEOREM 1. *Problem ($\mathcal{P}$) is feasible if and only if $\tau > 0$.*

Proof. Theorem 1 utilizes Proposition 1. We refer our readers to Appendix B for more details. □

Due to Theorem 1, we henceforth always assume that $\tau > 0$ to avoid trivialities.

3. Optimal mechanisms

This section contains our main results which are the analytical derivations of the optimal solutions of Problem ($\mathcal{P}$) for different values of the seller's target $\tau > 0$. We will consider in total four cases depending on the relationship between τ and the other inputs to the problem: $\hat{\nu} \in (0, \overline{\nu})$ and $\overline{\nu} > 0$.

Case I: $\tau \geq \dfrac{\overline{\nu}}{e}$

Case II: $\tau < \dfrac{\overline{\nu}}{e}$ and $\tau > \hat{\nu}$

Case III: $\tau < \dfrac{\overline{\nu}}{e}$, $\tau > \left(2e^{-1/2} - 1\right)\hat{\nu}$ and $\tau \leq \hat{\nu}$

Case IV: $\tau \leq \left(2e^{-1/2} - 1\right)\hat{\nu}$

Note that since $\hat{\nu} < \overline{\nu}$, the condition of Case IV necessarily implies that $\tau < \frac{\overline{\nu}}{e}$. Together, these four cases are thus collectively exhaustive.

3.1. Analysis of Case I.

PROPOSITION 2. *If $\tau \geq \frac{\overline{\nu}}{e}$, then $(q^\star, m^\star, 0)$ with*

$$q^\star(\nu) = \left(1 + \log\left(\frac{\nu}{\overline{\nu}}\right)\right)^+ \quad \text{and} \quad m^\star(\nu) = \left(\nu - \frac{\overline{\nu}}{e}\right)^+ \tag{2}$$

is optimal in Problem $(\mathcal{P})$.

Note that this optimal mechanism depends on neither τ, as long as it is sufficiently large, nor $\hat{\nu}$. This mechanism has also been investigated and been shown by Koçyiğit et al. (2021) to minimize the seller's worst-case regret.

Proof. We will establish the feasibility of the suggested mechanism $(q^\star, m^\star)$ by firstly showing that it is incentive compatible. Due to Proposition 1, it suffices to show that

$$m^\star(\nu) = q^\star(\nu)\nu - \int_0^\nu q^\star(x)\ \mathrm{d}x \quad \forall \nu \in [0, \overline{\nu}] \tag{3}$$

as $m^\star(0) = 0$ and $q^\star(\nu)$ is increasing in $\nu \in [0, \overline{\nu}]$ by construction. Indeed, the above equation holds trivially when $\nu \leq \frac{\overline{\nu}}{e}$ and both sides vanish. When $\nu > \frac{\overline{\nu}}{e}$, on the other hand, we observe that

$$m^\star(\nu) = \int_{\frac{\overline{\nu}}{e}}^{\nu} \mathrm{d}m^\star(x) = \int_{\frac{\overline{\nu}}{e}}^{\nu} \ \mathrm{d}x = \int_{\frac{\overline{\nu}}{e}}^{\nu} x\ \mathrm{d}q^\star(x),$$

which in turn implies that

$$m^\star(\nu) = q^\star(\nu)\nu - \int_{\frac{\overline{\nu}}{e}}^{\nu} q^\star(x)\ \mathrm{d}x = q^\star(\nu)\nu - \int_0^\nu q^\star(x)\ \mathrm{d}x,$$

and again (3) holds. In addition to being incentive compatible, $(q^\star, m^\star)$ is also individually rational as (3) further implies that $q^\star(\nu)\nu - m^\star(\nu) = \int_0^\nu q^\star(x)\ \mathrm{d}x \geq 0$ for all $\nu \in [0, \overline{\nu}]$. Besides, it is straightforward to verify that $q^\star \in \mathcal{L}([0, \overline{\nu}], [0, 1])$.

Under this mechanism, we also note that

$$\nu - m^\star(\nu) = \min\left\{\nu, \frac{\overline{\nu}}{e}\right\} \leq \tau \quad \forall \nu \in [0, \overline{\nu}],$$

and consequently the satisficing constraint in Problem $(\mathcal{P})$ is satisfied by the suggested solution. As the objective is to minimize k and as k is bounded from below by zero, the proof is completed. $\square$

In the remainder of the section, we exclusively focus on Cases II-IV, where $\tau < \frac{\overline{\nu}}{e}$.

3.2. Analysis of Case II.

PROPOSITION 3. *If* $\hat{\nu} < \tau < \frac{\overline{\nu}}{e}$*, then* $(q^\star, m^\star, k^\star)$*, where*

$$q^\star(\nu) = \left(1 + (1-k^\star)\log\left(\frac{\nu}{\overline{\nu}}\right)\right)^+ \quad \text{and} \quad m^\star(\nu) = (1-k^\star)\left(\nu - e^{1/(k^\star-1)}\overline{\nu}\right)^+ \tag{4}$$

and $k^\star \in (0,1)$ *is a solution of* $k\hat{\nu} + (1-k)e^{1/(k-1)}\overline{\nu} = \tau$*, is feasible in Problem* $(\mathcal{P})$*.*

Note that $q^\star(\nu)$ and $m^\star(\nu)$ are continuous in ν, and they both vanish when $\nu \le e^{1/(k^\star-1)}\overline{\nu}$. Besides, the expression $k\hat{\nu} + (1-k)e^{1/(k-1)}\overline{\nu}$ evaluates to $\frac{\overline{\nu}}{e}$ when $k = 0$ and converges to $\hat{\nu}$ when k approaches one from below. By the intermediate value theorem, for any $\tau \in \left(\hat{\nu}, \frac{\overline{\nu}}{e}\right)$, there must exist $k \in (0,1)$ such that $f(k) \triangleq k\hat{\nu} + (1-k)e^{1/(k-1)}\overline{\nu} = \tau$, which we denote by $k^\star$. In fact, $k^\star$ is unique. To see this, we note that $\frac{\partial^2}{\partial k^2} f(k) = \frac{e^{1/(k-1)}}{(1-k)^3}\overline{\nu} > 0$, and hence $f(k)$ is convex over the open unit interval. Suppose for the sake of a contradiction that there are two different solutions k_1 and k_2 from $(0,1)$ of $f(k) = \tau$ and without loss of generality assume that $k_1 < k_2$. Then, consider a sequence $\{k_i^\dagger\}_{i\in\mathbb{N}}$ from $(k_2, 1)$ which converges to one from below. It follows from the convexity of f that

$$\frac{k_i^\dagger - k_2}{k_i^\dagger - k_1} f(k_1) + \frac{k_2 - k_1}{k_i^\dagger - k_1} f(k_i^\dagger) \ge f\left(\frac{k_i^\dagger - k_2}{k_i^\dagger - k_1}\cdot k_1 + \frac{k_2 - k_1}{k_i^\dagger - k_1}\cdot k_i^\dagger\right) = f(k_2) \quad \forall i \in \mathbb{N},$$

and hence $f(k_i^\dagger) \ge \tau$, $\forall i \in \mathbb{N}$. This observation contradicts with $\lim_{i\uparrow\infty} f(k_i^\dagger) = \hat{\nu} < \tau$. As a result, $k^\star$ is unique as postulated.

Proof. Borrowing the idea from the proof of Proposition 2, we will first show that (3) robustly holds with this new choice of $q^\star$ and $m^\star$ to establish the incentive compatibility and the individually rationality of the suggested mechanism. Note that (3) automatically holds when $\nu \in [0, e^{1/(k^\star-1)}\overline{\nu}]$ and both $q^\star(\nu)$ and $m^\star(\nu)$ are equal to zero. Suppose now that $\nu > e^{1/(k^\star-1)}\overline{\nu}$. In this case, we have

$$m^\star(\nu) = \int_{e^{1/(k^\star-1)}\overline{\nu}}^{\nu} \mathrm{d}m^\star(x) = \int_{e^{1/(k^\star-1)}\overline{\nu}}^{\nu} (1-k^\star)\ \mathrm{d}x = \int_{e^{1/(k^\star-1)}\overline{\nu}}^{\nu} x\ \mathrm{d}q^\star(x),$$

which in turn implies that

$$m^\star(\nu) = q^\star(\nu)\nu - \int_{e^{1/(k^\star-1)}\overline{\nu}}^{\nu} q^\star(x)\ \mathrm{d}x = q^\star(\nu)\nu - \int_0^{\nu} q^\star(x)\ \mathrm{d}x.$$

Hence, (3) follows.

Next, it remains to show that the last constraint of Problem ($\mathcal{P}$) is also satisfied robustly for any $\nu \in [0, \overline{\nu}]$. This constraint trivially holds whenever $\nu \in [0, \hat{\nu}]$ because then $\nu - m^\star(\nu) \leq \nu \leq \hat{\nu} < \tau \leq \tau + k^\star|\nu - \hat{\nu}|$. When $\nu > \hat{\nu}$, on the other hand, we have that

$$\nu - m^\star(\nu) = \min\left\{\nu, k^\star\nu + (1-k^\star)e^{1/(k^\star-1)}\overline{\nu}\right\} \leq k^\star\nu + (1-k^\star)e^{1/(k^\star-1)}\overline{\nu} = \tau + k^\star(\nu - \hat{\nu}).$$

Finally, one can readily verify that $q^\star \in \mathcal{L}([0,\overline{\nu}],[0,1])$, and the proposition therefore follows. □

To establish the optimality of mechanism in (4), our strategy is to construct a tight lower bound of Problem ($\mathcal{P}$). This lower bound is expressed as the optimal objective value of the following maximization problem

$$\begin{array}{ll} \text{maximize} & \displaystyle\int_0^{\overline{\nu}} (\nu - \tau)\,\beta(\nu)\ \mathrm{d}\nu - \tau\overline{\nu}\beta(\overline{\nu}) \\ \text{subject to} & \beta \in \mathcal{L}([0,\overline{\nu}], \mathbb{R}_+) \\ & \displaystyle\int_0^{\overline{\nu}} |\nu - \hat{\nu}|\beta(\nu)\ \mathrm{d}\nu + (\overline{\nu} - \hat{\nu})\overline{\nu}\beta(\overline{\nu}) = 1 \\ & \nu\beta(\nu) \leq \overline{\nu}\beta(\overline{\nu}) + \displaystyle\int_\nu^{\overline{\nu}} \beta(x)\ \mathrm{d}x \quad \forall \nu \in [0,\overline{\nu}]. \end{array} \tag{$\mathcal{D}$}$$

Note that Problem ($\mathcal{D}$) implicitly imposes that any feasible β must be integrable; however, β is not necessarily continuous or monotonic.

Proposition 4. *Problem ($\mathcal{P}$) is lower bounded by Problem ($\mathcal{D}$).*

Proof. We first use Proposition 1 to equivalently express Problem ($\mathcal{P}$) as

$$\begin{array}{ll} \text{minimize} & k \\ \text{subject to} & q \in \mathcal{L}([0,\bar{\nu}],[0,1]),\ m^0 \in \mathbb{R}_-,\ k \in \mathbb{R}_+ \\ & q(\nu) \text{ is increasing in } \nu \in [0,\bar{\nu}] \\ & \nu - m^0 - q(\nu)\nu + \displaystyle\int_0^\nu q(x)\ \mathrm{d}x \leq \tau + k|\nu - \hat{\nu}| \quad \forall \nu \in [0,\overline{\nu}], \end{array} \tag{5}$$

where m^0 denotes the value of $m(0)$ and it has to be non-positive because of the individual rationality constraint ($q(0)\times 0 - m(0) \geq 0$). Note that when $m(0)$ is non-positive, the mechanism is automatically individually rational for all values of $\nu \in [0,\overline{\nu}]$ because

$$q(\nu)\nu - m(\nu) = -m(0) + \int_0^\nu q(x)\ \mathrm{d}x \geq 0.$$

As $-m^0$ appears at the left-hand side of the smaller-than-or-equal-to inequality constraint, we may however assume that m^0 is zero without any loss of optimality. We then find

$$k|\nu-\hat{\nu}| \geq \nu-\tau-q(\nu)\nu+\int_0^\nu q(x)\ \mathrm{d}x \quad \forall \nu \in [0,\overline{\nu}] \tag{6}$$

for any feasible k and q. Hence, for any integrable $\beta \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}_+)$, it follows that

$$\begin{aligned} k\int_0^{\overline{\nu}} |\nu-\hat{\nu}|\beta(\nu)\ \mathrm{d}\nu &\geq \int_0^{\overline{\nu}} \left(\nu-\tau-q(\nu)\nu+\int_0^\nu q(x)\ \mathrm{d}x\right)\beta(\nu)\ \mathrm{d}\nu \\ &= \int_0^{\overline{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \int_0^{\overline{\nu}} q(\nu)\nu\beta(\nu)\ \mathrm{d}\nu + \int_0^{\overline{\nu}}\left(\int_0^\nu q(x)\ \mathrm{d}x\right)\beta(\nu)\ \mathrm{d}\nu \\ &= \int_0^{\overline{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \int_0^{\overline{\nu}} q(\nu)\nu\beta(\nu)\ \mathrm{d}\nu + \int_0^{\overline{\nu}}\left(\int_\nu^{\overline{\nu}} \beta(x)\ \mathrm{d}x\right)q(\nu)\ \mathrm{d}\nu \end{aligned} \tag{7}$$

and that

$$k\,(\overline{\nu}-\hat{\nu})\,\overline{\nu}\beta(\overline{\nu}) \geq \left(\overline{\nu}-\tau-q(\overline{\nu})\overline{\nu}+\int_0^{\overline{\nu}} q(x)\ \mathrm{d}x\right)\overline{\nu}\beta(\overline{\nu}) \geq \left(\int_0^{\overline{\nu}} q(\nu)\ \mathrm{d}\nu-\tau\right)\overline{\nu}\beta(\overline{\nu}). \tag{8}$$

Specifically, for any β that is feasible in Problem $(\mathcal{D})$, summing up (7) and (8) yields

$$\begin{aligned} k &\geq \int_0^{\overline{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \tau\overline{\nu}\beta(\overline{\nu}) + \int_0^{\overline{\nu}} q(\nu)\left(\overline{\nu}\beta(\overline{\nu})-\nu\beta(\nu)+\int_\nu^{\overline{\nu}}\beta(x)\ \mathrm{d}x\right)\mathrm{d}\nu \\ &\geq \int_0^{\overline{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \tau\overline{\nu}\beta(\overline{\nu}), \end{aligned}$$

which completes the proof. □

We next establish the tightness of the proposed lower bound, which will in turn imply the optimality of the mechanism defined in (4).

PROPOSITION 5. *If $\hat{\nu} < \tau < \frac{\overline{\nu}}{e}$, then $\beta^\star$ which is defined through*

$$\beta^\star(\nu) = \begin{cases} \frac{c}{\nu^2} & \text{if } \nu \in \left[e^{1/(k^\star-1)}\overline{\nu},\overline{\nu}\right], \\ 0 & \text{if } \nu \in \left[0, e^{1/(k^\star-1)}\overline{\nu}\right), \end{cases}$$

where $k^\star \in (0,1)$ is defined as in Proposition 3 and

$$c = \left[\frac{2-k^\star}{1-k^\star} - e^{1/(1-k^\star)}\frac{\hat{\nu}}{\overline{\nu}}\right]^{-1},$$

is feasible in Problem $(\mathcal{D})$ and it attains the objective value of $k^\star$.

Proof. First, we note that the constant c is strictly positive. To see this, recall that

$$k^\star \hat{\nu} + (1-k^\star) e^{1/(k^\star-1)} \overline{\nu} = \tau > \hat{\nu} \implies \left(\frac{2-k^\star}{1-k^\star} \right) \overline{\nu} > \overline{\nu} > e^{1/(1-k^\star)} \hat{\nu}, \tag{9}$$

and hence $\beta^\star(\nu) \geq 0$ for all $\nu \in [0, \overline{\nu}]$. To verify that $\beta^\star$ satisfies the first constraint of Problem ($\mathcal{D}$), we evaluate the quantity on its left-hand side:

$$\begin{aligned} \int_0^{\overline{\nu}} |\nu - \hat{\nu}| \beta^\star(\nu) \, \mathrm{d}\nu + (\overline{\nu} - \hat{\nu}) \overline{\nu} \beta^\star(\overline{\nu}) &= c \left[\int_{e^{1/(k^\star-1)}\overline{\nu}}^{\overline{\nu}} \left(\frac{1}{\nu} - \frac{\hat{\nu}}{\nu^2} \right) \mathrm{d}\nu + 1 - \frac{\hat{\nu}}{\overline{\nu}} \right] \\ &= c \left[\frac{1}{1-k^\star} + \hat{\nu} \left(\frac{1}{\overline{\nu}} - \frac{1}{e^{1/(k^\star-1)}\overline{\nu}} \right) + 1 - \frac{\hat{\nu}}{\overline{\nu}} \right] = 1, \end{aligned}$$

where the first equality follows from the construction of $\beta^\star$ and from (9) which ensures that the absolute value function can be removed and the last equality follows from the definition of c. Next, we observe that the second constraint in Problem ($\mathcal{D}$) trivially holds when $\nu < e^{1/(k^\star-1)}\overline{\nu}$. When $\nu \geq e^{1/(k^\star-1)}\overline{\nu}$, on the other hand, we have that

$$\int_\nu^{\overline{\nu}} \beta^\star(x) \, \mathrm{d}x = \frac{c}{\nu} - \frac{c}{\overline{\nu}} = \nu \beta^\star(\nu) - \overline{\nu} \beta^\star(\overline{\nu}).$$

We thus establish the feasibility of $\beta^\star$.

To finally complete the proof, we evaluate the objective function of Problem ($\mathcal{D}$) when $\beta = \beta^\star$:

$$\begin{aligned} \int_0^{\overline{\nu}} (\nu - \tau) \beta^\star(\nu) \, \mathrm{d}\nu - \tau \overline{\nu} \beta^\star(\overline{\nu}) &= c \left[\int_{e^{1/(k^\star-1)}\overline{\nu}}^{\overline{\nu}} \left(\frac{1}{\nu} - \frac{\tau}{\nu^2} \right) \mathrm{d}\nu - \frac{\tau}{\overline{\nu}} \right] \\ &= c \left[\frac{1}{1-k^\star} + \tau \left(\frac{1}{\overline{\nu}} - \frac{1}{e^{1/(k^\star-1)}\overline{\nu}} \right) - \frac{\tau}{\overline{\nu}} \right] \\ &= c \left[\frac{1}{1-k^\star} - \frac{\tau}{e^{1/(k^\star-1)}\overline{\nu}} \right] \\ &= c \left[\frac{1}{1-k^\star} - 1 + k^\star - \frac{k^\star \hat{\nu}}{e^{1/(k^\star-1)}\overline{\nu}} \right] \\ &= c k^\star \left[\frac{2-k^\star}{1-k^\star} - e^{1/(1-k^\star)} \frac{\hat{\nu}}{\overline{\nu}} \right] = k^\star, \end{aligned}$$

where the first equality follows from the construction of $\beta^\star$, the fourth equality from expressing τ as a function of $k^\star, \hat{\nu}, \overline{\nu}$ (*cf.* Proposition 3), and the last equality from our selection of c. The proof is now completed. □

3.3. Analysis of Case III.

Proposition 6. *If* $\tau < \frac{\overline{\nu}}{e}$ *and* $\left(2e^{-1/2}-1\right)\hat{\nu} < \tau \le \hat{\nu}$*, then* $(q^\star, m^\star, k^\star)$*, where*

$$q^\star(\nu) = \begin{cases} 1 + (1-k^\star)\log\left(\frac{\nu}{\overline{\nu}}\right) & \text{if } \nu \in (\hat{\nu}, \overline{\nu}], \\ 1 + \log\left(\frac{\nu}{\hat{\nu}}\right) + k^\star \log\left(\frac{\nu\overline{\nu}}{\hat{\nu}^2}\right) & \text{if } \nu \in \left[\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}, \hat{\nu}\right], \\ 0 & \text{otherwise,} \end{cases} \tag{10a}$$

and

$$m^\star(\nu) = \begin{cases} (1-k^\star)(\nu-\hat{\nu}) + (1+k^\star)\left(\hat{\nu} - \left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}\right) & \text{if } \nu \in (\hat{\nu}, \overline{\nu}], \\ (1+k^\star)\left(\nu - \left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}\right) & \text{if } \nu \in \left[\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}, \hat{\nu}\right], \\ 0 & \text{otherwise,} \end{cases} \tag{10b}$$

and $k^\star \in (0,1)$ *is a solution of* $\tau + k\hat{\nu} = (1+k)\left(\frac{\hat{\nu}^{2k}}{e\overline{\nu}^{k-1}}\right)^{1/(k+1)}$*, is feasible in Problem* $(\mathcal{P})$*.*

By construction the allocation rule $q^\star(\nu)$ is continuous, is increasing in ν and is bounded above by one provided that $k^\star \in (0,1)$. Likewise, under the same condition on $k^\star$ the payment rule $m^\star(\nu)$ is also increasing and continuous. Besides, Proposition 6 essentially imposes that

$$(1+k^\star)\hat{\nu} \ge \tau + k^\star\hat{\nu} = (1+k^\star)\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)},$$

and therefore the second line in (10a) and (10b) involves a non-empty interval of ν.

To warrant the existence of $k^\star$, we denote $(1+k)\left(\frac{\hat{\nu}^{2k}}{e\overline{\nu}^{k-1}}\right)^{1/(k+1)} - k\hat{\nu}$ by $f(k)$ and note that

$$f(0) = \frac{\overline{\nu}}{e} > \tau \quad \text{and} \quad f(1) = \left(2e^{-1/2}-1\right)\hat{\nu} < \tau.$$

Thus, there always exists $k^\star \in (0,1)$ such that $f(k^\star) = \tau$. Furthermore, we introduce

$$\begin{aligned} g(k) \triangleq \log(f(k) + k\hat{\nu}) &= \log(1+k) - \frac{1}{1+k} + \frac{2k}{1+k}\log(\hat{\nu}) + \frac{1-k}{1+k}\log(\overline{\nu}) \\ &= \log(1+k) - \frac{1}{1+k} + \log(\hat{\nu}) + \frac{1-k}{1+k}\log\left(\frac{\overline{\nu}}{\hat{\nu}}\right). \end{aligned}$$

We then find

$$\frac{\partial}{\partial k} g(k) = \frac{1}{1+k} + \frac{1}{(1+k)^2} - \frac{2}{(1+k)^2}\log\left(\frac{\overline{\nu}}{\hat{\nu}}\right)$$

and

$$\frac{\partial^2}{\partial k^2} g(k) = -\frac{1}{(1+k)^2} - \frac{2}{(1+k)^3} + \frac{4}{(1+k)^3} \log\left(\frac{\overline{\nu}}{\hat{\nu}}\right).$$

As a result,

$$\frac{\partial^2}{\partial k^2}(f(k)+k\hat{\nu}) = \frac{\partial^2}{\partial k^2} e^{g(k)} = e^{g(k)}\left(\frac{\partial^2}{\partial k^2} g(k) + \left(\frac{\partial}{\partial k} g(k)\right)^2\right) = \frac{e^{g(k)}}{(1+k)^4}\left(1-2\log\left(\frac{\overline{\nu}}{\hat{\nu}}\right)\right)^2 \geq 0.$$

Hence, $f(k)+k\hat{\nu}$ and $f(k)$ is convex in $k \in (0,1)$, and by a familiar argument following Proposition 3, we have that $k^\star$ is unique.

Proof. Borrowing the idea from the proof of Proposition 2, we will first show that (3) robustly holds with this new choice of $q^\star$ and $m^\star$ to establish the incentive compatibility and the individually rationality of the suggested mechanism. Note that (3) automatically holds when ν is sufficiently small and both $q^\star(\nu)$ and $m^\star(\nu)$ vanish. Suppose now that $\nu \geq \left(\frac{\hat{\nu}2k^\star}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}$ and $\nu \leq \hat{\nu}$. In this case, we have

$$m^\star(\nu) = \int_{\left(\frac{\hat{\nu}2k^\star}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}}^{\nu} \mathrm{d}m^\star(x) = \int_{\left(\frac{\hat{\nu}2k^\star}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}}^{\nu} (1+k^\star)\ \mathrm{d}x = \int_{\left(\frac{\hat{\nu}2k^\star}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}}^{\nu} x\ \mathrm{d}q^\star(x),$$

which in turn implies that

$$m^\star(\nu) = q^\star(\nu)\nu - \int_{\left(\frac{\hat{\nu}2k^\star}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}}^{\nu} q^\star(x)\ \mathrm{d}x = q^\star(\nu)\nu - \int_0^{\nu} q^\star(x)\ \mathrm{d}x.$$

Thus, (3) holds for any ν in the imposed range, and in particular,

$$m^\star(\hat{\nu}) = q^\star(\hat{\nu})\hat{\nu} - \int_0^{\hat{\nu}} q^\star(x)\ \mathrm{d}x. \tag{11}$$

Similarly, when $\nu \in (\hat{\nu}, \overline{\nu}]$, it follows that

$$m^\star(\nu) - m^\star(\hat{\nu}) = \int_{\hat{\nu}}^{\nu} \mathrm{d}m^\star(x) = \int_{\hat{\nu}}^{\nu} (1-k^\star)\ \mathrm{d}x = \int_{\hat{\nu}}^{\nu} x\ \mathrm{d}q^\star(x),$$

which in turn implies that

$$m^\star(\nu) - m^\star(\hat{\nu}) = q^\star(\nu)\nu - q^\star(\hat{\nu})\hat{\nu} - \int_{\hat{\nu}}^{\nu} q^\star(x)\ \mathrm{d}x = q^\star(\nu)\nu - q^\star(\hat{\nu})\hat{\nu} - \int_0^{\nu} q^\star(x)\ \mathrm{d}x + \int_0^{\hat{\nu}} q^\star(x)\ \mathrm{d}x.$$

Once again, (3) holds robustly for any $\nu \in (\hat{\nu}, \overline{\nu}]$ because of (11). As a result, we can now conclude that the mechanism $(q^\star, m^\star)$ is incentive compatible and individually rational.

Using $k^\star$ suggested by the proposition, we can express the payment rule (10b) as

$$m^\star(\nu) = \begin{cases} (1-k^\star)\nu - \tau + k^\star \hat{\nu} & \text{if } \nu \in (\hat{\nu}, \overline{\nu}], \\ (1+k^\star)\nu - \tau - k^\star \hat{\nu} & \text{if } \nu \in \left[\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}} \right)^{1/(k^\star+1)}, \hat{\nu} \right], \\ 0 & \text{otherwise,} \end{cases}$$

from which it is readily seen that the mechanism suggested satisfies

$$\nu - m^\star(\nu) = \begin{cases} \tau + k^\star(\nu - \hat{\nu}) & \text{if } \nu \in (\hat{\nu}, \overline{\nu}], \\ \tau + k^\star(\hat{\nu} - \nu) & \text{if } \nu \in \left[\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}} \right)^{1/(k^\star+1)}, \hat{\nu} \right], \\ \nu & \text{otherwise,} \end{cases}$$

and consequently the satisficing constraint in Problem $(\mathcal{P})$ for the first two cases when ν is sufficiently large. For a small ν, on the other hand, we have

$$(k^\star+1)\nu < (k^\star+1)\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}} \right)^{1/(k^\star+1)} = \tau + k^\star \hat{\nu} \quad \Longrightarrow \quad \nu - m^\star(\nu) = \nu < \tau + k^\star(\hat{\nu} - \nu).$$

Finally, one can readily verify that $q^\star \in \mathcal{L}([0,\overline{\nu}],[0,1])$ to complete the proof. □

PROPOSITION 7. *If* $\tau < \frac{\overline{\nu}}{e}$ *and* $\left(2e^{-1/2}-1\right)\hat{\nu} < \tau \le \hat{\nu}$, *then* $\beta^\star$ *which is defined through*

$$\beta^\star(\nu) = \begin{cases} \frac{c}{\nu^2} & \text{if } \nu \in \left[\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}} \right)^{1/(k^\star+1)}, \overline{\nu} \right], \\ 0 & \text{if } \nu \in \left[0, \left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}} \right)^{1/(k^\star+1)} \right), \end{cases}$$

where $k^\star \in (0,1)$ *is defined as in Proposition 6 and*

$$c = \left[e^{1/(1+k^\star)} \left(\frac{\hat{\nu}}{\overline{\nu}} \right)^{\frac{1-k^\star}{1+k^\star}} - \frac{2}{1+k^\star} \log\left(\frac{\hat{\nu}}{\overline{\nu}} \right) - \frac{2+k^\star}{1+k^\star} \right]^{-1},$$

is feasible in Problem $(\mathcal{D})$ *and it attains the objective value of* $k^\star$.

Proof. First, we will show that the constant c chosen here is positive, which in turns will ensure that $\beta^\star(\nu)$ is non-negative for all $\nu \in [0,\overline{\nu}]$. To this end, we introduce a constant $a = \log\left(\frac{\hat{\nu}}{\overline{\nu}}\right) < 0$ and note that we can express $c = (h(a))^{-1}$, where

$$h(a) = e^{\frac{1+a(1-k^\star)}{1+k^\star}} - \frac{2a}{1+k^\star} - \frac{2+k^\star}{1+k^\star}.$$

Besides, we note that our selection of $k^\star$ implies that

$$(1+k^\star)\hat{\nu} \geq \tau + k^\star\hat{\nu} = (1+k^\star)\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)} \iff \left(\frac{\hat{\nu}}{\overline{\nu}}\right)^{1-k^\star} \geq \frac{1}{e} \iff a \geq \frac{1}{k^\star-1}.$$

Since $h(a)$ is convex in a, its minimum over $a \in \left[\frac{1}{k^\star-1}, 0\right]$ is attained when a is a stationary point if it resides within the bound or else when a is either of the two boundary points. To this end, we first evaluate $h(a)$ at the boundary points and find:

$$h\left(\frac{1}{k^\star-1}\right) = 1 + \frac{2}{(1-k^\star)(1+k^\star)} - \frac{2+k^\star}{1+k^\star} = \frac{2}{(1-k^\star)(1+k^\star)} - \frac{1}{1+k^\star} = \frac{1}{1-k^\star} > 0,$$

and

$$h(0) = e^{1/(1+k^\star)} - \frac{2+k^\star}{1+k^\star} > 0,$$

where the inequality follows because $1+x$ is the first-order Taylor approximation of e^x around $x=0$, hence $e^x > 1+x$ for all $x \neq 0$, and we can substitute $\frac{1}{1+k^\star}$ for x. Next, we determine the stationary point $a^\star$ of $h(a)$ which satisfies

$$\left(\frac{1-k^\star}{1+k^\star}\right) e^{\frac{1+a^\star(1-k^\star)}{1+k^\star}} - \frac{2}{1+k^\star} = 0 \iff e^{\frac{1+a^\star(1-k^\star)}{1+k^\star}} = \frac{2}{1-k^\star}.$$

If $a^\star > 0$ or $a^\star < \frac{1}{k^\star-1}$, we can safely disregard it. Else if $a^\star \in \left[\frac{1}{k^\star-1}, 0\right]$, we find

$$h(a^\star) = \frac{2}{1-k^\star} - \frac{2a^\star}{1+k^\star} - \frac{2+k^\star}{1+k^\star} \geq \frac{2}{1-k^\star} - \frac{2+k^\star}{1+k^\star} > 2-2 = 0.$$

As a result, c is positive regardless of the value of τ, $\overline{\nu}$ and $\hat{\nu}$ as postulated.

Next, to verify that $\beta^\star$ satisfies the first constraint of Problem ($\mathcal{D}$), we observe that

$$\begin{aligned}
&\int_0^{\overline{\nu}} |\nu-\hat{\nu}|\beta^\star(\nu)\ \mathrm{d}\nu + (\overline{\nu}-\hat{\nu})\overline{\nu}\beta^\star(\overline{\nu}) \\
&= c\left[\int_{\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}}^{\hat{\nu}} \left(-\frac{1}{\nu}+\frac{\hat{\nu}}{\nu^2}\right)\ \mathrm{d}\nu + \int_{\hat{\nu}}^{\overline{\nu}} \left(\frac{1}{\nu}-\frac{\hat{\nu}}{\nu^2}\right)\ \mathrm{d}\nu + 1 - \frac{\hat{\nu}}{\overline{\nu}}\right] \\
&= c\left[\left(\log\left(\frac{1}{\hat{\nu}}\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}\right) + \hat{\nu}\left(\left(\frac{e\overline{\nu}^{k^\star-1}}{\hat{\nu}^{2k^\star}}\right)^{1/(k^\star+1)} - \frac{1}{\hat{\nu}}\right)\right) + \right. \\
&\qquad\qquad \left.\left(\log\left(\frac{\overline{\nu}}{\hat{\nu}}\right) + \hat{\nu}\left(\frac{1}{\overline{\nu}}-\frac{1}{\hat{\nu}}\right)\right) + 1 - \frac{\hat{\nu}}{\overline{\nu}}\right] \\
&= c\left[e^{1/(1+k^\star)}\left(\frac{\hat{\nu}}{\overline{\nu}}\right)^{\frac{1-k^\star}{1+k^\star}} - \frac{2}{1+k^\star}\log\left(\frac{\hat{\nu}}{\overline{\nu}}\right) - \frac{2+k^\star}{1+k^\star}\right] = 1.
\end{aligned}$$

The feasibility of $\beta^\star$ in view of the second constraint of Problem $(\mathcal{D})$ could be similarly established due to a familiar argument from the proof of Proposition 5. It remains to show that $\beta^\star$ attains the objective value that is equal to $k^\star$. To see this, observe that

$$\begin{aligned}
&\int_0^{\overline{\nu}} (\nu-\tau)\,\beta^\star(\nu)\;\mathrm{d}\nu - \tau\overline{\nu}\beta^\star(\overline{\nu})\\
&= c\left[\int_{\left(\frac{\hat{\nu}^{2k^\star}}{e\overline{\nu}^{k^\star-1}}\right)^{1/(k^\star+1)}}^{\overline{\nu}} \left(\frac{1}{\nu}-\frac{\tau}{\nu^2}\right)\;\mathrm{d}\nu - \frac{\tau}{\overline{\nu}}\right]\\
&= c\left[\frac{1}{1+k^\star} - \frac{2k^\star}{1+k^\star}\log\left(\frac{\hat{\nu}}{\overline{\nu}}\right) + \tau\left(\frac{1}{\overline{\nu}} - e^{1/(1+k^\star)}\left(\frac{\overline{\nu}^{k^\star-1}}{\hat{\nu}^{2k^\star}}\right)^{1/(1+k^\star)}\right) - \frac{\tau}{\overline{\nu}}\right]\\
&= c\left[\frac{1}{1+k^\star} - \frac{2k^\star}{1+k^\star}\log\left(\frac{\hat{\nu}}{\overline{\nu}}\right) - e^{1/(1+k^\star)}\tau\left(\frac{\overline{\nu}^{k^\star-1}}{\hat{\nu}^{2k^\star}}\right)^{1/(1+k^\star)}\right]\\
&= c\left[\frac{1}{1+k^\star} - \frac{2k^\star}{1+k^\star}\log\left(\frac{\hat{\nu}}{\overline{\nu}}\right) - (1+k^\star) + k^\star\hat{\nu}e^{1/(1+k^\star)}\left(\frac{\overline{\nu}^{k^\star-1}}{\hat{\nu}^{2k^\star}}\right)^{1/(1+k^\star)}\right]\\
&= ck^\star\left[e^{1/(1+k^\star)}\left(\frac{\hat{\nu}}{\overline{\nu}}\right)^{\frac{1-k^\star}{1+k^\star}} - \frac{2}{1+k^\star}\log\left(\frac{\hat{\nu}}{\overline{\nu}}\right) - \frac{2+k^\star}{1+k^\star}\right] = k^\star,
\end{aligned}$$

where the first equality follows from the construction of $\beta^\star$, the fourth equality from expressing τ as a function of $k^\star, \hat{\nu}, \overline{\nu}$ (*cf.* Proposition 6), and the last equality from our selection of c. The proof is now completed. □

3.4. Analysis of Case IV.

PROPOSITION 8. *If* $\tau \le \left(2e^{-1/2}-1\right)\hat{\nu}$*, then* $(q^\star, m^\star, k^\star)$*, where*

$$q^\star(\nu) = \begin{cases} 1 & \text{if } \nu \in (\hat{\nu}, \overline{\nu}],\\ 1 + (1+k^\star)\log\left(\frac{\nu}{\hat{\nu}}\right) & \text{if } \nu \in \left[e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}\right],\\ 0 & \text{otherwise,}\end{cases} \tag{12a}$$

and

$$m^\star(\nu) = \begin{cases} (1+k^\star)\left(1-e^{-1/(k^\star+1)}\right)\hat{\nu} & \text{if } \nu \in (\hat{\nu}, \overline{\nu}],\\ (1+k^\star)\left(\nu - e^{-1/(k^\star+1)}\hat{\nu}\right) & \text{if } \nu \in \left[e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}\right],\\ 0 & \text{otherwise,}\end{cases} \tag{12b}$$

and $k^\star \in [1,\infty)$ *is a solution of* $\tau + k\hat{\nu} = (1+k)e^{-1/(k+1)}\hat{\nu}$*, is feasible in Problem* $(\mathcal{P})$*.*

Observe that $f(k) = (1+k)e^{-1/(k+1)} - k$ is decreasing with $f(1) = 2e^{-1/2} - 1$ and $\lim_{k\uparrow\infty} f(k) = 0$. Thus, $f(k)\hat{\nu}$ continuously changes from $\left(2e^{-1/2} - 1\right)\hat{\nu}$ to zero when k continuously increases from unity, and consequently $k^\star$ exists (whenever $\tau \leq \left(2e^{-1/2} - 1\right)\hat{\nu}$) and is unique.

Proof. Borrowing the idea from the proof of Proposition 2, we will first show that (3) robustly holds with this new choice of $q^\star$ and $m^\star$ to establish the incentive compatibility and the individually rationality of the suggested mechanism. Note that (3) automatically holds when $\nu < e^{-1/(k^\star+1)}\hat{\nu}$ and both $q^\star(\nu)$ and $m^\star(\nu)$ vanish. Suppose now that $\nu \in \left[e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}\right]$. In this case, we have

$$m^\star(\nu) = \int_{e^{-1/(k^\star+1)}\hat{\nu}}^{\nu} \mathrm{d}m^\star(x) = \int_{e^{-1/(k^\star+1)}\hat{\nu}}^{\nu} (1+k^\star)\ \mathrm{d}x = \int_{e^{-1/(k^\star+1)}\hat{\nu}}^{\nu} x\ \mathrm{d}q^\star(x),$$

which in turn implies that

$$m^\star(\nu) = q^\star(\nu)\nu - \int_{e^{-1/(k^\star+1)}\hat{\nu}}^{\nu} q^\star(x)\ \mathrm{d}x = q^\star(\nu)\nu - \int_0^{\nu} q^\star(x)\ \mathrm{d}x.$$

As a result, (3) holds for any $\nu \in \left[e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}\right]$. Finally, when $\nu > \hat{\nu}$,

$$m^\star(\nu) = m^\star(\hat{\nu}) = q^\star(\hat{\nu})\hat{\nu} - \int_0^{\hat{\nu}} q^\star(x)\,\mathrm{d}x = \hat{\nu} - \int_0^{\hat{\nu}} q^\star(x)\,\mathrm{d}x = \nu - \int_0^{\nu} q^\star(x)\,\mathrm{d}x = q^\star(\nu)\nu - \int_0^{\nu} q^\star(x)\,\mathrm{d}x,$$

where the second equality follows from (3), which has already been shown to hold when $\nu = \hat{\nu}$, and the remaining equalities hold due to (12a) and (12b). As a result, we can now conclude that the mechanism $(q^\star, m^\star)$ is incentive compatible and individually rational.

Using $k^\star$ suggested by the proposition, we can express the payment rule (12b) as

$$m^\star(\nu) = \begin{cases} \hat{\nu} - \tau & \text{if } \nu \in (\hat{\nu}, \overline{\nu}], \\ (1+k^\star)\nu - \tau - k^\star\hat{\nu} & \text{if } \nu \in \left[e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}\right], \\ 0 & \text{otherwise,} \end{cases}$$

from which it is readily seen that the mechanism suggested satisfies

$$\nu - m^\star(\nu) = \begin{cases} \tau + (\nu - \hat{\nu}) & \text{if } \nu \in (\hat{\nu}, \overline{\nu}], \\ \tau + k^\star(\hat{\nu} - \nu) & \text{if } \nu \in \left[e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}\right], \\ \nu & \text{otherwise,} \end{cases}$$

and consequently the satisficing constraint in Problem $(\mathcal{P})$ for the first two cases where ν is larger than or equal to $e^{-1/(k^\star+1)}\hat{\nu}$. For $\nu < e^{-1/(k^\star+1)}\hat{\nu}$, on the other hand, we have

$$(k^\star+1)\nu < (k^\star+1)e^{-1/(k^\star+1)}\hat{\nu} = \tau + k^\star\hat{\nu} \quad \Longrightarrow \quad \nu - m^\star(\nu) = \nu < \tau + k^\star(\hat{\nu}-\nu).$$

Finally, as $q^\star \in \mathcal{L}([0,\overline{\nu}],[0,1])$, the proof is completed. □

To establish the optimality of the mechanism in (12), we need a different lower bound which is

$$\begin{aligned}
\text{maximize} \quad & \int_0^{\hat{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \tau\hat{\nu}\beta(\hat{\nu}) \\
\text{subject to} \quad & \beta \in \mathcal{L}([0,\hat{\nu}],\mathbb{R}_+) \\
& \int_0^{\hat{\nu}} (\hat{\nu}-\nu)\beta(\nu)\ \mathrm{d}\nu = 1 \\
& \nu\beta(\nu) \le \hat{\nu}\beta(\hat{\nu}) + \int_\nu^{\hat{\nu}} \beta(x)\ \mathrm{d}x \quad \forall \nu \in [0,\hat{\nu}].
\end{aligned} \tag{$\mathcal{D}'$}$$

Note that Problem $(\mathcal{D}')$ implicitly imposes that any feasible β must be integrable.

PROPOSITION 9. *Problem $(\mathcal{P})$ is lower bounded by Problem $(\mathcal{D}')$.*

Proof. The proof widely parallels to that of Proposition 4 in that, for any (k,q) that is feasible in Problem $(\mathcal{P})$ and any integrable $\beta \in \mathcal{L}([0,\hat{\nu}],\mathbb{R}_+)$, we have

$$\begin{aligned}
k\int_0^{\hat{\nu}} (\hat{\nu}-\nu)\beta(\nu)\ \mathrm{d}\nu &\ge \int_0^{\hat{\nu}} \left(\nu - \tau - q(\nu)\nu + \int_0^{\nu} q(x)\ \mathrm{d}x\right)\beta(\nu)\ \mathrm{d}\nu \\
&= \int_0^{\hat{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \int_0^{\hat{\nu}} q(\nu)\nu\beta(\nu)\ \mathrm{d}\nu + \int_0^{\hat{\nu}} \left(\int_0^{\nu} q(x)\ \mathrm{d}x\right)\beta(\nu)\ \mathrm{d}\nu \\
&= \int_0^{\hat{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \int_0^{\hat{\nu}} q(\nu)\nu\beta(\nu)\ \mathrm{d}\nu + \int_0^{\hat{\nu}} \left(\int_\nu^{\hat{\nu}} \beta(x)\ \mathrm{d}x\right)q(\nu)\ \mathrm{d}\nu
\end{aligned} \tag{13}$$

and

$$0 \ge \left(\hat{\nu} - \tau - q(\hat{\nu})\hat{\nu} + \int_0^{\hat{\nu}} q(x)\ \mathrm{d}x\right)\hat{\nu}\beta(\hat{\nu}) \ge \left(\int_0^{\hat{\nu}} q(\nu)\ \mathrm{d}\nu - \tau\right)\hat{\nu}\beta(\hat{\nu}), \tag{14}$$

both of which are direct consequences of (6), which continues to hold true. Specifically, for any β that is feasible in Problem $(\mathcal{D}')$, summing up the inequalities (13) and (14) yields

$$\begin{aligned}
k &\ge \int_0^{\hat{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \tau\hat{\nu}\beta(\hat{\nu}) + \int_0^{\hat{\nu}} q(\nu)\left(\hat{\nu}\beta(\hat{\nu}) - \nu\beta(\nu) + \int_\nu^{\hat{\nu}} \beta(x)\ \mathrm{d}x\right)\mathrm{d}\nu \\
&\ge \int_0^{\hat{\nu}} (\nu-\tau)\,\beta(\nu)\ \mathrm{d}\nu - \tau\hat{\nu}\beta(\hat{\nu}),
\end{aligned}$$

which completes the proof. □

Proposition 10. *If $\tau \le \left(2e^{-1/2} - 1\right)\hat{\nu}$, then $\beta^\star$ which is defined through*

$$\beta^\star(\nu) = \begin{cases} \frac{c}{\nu^2} & \text{if } \nu \in \left[e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}\right], \\ 0 & \text{if } \nu \in \left[0, e^{-1/(k^\star+1)}\hat{\nu}\right), \end{cases}$$

where $k^\star \in [1, \infty)$ is defined as in Proposition 8 and

$$c = \left[e^{1/(k^\star+1)} - \frac{2+k^\star}{1+k^\star}\right]^{-1},$$

is feasible in Problem $(\mathcal{D}')$ and it attains the objective value of $k^\star$.

Proof. First of all, we note that $c > 0$ because e^x is lower bounded by $1 + x$ everywhere including where $x = \frac{1}{k^\star+1} > 0$, which in turn ensures that $\beta^\star(\nu) \ge 0$ for all $\nu \in [0, \hat{\nu}]$. In terms of the feasibility of $\beta^\star$ in Problem $(\mathcal{D}')$, we observe that

$$\int_0^{\hat{\nu}} (\hat{\nu} - \nu)\beta^\star(\nu) \, \mathrm{d}\nu = c\left[\int_{e^{-1/(k^\star+1)}\hat{\nu}}^{\hat{\nu}} \left(\frac{\hat{\nu}}{\nu^2} - \frac{1}{\nu}\right) \mathrm{d}\nu\right] = c\left[\left(e^{1/(k^\star+1)} - 1\right) - \frac{1}{k^\star+1}\right] = 1,$$

where the first and the third equalities follow from the construction of $\beta^\star$ and c, respectively. Hence, $\beta^\star$ satisfies the first constraint of Problem $(\mathcal{D}')$. For the second constraint, we note that

$$\int_\nu^{\hat{\nu}} \beta^\star(x) \, \mathrm{d}x = \frac{c}{\nu} - \frac{c}{\hat{\nu}} = \nu\beta^\star(\nu) - \hat{\nu}\beta^\star(\hat{\nu}),$$

for any $\nu \in [e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}]$, and the same constraint trivially holds when $\nu \notin [e^{-1/(k^\star+1)}\hat{\nu}, \hat{\nu}]$. Finally, we ascertain that $\beta^\star$ attains an objective function value equal to $k^\star$:

$$\begin{aligned} \int_0^{\hat{\nu}} (\nu - \tau)\,\beta^\star(\nu) \, \mathrm{d}\nu - \tau\hat{\nu}\beta^\star(\hat{\nu}) &= c\left[\int_{e^{-1/(k^\star+1)}\hat{\nu}}^{\hat{\nu}} \left(\frac{1}{\nu} - \frac{\tau}{\nu^2}\right) \, \mathrm{d}\nu - \frac{\tau}{\hat{\nu}}\right] \\ &= c\left[\frac{1}{k^\star+1} + \tau\left(\frac{1}{\hat{\nu}} - \frac{1}{e^{-1/(k^\star+1)}\hat{\nu}}\right) - \frac{\tau}{\hat{\nu}}\right] \\ &= c\left[\frac{1}{k^\star+1} - e^{1/(k^\star+1)}\frac{\tau}{\hat{\nu}}\right] \\ &= c\left[\frac{1}{k^\star+1} - (k^\star+1) + k^\star e^{1/(k^\star+1)}\right] \\ &= ck^\star\left[e^{1/(k^\star+1)} - \frac{2+k^\star}{1+k^\star}\right] = k^\star, \end{aligned}$$

where the first equality follows from the construction of $\beta^\star$, the fourth equality from expressing τ as a function of $k^\star$ and $\hat{\nu}$ (*cf.* Proposition 8), and the last equality from our selection of c. The proof is now completed. □

We summarize the analyses of the four cases in the theorem below.

Theorem 2. *Problem ($\mathcal{P}$) is solved by the mechanism*

$$\begin{cases} \text{from Proposition 2} & \text{if } \tau \geq \frac{\overline{\nu}}{e}, \\ \text{from Proposition 3} & \text{if } \tau < \frac{\overline{\nu}}{e} \text{ and } \tau > \hat{\nu}, \\ \text{from Proposition 6} & \text{if } \tau < \frac{\overline{\nu}}{e},\ \tau > \left(2e^{-1/2}-1\right)\hat{\nu} \text{ and } \tau \leq \hat{\nu}, \\ \text{from Proposition 8} & \text{if } \tau \leq \left(2e^{-1/2}-1\right)\hat{\nu}. \end{cases}$$

Proof. This is a consequence of Propositions 2–10. □

All in all, we show that the optimal solution of Problem ($\mathcal{P}$) can be determined by using a simple line search to find a value of $k^\star$ from a certain interval that solves a given characteristic equation. For Case IV, the upperbound on $k^\star$ is not explicitly given. However, we can without any loss impose that $k^\star \leq \max\left\{\frac{\hat{\nu}}{\tau}-2, 1\right\}$ which is a valid inequality known from the proof of Theorem 1 (see Appendix B).

4. Optimal deterministic posted price mechanisms

Our aim for this section is to derive an optimal deterministic posted price mechanism. We denote by $p \in [0, \overline{\nu}]$ the posted price which is to be optimized, and we restrict the allocation and payment rule to $q(\nu) = \mathbb{1}(\nu \geq p)$ and $m(\nu) = p\mathbb{1}(\nu \geq p)$. Under this restriction, Problem ($\mathcal{P}$) reduces to

$$\begin{array}{ll} \text{minimize} & k \\ \text{subject to} & p \in [0, \overline{\nu}],\ k \in \mathbb{R}_+ \\ & \nu - p\mathbb{1}(\nu \geq p) \leq \tau + k|\nu - \hat{\nu}| \quad \forall \nu \in [0, \overline{\nu}]. \end{array} \tag{15}$$

Unlike Problem ($\mathcal{P}$), Problem (15) involves a finite number of decision variables (*i.e.,* p and k). It however still contains an infinite number of constraints, each of which is neither convex nor concave in the posted price p.

THEOREM 3. *Problem* (15) *is solved by*

$$p^\star = \begin{cases} \tau & \text{if } \tau \geq \frac{\overline{\nu}}{2}, \\ \text{a root of } \frac{p-\tau}{p-\hat{\nu}} = \frac{\overline{\nu}-p-\tau}{\overline{\nu}-\hat{\nu}} \text{ from } (\tau, \overline{\nu}-\tau) & \text{if } \tau < \frac{\overline{\nu}}{2} \text{ and } \tau > \hat{\nu}, \\ \hat{\nu} & \text{if } \tau < \frac{\overline{\nu}}{2} \text{ and } \tau = \hat{\nu}, \\ \max\left\{\hat{\nu}-\tau,\ \text{a root of } \frac{p-\tau}{\hat{\nu}-p} = \frac{\overline{\nu}-p-\tau}{\overline{\nu}-\hat{\nu}} \text{ from } (\tau,\hat{\nu})\right\} & \text{if } \tau < \frac{\overline{\nu}}{2} \text{ and } \tau < \hat{\nu}. \end{cases}$$

The existence and the uniqueness of the supposed roots will be established during the course of the proof. We also note that, by a simple algebraic rearrangement, these roots can be determined by solving a quadratic equation in p.

Proof. First, we observe that Problem (15) is equivalent to

$$\begin{array}{ll} \text{minimize} & k \\ \text{subject to} & p \in [0, \overline{\nu}],\ k \in \mathbb{R}_+ \\ & \nu \leq \tau + k|\nu - \hat{\nu}| \qquad \forall \nu \in [0, p] \\ & \nu - p \leq \tau + k|\nu - \hat{\nu}| \quad \forall \nu \in [p, \overline{\nu}]. \end{array} \tag{16}$$

Next, we can reduce the infinite uncertainty set $[0, \overline{\nu}] = [0, p] \cup [p, \overline{\nu}]$ to $\{0, p, \hat{\nu}, \overline{\nu}\}$. As a result, we have a problem with only two decision variables and a few constraints, and we find that it is analytically solvable. Detailed arguments are omitted for brevity and can be found in Appendix B. □

Based on Theorem 3 and our earlier results, we can now establish the following observations.

- The mechanism that is optimal in Problem ($\mathcal{P}$) is not necessarily unique. Indeed, when $\tau \geq \frac{\overline{\nu}}{2}$, the optimal objective value of Problem ($\mathcal{P}$) vanishes and it could be solved by a mechanism with either a randomized or a deterministic allocation; see Theorems 2 and 3.
- The restriction to a class of deterministic posted price mechanisms does not impact the feasibility of Problem ($\mathcal{P}$). That is Problem ($\mathcal{P}$) is feasible if and only if Problem (15) is. Particularly, both problems are feasible if and only if $\tau > 0$.
- Even though Problems ($\mathcal{P}$) and (15) share the same necessary and sufficient condition for feasibility, the latter can result in a mechanism that is arbitrarily worse than the former. To see this, we may consider a target $\tau \in [\frac{\overline{\nu}}{e}, \frac{\overline{\nu}}{2})$. The optimal objective value of Problem ($\mathcal{P}$) is zero,

whereas that of the restricted problem (15) is strictly positive. To see this, we may suppose otherwise that $k=0$ is feasible in Problem (15). Its satisficing constraint evaluated at $\nu \uparrow p$ and $\nu = \overline{\nu}$ would then imply that both p and $\overline{\nu} - p$ are smaller than or equal to τ. Therefore, $\tau \geq \frac{\overline{\nu}}{2}$ and this observation contradicts with the admissible range of τ currently considered.

5. Target-free formulations

We will now consider another variant of Problem ($\mathcal{P}$) where the seller is relieved from choosing τ:

$$
\begin{array}{lll}
\text{minimize} & k & \\
\text{subject to} & q \in \mathcal{L}([0,\overline{\nu}],[0,1]),\ m \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}),\ k \in \mathbb{R}_+ & \\
& q(\nu)\nu - m(\nu) \geq 0 & \forall \nu \in [0,\overline{\nu}] \\
& q(\nu)\nu - m(\nu) \geq q(\omega)\nu - m(\omega) & \forall \nu, \omega \in [0,\overline{\nu}] \\
& \nu - m(\nu) \leq \hat{\nu} - m(\hat{\nu}) + k|\nu - \hat{\nu}| & \forall \nu \in [0,\overline{\nu}].
\end{array}
\tag{17}
$$

In other words, Problem (17) is obtained by replacing the target regret τ which is an explicit input parameter of Problem ($\mathcal{P}$) by $\hat{\nu} - m(\hat{\nu})$, which represents the regret of the mechanism (q,m) under the nominal scenario $\nu = \hat{\nu}$.

Proposition 11. *If $\hat{\nu} \geq \frac{\overline{\nu}}{e}$, then Problem* (17) *is solved by the mechanism defined in* (2)*.*

Proof. By Proposition 2, the mechanism $(q^\star, m^\star)$ is incentive compatible and individually rational. Besides,

$$
\nu - m^\star(\nu) = \min\left\{\nu, \frac{\overline{\nu}}{e}\right\} \leq \frac{\overline{\nu}}{e} = \min\left\{\hat{\nu}, \frac{\overline{\nu}}{e}\right\} = \hat{\nu} - m^\star(\hat{\nu}).
$$

Hence, $(q^\star, m^\star, 0)$ is feasible in Problem (17). As the objective is to minimize k and as k has an explicit lower bound of zero, $(q^\star, m^\star, 0)$ must be optimal and the proof is completed. □

Proposition 12. *If $\hat{\nu} < \frac{\overline{\nu}}{e}$, then Problem* (17) *is solved by the mechanism defined in* (4) *where $k^\star \in (0,1)$ satisfies $e^{1/(k^\star - 1)}\overline{\nu} = \hat{\nu}$.*

Proof. By an argument parallel to the proof of Proposition 3, the mechanism $(q^\star, m^\star)$ is incentive compatible and individually rational. Besides,

$$
\nu - m^\star(\nu) = \min\left\{\nu, k^\star\nu + (1-k^\star)e^{1/(k^\star-1)}\overline{\nu}\right\} = \min\left\{\nu, k^\star\nu + (1-k^\star)\hat{\nu}\right\} \leq \hat{\nu} + k^\star\left|\nu - \hat{\nu}\right|,
$$

where the inequality holds regardless of whether $\nu \geq \hat{\nu}$ or $\nu < \hat{\nu}$. Since $m^\star(\hat{\nu}) = 0$ by construction, we can conclude that the $(q^\star, m^\star, k^\star)$ is feasible in Problem (17).

To establish optimality, we first note that, after invoking Proposition 1 to eliminate m, any (q,k) that is feasible in Problem (17) must satisfy

$$
\begin{aligned}
& \nu - q(\nu)\nu + \int_{\hat{\nu}}^{\nu} q(x)\ \mathrm{d}x \leq \hat{\nu} - q(\hat{\nu})\hat{\nu} + k(\nu - \hat{\nu}) && \forall \nu \in [\hat{\nu}, \overline{\nu}] \\
\iff\ & (1-k)(\nu - \hat{\nu}) \leq q(\nu)\nu - q(\hat{\nu})\hat{\nu} - \int_{\hat{\nu}}^{\nu} q(x)\ \mathrm{d}x && \forall \nu \in [\hat{\nu}, \overline{\nu}] \\
\iff\ & (1-k)(\nu - \hat{\nu}) \leq \int_{\hat{\nu}}^{\nu} x\ \mathrm{d}q(x) && \forall \nu \in [\hat{\nu}, \overline{\nu}],
\end{aligned}
\tag{18}
$$

and we will argue that k cannot be smaller than $k^\star$. To simplify the remaining parts of the proof, we introduce $c \triangleq \left(\log\left(\frac{\overline{\nu}}{\hat{\nu}}\right)\right)^{-1} > 0$. The last line of (18) has the following consequence

$$
\int_{\hat{\nu}}^{\overline{\nu}} (1-k)(\nu - \hat{\nu}) \frac{c}{\nu^2}\ \mathrm{d}\nu + (1-k)(\overline{\nu} - \hat{\nu}) \frac{c}{\overline{\nu}} \ \leq\ \int_{\hat{\nu}}^{\overline{\nu}} \frac{c}{\nu^2} \int_{\hat{\nu}}^{\nu} x\ \mathrm{d}q(x) \mathrm{d}\nu + \frac{c}{\overline{\nu}} \int_{\hat{\nu}}^{\overline{\nu}} x\ \mathrm{d}q(x). \tag{19}
$$

Observe that the left-hand side of (19) evaluates to

$$
(1-k)c\left[\int_{\hat{\nu}}^{\overline{\nu}} \left(\frac{1}{\nu} - \frac{\hat{\nu}}{\nu^2}\right) \mathrm{d}\nu + 1 - \frac{\hat{\nu}}{\overline{\nu}}\right] \ =\ (1-k)c\log\left(\frac{\overline{\nu}}{\hat{\nu}}\right) \ =\ 1-k,
$$

where the final equality holds by our construction of c. Likewise, the right-hand side of (19) evaluates to

$$
\begin{aligned}
c\left[\int_{\hat{\nu}}^{\overline{\nu}} x \int_{x}^{\overline{\nu}} \frac{1}{\nu^2}\ \mathrm{d}\nu\ \mathrm{d}q(x) + \frac{1}{\overline{\nu}} \int_{\hat{\nu}}^{\overline{\nu}} x\ \mathrm{d}q(x)\right] &= \ c\left[\int_{\hat{\nu}}^{\overline{\nu}} x \left(\frac{1}{x} - \frac{1}{\overline{\nu}}\right)\ \mathrm{d}q(x) + \frac{1}{\overline{\nu}} \int_{\hat{\nu}}^{\overline{\nu}} x\ \mathrm{d}q(x)\right] \\
&= c\int_{\hat{\nu}}^{\overline{\nu}} \mathrm{d}q(x) = c\left(q(\overline{\nu}) - q(\hat{\nu})\right).
\end{aligned}
$$

As a result, it necessarily holds that $1-k \leq c$, that is, $k \geq 1-c$. Finally, by noting that c is in fact equal to $1-k^\star$, the proof is completed. □

We remark that there are infinitely many mechanisms that solve Problem (17). Indeed, for any $(q^\star, m^\star, k^\star)$ that is optimal, we can construct a family of optimal mechanisms of the form $(q^\star, m^\star - \delta, k^\star)$, which are parametrized by $\delta \geq 0$. However, when $\delta > 0$, these mechanisms are Pareto inefficient as to the seller they would earn a strictly smaller expected revenue (regardless of the probability distribution that governs ν).

6. Statistics of the optimal mechanisms

In all cases, the optimal allocation rule of Problem ($\mathcal{P}$) is continuous and non-decreasing as well as satisfies $q^\star(0)=0$ and $q^\star(\overline{\nu})=1$. Hence, $q^\star$ admits an interpretation as a cumulative distribution function that is supported on $[0,\overline{\nu}]$, and there exists a random variable $\tilde{p}\sim\mathbb{P}$ such that

$$q^\star(\nu)=\mathbb{P}\left(\tilde{p}\le\nu\right)=\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}(\tilde{p}\le\nu)\right)\quad\forall\nu\in[0,\overline{\nu}].$$

Moreover, the optimal mechanism always satisfies (3), and hence

$$m^\star(\nu)=q^\star(\nu)\nu-\int_0^\nu q^\star(x)\ \mathrm{d}x=\int_0^\nu x\ \mathrm{d}q^\star(x)=\int_0^{\overline{\nu}} x\mathbb{1}(x\le\nu)\ \mathrm{d}q^\star(x)=\mathbb{E}_{\mathbb{P}}\left(\tilde{p}\mathbb{1}(\tilde{p}\le\nu)\right)\quad\forall\nu\in[0,\overline{\nu}].$$

We can thus interpret the optimal mechanism $(q^\star,m^\star)$ as a *'randomized posted price mechanism'* where the price $\tilde{p}$ is drawn from the probability distribution $\mathbb{P}$. Using randomized posted prices offers a distinct advantage to the seller in the sense that it increases the willingness of the buyer to engage in the sale transaction as the buyer has to pay only when he is actually given the item. In contrast, directly implementing the mechanism $(q^\star,m^\star)$ entails offering a lottery that requires the buyer with a value ν to make a payment of amount $m^\star(\nu)$ regardless of whether or not he will obtain the item, which to him can only happen favourably with a probability of $q^\star(\nu)$.

We will now carry out a sensitivity analysis to see how the mean price $\mathbb{E}_{\mathbb{P}}\left(\tilde{p}\right)$ changes with the values of τ and $\hat{\nu}$, and simultaneously we will compare it with the optimal deterministic posted price $p^\star$ derived in Section 4. Throughout the experiment, we assume that $\overline{\nu}=1\$=100¢$. From Figure 1, it is seen that when τ is small, $\mathbb{E}(\tilde{p})$ and $p^\star$ coincide. As the target τ gets increasingly larger (*i.e.,* as the seller becomes less conservative), the variance of the optimal random price becomes more sizeable, which justifies the implementation of the more convoluted mechanism we propose. When τ reaches a certain threshold, the variance of $\tilde{p}$ may drop but stay significant nonetheless. Overall, we find that the optimal random price is, on average, at least as large as the optimal deterministic price; hence, the randomized strategy does not only incur a smaller regret but it also has a potential to extract higher revenue from the buyer. Last but not least, to appreciate the benefit of the price dispersion from another angle, we compare the optimal objective value of Problem ($\mathcal{P}$), denoted

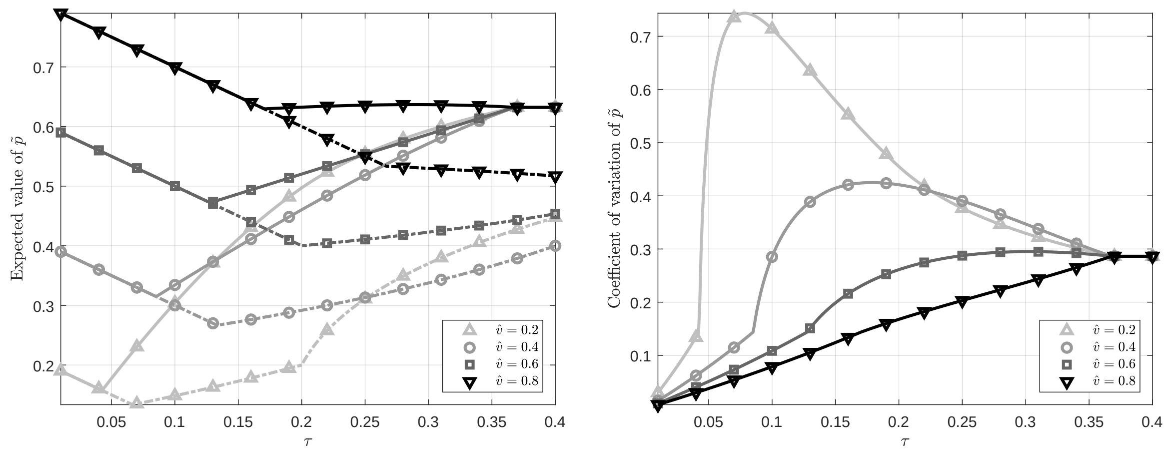

Figure 1 **The mean (left) and the coefficient of variation (right) of the optimal random price $\tilde{p}$ under different combinations of τ and $\hat{\nu}$. The accompanied dashed curves (left) show the optimal deterministic prices $p^\star$.**

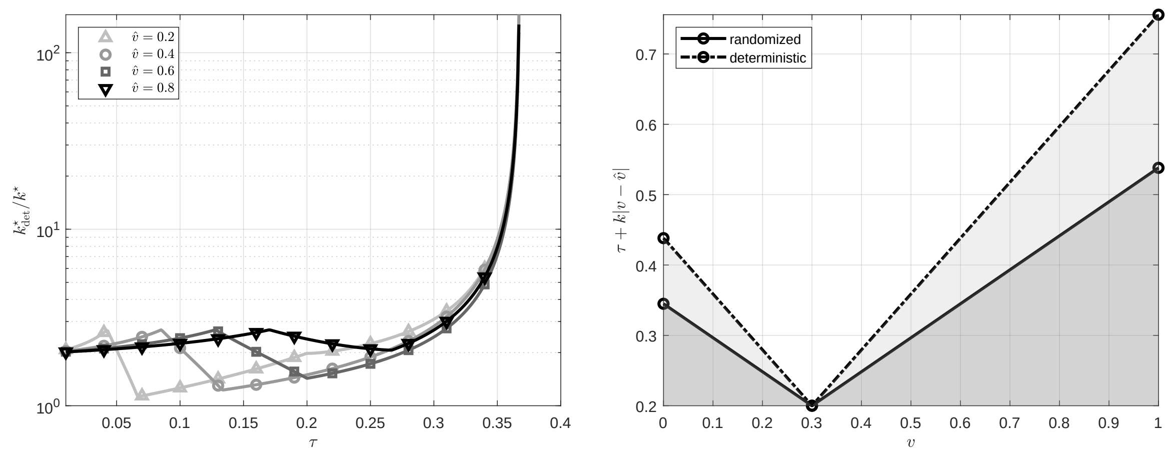

Figure 2 **The comparison between $k^\star$ and $k^\star_{\text{det}}$ (in a logarithmic scale) under different combinations of τ and $\hat{\nu}$ as well as the comparison between the corresponding regret upper bounds.**

by $k^\star$, and that of Problem (15), denoted by $k^\star_{\text{det}}$. These optimal objective values characterize the sensitivity of the regret upper bound with respect to the departure of the buyer's value ν from $\hat{\nu}$. Figure 2 (left) shows that the randomized pricing strategy can lead to a substantial reduction of this sensitivity of at least 12%, and oftentimes the difference between the two strategies is considerably more pronounced, when $\hat{\nu} \in \{20¢, 40¢, 60¢, 80¢\}$. Figure 2 (right) exemplary visualizes the regret bound $\tau + k|\nu - \hat{\nu}|$, $k \in \{k^\star, k^\star_{\text{det}}\}$, of the two pricing strategies when $\tau = 20¢$ and $\hat{\nu} = 30¢$.

Next, we make a statistical comparison between the worst-case regret minimization (Koçyiğit et al. 2021) and the proposed robust regret satisficing frameworks. Retaining the names of the optimization techniques adopted, we shall refer to their recommended mechanisms as *'robust'* and *'satisficing'* solutions, respectively. Specifically for this experiment, we work with a target-free model proposed in Section 5, we normalize $\overline{\nu}$ to 1\$, and we model the buyer's value as a Beta random variable $\tilde{\nu}$ with shape parameters $s_1 > 0$ and $s_2 > 0$, and we choose $\hat{\nu} = \mathbb{E}[\tilde{\nu}]$. We consider various combinations of the shape parameters that are consistent with different means from $\{5¢, 10¢, \ldots, 35¢\}$ (as when $\hat{\nu} = \mathbb{E}[\tilde{\nu}] > \frac{1}{e} \approx 0.368$, the solutions of both robust approaches coincide), and coefficients of variation from $\{0.1, 0.2, \ldots, 2.0\}$. It should be noted that there may be no shape parameters that are compatible with a certain combination of the mean and the coefficient of variation of $\tilde{\nu}$ from their stipulated ranges. In such cases, the corresponding entries in Table 1 are marked with a hyphen. Table 1 (left) reports the relative improvement (in %) in terms of the expected regret of the satisficing solution from Proposition 12 over the robust solution from Koçyiğit et al. (2021), whereas Table 1 (right) reports the relative improvement in terms of the expected revenue. We observe that, in an overwhelming majority of instances, our target-free robust satisficing mechanism makes a significant improvement. Though, as one would expect, if the variance of $\tilde{\nu}$ becomes exceptionally large, the seller should carefully and cautiously hedge against this uncertainty and the standing of the robust solution from Koçyiğit et al. (2021) remains. In light of this, Table 1 markedly defines the region where the satisficing solution is superior to the robust solution and vice versa.

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CV of $\tilde{\nu}$ \ $\hat{\nu}$	5¢	10¢	15¢	20¢	25¢	30¢	35¢
0.1	1.3	1.7	2.1	2.5	2.9	3.3	1.9
0.2	2.7	3.5	4.2	5.0	5.7	5.3	2.0
0.3	4.0	5.2	6.3	7.4	7.7	6.0	1.9
0.4	5.3	6.9	8.4	9.4	8.7	6.1	1.8
0.5	6.6	8.6	10.3	10.7	9.2	5.9	1.6
0.6	7.8	10.2	11.8	11.5	9.1	5.5	1.4
0.7	9.0	11.8	12.9	11.7	8.7	4.9	1.1
0.8	10.2	13.1	13.6	11.5	7.9	4.1	0.8
0.9	11.4	14.2	13.8	10.9	6.9	3.1	0.5
1.0	12.5	15.0	13.6	9.9	5.6	1.8	0.0
1.1	13.5	15.5	13.0	8.5	3.8	0.3	−0.7
1.2	14.5	15.6	12.1	6.8	1.7	−1.7	−1.5
1.3	15.3	15.4	10.7	4.6	−1.0	−4.2	−2.5
1.4	16.0	14.9	8.9	1.8	−4.4	−7.6	-
1.5	16.5	14.0	6.7	−1.5	−8.7	−11.3	-
1.6	16.8	12.8	4.0	−5.7	−14.2	-	-
1.7	17.0	11.3	0.8	−10.8	−20.3	-	-
1.8	17.0	9.4	−3.1	−17.3	-	-	-
1.9	16.7	7.2	−7.7	−25.3	-	-	-
2.0	16.3	4.6	−13.3	-	-	-	-

CV of $\tilde{\nu}$ \ $\hat{\nu}$	5¢	10¢	15¢	20¢	25¢	30¢	35¢
0.1	⋆	⋆	⋆	⋆	⋆	⋆	92.6
0.2	⋆	⋆	⋆	⋆	⋆	⋆	32.1
0.3	⋆	⋆	⋆	⋆	⋆	⋆	17.5
0.4	⋆	⋆	⋆	⋆	⋆	71.4	11.0
0.5	⋆	⋆	⋆	⋆	⋆	44.2	7.2
0.6	⋆	⋆	⋆	⋆	88.9	29.0	4.8
0.7	⋆	⋆	⋆	⋆	58.1	19.2	3.1
0.8	⋆	⋆	⋆	⋆	38.8	12.4	1.8
0.9	⋆	⋆	⋆	73.0	25.6	7.4	0.8
1.0	⋆	⋆	⋆	49.4	16.1	3.5	0.0
1.1	⋆	⋆	⋆	32.8	8.8	0.4	−0.7
1.2	⋆	⋆	73.3	20.5	3.1	−2.1	−1.3
1.3	⋆	⋆	50.2	11.1	−1.5	−4.2	−1.7
1.4	⋆	⋆	33.2	3.6	−5.3	−6.0	-
1.5	⋆	⋆	20.2	−2.5	−8.5	−7.4	-
1.6	⋆	79.9	9.9	−7.5	−11.3	-	-
1.7	⋆	56.4	1.7	−11.8	−13.3	-	-
1.8	⋆	38.4	−5.2	−15.4	-	-	-
1.9	⋆	24.2	−10.9	−18.5	-	-	-
2.0	⋆	12.8	−15.7	-	-	-	-

Table 1 Relative improvement (in %) of the satisficing mechanism over its robust counterpart in terms of the expected regret (left) and the expected revenue (right) under different Beta distributions of the buyer's value. The entries marked with a star symbol indicate that the improvement exceeds 100%.

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Appendix A: Duality perspective for the lower bound problems $(\mathcal{D})$ and $(\mathcal{D}')$

We first argue below that Problem $(\mathcal{D})$ has an intimate relationship with the restriction of the dual of a certain relaxation of Problem $(\mathcal{P})$ and that Proposition 4 can be viewed as a consequence of weak duality. We start from relaxing Problem (5), which is equivalent to Problem $(\mathcal{P})$ even when m^0 is fixed at zero, to

$$\begin{aligned} \text{minimize} \quad & k \\ \text{subject to} \quad & q \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}_+),\ k \in \mathbb{R}_+ \\ & q(\overline{\nu}) \le 1 \\ & \nu - q(\nu)\nu + \int_0^\nu q(x)\ \mathrm{d}x \le \tau + k|\nu - \hat{\nu}| \quad \forall \nu \in [0,\overline{\nu}]. \end{aligned} \tag{20}$$

Assigning a dual variable $\alpha \in \mathbb{R}_+$ to the first constraint and a dual measure $\mathfrak{P} \in \mathcal{M}_+([0,\overline{\nu}])$ to the second constraint, the associated Lagrangian function reads

$$L(k,q,\alpha,\mathfrak{P}) = k + \alpha\left(q(\overline{\nu}) - 1\right) + \int_0^{\overline{\nu}} \left(\nu - q(\nu)\nu - \tau - k|\nu - \hat{\nu}| + \int_0^\nu q(x)\ \mathrm{d}x\right)\mathfrak{P}(\mathrm{d}\nu),$$

and consequently the dual problem is

$$\max_{\alpha,\mathfrak{P}} \left\{ \min_{k,q} \left\{ L(k,q,\alpha,\mathfrak{P}) : k \in \mathbb{R}_+, q \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}_+) \right\} : \alpha \in \mathbb{R}_+, \mathfrak{P} \in \mathcal{M}_+([0,\overline{\nu}]) \right\}. \tag{21}$$

By restricting the dual measure $\mathfrak{P}$ to

$$\mathfrak{P}(\{\overline{\nu}\}) = \overline{\nu}\beta(\overline{\nu}) \quad \text{and} \quad \mathfrak{P}([a,b]) = \int_a^b \beta(x)\ \mathrm{d}x \text{ for any } a \text{ and } b \text{ such that } 0 \le a \le b < \overline{\nu}$$

and the dual variable α to $\alpha = \overline{\nu}^2\beta(\overline{\nu})$, for some integrable non-negative function β, we have

$$\begin{aligned} L(k,q,\alpha,\mathfrak{P}) = & \int_0^{\overline{\nu}} (v - \tau)\beta(\nu)\ \mathrm{d}\nu - \tau\overline{\nu}\beta(\overline{\nu}) + k\left(1 - \int_0^{\overline{\nu}} |\nu - \hat{\nu}|\beta(\nu)\ \mathrm{d}\nu - (\overline{\nu} - \hat{\nu})\overline{\nu}\beta(\overline{\nu})\right) + \\ & \int_0^{\overline{\nu}} q(\nu)\left(\int_\nu^{\overline{\nu}} \beta(x)\ \mathrm{d}x - \nu\beta(\nu) + \overline{\nu}\beta(\overline{\nu})\right)\ \mathrm{d}\nu. \end{aligned}$$

In other words, the Lagrangian function now consists of three parts. The first part is the objective function of Problem $(\mathcal{D})$, the second part is a product of the primal k and the first constraint of Problem $(\mathcal{D})$, whereas the last part is a continuum sum of the primal $q(v)$ multiplied by the second constraint of Problem $(\mathcal{D})$.

Analogously to Proposition 4, Proposition 9 is also tightly connected with duality theory. To see this, we consider another relaxation of Problem $(\mathcal{P})$:

$$\begin{aligned} \text{minimize} \quad & k \\ \text{subject to} \quad & q \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}_+),\ k \in \mathbb{R}_+ \\ & q(\hat{\nu}) \le 1 \\ & \nu - q(\nu)\nu + \int_0^\nu q(x)\ \mathrm{d}x \le \tau + k|\nu - \hat{\nu}| \quad \forall \nu \in [0,\overline{\nu}]. \end{aligned} \tag{22}$$

Problem (22) is almost identical to Problem (20) except that the constraint $q(\overline{\nu}) \le 1$ in the latter is replaced by $q(\hat{\nu}) \le 1$ in the former. Assigning a dual variable $\alpha \in \mathbb{R}_+$ to the first constraint and a dual measure $\mathfrak{P} \in \mathcal{M}_+([0,\overline{\nu}])$ to the second constraint, the Lagrangian function associated with Problem (22) reads

$$L(k,q,\alpha,\mathfrak{P}) = k + \alpha\left(q(\hat{\nu}) - 1\right) + \int_0^{\overline{\nu}} \left(\nu - q(\nu)\nu - \tau - k|\nu - \hat{\nu}| + \int_0^{\nu} q(x)\ \mathrm{d}x\right) \mathfrak{P}(\mathrm{d}\nu),$$

and consequently the dual problem is

$$\max_{\alpha,\mathfrak{P}} \left\{ \min_{k,q} \left\{ L(k,q,\alpha,\mathfrak{P}) : k \in \mathbb{R}_+, q \in \mathcal{L}([0,\overline{\nu}],\mathbb{R}_+) \right\} : \alpha \in \mathbb{R}_+, \mathfrak{P} \in \mathcal{M}_+([0,\overline{\nu}]) \right\}. \tag{23}$$

By restricting the dual measure $\mathfrak{P}$ to

$$\mathfrak{P}(\{\hat{\nu}\}) = \hat{\nu}\beta(\hat{\nu}) \quad \text{and} \quad \mathfrak{P}([a,b]) = \begin{cases} \int_a^b \beta(x)\ \mathrm{d}x & \text{for any } a \text{ and } b \text{ such that } 0 \le a \le b < \hat{\nu} \\ 0 & \text{for any } a \text{ and } b \text{ such that } \hat{\nu} < a \le b \le \overline{\nu} \end{cases}$$

and the dual variable α to $\alpha = \hat{\nu}^2\beta(\hat{\nu})$, for some integrable non-negative function β, we have

$$\begin{aligned} L(k,q,\alpha,\mathfrak{P}) = \int_0^{\hat{\nu}} (v - \tau)\beta(\nu)\ \mathrm{d}\nu - \tau\hat{\nu}\beta(\hat{\nu}) + k\left(1 - \int_0^{\hat{\nu}} (\hat{\nu} - \nu)\beta(\nu)\ \mathrm{d}\nu\right) + \\ \int_0^{\hat{\nu}} q(\nu)\left(\int_{\nu}^{\hat{\nu}} \beta(x)\ \mathrm{d}x - \nu\beta(\nu) + \hat{\nu}\beta(\hat{\nu})\right)\ \mathrm{d}\nu. \end{aligned}$$

The above expression consists of three parts which give rise to the objective function and the two constraints of Problem ($\mathcal{D}'$), respectively, which validates Proposition 9 through weak duality.

Appendix B: Omitted proofs

Proof of Theorem 1. We divide the proof into two steps. The first step aims at showing that, if $\tau \le 0$, then Problem ($\mathcal{P}$) is infeasible. Subsequently, we argue in the second step that, if on the other hand $\tau > 0$, then the Problem ($\mathcal{P}$) becomes feasible.

Step 1: Suppose first that $\tau < 0$ and consequently $\hat{\nu} - m(\hat{\nu}) \le \tau < 0$. This inequality is manifestly impossible to satisfy as the individual rationality constraint implies that $\hat{\nu} - m(\hat{\nu}) \ge q(\hat{\nu})\hat{\nu} - m(\hat{\nu}) \ge 0$. Suppose next that $\tau = 0$, it similarly follows from the first and the last constraints in Problem ($\mathcal{P}$) that

$$0 \le q(\hat{\nu})\hat{\nu} - m(\hat{\nu}) \le \hat{\nu} - m(\hat{\nu}) \le 0,$$

and hence all of the above inequalities must be satisfied as equalities. We therefore find $q(\hat{\nu}) = 1$ and $m(\hat{\nu}) = \hat{\nu}$. From Proposition 1 (ii), we further find

$$\int_0^{\hat{\nu}} q(x)\ \mathrm{d}x = m(0) + q(\hat{\nu})\hat{\nu} - m(\hat{\nu}) = m(0) \le 0,$$

where the inequality is a consequence of individual rationality. As q is a non-negative and a non-decreasing function (see Proposition 1 (i)), it must hold that $q(x)=0$ for all $x\in[0,\hat{\nu})$. We next turn our attention to the satisficing constraint in Problem ($\mathcal{P}$), which implies that

$$k \geq \frac{\nu - m(\nu)}{\hat{\nu}-\nu} \geq \frac{\nu - q(\nu)\nu}{\hat{\nu}-\nu} \quad \forall \nu \in [0,\hat{\nu}) \quad \Longrightarrow \quad k \geq \lim_{\nu \uparrow \hat{\nu}} \frac{\nu - q(\nu)\nu}{\hat{\nu}-\nu} = \infty,$$

and there is therefore no finite k that is feasible in Problem ($\mathcal{P}$).

Step 2: We divide the proof into two cases depending on the value of the target regret τ. Suppose first that $\tau \geq \hat{\nu}$ and consider a deterministic posted price mechanism $q(\nu)=\mathbb{1}(\nu\geq\hat{\nu})$ and $m(\nu)=\hat{\nu}\mathbb{1}(\nu\geq\hat{\nu})$, which is known to satisfy the first two constraints of Problem ($\mathcal{P}$). Furthermore, the last constraint of Problem ($\mathcal{P}$) trivially holds for any $k\geq 1$. Hence, the constructed mechanism is indeed feasible.

Suppose now that $\tau<\hat{\nu}$ and consider another deterministic posted price mechanism $q(\nu)=\mathbb{1}(\nu\geq\hat{\nu}-\tau)$ and $m(\nu)=(\hat{\nu}-\tau)\mathbb{1}(\nu\geq\hat{\nu}-\tau)$. For this case, we choose $k=\max\left\{\frac{\hat{\nu}}{\tau}-2,1\right\}$. To verify that the constructed (q,m,k) satisfies the last constraint in Problem ($\mathcal{P}$), we note that

$$\tau + k|\nu-\hat{\nu}| \geq \tau + \frac{(\hat{\nu}-2\tau)^{+}(\hat{\nu}-\nu)}{\tau} \geq \tau + \frac{(\hat{\nu}-2\tau)^{+}(\hat{\nu}-\nu)}{\hat{\nu}-\nu} \geq \hat{\nu}-\tau > \nu = \nu - m(\nu) \quad \forall \nu\in[0,\hat{\nu}-\tau),$$

and

$$\tau + k|\nu-\hat{\nu}| \geq \tau + |\nu-\hat{\nu}| \geq \tau + \nu - \hat{\nu} = \nu - m(\nu) \quad \forall \nu \in [\hat{\nu}-\tau,\overline{\nu}].$$

The proof is now completed. □

Proof of Theorem 3. When $\tau \geq \frac{\overline{\nu}}{2}$, Problem (16) admits an analytical solution with $(p^\star,k^\star)=(\tau,0)$. Indeed, the proposed solution is feasible as, for all $\nu\in[0,p^\star]$, we find $\nu\leq p^\star=\tau$, and for all $\nu\in[p^\star,\overline{\nu}]$, we find $\nu-p^\star\leq\overline{\nu}-\tau\leq\tau$. This solution is not only feasible but it must also be optimal as k cannot be lower than zero. Henceforth, in the remainder of the proof we shall assume that $\tau<\frac{\overline{\nu}}{2}$. We distinguish two (overlapping) cases depending on the value of the price p: (i) $p\geq\hat{\nu}$ and (ii) $p\leq\hat{\nu}$. Note that instead of keeping all constraints that are parametrized by $\nu\in[0,\overline{\nu}]$ in Problem (16), it is sufficient to keep only the constraints that correspond to $\nu\in\{0,p,\hat{\nu},\overline{\nu}\}$, which constitutes either a point at the kink of the absolute value function or at the respective boundary of each of the two constraints.

Case 1: When $p \geq \hat{\nu}$, the first constraint of Problem (16) evaluated at $\nu = \hat{\nu}$ implies that it can only be feasible when $\tau \geq \hat{\nu}$, and when it is feasible, it is equivalent to

$$\begin{array}{ll} \text{minimize} & k \\ \text{subject to} & p \in [\hat{\nu}, \overline{\nu}],\ k \in \mathbb{R}_+ \\ & p \leq \tau + k(p - \hat{\nu}) \\ & \overline{\nu} - p \leq \tau + k(\overline{\nu} - \hat{\nu}). \end{array} \tag{24}$$

Notice that if τ and $\hat{\nu}$ coincide but p is chosen differently, then k cannot be smaller than unity. On the other hand, if p is chosen as $\hat{\nu}$ or equivalently as τ, then k can be as small as $\frac{\overline{\nu}-2\tau}{\overline{\nu}-\tau} < 1$. In view of this, we may thus conclude that the optimal price must be $\hat{\nu}$. In the remainder of this case, we suppose that $\tau > \hat{\nu}$, and we observe that each of the two constraints of Problem (24) characterizes the following p-dependent lower bound on k:

$$\ell_1(p) = \frac{(p-\tau)^+}{p-\hat{\nu}} \quad \text{and} \quad \ell_2(p) = \frac{(\overline{\nu}-p-\tau)^+}{\overline{\nu}-\hat{\nu}}.$$

Solving Problem (24) is therefore tantamount to solving

$$\min_{p \in [\hat{\nu}, \overline{\nu}]} \max \left\{ \ell_1(p), \ell_2(p) \right\}. \tag{25}$$

By construction, $\ell_1(p) = \left(\frac{\hat{\nu}-\tau}{p-\hat{\nu}} + 1 \right)^+$ is increasing and $\ell_2(p)$ is decreasing in $p \in [\hat{\nu}, \overline{\nu}]$. Furthermore, on the restricted range $[\tau, \overline{\nu} - \tau] \subset [\hat{\nu}, \overline{\nu}]$ of p, we have $\ell_1(\tau) = 0$, $\ell_1(\overline{\nu} - \tau) > 0$, $\ell_2(\tau) > 0$ and $\ell_2(\overline{\nu} - \tau) = 0$. As a result, there is a unique $p^\dagger \in (\tau, \overline{\nu} - \tau)$ such that $\ell_1(p^\dagger) = \ell_2(p^\dagger)$. This choice of $p^\dagger$ must solve Problem (25) because any p that deviates from $p^\dagger$ yields a larger value of $\max\{\ell_1(p), \ell_2(p)\}$. Indeed, if $p < p^\dagger$, then $\ell_2(p) \geq \ell_2(p^\dagger)$. On the other hand, if $p > p^\dagger$, then $\ell_1(p) \geq \ell_1(p^\dagger)$. Hence, any $p \neq p^\dagger$ is weakly dominated by $p^\dagger$.

Case 2: When $p \leq \hat{\nu}$, Problem (16) on the other hand becomes

$$\begin{array}{ll} \text{minimize} & k \\ \text{subject to} & p \in [(\hat{\nu} - \tau)^+, \hat{\nu}],\ k \in \mathbb{R}_+ \\ & p \leq \tau + k(\hat{\nu} - p) \\ & \overline{\nu} - p \leq \tau + k(\overline{\nu} - \hat{\nu}), \end{array} \tag{26}$$

where this time the first constraint characterizes the following lower bound on k:

$$\ell_3(p) = \frac{(p-\tau)^+}{\hat{\nu}-p},$$

and the second constraint yields a lower bound $\ell_2(p)$ of k. Solving Problem (26) is thus tantamount to solving

$$\min_{p\in[(\hat{\nu}-\tau)^+,\hat{\nu}]} \max\{\ell_2(p),\ell_3(p)\}.$$

We can then consider two subcases. First, if $\hat{\nu}\leq\tau$, then $\ell_3(p)=0$ for all $p\in[(\hat{\nu}-\tau)^+,\hat{\nu}]$ ($\implies p\leq\tau$). In this case, though, Problem (26) further reduces to $\min_{p\in[(\hat{\nu}-\tau)^+,\hat{\nu}]}\ell_2(p)$, and the optimal price is $\hat{\nu}$ because ℓ_2 is monotonically decreasing. This characterization of the optimal price however is already subsumed by the analysis in Case 1.

Supposing finally that $\hat{\nu}>\tau$, we can express $\ell_3(p)=\left(\frac{\hat{\nu}-\tau}{\hat{\nu}-p}-1\right)^+$, which is increasing in $p\in[0,\hat{\nu}]$ and strictly increasing in $p\in[\tau,\hat{\nu}]$. Since $\ell_3(\tau)=0$, $\lim_{p\uparrow\hat{\nu}}\ell_3(p)=\infty$, $\ell_2(\tau)>0$ and $\ell_2(\hat{\nu})<\infty$, it follows that there must be a unique $p'\in(\tau,\hat{\nu})$ such that $\ell_2(p')$ and $\ell_3(p')$ coincide. By a familiar argument from Case 1, if $p'\geq\hat{\nu}-\tau$, then it solves Problem (26). Otherwise, if $p'<\hat{\nu}-\tau$, then it is not feasible (let alone optimal) in Problem (26). In this case, the optimal price from $[\hat{\nu}-\tau,\hat{\nu}]$, in view of Problem (26), is $\hat{\nu}-\tau$ because $\ell_3(p)\geq\ell_3(\hat{\nu}-\tau)$ and $\ell_3(p)\geq\ell_2(p)$ for all $p\in[\hat{\nu}-\tau,\hat{\nu}]$. All in all, for this case, we have $p^\star=\max\{\hat{\nu}-\tau,p'\}$. $\square$