Superadditive duality and convex hulls for mixed-integer conic optimization

Temitayo Ajayi* Akshay Gupte† Amin Khademi‡ Andrew Schaefer*

Abstract

We present an infinite family of linear valid inequalities for a mixed-integer conic program, and prove that these inequalities describe the convex hull of the feasible set when this set is bounded and described by integral data. The main element of our proof is to establish a new strong superadditive dual for mixed-integer conic programming that, unlike the existing dual from literature, is much cleaner to describe since it does not include directional derivative constraints, and becomes a finite-dimensional problem when the input data is integral.

Keywords. Superadditive duality · Convex hull · Extended formulations · Valid inequalities

1 Introduction

Let $\mathcal{C}_1 \subseteq \mathbb{R}^p$, $\mathcal{C}_2 \subseteq \mathbb{R}^{n-p}$ and $K \subseteq \mathbf{E}$ be nonempty closed convex cones, where $\mathbf{E}$ is a Euclidean space, and $\mathcal{A}\colon \mathbb{R}^n \to \mathbf{E}$ be a linear map and $b \in \mathbf{E}$ a vector. Consider the closed convex set $X := \{x \in \mathcal{C}_1 \times \mathcal{C}_2 :\ \mathcal{A}x \preccurlyeq_K b\}$ where the constraints $\mathcal{A}x \preccurlyeq_K b$ are defined as $b - \mathcal{A}x \in K$. A mixed-integer conic program (conic MIP) with $p \geqslant 1$ integer variables $(x_1, \ldots, x_p)$ and linear objective $c \in \mathbb{R}^n$ is

$$z^* = \sup\left\{c^\top x :\ x \in X_{\text{int}}\right\} \quad \text{where } X_{\text{int}} := X \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}). \tag{MICP}$$

We assume that $X_{\text{int}} \neq \emptyset$. The conic constraint $x \in \mathcal{C} := \mathcal{C}_1 \times \mathcal{C}_2$ is kept separate because it involves a special linear map (identity) and constant r.h.s. vector being all zeros. The cone K could be a Cartesian product of finitely many cones, some of which could be in $\mathbb{R}^m$ and others could be matrix cones in $\mathbb{R}^{k\times k}$. Equality constraints $Bx = d$ can be handled by writing as $Bx \preccurlyeq_{\{\mathbf{0}\}} d$. We denote $A_j := \mathcal{A}\mathbf{e}_j$ where $\mathbf{e}_j$ is a unit coordinate vector in $\mathbb{R}^n$. Besides mixed-integer linear programming (MILP), other commonly occurring special cases of MICP are with Lorentz cones (MISOCP) and semidefinite cones (MISDP), and there are a plethora of applications for these (see surveys [Bel+13, BS13]).

Derivation of valid inequalities and branch-and-cut algorithms for MICP has gained traction in recent years, mostly concentrated on extending ideas from MILP and generalising the cutting plane families of Chvatal-Gomory [ÇI05, DDV14], split disjunctions [MKV15, Bel+17, LTV20], mixed-integer rounding [AN10], and intersection cuts [AJ13, MKV16]. Other approaches include lifting-based methods [AN11], polymatroid inequalities by exploiting submodular structure for MISOCPs [AG20] and building tractable outer-approximations in possibly higher-dimensional space [Vie+17, Lub+18, KT20].

*Department of Computational and Applied Mathematics, Rice University, USA .
†School of Mathematics, University of Edinburgh, UK.
‡Department of Industrial Engineering, Clemson University, USA.

The focus of this paper is on characterising the mixed-integer hull X_{hull} of MICP, which is the closure convex hull of X_{int}, in the original space (x-space). In this regard, there are a few structural convexification results by Klnç-Karzan et al. [Kl16, KS16, KKL20] for exact representations of X_{hull}. In particular, [Kl16, KS16] derive infinitely many cone-minimal and cone-sublinear inequalities that describe the mixed-integer hull of a disjunctive conic set under a technical condition, and [KKL20] studies X_{int} intersected with the epigraph of a submodular function (so $x_1, \ldots, x_p \in \{0,1\}$) with $b = \mathbf{0}$ for the conic constraints.

1.1 Our Contributions

We present an infinite family of linear valid inequalities for X_{hull} with the input data $(\mathcal{A}, b, c)$ having integer elements, each such inequality is generated by a ray of a closed convex cone (projection cone) that resides in an exponential-sized space. These inequalities are in the x-space and obtained by projecting a nontrivial high-dimensional conic set that is a valid relaxation of the mixed-integer hull. When X is a bounded set, we show that our inequalities (in particular, the ones corresponding to extreme rays of the projection cone) are sufficient to describe X_{hull}. The minimality of this description (and whatever minimality means with nonlinear cones) is not addressed in this paper. Thus, the main contribution is to establish new polyhedral relaxations of the mixed-integer hull that are tight for bounded sets with integral data.

The high-dimensional set that serves as an extended formulation for our projection step is constructed through a new superadditive dual for MICP. This dual is always a weak dual for our primal problem without any assumptions. Construction of an extended formulation from our dual requires integrality of data. Proving sufficiency of the projected inequalities to describe X_{hull} requires the extended formulation to be tight, which requires the dual to be strong, and our arguments for generating a certificate for strong duality make use of a packing transformation which can only be applied to a bounded X_{int}.

Our linear inequalities are amenable to use in branch-and-cut algorithms that use polyhedral relaxations in the search tree. We leave their efficient implementation through separation algorithms etc., for future work.

1.2 Related Work

The method of describing X_{hull} by projecting an extended formulation derived from a strong superadditive dual connects the two areas of strong duality and tight convexifications, and is inspired by [Las04] who gives a projection-type result for integer hulls ($p = n$) of packing polytopes. Since we work with general convex cones in MICP and handle mixed-integer problems for which duality is more complicated than pure integer problems, our results are a generalisation of Lasserre's work. When the projection of X onto $(x_1, \ldots, x_p)$ is a bounded set, X can be reformulated as a disjunctive conic set by taking the finite union over slices of X obtained by fixing integer variables [cf. Kl16, Example 1]. Hence, the convex hull characterisations of [Kl16] for disjunctive conic sets are related, but since these were derived using the cone-minimality approach that is different than our duality-extension-projection approach, our results are not an immediate consequence nor does there seem to be a direct correspondence between our inequalities and those of [Kl16]. We also remark that the strong superadditive dual from literature implies valid inequalities that do correspond to or dominate every inequality defining the mixed-integer hull [MDV12, Corollary 6.1], but they are of the form $\sum_{i=1}^{p} f(A_i)x_i + \sum_{i=p+1}^{n} \bar{f}(A_i)x_i \geqslant f(b)$ which involves finding a superadditive functional f and its directional derivative $\bar{f}$, and hence are in a different vein than ours. Nonetheless, establishing connections to the inequalities of

[MDV12, Kl16] is an interesting question for the future.

1.3 Background on Superadditive Duality

Strong superadditive duals for MICP were recently generalised from the classical ones for MILP by [MDV12, KM19] with a full-dimensional and pointed K (although X_{int} can be low-dimensional). Their strong duality has been established under different conditions such as mixed-integer Slater CQ, or feasibility of conic dual of the relaxation X. We state this dual here. For any $S \subseteq \mathbf{E}$ with $A_j \in S$ for $j = 1, \dots, p$, denote the set of functionals that are superadditive and K-monotone over S by

$$\begin{aligned}\mathcal{F}(S) \coloneqq \big\{ f \colon S \to \mathbb{R} :\ & f \text{ is superadditive and } K\text{-monotone over } S, \\ & f(A_j) \geqslant c_j,\ j = 1, \dots, p \big\}.\end{aligned} \tag{1}$$

Superadditivity of $f \colon \mathbf{E} \to \mathbb{R} \cup \{\pm\infty\}$ over $S \subseteq \mathbf{E}$ means that $f(\beta + \beta') \geqslant f(\beta) + f(\beta')$ for any $\beta, \beta' \in S$ with $\beta + \beta' \in S$. Monotonicity w.r.t. the cone K means that $f(\beta) \leqslant f(\beta')$ for any $\beta, \beta' \in S$ with $\beta \preccurlyeq_K \beta'$. The superadditive dual of [MDV12, KM19] takes the infimum of $f(b)$ over $f \in \mathcal{F}(\mathbf{E})$ ([MDV12] imposes $f(A_j) \geqslant c_j$ for $j = 1, \dots, n$) and $f(\mathbf{0}) = 0$ and certain directional derivative constraints corresponding to continuous variables. This form is intractable for use since the derivative constraints cannot be written easily. Also, the problem is infinite-dimensional since the functions are taken over the entire ambient space. We derive a more tractable form for a strong superadditive dual that is finite-dimensional with integral data and does not have derivative constraints.

2 Main Results and Outline

Let $\mathbb{1}\{\beta = \mathcal{A}'_j, \eta = B_j\}$ denote the indicator function which is equal to 1 if $\beta = \mathcal{A}'_j$ and $\eta = B_j$, otherwise it is 0. Let $\mathcal{D} = (\mathcal{S}_{b'}(K') \cap \mathbf{E}_{\text{int}}) \times (\mathcal{S}_d(\mathbb{R}^m_+) \cap \mathbb{Z}^m)$, where $\mathbf{E}_{\text{int}}$ are the integer elements in $\mathbf{E}$. Consider the following feasible set in an exponential-sized space $(x, u^2, u^3, \dots, u^6)$.

$$\begin{aligned}
&\sum_{j=1}^{p} \mathbb{1}\{\beta = \mathcal{A}'_j, \eta = B_j\} x_j \;-\; \sum_{(\beta',\eta') \in \mathcal{D} : (\beta+\beta', \eta+\eta') \in \mathcal{D}} u^2_{(\beta,\beta',\eta,\eta')} \\
&\quad + \sum_{(\beta',\eta') \in \mathcal{D} : \beta' \preccurlyeq_K \beta, \eta' \leqslant \eta} u^2_{(\beta',\beta-\beta',\eta',\eta-\eta')} \\
&\quad - \sum_{\beta' \in \mathcal{S}_{b'}(K') \cap \mathbf{E}_{\text{int}} : \beta \preccurlyeq_K \beta'} u^3_{(\beta,\beta',\eta)} \;+\; \sum_{\beta' \in \mathcal{S}_{b'}(K') \cap \mathbf{E}_{\text{int}} : \beta' \preccurlyeq_K \beta} u^3_{(\beta',\beta,\eta)} \\
&\quad + \mathbb{1}\{\beta = \mathbf{0}, \eta = \mathbf{0}\} u^4 \;-\; u^5_{(\beta,\eta)} \;=\; 0, \qquad \forall (\beta, \eta) \in \mathcal{D},
\end{aligned} \tag{2a}$$

$$\sum_{(\beta,\eta) \in \mathcal{D}} u^5_{(\beta,\eta)} \;=\; 1, \tag{2b}$$

$$\mathcal{A}'_C(u^6) \preccurlyeq_K u^5_{(\beta,\eta)}[b' - \beta], \qquad \forall (\beta, \eta) \in \mathcal{D}, \tag{2c}$$

$$B_C u^6_{(\beta,\eta)} \;=\; u^5_{(\beta,\eta)}[d - \eta], \qquad \forall (\beta, \eta) \in \mathcal{D}, \tag{2d}$$

$$x_C \;=\; \sum_{(\beta,\eta) \in \mathcal{D}} u^6_{(\beta,\eta)},\ x \geqslant \mathbf{0},\ u^k \geqslant \mathbf{0}, k \in \{2,3,5\},\ u^6_{\beta,\eta} \in \mathcal{C}_2,\ u^4 \in \mathbb{R}. \tag{2e}$$

Write this set as $\mathscr{A} x + \mathscr{B} \boldsymbol{u} \preccurlyeq_{\mathscr{K}} h$ where $h = (\mathbf{0}, 1, \mathbf{0})$ and $\mathscr{K}$ is the Cartesian product of $\{\mathbf{0}\}$ (for linear equations) and K', with $x \in \mathcal{C}$ and $\boldsymbol{u} = (u^2, u^3, u^4, u^5, u^6)$ in the cone $\mathcal{C}'$ which is the Cartesian product of nonnegative orthants of appropriate dimensions (for u^2, u^3, u^5), of $\mathbb{R}$ (for u^4), and of the cone $\mathcal{C}_2$ (for u^6). Note that $\mathscr{K}^\star$ is the Cartesian product of $K'^\star$ and $\mathbb{R}^k$ for some finite k.

Theorem 2.1. *Suppose $\mathcal{C} = \mathbb{R}^n_+$, and $\{A_1, \ldots, A_p, b\}$ are integer vectors. Then for any ray π of the closed convex cone $\{\pi \in \mathscr{K}^\star \colon \mathscr{B}^\star \pi \succcurlyeq_K \mathbf{0}\}$, we have a valid inequality*

$$\left\langle \mathscr{A}^\star \pi,\, x \right\rangle \leqslant \langle \pi,\, h \rangle$$

for X_{hull}. Furthermore, if X_{int} is bounded, then these inequalities, or a transformation of them, define the mixed-integer hull X_{hull}.

The high-dimensional set in (2) is derived using our dual problem which can be written for $S \subset \mathbf{E}$ as

$$\begin{aligned} \nu_S^* = \inf_{f,y} \sup_{\beta \in S} & \left[f(\beta) + \left\langle b - \beta,\, y_\beta \right\rangle \right] \\ & f \in \mathcal{F}(S), \quad f(\mathbf{0}) = 0, \\ & y_\beta \in Y_C, \quad \forall \beta \in S, \end{aligned} \tag{3}$$

where the set $\mathcal{F}(S)$ is from (1) and Y_C is the conic dual of the truncated continuous problem obtained by fixing $(x_1, \ldots, x_p) = \mathbf{0}$,

$$Y_C \coloneqq \left\{ y \in K^\star : \, \mathcal{A}_C^\star y \succcurlyeq_{\mathcal{C}_2^\star} c_C \right\}, \tag{4}$$

Here, $K^\star$ and $\mathcal{C}_2^\star$ are dual cones, $\mathcal{A}_C^\star \colon y \mapsto (\left\langle A_{p+1},\, y \right\rangle, \ldots, \langle A_n,\, y \rangle)$ is the adjoint of the map $\mathcal{A}_C \colon (x_{p+1}, \ldots, x_n) \mapsto \sum_{j>p} A_j x_j$, and $c_C = (c_{p+1}, \ldots, c_n)$.

Notation. Consider the following closed convex sets in $\mathbf{E}$ for a given vector $v \in \mathbf{E}$ and closed convex cones $K \subseteq K' \subset \mathbf{E}$:

$$\mathcal{S}_v(K, K') \coloneqq \{ y \in \mathbf{E} : \, \mathbf{0} \preccurlyeq_{K'} y \preccurlyeq_K v \}, \quad \mathcal{S}_v(K) \coloneqq \mathcal{S}_v(K, K). \tag{5}$$

The set $\mathcal{S}_v(K)$ is equal to $\mathtt{sub}_K\{v\}$, the submissive of the point v w.r.t. K and with $p = 0$ (cf. Definition 1), whereas the set $\mathcal{S}_v(K, K')$ can be thought of as an extension of this notion. It will also be useful to write

$$X = X' \cap \{x : \, Bx = d\}, \qquad \text{where } X' \coloneqq \left\{ x \in \mathcal{C} : \, \mathcal{A}' x \preccurlyeq_{K'} b' \right\}, \tag{6}$$

for some full-dimensional set X', with the linear equations giving the affine hull of X. Note that due to the facial reduction technique, full-dimensionality of X' means that we can assume that K' is also full-dimensional.

Theorem 2.2. *Assume $\mathcal{C}_1 = \mathbb{R}^p_+$ and $\mathcal{C}_2$ is pointed and c_C is such that $Y_C \neq \emptyset$.*

1. *$\nu^*_{\mathcal{S}_b(K)}$ is a strong dual when X is full-dimensional and $\mathcal{A}(\mathcal{C}) \subseteq K$ and $b \in K$.*
2. *Otherwise, for every $\mathbf{0} \neq \xi \in \mathrm{ri}\, K'$ there is a large enough scalar $\rho > 0$ such that $\nu^*_{\mathcal{D}_{\xi,\rho}}$ is a strong dual with*

$$\mathcal{D}_{\xi,\rho} \coloneqq \mathcal{S}_{b'+\rho\xi}(K') \times \mathcal{S}_{d+\rho\mathbf{1}}(\mathbb{R}^m_+). \tag{7}$$

Furthermore, when the vectors $A_1, \ldots, A_p$ and b are integral, then the above strong duals hold even with considering only the integer elements in the respective sets $\mathcal{S}_b(K)$ and $\mathcal{D}_{\xi,\rho}$.

The first case in this theorem is what we refer to as the packing condition for X. If that does not hold then the second case essentially transforms the set using ξ and ρ.

2.1 Outline of our Proofs

The extension in the first theorem is a conic set in a high-dimensional space. We use a projection result from [Aja+21] for projecting conic sets onto subspaces. This is analogous to projecting polyhedral sets using projection cones. In our case, the variables to project are u^2, u^3, u^4, u^5, u^6 using extreme rays of the cone associated with these variables in the constraints of the extension. Characterizing extreme rays of projection cone is a nontrivial task and hard in general, but at least one can use the fact that each ray (not necessarily extreme) of the projection cone yields a valid inequality in the x-space to the mixed-integer hull. To construct the extension from the dual, we use finiteness of the dual when the data is integral, which means that the dual problem can be written as a big LP, and then invoking LP duality gives us the extension stated at the beginning of this section. Thus the main goal of the rest of the paper is to establish a new strong dual that does not involve derivative constraints. The main ingredient of deriving this dual is to decompose the value function of MICP into its integer and continuous truncations and establishing lower and upper bounds on the value functions through this decomposition. This allows us to treat the integer and continuous parts separately and glue the results together into one strong dual, and also helps us to avoid derivative constraints on the continuous part by invoking strong duality for continuous conic programs. We present our proof for the strong dual using the case of packing problems but this is wlog as we show that every bounded set can be transformed to satisfy the structural property of being packing.

Comparison to Disjunctive Programming Although when X_{int} is bounded one can enumerate all the integer feasible vectors and write a high-dimensional disjunctive conic formulation for X_{hull} that can be projected to x-space, one can verify after writing this disjunction that the extended formulation is not the same as our proposed extension, and so indeed we have a different representation for the mixed-integer hull. The projected inequalities are also different since the projection cones for the two systems are different.

3 Value Function

Consider MICP parametrized by its right-hand side. For $\beta \in \mathbf{E}$, we have

$$X_{\text{int}}(\beta) := \left\{ x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} : \mathcal{A}x \preccurlyeq_K \beta, \ x \in \mathcal{C} \right\}.$$

The mixed-integer hull and the continuous relaxation of the parametric problem are $X_{\text{hull}}(\beta)$ and $X(\beta)$, respectively. The integer restriction is $X_I(\beta) := \left\{ x \in X_{\text{int}}(\beta) : \ x_j = 0, \ p+1 \leqslant j \leqslant n \right\}$ and continuous restriction is $X_C(\beta) := \left\{ x \in X_{\text{int}}(\beta) : \ x_j = 0, \ 1 \leqslant j \leqslant p \right\}$. Thus, $X_I(\beta) \cup X_C(\beta) \subseteq X_{\text{int}}(\beta)$. Denote the sets of feasible right-hand sides by

$$\mathcal{R} := \left\{ \beta \in \mathbf{E} : \ X_{\text{int}}(\beta) \neq \emptyset \right\}, \quad \mathcal{R}_I := \left\{ \beta \in \mathbf{E} : \ X_I(\beta) \neq \emptyset \right\},$$

and similarly define $\mathcal{R}_C$ with respect to X_C. Note that $b \in \mathcal{R}$ due to $X_{\text{int}} \neq \emptyset$. Observe that $K \subseteq \mathcal{R} \cap \mathcal{R}_I \cap \mathcal{R}_C$ because $x = \mathbf{0}$ is feasible for any $\beta \in K$. Linearity of $\mathcal{A}$ and additive equivariance of $\preccurlyeq_K$ makes it easy to verify that these sets are closed under addition, and related as $\mathcal{R}_I \cup \mathcal{R}_C \subseteq \mathcal{R}_I + \mathcal{R}_C = \mathcal{R}$.

Lemma 3.1. *$(\mathcal{R}, +)$, $(\mathcal{R}_I, +)$ and $(\mathcal{R}_C, +)$ are monoids with $\mathcal{R}_I \cup \mathcal{R}_C \subseteq \mathcal{R}_I + \mathcal{R}_C \subseteq \mathcal{R}$. Furthermore, $\mathcal{R}_I + \mathcal{R}_C = \mathcal{R}$ when $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$.*

Proof. Under the additive identity element $\mathbf{0}$, the monoid property is $X_{\text{int}}(\beta) + X_{\text{int}}(\beta') \subseteq X_{\text{int}}(\beta+\beta')$ for any $\beta, \beta' \in \mathcal{R}$, which is elementary to argue. The inclusion $\mathcal{R}_I \cup \mathcal{R}_C \subseteq \mathcal{R}_I + \mathcal{R}_C$ follows from the fact that $\mathbf{0} \in \mathcal{R}_I \cap \mathcal{R}_C$. The sets X_I and X_C are in orthogonal subspaces $x_I := (x_1, \dots, x_p, \mathbf{0})$ and $x_C := (\mathbf{0}, x_{p+1}, \dots, x_n)$. Therefore, additive equivariance of $\preccurlyeq_K$ implies that for any $\beta \in \mathcal{R}_I$ and $\beta' \in \mathcal{R}_C$, we have $X_I(\beta) + X_C(\beta') \subseteq X_{\text{int}}(\beta + \beta')$. This gives us $\mathcal{R}_I + \mathcal{R}_C \subseteq \mathcal{R}$. For the reverse inclusion, assume $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ where $\mathcal{C}_1 \subset \mathbb{R}^p$ and $\mathcal{C}_2 \subset \mathbb{R}^{n-p}$. Take any $x \in X_{\text{int}}(\beta)$ and define $\beta' = \sum_{j=1}^{p} A_j x_j$. Since $x = x_I + x_C$ and $(x_1, \dots, x_p) \in \mathcal{C}_1$ and $\mathbf{0} \in \mathcal{C}_2$, we have $x_I \in X_I(\beta')$ and so $\beta' \in \mathcal{R}_I$. We also have $\sum_{j=p+1}^{n} A_j x_j \preccurlyeq_K \beta - \beta'$ due to $\mathcal{A}x \preccurlyeq_K \beta$, and since $\preccurlyeq_K$ is transitive, it follows that $\beta'' \in \mathcal{R}_C$ for any $\beta'' \succcurlyeq_K \beta - \beta'$. Therefore, $\beta = \beta' + (\beta - \beta')$ gives us $\mathcal{R} \subseteq \mathcal{R}_I + \mathcal{R}_C$, thereby leading to equality. □

The recession cone of a nonempty parametric set $X(\beta)$ is invariant to the right-hand side and equal to the cone $X(\mathbf{0})$ [cf. Aja+21]. Since boundedness of a nonempty closed convex set is equivalent to its recession cone being equal to $\{\mathbf{0}\}$, it follows that ?THM? **??** gives us $X_{\text{int}}(\beta)$ is bounded for $\beta \in \mathcal{R}$, $X_I(\beta)$ is bounded for $\beta \in \mathcal{R}_I$ and $X_C(\beta)$ is bounded for $\beta \in \mathcal{R}_C$.

The value function of a conic MIP is the functional $z \colon \mathbf{E} \to \mathbb{R} \cup \{\pm\infty\}$ which gives the optimal value over the parametric set $X_{\text{int}}(\beta)$,

$$z(\beta) \;=\; \sup\left\{c^\top x :\; x \in X_{\text{int}}(\beta)\right\}. \tag{8a}$$

The values of $-\infty$ and ∞ correspond to infeasibility and unboundedness, respectively, and so the domain of this function (the values of β where $z(\beta) > -\infty$) is the set $\mathcal{R}$. It is not difficult to argue that the value at origin determines finiteness over its domain. We also have that the value function z belongs to $\mathcal{F}(\mathcal{R})$.

Lemma 3.2 ([cf. MDV12, Proposition 4.8])**.** *$z \in \mathcal{F}(\mathcal{R})$.*

Let us also note that $\mathcal{F}(S)$ is a convex set in the functional space.

Lemma 3.3. *$\mathcal{F}(S)$ is a closed convex set that is a polyhedron when $|S| < \infty$. Also, $\mathcal{F}(S') \supseteq \mathcal{F}(S)$ for $\emptyset \neq S' \subset S$.*

Proof. The fact that $\mathcal{F}(S)$ is closed and convex follows from its definition as the intersection of closed halfspaces. If $|S|$ is finite, then $\mathcal{F}(S) \subseteq \mathbb{R}^{n'}$, for some positive integer n' and so the set is formed by the intersection of finitely many closed half-spaces, which is a polyhedron. If $\emptyset \neq S' \subset S$ and $f \in \mathcal{F}(S)$, then f is also superadditive and K-monotone over S', which implies that $f \in \mathcal{F}(S')$. □

Now we derive bounds for $z(\beta)$ using the truncated value functions

$$z_I(\beta) \;=\; \sup\{c^\top x \colon x \in X_I(\beta)\}, \quad z_C(\beta) \;=\; \sup\{c^\top x \colon x \in X_C(\beta)\}, \tag{8b}$$

corresponding to the integer and continuous restrictions and whose domains are $\mathcal{R}_I$ and $\mathcal{R}_C$. Also consider for any $\beta \in \mathbf{E}$ the function

$$Z_\beta \colon \beta' \in \mathbf{E} \;\mapsto\; z_I(\beta') + z_C(\beta - \beta'). \tag{8c}$$

The domain of Z_β is $\mathcal{R}_I \cap (\beta - \mathcal{R}_C)$.

The pointwise supremum of Z_β is a lower bound on $z(\beta)$.

Lemma 3.4. *$z(\beta) \geqslant \sup_{\beta' \in \mathbf{E}} Z_\beta(\beta') \geqslant \max\left\{z_I(\beta), z_C(\beta)\right\}$ for any $\beta \in \mathcal{R}$.*

Proof. Let us first note the trivial lower bound $z(\beta) \geqslant \max\left\{z_I(\beta),\, z_C(\beta)\right\}$ for any $\beta \in \mathbf{E}$. This is because we have $X_I(\beta) \bigcup X_C(\beta) \subseteq X_{\text{int}}(\beta)$ and the value functions are equal to $-\infty$ outside their respective domains.

We have to prove something stronger, that the trivial lower bound is dominated by the supremum of Z_β. This supremum can be considered over $\mathcal{R}_I \cap (\beta - \mathcal{R}_C)$ which is the domain of Z_β. Take any $\beta' \in \mathcal{R}_I \cap (\beta - \mathcal{R}_C)$. Since $\mathcal{R}_I \cup \mathcal{R}_C \subseteq \mathcal{R}$ from Lemma 3.1, we get $\beta' \in \mathcal{R}$ and $\beta - \beta' \in \mathcal{R}$. Superadditivity in Lemma 3.2 gives us $z(\beta) = z(\beta' + \beta - \beta') \geqslant z(\beta') + z(\beta - \beta')$. For the two terms $z(\beta')$ and $z(\beta - \beta')$ on the right-hand side, we apply the trivial lower bound with β' and $\beta - \beta'$ to get $z(\beta') \geqslant z_I(\beta')$ and $z(\beta - \beta') \geqslant z_C(\beta - \beta')$. Substituting this into the inequality from superadditivity yields $z(\beta) \geqslant Z_\beta(\beta')$ for any $\beta' \in \mathcal{R}_I \cap (\beta - \mathcal{R}_C)$. Lower bounds on z are only interesting when $z(\mathbf{0}) = 0$, because of the following claim.

Lemma 3.5. *$z(\mathbf{0}) \in \{0, \infty\}$, and $z(\mathbf{0}) = \infty$ implies $z(\beta) = \infty$ for all $\beta \in \mathcal{R}$.*

Proof. It is clear that $z(\mathbf{0}) \geqslant 0$ due to $\mathbf{0} \in X_{\text{int}}(\mathbf{0})$. Note that $\mathbb{Z}_{\geqslant 1} X_{\text{int}}(\mathbf{0}) \subset X_{\text{int}}(\mathbf{0})$, i.e., $X_{\text{int}}(\mathbf{0})$ is closed under positive integer scaling. If there exists an $x \in X_{\text{int}}(\mathbf{0})$ with $c^\top x > 0$, we have $c^\top(\lambda x) \to \infty$ as $\lambda \to \infty$ and then $\lambda x \in X_{\text{int}}(\mathbf{0})$ for all integer $\lambda \geqslant 1$ implies that $z(\mathbf{0}) = \infty$. If $z(\mathbf{0}) = \infty$, then there exists $x \in X_{\text{int}}(\mathbf{0})$ such that $c^\top x > 0$. The monoid property of $X_{\text{int}}(\beta)$ from Lemma 3.1 and $\mathbf{0} \in \mathcal{R}$ gives us $X_{\text{int}}(\beta) + X_{\text{int}}(\mathbf{0}) \subseteq X_{\text{int}}(\beta)$ for any $\beta \in \mathcal{R}$. Since $\lambda x \in X_{\text{int}}(\mathbf{0})$ for $\lambda \in \mathbb{Z}_{\geqslant 1}$, we have $x' + \lambda x \in X_{\text{int}}(\beta)$ for any $x' \in X_{\text{int}}(\beta)$. Now $c^\top x > 0$ implies that $c^\top(x' + \lambda x) \to \infty$ as $\lambda \to \infty$, yielding $z(\beta) = \infty$.

If $X_{\text{int}}(\beta)$ is bounded for some $\beta \in \mathcal{R}$ then the above part implies that $z(\mathbf{0})$ must be equal to zero. ?THM? **??** guarantees boundedness for *every* $\beta \in \mathcal{R}$. □

Since $z_I(\mathbf{0}), z_C(\mathbf{0}) \geqslant 0$ due to $x = \mathbf{0}$ being feasible to both the restrictions, $z(\mathbf{0}) = 0$ and the trivial lower bound implies that $z_I(\mathbf{0}) = z_C(\mathbf{0}) = 0$. This leads to $Z_\beta(\mathbf{0}) = z_C(\beta)$ and $Z_\beta(\beta) = z_I(\beta)$. Therefore, $\sup_{\beta' \in \mathbf{E}} Z_\beta(\beta') \geqslant \max\left\{z_I(\beta), z_C(\beta)\right\}$. □

The second lower bound in the above lemma is loose in general since optimal solutions of $z(\beta)$ can have support over both integer and continuous variables.

To describe upper bounds on z, we need to define the submissive of a set and note some of its basic properties.

Definition 1. The submissive of a set $S \subset \mathcal{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is the set $\operatorname{sub} S = \operatorname{sub}_{\mathcal{C}} S := \{y \in \mathbb{Z}^p \times \mathbb{R}^{n-p} :\ \mathbf{0} \preccurlyeq_{\mathcal{C}} y \preccurlyeq_{\mathcal{C}} x,\ \text{for some } x \in S\}$. A subset of $\mathcal{C}$ that is equal to its submissive is said to be *$\mathcal{C}$-packing*.

This definition generalises what is commonly used in polyhedral combinatorics for packing polyhedra S with $\mathcal{C} = \mathbb{R}^n_+$ and $p = 0$. We allow S to be low-dimensional. It is obvious that $S \subseteq \operatorname{sub} S$ and easy to verify properties of inclusion-preserving, sub-distributivity over intersection, and distributivity over Cartesian product. Let us note a relationship between the submissives of $X_{\text{int}}(\beta)$ and $X_I(\beta)$.

Lemma 3.6. *Suppose $\mathcal{C}_2$ is pointed. Then, $\operatorname{sub} X_I(\beta) \subseteq \{x \in \operatorname{sub} X_{\text{int}}(\beta) :\ x_j = 0,\, p+1 \leqslant j \leqslant n\} \subset \mathcal{C}_1 \times \{\mathbf{0}\}$.*

Proof. By definition, $X_I(\beta) = X_{\text{int}}(\beta) \cap \mathcal{H}$ where we denote $\mathcal{H} := \{x \in \mathcal{C} :\ x_j = 0,\, p+1 \leqslant j \leqslant n\}$. The following claim about submissive is easy to verify.

Claim 1. *For any two sets $S_1, S_2 \subset \mathcal{C}$, we have that $\operatorname{sub}(S_1 \cap S_2) \subseteq \operatorname{sub} S_1 \cap \operatorname{sub} S_2$, and $S_1 \subset S_2$ implies $\operatorname{sub} S_1 \subseteq \operatorname{sub} S_2$. If $S_1 \subset \mathcal{C}_1$ and $S_2 \subset \mathcal{C}_2$ for two closed convex cones $\mathcal{C}_1$ and $\mathcal{C}_2$, then $\operatorname{sub}_{\mathcal{C}_1 \times \mathcal{C}_2}(S_1 \times S_2) = \operatorname{sub}_{\mathcal{C}_1} S_1 \times \operatorname{sub}_{\mathcal{C}_2} S_2$.*

This gives us $\mathtt{sub}\, X_I(\beta) \subseteq \mathtt{sub}\, X_{\text{int}}(\beta) \cap \mathtt{sub}\, \mathcal{H}$. The distributivity property from this lemma with the assumption $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ yields $\mathtt{sub}\, \mathcal{H} = \mathtt{sub}_{\mathcal{C}_1} \mathcal{C}_1 \times \mathtt{sub}_{\mathcal{C}_2}\{\mathbf{0}\} = \mathcal{C}_1 \times \mathtt{sub}_{\mathcal{C}_2}\{\mathbf{0}\}$. Since $\mathtt{sub}_{\mathcal{C}_2}\{\mathbf{0}\} = \mathcal{C}_2 \cap -\mathcal{C}_2$, the pointedness assumption on $\mathcal{C}_2$ implies that $\mathtt{sub}_{\mathcal{C}_2}\{\mathbf{0}\} = \{\mathbf{0}\}$, which leads to $\mathtt{sub}\, X_I(\beta) \subseteq \mathtt{sub}\, X_{\text{int}}(\beta) \cap (\mathcal{C}_1 \times \{\mathbf{0}\}) = \{x \in \mathtt{sub}\, X_{\text{int}}(\beta) :\ x_j = 0,\, p+1 \leqslant j \leqslant n\}$, as desired. □

The next result shows that the value function z can be decomposed into the integer and continuous truncated value functions if we take a pointwise supremum of Z_β over linear image of the submissive of X_{int}.

Lemma 3.7. *For any $\beta \in \mathcal{R}$ and set $S \subset \mathbf{E}$ such that $S \supseteq \mathcal{A}(\mathcal{S})$ where $\mathcal{S} := \{x \in \mathtt{sub}\, X_{\text{int}}(\beta) :\ x_j = 0,\, p+1 \leqslant j \leqslant n\}$, we have $z(\beta) = \sup_{\beta' \in S} Z_\beta(\beta')$.*

Proof. Lemma 3.4 immediately gives us $z(\beta) \geqslant \sup_{\beta' \in S} Z_\beta(\beta')$. We argue that for every $x \in X_{\text{int}}(\beta)$ there exists some $\beta' \in \mathcal{A}(\mathcal{S}) \cap \mathcal{R}_I$ such that $c^\top x \leqslant Z_\beta(\beta')$. This implies that $z(\beta) = \sup\left\{c^\top x :\ x \in X_{\text{int}}(\beta)\right\} \leqslant \sup\left\{Z_\beta(\beta') :\ \beta' \in \mathcal{A}(\mathcal{S}) \cap \mathcal{R}_I\right\}$, and since $\mathcal{A}(\mathcal{S}) \cap \mathcal{R}_I \subseteq S \cap \mathcal{R}_I \subseteq S$, we obtain $z(\beta) \leqslant \sup_{\beta' \in S} Z_\beta(\beta')$, thereby leading to the claimed equality.

Take any $x \in X_{\text{int}}(\beta)$ and partition it into orthogonal subspaces as $x_I := (x_1, \dots, x_p, \mathbf{0})$ and $x_C := (\mathbf{0}, x_{p+1}, \dots, x_n)$. Define $\beta' := \mathcal{A}(x_I) = \sum_{j \leqslant p} A_j x_j$. Note that $(x_1, \dots, x_p) \in \mathcal{C}_1$ and $\mathbf{0} \in \mathcal{C}_2$. This gives us $x_I \in X_I(\beta')$ and so $\beta' \in \mathcal{R}_I$. Since the partial order $\preccurlyeq_{\mathcal{C}_1 \times \mathcal{C}_2}$ distributes over the direct product, we also get $x_I \preccurlyeq_{\mathcal{C}} x$. Therefore, $x \in X_{\text{int}}(\beta)$ implies that $x_I \in \mathcal{S}$, and then $\beta' = \mathcal{A}(x_I)$ leads to $\beta' \in \mathcal{A}(\mathcal{S})$. Thus, we have $\mathcal{A}(x_I) \in \mathcal{A}(\mathcal{S}) \cap \mathcal{R}_I$. The construction of x_C yields $x_C \in X_C(\beta - \beta')$ due to $\sum_{j > p} A_j x_j \preccurlyeq_K \beta - \sum_{j \leqslant p} A_j x_j = \beta - \beta'$ and $(x_{p+1}, \dots, x_n) \in \mathcal{C}_2$. Hence, $c^\top x_I \leqslant z_I(\beta')$ and $c^\top x_C \leqslant z_C(\beta - \beta')$. Linearity of the objective and $x = x_I + x_C$ means that $c^\top x = c^\top x_I + c^\top x_C$, which leads to $c^\top x \leqslant z_I(\beta') + z_C(\beta - \beta') = Z_\beta(\beta')$.

□

Now we give an upper bound on the truncated value function z_I as a pointwise infimum of functionals in $\mathcal{F}(S)$ where S contains the linear image of the submissive of X_I. This claim holds when the integer variables are in the nonnegative orthant. It is obvious that $\mathcal{A}(\mathbb{Z}^p_+ \times \{\mathbf{0}\}) \subseteq \mathcal{R}_I$. Lemma 3.6 then tells us that $\mathcal{A}(\mathtt{sub}\, X_I(\beta)) \subseteq \mathcal{R}_I$, and so there always exists a set S sandwiched between these two sets. This lemma also implies that the condition required on S here is weaker than the one in Lemma 3.7.

Lemma 3.8. *Suppose $\mathcal{C}_1 = \mathbb{R}^p_+$. For $\beta \in \mathcal{R}_I$ and $S \subset \mathbf{E}$ such that $\beta \in S$ and $S \supseteq \mathcal{A}(\mathtt{sub}\, X_I(\beta))$, we have $z_I(\beta) \leqslant \inf_f \left\{f(\beta) :\ f \in \mathcal{F}(S)\right\}$, with equality holding when we also have $S \subseteq \mathcal{R}_I$.*

Proof. Take any $x \in X_I(\beta)$ and $f \in \mathcal{F}(S)$. We argue that $c^\top x \leqslant f(\beta)$; this implies the upper bound on $z_I(\beta)$ since $z_I(\beta)$ is the supremum of $c^\top x$ over $X_I(\beta)$. Let $\beta' := \mathcal{A}x$. Since $x \in X_I(\beta) \subset \mathtt{sub}\, X_I(\beta)$, the assumption on S gives us $\beta' \in S$. Also, $\beta' \preccurlyeq_K \beta$ due to feasibility of x. Applying K-monotonicity of $f(\cdot)$ over S and using $x = \sum_{j=1}^p x_j \mathbf{e}_j$ and linearity of $\mathcal{A}$ leads to $f(\beta) \geqslant f(\beta') = f(\mathcal{A}x) = f\left(\sum_{j=1}^p \mathcal{A}(x_j \mathbf{e}_j)\right)$. For any $J \subseteq \{1, \dots, p\}$, we have $\sum_{j \in J} x_j \mathbf{e}_j \in \mathtt{sub}\, X_I(\beta)$ because $x_j \geqslant 0$ for $j \leqslant p$ implies $\sum_{j \in J} x_j \mathbf{e}_j \preccurlyeq_{\mathbb{R}^p_+ \times \mathcal{C}_2} x$. Then the assumption on S implies that $\mathcal{A}(\sum_{j \in J} x_j \mathbf{e}_j) \in S$, which by linearity of the map means that $\sum_{j \in J} \mathcal{A}(x_j \mathbf{e}_j) \in S$. Therefore, we can use superadditivity of f over S to split the term $f(\mathcal{A}x)$ and obtain

$$f\left(\sum_{j=1}^p \mathcal{A}(x_j \mathbf{e}_j)\right) \geqslant \sum_{j=1}^p f\left(\mathcal{A}(x_j \mathbf{e}_j)\right) = \sum_{j=1}^p f\left(x_j \mathcal{A}\mathbf{e}_j\right) = \sum_{j=1}^p f\Big(\underbrace{\mathcal{A}\mathbf{e}_j + \cdots + \mathcal{A}\mathbf{e}_j}_{x_j \text{times}}\Big),$$

where the second equality is due to $x_j \in \mathbb{Z}_+$. Since $\mathbf{0} \leqslant k\mathbf{e}_j \leqslant x$ for any integer k with $0 \leqslant k \leqslant x_j$, assumption on S and linearity of $\mathcal{A}$ gives us $k\mathcal{A}\mathbf{e}_j \in S$. Therefore, superadditivity of f over S can be applied to split the summands on the right-hand side, leading to $f(\beta) \geqslant f\left(\sum_{j=1}^{p} \mathcal{A}(x_j\mathbf{e}_j)\right) \geqslant \sum_{j=1}^{p} x_j f\left(\mathcal{A}\mathbf{e}_j\right)$. Finally noting that $\mathcal{A}\mathbf{e}_j = A_j$, the constraint $f(A_j) \geqslant c_j$ for $f \in \mathcal{F}(S)$ leads to $f(\beta) \geqslant \sum_{j=1}^{p} x_j c_j = c^\top x$ (the equality is due to $x_j = 0$, $p < j \leqslant n$), which is what we had claimed to prove. For the upper bound to become an equality, it suffices to argue that $z_I \in \mathcal{F}(S)$. Lemma 3.2 applied to $X_I(\beta)$ gives us $z_I \in \mathcal{F}(\mathcal{R}_I)$, and then $S \subseteq \mathcal{R}_I$ and Lemma 3.3 lead to $z_I \in \mathcal{F}(S)$. ☐

4 Weak and Strong Duality

The analysis for the rest of this paper makes the following assumption that is required in Theorem 2.2.

Assumption 1. Henceforth, $\mathcal{C}_1 = \mathbb{R}^p_+$ and $\mathcal{C}_2$ is pointed.

Let us argue weak duality first.

Proposition 4.1. *$z^* \leqslant \nu^*_S$ for any set $S \subset \mathbf{E}$ with $b \in S$ and $S \supseteq \{\mathcal{A}x : x \in \mathtt{sub}\, X_{\text{int}},\ x_j = 0,\ p+1 \leqslant j \leqslant n\}$.*

Proof. $X_{\text{int}} \neq \emptyset$ implies that $b \in \mathcal{R}$. We have $z^* = z(b) = \mathtt{sup}_{\beta \in S}\, z_I(\beta) + z_C(b - \beta)$, where the second equality is from Lemma 3.7. Applying weak duality of conic programming to the truncated problem $z_C(b-\beta)$ yields $z^* \leqslant \mathtt{sup}_{\beta\in S}\left[z_I(\beta) + \mathtt{inf}_{y\in Y_C}\langle b-\beta,\, y\rangle\right]$. Lemma 3.6 implies that $S \supseteq \mathcal{A}(\mathtt{sub}\, X_I)$ and then Lemma 3.8 gives us $z^* \leqslant \mathtt{sup}_{\beta\in S}\Big[\mathtt{inf}_{f\in\mathcal{F}(S)} f(\beta) + \mathtt{inf}_{y\in Y_C}\langle b-\beta,\, y\rangle\Big]$. The two infimums are independent problems and so combining them into one infimum and interchanging $\mathtt{inf}$ and $\mathtt{sup}$ leads to

$$z^* \leqslant \inf_{\substack{f\in\mathcal{F}(S)\\ y_\beta\in Y_C,\,\forall\beta\in S}} \ \sup_{\beta\in S}\Big[f(\beta) + \big\langle b-\beta,\, y_\beta\big\rangle\Big] \leqslant \nu^*_S,$$

where the last inequality is because the dual problem has an $f(\mathbf{0}) = 0$ which can only make the infimum worse. Note that when interchanging the $\mathtt{sup}$ and $\mathtt{inf}$ we create a copy of the y variable for every $\beta \in S$. ☐

4.1 Optimality Certificates for Packing Problems

?THM? **??** and pointedness of the cone make this set always feasible.

Lemma 4.2. *If $\mathcal{C}_2$ is pointed, then $Y_C \neq \emptyset$ for all c.*

Proof. [Aja+21, Lemma 6.8] tells us that for a continuous conic program whose primal has a trivial recession cone (equal to $\{\mathbf{0}\}$), the recession cone of the dual is strictly feasible. The dual analogue of [Aja+21, Proposition 4.6] gives us that this strict feasibility implies strict feasibility of the parametric dual feasible set for every right hand side in the affine hull of the dual cone (${\mathcal{C}_2}^\star$ for us). The dual cone of a pointed cone is full-dimensional, which means that its affine hull is the entire ambient space. Applying these arguments to the truncated problem $z_C(\cdot)$, which has a trivial recession cone because $X_C(\mathbf{0}) \subseteq X(\mathbf{0})$ and $X(\mathbf{0}) = \{\mathbf{0}\}$ by ?THM? **??**, gives us $Y_C \neq \emptyset$ for all c. ☐

Denote $\boldsymbol{y} := \{y_\beta\}_{\beta\in S}$ and define the function

$$\Psi\colon\ (f,\boldsymbol{y}) \in \mathcal{F}(S) \times \bigoplus_{\beta\in S} Y_C \ \mapsto\ \sup_{\beta\in S}\left[f(\beta) + \langle b-\beta,\, y_\beta\rangle\right].$$

Thus, our dual problem is to find the infimum of $\Psi(f,\boldsymbol{y})$ over its domain under the additional restriction that $f(\mathbf{0}) = 0$. An optimality certificate for the dual problem would be a tuple $(f^*,\boldsymbol{y}^*)$ satisfying dual feasibility conditions — $f^* \in \mathcal{F}(S)$ with $f^*(\mathbf{0}) = 0$ and $y^*_\beta \in Y_C$ for all $\beta \in S$, and tightness of the bound $z^* = \Psi(f^*,\boldsymbol{y}^*)$. Since continuous conic problems have only ε-tight certificates for arbitrary small $\varepsilon > 0$ (unlike linear programming), we can only hope to find dual feasible $(f^*,\boldsymbol{y}^*)$ satisfying $\Psi(f^*,\boldsymbol{y}^*) \leqslant z^* + \varepsilon$. We provide a dual feasible function f^* and dual feasible vectors y^ε_β for all $\beta \in S$ and arbitrarily small $\varepsilon > 0$ such that $\Psi(f^*,\boldsymbol{y}^*) \leqslant z^* + \varepsilon$. The construction we describe in this section is for the case of conic packing problems, a structural property which we define next, and we will see in the next section that this assumption is wlog upto performing a certain transformation.

Consider the primal continuous relaxation X written as in (6).

Definition 2. The set X is *packing* if it is full-dimensional and $\mathcal{A}(\mathcal{C}) \subseteq K$, otherwise it is *weakly packing* if $\mathcal{A}'(\mathcal{C}) \subseteq K'$ and matrix B has nonnegative entries, where we write $X = \{x \in X' :\ Bx = d\}$ for a full-dimensional set $X' := \{x \in \mathcal{C} :\ \mathcal{A}'x \preccurlyeq_{K'} b'\}$ and $Bx = d$ representing the affine hull of X.

A conic packing set (either X or X') is nonempty if and only if the right-hand side in the conic inequalities belongs to the associated cone (e.g., $b \in K$ for X) and is also bounded under mild conditions. We cannot expect a low-dimensional X to be packing because if $K = K' \times \{\mathbf{0}\}$ (where the cone $\{\mathbf{0}\}$ is used to model linear equations), the packing property would require the cone $\mathcal{C}$ to be contained in the kernel of B, which is a very special case and uninteresting. Modelling equations with two inequalities also does not help.

A first case of strong duality is for full-dimensional sets that are packing.

Proposition 4.3. *If X is a packing set with $b \in K$, $\nu^*_{\mathcal{S}_b(K)}$ is a strong dual.*

Proof. A feasible packing set X has $\mathcal{A}(\mathcal{C}) \subseteq K$ and $b \in K$. ?THM? **??** gives us $\mathcal{S}_b(K) \neq \emptyset$. Since $K \subseteq \mathcal{R}_I$ and $\mathcal{S}_b(K) = K \cap (b - K) \subseteq K$, we have $\mathcal{S}_b(K) \subseteq \mathcal{R}_I$. Definition 2 imparts a stronger property than Definition 1, as claimed below.

Claim 2. *If X is packing, then both X and X_{int} are $\mathcal{C}$-packing.*

Proof of Claim. We have to argue $\mathtt{sub}\, X \subseteq X$, similar arguments hold for X_{int}. Take any $y \in \mathtt{sub}\, X$. Since $x - y \in \mathcal{C}$ for some $x \in X$, we have $b - \mathcal{A}y = b - \mathcal{A}x + \mathcal{A}(x-y) \in K + K = K$. □

This implies that $\mathtt{sub}\, X_{\text{int}} = X_{\text{int}}$. The packing property of X leads to $\mathcal{A}(X_I) \subseteq \mathcal{S}_b(K)$ because for every $x \in X_I$, $\mathcal{A}x \in \mathcal{A}(\mathcal{C}) \subseteq K$ and $\mathcal{A}x \preccurlyeq_K b$. Hence, we can apply Proposition 4.1 with $S = \mathcal{S}_b(K)$ to get $z^* \leqslant \nu^*_{\mathcal{S}_b(K)}$.

Now take any small $\varepsilon > 0$. We have to construct f and y_β for $\beta \in \mathcal{S}_b(K)$ such that $\Psi(f,\boldsymbol{y}) \leqslant z^* + \varepsilon$. Set $f(\beta) = z_I(\beta)$ for all $\beta \in \mathcal{S}_b(K)$. Since $\mathcal{S}_b(K) \subseteq \mathcal{R}_I$, Lemma 3.2 and Lemma 3.3 yield $f \in \mathcal{F}(\mathcal{S}_b(K))$. We showed in the proof of Lemma 3.4 that $z(\mathbf{0}) = 0$ implies $z_I(\mathbf{0}) = 0$. Hence, this f is a dual feasible function. For the construction of y_β, we only need to consider those β for which $z_C(b-\beta)$ is finite, which is equivalent to $X_C(b-\beta) \neq \emptyset$ due to boundedness of the sets (?THM? **??**). This is because z^* is finite and the proof of Proposition 4.1 tells us that $z^* = \sup_{\beta\in\mathcal{S}_b(K)} z_I(\beta) + z_C(b-\beta)$. For every $\beta \in \mathcal{S}_b(K)$ such that $X_C(b-\beta) \neq \emptyset$, set y_β to be an ε-optimal solution to the dual problem $\inf_{y_\beta\in Y_C} \langle b-\beta,\, y_\beta\rangle$. Continuous conic programs have strong duality when the feasible region is nonempty and bounded inside a pointed cone [cf. Aja+21]. Applying this to $z_C(b-\beta)$ and using pointedness of $\mathcal{C}_2$ implies that $\langle b-\beta,\, y_\beta\rangle \leqslant z_C(b-\beta)+\varepsilon$, and so we have $\Psi(f,\boldsymbol{y}) \leqslant \sup_{\beta\in\mathcal{S}_b(K)} \Big[z_I(\beta) + z_C(b-\beta)\Big] + \varepsilon = z^* + \varepsilon$. □

Now consider X to be low-dimensional written as in (6) with $m \geqslant 1$ linear equations. In this case, not only should we not expect X to be packing as commented earlier, but also the set $\mathcal{S}_b(K' \times \{\mathbf{0}\})$ could be empty if some of the equations have a nonzero right-hand side.

Proposition 4.4. *If X is a low-dimensional set and is weakly packing, $\nu^*_{\mathcal{D}}$ is a strong dual for the set $\mathcal{D} = \mathcal{S}_{b'}(K') \times \mathcal{S}_d(\mathbb{R}^m_+)$.*

Weak duality arguments are similar to those in Proposition 4.3, but in the presence of linear equations, the construction of optimality certificates is not as straightforward. In particular, the value function z_I could equal $-\infty$ at most points in $\mathcal{S}_b(K)$ (infeasibility). To circumvent this infeasibility issue, we consider the value function of the slack problem over X_I:

$$z_\Delta(\beta) \;=\; \sup\left\{\sum_{j\leqslant p} c_j x_j - 2\Delta\sum_{j=1}^{m} s_j \,:\, (x,s)\in X_\Delta(\beta,\eta)\right\},$$

where $X_\Delta(\beta,\eta) := \{(x,s)\in\mathbb{Z}^{p+m}_+ :\ \sum_{j\leqslant p} A_j x_j \preccurlyeq_{K'} \beta,\ \sum_{j\leqslant p} B_j x_j + s = \eta\}$. We argue that this perturbed value function is dual feasible for large Δ. The values for y_β are ε-optima to the truncated continuous problem as in the proof of Proposition 4.3. Our proof shows that for large enough Δ (with a precise lower threshold), setting $f = z_\Delta$ and y_β as described above leads to ε-tight dual certificates.

***Proof of Proposition** 4.4.* The arguments for $z^* \leqslant \nu^*_{\mathcal{D}}$ are similar to those in Proposition 4.3. Now take any small $\varepsilon > 0$. We have to construct f and $y_{\beta,\eta}$ for $(\beta,\eta)\in\mathcal{D}$ such that $\Psi(f,\boldsymbol{y}) \leqslant z^*+\varepsilon$. Set $f(\beta,\eta) = z_\Delta(\beta,\eta)$ for $(\beta,\eta)\in\mathcal{D}$. For $y_{\beta,\eta}$, as in the proof of Proposition 4.3, we only need to consider those (β,η) for which $X_C(b-\beta, d-\eta)\neq\emptyset$ and for these, set $y_{\beta,\eta}$ to be an ε-optimal solution to the dual problem.

For $(\beta,\eta)\in\mathcal{D}$, we have $(\mathbf{0},\eta)\in X_\Delta(\beta,\eta)$ and $X_\Delta(\beta,\eta)$ is a bounded set due to the the packing structure of X, thereby making z_Δ finite-valued over $\mathcal{D}$. By construction, z_Δ is the value function of a conic MIP and since $X_\Delta(\beta,\eta)\neq\emptyset$ for every $(\beta,\eta)\in\mathcal{D}$, Lemma 3.2 and Lemma 3.3 yield $f\in\mathcal{F}(\mathcal{D})$. Consider any $(\beta,\eta)\in\mathcal{D}$ such that $z_I(\beta,\eta)$ is finite. We claim that $z_I(\beta,\eta) = z_\Delta(\beta,\eta)$ for $\Delta \geqslant 2|\tilde z|$ where $\tilde z := \sup\{c^\top x \colon \mathcal{A}x \preccurlyeq_K b, x\in\mathbb{R}^n_+\}$. The packing structure implies that there exists an optimal solution $x^*\in X_I(\beta,\eta)$. Observe that $(x^*,\mathbf{0})\in X_\Delta(\beta,\eta)$ and so $z_\Delta(\beta,\eta)\geqslant z_I(\beta,\eta)$. Suppose there exists $(x',s')\in X_\Delta(\beta,\eta)$ with $\|s'\|_\infty \geqslant 1$. Then, $c_I^\top x' - 2\Delta\sum_j s'_j \leqslant \tilde z - \Delta \leqslant -\Delta/2 \leqslant c_I^\top x^* = z_I(\beta,\eta)$. Thus, we get $z_I(\beta,\eta) = z_\Delta(\beta,\eta)$ for $\Delta\geqslant 2|\tilde z|$.

For all $(\beta,\eta)\in\mathcal{D}$ such that $X_C(b-\beta,d-\eta)\neq\emptyset$, we have $\Psi(f,\boldsymbol{y}) \leqslant z_\Delta(\beta,\eta) + z_C(b-\beta, d-\eta)+\varepsilon$. If $X_I(\beta,\eta)\neq\emptyset$, then using $z_\Delta(\beta,\eta) = z_I(\beta,\eta)$ argued earlier and $z^* = z(b,d)\geqslant z_I(\beta,\eta) + z_C(b-\beta,d-\eta)$ from Lemma 3.7 leads to $\Psi(f,\boldsymbol{y})\leqslant z^*+\varepsilon$. For $X_I(\beta,\eta)=\emptyset$, take $\Delta\geqslant\Delta_1 := \Big|\sup_{(\beta,\eta)\in\mathcal{D}} z(\beta,\eta)\Big| + \Big|\sup_{(\beta,\eta)\in\mathcal{D}} z_C(b-\beta,d-\eta)\Big|$. The first term is lower bounded by $\big|z(b,d)\big|$ due to $(b,d)\in\mathcal{D}$. Then, it is easy to deduce that $\Delta_1 \geqslant -z(b,d) + z_C(b-\beta,d-\eta)$

using the fact that $|\tau| = \max\{\tau, -\tau\}$ for any $\tau \in \mathbb{R}$. Now we have

$$\begin{aligned}\Psi(f, \boldsymbol{y}) &\leqslant -\Delta + z_C(b-\beta, d-\eta) + \varepsilon \\ &\leqslant -\Delta_1 + z_C(b-\beta, d-\eta) + \varepsilon \\ &\leqslant z(b,d) - z_C(b-\beta, d-\eta) + z_C(b-\beta, d-\eta) + \varepsilon \\ &= z(b,d) + \varepsilon \\ &= z^* + \varepsilon,\end{aligned}$$

and the proof is complete. □

4.2 Transforming to a Packing Problem

The assumption of weakly packing property from Definition 2 is wlog since every bounded set can be transformed to a weakly packing set that has one additional variable and one additional linear equation, such that the mixed-integer points in this set project onto the mixed-integer points in the original set. For a vector v, let $v^- = (\min\{0, v_j\})_j$. For $v \in \mathbf{E}$ and $\xi \in \operatorname{ri} K'$ (relative interior of K'), let $\delta_\xi(v)$ be the smallest stepsize δ such that $v + \delta\xi \in K'$ (such a value always exists for convex cones and since K' is full-dimensional), and define $\alpha_j = \max\{\delta_\xi(\mathcal{A}'_j), \|B_j\|_\infty\}$ for $j = 1, \dots, n$.

Here we need to assume boundedness of X_{int} so that α achieves a finite optimum over this set.

Proposition 4.5. *Assume X_{int} is bounded. For any $\xi \in \operatorname{ri} K'$ and $\rho \geqslant \max\{\delta_\xi(b'), \|d^-\|_\infty, \sup_{x \in X_{\text{int}}} \alpha^\top x\}$,*

$$\begin{aligned}\mathcal{X}(\xi, \rho) \coloneqq \Big\{(x, s) \in \mathcal{C} \times \mathbb{R}_+ :\ &\sum_{j=1}^n (\mathcal{A}'_j + \alpha_j \xi) x_j + s\xi \preccurlyeq_{K'} b' + \rho\xi, \\ &\sum_{j=1}^n (B_j + \alpha_j \mathbf{1}) x_j + s\mathbf{1} = d + \rho\mathbf{1}, \\ &\alpha^\top x + s = \rho\Big\}\end{aligned}$$

is weakly packing and the mixed-integer points in it project onto X_{int}.

Furthermore if K has a polynomial-time separation oracle, then $\mathcal{X}(\xi, \rho)$ can be constructed in polynomial-time.

The idea behind this transformation is that the generators $\mathcal{A}'_j$ are projected onto the cone K' via the operation $\mathcal{A}'_j + \alpha_j \xi$ which can be shown to belong to K' for the given choices of α and ξ. Similarly, the generators B_j are projected onto the nonnegative orthant. The right-hand sides are also projected and adjusted accordingly to preserve feasibility. The new constraint $\alpha^\top x + s = \rho$ in some sense encodes the optimal value in the direction of α.

Proof of Proposition 4.5. Since $\alpha_j \geqslant \|B_j\|_\infty \geqslant 0$, we have that $B_j + \alpha_j \mathbf{1} \geqslant \mathbf{0}$. Since $\alpha_j \geqslant \delta_\xi(\mathcal{A}'_j) \geqslant 0$ and $\xi \in \operatorname{ri} K'$, we have that $\mathcal{A}'_j + \alpha_j \xi \in K'$. Using Assumption 1 and assuming wlog that $\mathcal{C}_2 \subseteq \mathbb{R}^{n-p}_+$ (otherwise $\mathcal{C}_2$ can be tilted to be in the nonnegative orthant, or continuous variables can be written as difference of two nonnegative variables), we get that the set $\mathcal{X}(\xi, \rho)$ satisfies Definition 2 for a weakly packing set.

Now we need ρ to be large enough, as prescribed, to obtain that the projection of $\mathcal{X}(\xi, \rho) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p+1})$ is equal to X_{int}. This projection claim can be argued as follows. Additive invariance of $\preccurlyeq_K$ means that $\mathcal{A}'x \preccurlyeq_K b' \iff \mathcal{A}'x + \rho\xi \preccurlyeq_K b' + \rho\xi \iff \sum_{j=1}^n (\mathcal{A}'_j + \alpha_j\xi)x_j + (\rho - \alpha^\top x)\xi \preccurlyeq_K b' + \rho\xi$. The condition $\rho \geqslant \delta_\xi(b')$ is necessary for feasibility, since it is easy to see that for a packing set to be feasible we need the right-hand side to be in the cone associated with the conic partial order. Similar arguments hold for the linear equations with ξ replaced by the vector of ones $\mathbf{1}$. Taking $s = \rho - \alpha^\top x$ gives us that $\mathcal{X}(\xi, \rho)$ projects onto X. Hence, the projection of $\mathcal{X}(\xi,\rho) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p+1})$ is contained in X_{int}. To argue the reverse inclusion also holds, we need boundedness of X_{int} which implies that $\sup_{x \in X_{\text{int}}} \alpha^\top x < +\infty$. Then, a large enough ρ satisfies $\rho \geqslant \alpha^\top x$ for every $x \in X_{\text{int}}$. This leads to $s \geqslant 0$ when $s = \rho - \alpha^\top x$ and $x \in X_{\text{int}}$, thereby yielding the desired reverse inclusion.

The complexity statement is because a large enough ρ can also be computed in polynomial time with a separation/optimization oracle for K' since $\sup\{\alpha^\top x \colon x \in X\}$ is a convex problem, and the stepsize $\delta_\xi(\mathcal{A}'_j)$ for the projection can be computed with a line search. □

Remark 1. For the pure integer case ($p = n$), Proposition 4.5 would give us an extended formulation with n integer variables and 1 continuous variable, but we can modify this to get a pure integer extension. Taking $\alpha \geqslant \alpha_\xi$ with $\alpha \in \mathbb{Z}^n$ and ρ be a large enough integer, we would have $\rho - \alpha^\top x \in \mathbb{Z}$ for every $x \in \mathbb{Z}^n$, and hence the projection of $\mathcal{X}(\xi,\rho) \cap \mathbb{Z}^{n+1}$ would equal X_{int}.

4.2.1 Illustration of packing transformation for standard cones

Let us explicitly note the packing transformation for standard cases of K. In $\mathbb{R}^m$, we have two commonly used self-dual proper cones — the nonnegative orthant $\mathbb{R}^m_+$ and the Lorentz cone $\mathcal{SOCP}_m := \{(x, x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \|x\|_2 \leqslant x_m\}$, which is a special case (with $p = 2$) of the p-cone that considers the $\|\cdot\|_p$ norm. In the vector space of m-by-m matrices, the linear subspace of symmetric matrices $\mathcal{S}_m$ contains the cone of positive semidefinite matrices, denoted by $\mathcal{PSD}_m := \left\{ A \in \mathcal{S}_m : \left\langle A,\, xx^\top \right\rangle \geqslant 0,\ x \in \mathbb{R}^m \right\}$, and also the cones of doubly nonnegative matrices $\mathcal{DNN}_m$, copositive matrices $\mathcal{COP}_m$, and completely positive matrices $\mathcal{CP}_m$. The psd cone is self-dual and regular, whereas the other three cones are regular but not self-dual.

Orthant For $K = \mathbb{R}^m_+$, the vector of all ones is in the relative interior and so we can take $\xi = \mathbf{1}$. This implies that for $y \in \mathbb{R}^m$ we have $\delta_{\mathbf{1}}(y) = -\min\{0, y_1, \ldots, y_m\}$ and the projection onto $\mathbb{R}^m_+$ is $\phi_{\mathbf{1}}(y) = \big(y_j + \delta_{\mathbf{1}}(y)\big)_{j=1}^m$. Hence, $\alpha_j = -\min\{0, A_{1j}, \ldots, A_{mj}\}$ and we obtain the following extension,

$$\mathcal{X}(\alpha_{\mathbf{1}}) = \left\{ (x,s) \in \mathbb{R}^{n+1}_+ : \sum_{j=1}^n \big(A_j + \alpha_j \mathbf{1}\big)\, x_j + s\mathbf{1} \leqslant b + \rho\mathbf{1},\ \sum_{j=1}^n \alpha_j x_j + s = \rho \right\}.$$

A similar transformation for pure integer linear programs with feasible set defined by equality constraints was proposed in Lasserre [Las05, Appendix].

Lorentz cone For $K = \mathcal{SOCP}_m$, we can take $\xi = \mathbf{e}_m = (0, \ldots, 0, 1) \in \operatorname{ri} \mathcal{SOCP}_m$, giving us $\delta_{\mathbf{e}_m}(y) = -\min\{0, y_m - \|(y_1, \ldots, y_{m-1})\|\}$, which means that for $y \notin \mathcal{SOCP}_m$, the projection onto $\mathcal{SOCP}_m$ is $\phi_{\mathbf{e}_m}(y) = (y_1, \ldots, y_{m-1}, \|(y_1, \ldots, y_{m-1})\|)$. The extension $\mathcal{X}(\alpha_{\mathbf{e}_m})$ is then easy to construct. Similar arguments extend to K being a Cartesian product of Lorentz cones (or any p-norm cones).

Semidefinite and Copositive cones For $K = \mathcal{PSD}_m$, we have $\xi = \mathbf{I}_m$ (m-by-m identity matrix) in the relative interior of $\mathcal{PSD}_m$. Then for any $M \in \mathcal{S}_m$, we have $\delta_{\mathbf{I}_m}(M) = -\min\{0, \lambda_{\min}(M)\}$ ($\lambda_{\min}$ is the smallest eigenvalue). The extension $\mathcal{X}(\alpha_{\mathbf{I}_m})$ follows easily. The same arguments hold for mixed-integer copositive problems since $\mathbf{I}_m \in \operatorname{ri} \mathcal{COP}_m$.

Completely positive and DNN cones For any $M \in \mathcal{S}_m$, let $\ell = \min\{0, \lambda_{\min}(M)\}$, $r = -\min\{\min_{i,j}(M - \ell\mathbf{I}_m)_{ij}, 0\}$, and $R = [r\mathbf{1}, \ldots, r\mathbf{1}]$ which is a m-by-m matrix. Then $M - \ell\mathbf{I}_m \in \mathcal{PSD}_m$, and $(M - \ell\mathbf{I}_m)^{1/2} + R \geqslant \mathbf{0}$, which implies $M + (RR + 2(M - \ell\mathbf{I}_m)^{1/2}R - \ell\mathbf{I}_m) \in \mathcal{CP}_m$. The same transformation holds for $\mathcal{DNN}_m$ because every completely positive matrix is doubly nonnegative.

4.3 Proof of Theorem 2.2

Proposition 4.3 addresses the full-dimensional packing case. Proposition 4.4 addresses the low-dimensional packing case. If X is not weakly packing, we can employ Proposition 4.5 to obtain the weakly packing set $\mathcal{X}(\xi, \rho)$ and apply Proposition 4.4 to it, which gives us the set $\mathcal{D}_{\xi,\rho}$ as claimed. When the data is integral, we can restrict attention to the integer elements in $\mathcal{S}_b(K)$ and $\mathcal{D}_{\xi,\rho}$ because these sets are essentially trying to capture the set of all feasible right-hand sides for the truncated integer problem and for integral data, the left-hand side values will be integer vectors for the truncated problem, assuming that K has a rational linear subspace so that there exists an integral vector $\xi \in \operatorname{ri} K'$.

Remark 2. For the case of integral data, it is possible to derive upper bounds on the number of variables and constraints in the dual for special cones such as Lorentz cone, and also for semidefinite cone using Weyl's monotonicity theorem about semidefinite matrices.

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