Production Theory for Constrained Linear Activity Models

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1. The purpose of this paper is to generalize the framework of activity analysis discussed in Villar (2003) and obtain similar results concerning solvability. We generalize the model due to Villar (2003), without requiring any dimensional requirements on the activity matrices and by introducing a model of activity analysis in which each activity may (or may not) have a capacity constraint i.e. a maximum level at which the activity can operate. This may be one way to accommodate meaningful non-linearities similar to that considered for input-output analysis in Sandberg (1973).

In Sandberg (1973), the input-output coefficients were assumed to be differentiable, which is very likely an approximation of the more realistic and practically applicable representation in which the input-output coefficients are piecewise constant. Minor, technical increments on the solvability result in Sandberg (1973) can be found in Chander (1983) and some references therein. Undoubtedly, piecewise constant input-output constants are more difficult to deal with mathematically than differentiable ones. Non-linearities are much easier to represent in the framework of linear activity analysis by introducing upper bounds-whenever there is a capacity constraint-for the levels at which each such activity may operate. We do this by introducing a model of activity analysis in which each activity may (or may not) have a capacity constraint. In this paper we follow the usual nomenclature of input-output analysis for "the quantity of a good supplied to the consumers outside the production (or manufacturing) sector" and refer it as "final demand".

It seems that in this significantly more general framework we are able to obtain the desired results concerning solvability and existence of an equilibrium price vector under weaker assumptions than the corresponding requirements in Villar (2003). The property that guarantees solvability has the following (interesting?) economic interpretation: Given two price vectors p ,q a final demand vector and unit prices of operating capacity constrained activities, if the revenue from operating every unconstrained activity at unit level at price-vector p is no less than doing the same at price-vector q and if the revenue from operating a constrained activity at unit level at price-vector p is no less than doing the same at price-vector q plus the unit price of operating the activity then the value of the final demand at price vector p is no less than the value of the final demand at price vector q plus the total cost of the capacities. In the context of the constraint linear activity analysis model, we call this property "weakly proper". We also prove a version of the Non-Substitution Theorem that establishes the existence of "efficiency price-vectors" as a joint product. However, our Non-Substitutions Theorem- in spite of the generality of our model as compared to the one due to Villar (2003)- requires that if there are capacity constraints, then there is a minimal subset of the set of capacity constrained activities that are always used up to full capacity, for the production of all producible final demand vectors. Further, these capacity constrained activities are the only ones whose capacities are binding for some producible final demand vector. Hence there is a clear dependence of the equilibrium price vector on the final demand vector (i.e. the vector of quantities supplied for non-manufacturing prices), unlike the conclusion of Sraffian economics (see the sixth paragraph at: https://www.rethinkeconomics.org/journal/sraffian-economics/), and we are able to show this without using production functions. Our framework is a generalization of the one used in Sraffian economics.

In what follows we will be making extensive use of mathematical results in Topics 2 and 3- and therefore by implication results in Topic 1- of Lahiri (2022). Sometimes, when there is no scope for confusion, given two vector/points x and y in a real Euclidean space of the same dimension we use $x \geq y$ to denote that every co-ordinate of x is greater than or equal to the corresponding coordinate of y, $x > y$ to denote $x \geq y$ but $x \neq y$ (x is not equal to y), and $x >> y$ to denote every coordinate of x is strictly greater than the corresponding coordinate of y.

2. Consider a very simple production process which produces a single output ("corn") from a single input (once again "corn"). In the classical or Leontief Input-Output (IO) Model one assumes that there exists a fixed positive constant 'a' such that 'ax' units of corn are required to produce 'x' units of corn. The production process productive if and only if $0 < a < 1$. However, in reality the assumption of a fixed input-out coefficient 'a' is unrealistic. Fertility of the soil is not uniform. Hence, it is unreasonable to assume input-output co-efficient remains constant for all levels of output.

Sandberg (1973) suggested that the input-output coefficient(s) is a (are) differentiable function(s) of the gross output. Once again, differentiability of input-output coefficient appears to be an "unrealistic" assumption for actual production processes. Even if such an assumption is theoretically true, it is extremely difficult to invoke it for practical use in "production planning". It is perhaps more realistic to assume that input-output coefficients are piece-wise constant. In what follows we generalize such an assumption and develop the theory that follows from it to the case of production with possibly more than one good.

The Sraffian conclusion of market prices being independent of the quantity "supplied" (which is referred to in the literature on input-output analysis as "final demand") with fixed input-output coefficients and hence fixed unit costs of production with competitive factor markets, is not unreasonable at all in the "one good" setting, since that is precisely what is indicated by the intersection of any demand curve with a horizontal marginal cost (or perfectly competitive supply curve). However, in the more general setting with piece-wise constant input-out coefficients or piece-wise constant marginal cost functions, neither would the associated extension of the Sraffian linear model indicate such independence between market price and quantity supplied and nor would it be implied by the intersection of the market demand and marginal cost curves.

3. Consider an economy with 'm' produced goods indexed by $i = 1,\ldots,m$ and 'n' activities indexed by $j = 1,\ldots,n$.

An $m\times n$ matrix of real numbers is said to be non-zero, if it has "at least one non-zero entry".

Note: The rank of a non-zero matrix is a positive integer.

Let M be a non-zero $m\times n$ matrix of real numbers called an activity matrix whose j^{th} column denoted M^j for $j\in\{1,\ldots,n\}$ denotes the amount of net-output each good if the j^{th} activity is operated at unit level. Thus for $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, the i^{th} entry of M^j denoted m_{ij} denotes the "net-output" of the i^{th} produced good if the j^{th} activity (or activity j) is operated at unit level. If m_{ij} is negative, then $-m_{ij}$ is the amount of the i^{th} produced good used as "net-input", if the j^{th} activity is operated at unit level.

In what follows, unless otherwise stated, we shall use "net-output" and "output" interchangeably. The same applies for "net-input" and "input".

An activity-vector is a column vector $x \in \mathbb{R}^n_+$ such that for all $j \in \{1,\ldots,n\}$, the j^{th} row (or coordinate) of x, denoted x_j, is the level at which activity 'j' is operated.

A **constrained linear activity analysis model (CLAAM)** is a pair $(M, <\bar{x}_j | j \in W>)$ where M is an m×n activity matrix and if $W \neq \phi$ then $<\bar{x}_j | j \in W>$ is an array with $\bar{x}_j \in \mathbb{R}_{++}$ being the maximum level at which activity $j \in W \subset \{1,..,n\}$ can operate, the possible levels of operation for activities in $(1,\ldots,n\}\backslash W$ being unbonded above. If $W = \phi$, then a constrained linear activity analysis model with activity matrix M is (M,ϕ).

In the case of (M,ϕ) no activity has a capacity constraint.

Note: This is a very general formulation. In particular there could be non-negative m×n matrices B, A such that M = B - A.

A column vector $d \in \mathbb{R}^m_+ \backslash \{0\}$ is said to a final demand vector if for all $i \in \{1,\ldots,m\}$, the i^{th} row of d, i.e. d_i represents the quantity of i^{th} good supplied for non-manufacturing/non-production purposes, i.e. quantity of the i^{th} good supplied to the consumers outside the production process.

The following is an example of an activity matrix, different from any in Villar (2003) and any known to us otherwise (see chapters 6 and 7 of Lancaster (1968)).

Example 1: Let C be a n×n matrix of non-negative real numbers such that for all $i,j \in \{1,\ldots,n\}$, c_{ij} which is the $(i,j)^{th}$ element of C, is a non-negative real number that denotes the level at which activity i is required to be operated-to produce enough of the 'm' produced goods- so as to be able to operate activity j at unit level. Let B be a m×n matrix of non-negative real numbers such that for all $h \in \{1,\ldots,m\}$ and $i \in \{1,\ldots,n\}$, b_{hi} is the $(h,i)^{th}$ element of B, denoting the (net) amount of good 'h' that is produced with the purpose of satisfying final demand, if activity 'i' operates at unit level.

Let M = B(I-C).

If $x \in \mathbb{R}^n_+$ satisfying $x_j \leq \bar{x}_j$ for all $j \in W$, if $W \neq \phi$ and unconstrained otherwise, is an activity-vector then Cx denotes the level at which the activities are required to operate in order to "operationalize" the activity vector x. The remaining levels of activity vector (I-C)x are used to produce the "produced goods" in the amount B(I-C)x, "provided" $(I-C)x \in \mathbb{R}^n_+$.

In what follows, unless otherwise stated or required, we will write $(M, <\bar{x}_j | j \in W>)$ to represent a CLAAM, with the implicit understanding that if $W = \phi$, then the CLAAM reduces to or represents (M,ϕ).

The following definition will prove to be important in the analysis that follows.

A **price-vector** is a vector $p \in \mathbb{R}^m_+ \setminus \{0\}$ where the i^{th} coordinate of p, denoted p_i is the unit price at which the i^{th} produced good is sold in the market. Let $w > 0$ denote wage rate of labour.

4. Given a CLAAM (M, $<\bar{x}_j | j \in W>$) and a non-empty subset of activities J, we define the set **Constrained Span** (M, $<\bar{x}_j | j \in W>$,J), denoted CS(M, $<\bar{x}_j | j \in W>$,J) = $\{Mx \in \mathbb{R}^m | x \in \mathbb{R}^n$ with $x_j \leq \bar{x}_j$ for $j \in W \cap J$ and $x_j = 0$ for all $j \notin J\}$.

Thus, CS(M, $<\bar{x}_j | j \in W>$,J) $\subset$ Span (M,J).

If J = $\{1,\ldots,n\}$, then CS(M, $<\bar{x}_j | j \in W>$,J) is written simply as CS(M, $<\bar{x}_j | j \in W>$).

Let J be a non-empty subset of $\{1,..,n\}$.

The content of the following property is based on one due to Villar (2003).

A CLAAM (M, $<\bar{x}_j | j \in W>$) is said to be **weakly proper for** (a non-empty subset of activities) J, if for all $p,q \in \mathbb{R}^m_+$, any array of non-negative real numbers $<\alpha_j | j \in W \cap J>$ and $d \in$ CS(M, $<\bar{x}_j | j \in W>$, J)$\cap(\mathbb{R}^m_+ \setminus \{0\})$:[$p^T M^j - \alpha_j \geq q^T M^j$ for all $j \in W \cap J$ and $p^T M^j \geq q^T M^j$ for all $j \in J \setminus W$ implies $p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j \geq q^T d$].

We use "CLAAM (M, $<\bar{x}_j | j \in W>$) is weakly proper for J" and "(M, $<\bar{x}_j | j \in W>$) is a CLAAM weakly proper for J" or just "(M, $<\bar{x}_j | j \in W>$) is weakly proper for J" interchangeably.

"Weakly proper" in the context of CLAAM has the following (interesting?) economic interpretation if we interpret α_j as the unit cost of operating a capacity constrained activity $j \in W \cap J$: If the revenue from operating every unconstrained activity in J at unit level at price-vector p is no less than doing the same at price-vector q and if the revenue from operating a capacity constrained activity in J at unit level at price-vector p is no less than doing the same at price-vector q plus the unit price of the capacity then the value of the final demand d at price vector p is no less than the value of the final demand d at price vector q plus the total cost of the capacities.

In the interpretation provided above we are assuming that capacities have an imputed price/shadow price given by the alphas up to the maximum that is possible.

A CLAAM (M, $<\bar{x}_j | j \in W>$) is said to be **weakly proper** if it is weakly proper for $\{1,\ldots,n\}$.

In particular, (by setting $\alpha_j = 0$ for all $j \in W$) for all $p,q \in \mathbb{R}^m_+$ and $d \in$ CS(M, $<\bar{x}_j | j \in W>$)$\cap(\mathbb{R}^m_+ \setminus \{0\})$:[$p^T M \geq q^T M$ implies $p^T d \geq q^T d$].

Given a CLAAM (M, $<\bar{x}_j | j \in W>$) the activity matrix M is said to be **weakly proper** if for all $p,q \in \mathbb{R}^m_+$ and $d \in$ CS(M, $<\bar{x}_j | j \in W>$)$\cap(\mathbb{R}^m_+ \setminus \{0\})$:[$p^T M \geq q^T M$ implies $p^T d \geq q^T d$].

Clearly the activity matrix M is weakly proper if the CLAAM (M, $<\bar{x}_j | j \in W>$) is weakly proper.

Since any point in $\mathbb{R}^m$ can always be expressed as the difference between two points in $\mathbb{R}^m_+$ the following is an immediate consequence of the definition of a weakly proper activity matrix.

CLAAM (M, $<\bar{x}_j|j\in W>$) is weakly proper for J if and only if for all $p\in\mathbb{R}^m$, any array of non-negative real numbers $<\alpha_j|j\in W\cap J>$ and $d\in(CS(M, <\bar{x}_j|j\in W>,J))\cap(\mathbb{R}^m_+\backslash\{0\})$:[$p^TM^j - \alpha_j \geq 0$ for all $j\in W\cap J$, $p^TM^j \geq 0$ for all $j\in J\backslash W$, implies $p^Td - \sum_{j\in W\cap J}\alpha_j\bar{x}_j \geq 0$].

Note: The definition corresponding to weakly proper activity matrices in Villar (2003) is equivalent to our definition of weakly proper activity matrices of CLAAM's because Villar (2003) requires Span(M)$\cap\ \mathbb{R}^m_{++}\neq\phi$.

Lemma 1: Suppose CS(M, $<\bar{x}_j|j\in W>$,J)$\cap\ \mathbb{R}^m_{++}\neq\phi$. Then (M, $<\bar{x}_j|j\in W>$) is a weakly proper CLAAM for J <u>if and only if</u> for all $p,q\in\mathbb{R}^m_{++}$, any array of non-negative real numbers $<\alpha_j|j\in W\cap J>$ and $d\in CS(M)\cap\ \mathbb{R}^m_{++}$: [$p^TM^j - \alpha_j \geq q^TM^j$ for all $j\in W$ and $p^TM^j \geq q^TM^j$ for all $j\in J\backslash W$ implies $p^Td - \sum_{j\in W\cap J}\alpha_j\bar{x}_j \geq q^Td$].

Proof: If (M, $<\bar{x}_j|j\in W>$) is a weakly proper CLAAM for J, then it is easy to see that for all $p,q\in\mathbb{R}^m_{++}$, any array of non-negative real numbers $<\alpha_j|j\in W\cap J>$ and $d\in CS(M, <\bar{x}_j|j\in W>, J)\cap\ \mathbb{R}^m_{++}$: [$p^TM^j - \alpha_j \geq q^TM^j$ for all $j\in W\cap J$ and $p^TM^j \geq q^TM^j$ for all $j\in J\backslash W$ implies $p^Td - \sum_{j\in W\cap J}\alpha_j\bar{x}_j \geq q^Td$].

Hence suppose that for all $p,q\in\mathbb{R}^m_{++}$, any array of non-negative real numbers $<\alpha_j|j\in W\cap J>$ and $d\in CS(M, <\bar{x}_j|j\in W>,J)\cap\ \mathbb{R}^m_{++}$: [$p^TM^j - \alpha_j \geq q^TM^j$ for all $j\in W\cap J$ and $p^TM^j \geq q^TM^j$ for all $j\in J\backslash W$ implies $p^Td - \sum_{j\in W\cap J}\alpha_j\bar{x}_j \geq q^Td$].

Then as in the case of weakly proper CLAAM's for J, we get in this case that:

For all $p\in\mathbb{R}^m$, an array of non-negative real numbers $<\alpha_j|j\in W\cap J>$ and $d\in CS(M, <\bar{x}_j|j\in W>,J)\cap\ \mathbb{R}^m_{++}$:[$p^TM^j - \alpha_j \geq 0$ for all $j\in W\cap J$, $p^TM^j \geq 0$ for all $j\in J\backslash W$, implies $p^Td - \sum_{j\in W\cap J}\alpha_j\bar{x}_j \geq 0$].

Suppose $d\in CS(M, <\bar{x}_j|j\in W>,J)\cap(\mathbb{R}^m_+\backslash\{0\})$.

By hypothesis CS(M, $<\bar{x}_j|j\in W>$,J)$\cap\ \mathbb{R}^m_{++}\neq\phi$. Let $d^*\in CS(M, <\bar{x}_j|j\in W>,J)\cap\ \mathbb{R}^m_{++}$.

Now $d^*\in$ CS(M, $<\bar{x}_j|j\in W>$,J)$\cap\ \mathbb{R}^m_{++}$implies $td^*\in CS(M, <\bar{x}_j|j\in W>,J)\cap\ \mathbb{R}^m_{++}$for all $1 \geq t > 0$.

Similarly, $d\in CS(M, <\bar{x}_j|j\in W>,J)\cap\mathbb{R}^m_+$ implies $(1-t)d + td^*\in CS(M, <\bar{x}_j|j\in W>,J)\cap\ \mathbb{R}^m_{++}$ for all $1 \geq t > 0$.

Thus, $d\in CS(M, <\bar{x}_j|j\in W>,J)\cap\mathbb{R}^m_+$ implies $(1-t)d + td^*\in CS(M)\cap\ \mathbb{R}^m_{++}$ for all $1 \geq t > 0$.

Let $<t^{(h)}|h\in\mathbb{N}>$ be a sequence of positive real numbers less than or equal to 1, converging to 0.

Clearly, the sequence $<(1-t^{(h)})d + t^{(h)}d^*|h\in\mathbb{N}>$ converges to d and for all $h\in\mathbb{N}$, $p^T((1-t^{(h)})d + t^{(h)}d^*) \geq 0$.

Thus $p^Td \geq 0$.

Thus, (M, $<\bar{x}_j|j\in W>$) is a weakly proper CLAAM for J. Q.E.D.

Proposition 1: (M, $<\bar{x}_j|j\in W>$) is a weakly proper CLAAM for J if and only if for all final demand vectors $d\in CS(M, <\bar{x}_j|j\in W>, J)$ there exists $x\in\mathbb{R}^n_+$ satisfying $Mx = d$, $x_j \le \bar{x}_j$ for $j\in W\cap J$ and $x_j = 0$ for all $j\notin J$.

Proof: Let d be a final demand vector in $CS(M, <\bar{x}_j|j\in W>, J)$. For d = 0, clearly Mx = d, where x = 0 and further $x_j \le \bar{x}_j$ for $j\in W$. Hence, we may suppose that $d\in CS(M, <\bar{x}_j|j\in W>, J)\cap\mathbb{R}^m_+\backslash\{0\}$.

(M, $<\bar{x}_j|j\in W>$) is weakly proper for J if and only if there does not exist $p\in\mathbb{R}^m$, any array of non-negative real numbers $<\alpha_j|j\in W\cap J>$ and $d\in CS(M, <\bar{x}_j|j\in W>, J)\cap\mathbb{R}^m_{++}$ satisfying $p^TM^j - \alpha_j \ge 0$ for all $j\in W\cap J$, $p^TM^j \ge 0$ for all $j\in J\backslash W$ and $p^Td - \sum_{j\in W}\alpha_j\bar{x}_j < 0$]

Hence by Farka's lemma, (M, $<\bar{x}_j|j\in W>$) is weakly proper for J if and only if $Mx = d$, $x\in\mathbb{R}^n_+$, $x_j \le \bar{x}_j$ for $j\in W\cap J$ and $x_j = 0$ for all $j\notin J$ has a solution. Q.E.D.

Note: Nowhere have we invoked any restriction on the size of the activity matrix or its rank, except that the rank of the activity matrix is positive. That leaves out the uninteresting case of M = 0. Thus, our framework is considerably more general than that of Villar (2003).

An immediate consequence of the proposition above is that the requirement of $n \le m$ in Villar (2003) can be dispensed with not only for solvability problem in activity analysis, but also for the non-substitution theorem (theorem 5) in the same paper.

5. We will now present a similar generalization as above for the existence of an equilibrium price-vector for a CLAAM (M, $<\bar{x}_j|j\in W>$).

Let A_{m+1} be a row vector in $\mathbb{R}^n$ with all co-ordinates strictly positive, where the entry in the j^{th} column denoted $a_{m+1,j} > 0$, is the amount of the only non-produced good called "labour" that is used as input if activity j is operated at unit level. Let $\bar{L} > 0$ be the total initial amount of labour in the economy.

Recall that a price-vector is a vector $p\in\mathbb{R}^m_+\backslash\{0\}$. Let $w > 0$ denote wage rate of labour.

At price-vector p and wage rate w the <u>profit-vector</u> at the pair (p,w) denoted $\pi(p,w) = p^TM - wA_{m+1}$.

A row vector $v\in\mathbb{R}^n_+$ is said to be <u>profitable</u> at wage rate $w > 0$, if there exists $q\in\mathbb{R}^m$ such that $q^TM = wA_{m+1} + v$.

Note: The vector q in the definition of profitable vectors need not be non-negative.

A price-vector p is said to be an <u>equilibrium price-vector</u> at the wage rate $w > 0$ and row-vector $v\in\mathbb{R}^n_+$ if $v = \pi(p,w)$.

Recall that an activity matrix M is said to be weakly proper if for all $p,q\in\mathbb{R}^m_+$ and $d\in$ $d\in CS(M, <\bar{x}_j|j\in W>)\cap(\mathbb{R}^m_+\backslash\{0\})\cap(\mathbb{R}^m_+\backslash\{0\})$:[$p^TM \ge q^TM$ implies $p^Td \ge q^Td$].

Proposition 2: Given a CLAAM (M, $<\bar{x}_j|j\in W>$), suppose M is a weakly proper activity matrix and v is a profitable row vector at wage rate $w > 0$. Then there exists an equilibrium price-vector p at the wage rate w and row-vector v.

Proof: Since $w > 0$, $A_{m+1} >> 0$ and $v \geq 0$, we get $wA_{m+1} + v >> 0$.

Since $v \in \mathbb{R}^n_+$ if there exists $p \in \mathbb{R}^m_+$ such that $p^T M = wA_{m+1} + v$, then it must be the case that $p \in \mathbb{R}^m_+ \backslash \{0\}$.

Since v is profitable at wage rate $w > 0$, there exists $q \in \mathbb{R}^m$ such that $q^T M = wA_{m+1} + v$.

Towards a contradiction suppose that there does not exist $p \in \mathbb{R}^m_+$ such that $p^T M = wA_{m+1} + v$. By Farkas' lemma, there exists $x \in \mathbb{R}^m$ such that $Mx \in \mathbb{R}^m_+$ and $(wA_{m+1} + v)x < 0$.

Since M is weakly proper, $Mx \in \mathbb{R}^m_+$ implies that there exists $y \in \mathbb{R}^n_+$ with $y_j \leq \bar{x}_j$ for all $j \in W$ such that $My = Mx \geq 0$.

Now $q^T M = wA_{m+1} + v >> 0$ and $y \geq 0$ implies $q^T My = (wA_{m+1} + v)y \geq 0$.

Thus $My = Mx$ implies $q^T Mx = q^T My = (wA_{m+1} + v)y \geq 0$.

On the other hand $q^T M = wA_{m+1} + v$ implies $q^T Mx = (wA_{m+1} + v)x < 0$, contradicting $q^T Mx \geq 0$, that we obtained above.

Thus there exists $p \in \mathbb{R}^m_+$ such that $p^T M = wA_{m+1} + v$ and as we observed earlier this $p \in \mathbb{R}^m_+ \backslash \{0\}$. Q.E.D.

6. Recall that a final demand vector is a column vector $d \in \mathbb{R}^m_+ \backslash \{0\}$.

Let J be a non-empty subset of $\{1,..,n\}$.

Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$, a final demand vector d is said to be **producible by** (activities in) J if there exists $x \in \mathbb{R}^n_+$ satisfying $Mx = d$, $A_{m+1}x \leq \bar{L}$, $x_j \leq \bar{x}_j$ for all $j \in W$, and $x_j = 0$ for all $j \notin J$. Clearly any such x must belong to $\mathbb{R}^n_+ \backslash \{0\}$.

It follows from proposition 1, that if $(M, \langle \bar{x}_j | j \in W \rangle)$ is weakly proper for J, then any final demand vector $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J)$ is producible by J, provided the requirement of labour to produce it does not exceed $\bar{L}$.

If $J = \{1,\ldots,n\}$, then a final demand vector producible by J is said to be **producible**.

Hence the set of all final demand vectors producible by J is $\{Mx \in \mathbb{R}^m_+ \backslash \{0\} | x \in \mathbb{R}^n_+, x_j \leq \bar{x}_j$ for all $j \in W$, $A_{m+1}x \leq \bar{L}$, and $x_j = 0$ for all $j \notin J\}$.

Given a price-vector p, a wage rate $w > 0$, a "producible final demand vector" d, and $x \in \mathbb{R}^n_+$ satisfying $Mx = d$, the aggregate profit of the production sector is $p^T d - wA_{m+1}x$. If the production sector intended to maximize profit, then it would be required to solve the following profit maximization problem: Find x to solve

Maximize $p^T d - wA_{m+1}x$

s.t. $Mx = d$,

$A_{m+1}x \leq \bar{L}$,

$x_j \leq \bar{x}_j$ for all $j \in W$,

$x \geq 0$.

However, given the price-vector p and w, the above for a producible final demand vector is equivalent to solving the following linear programming problem denoted LP-d:

Minimize $wA_{m+1}x$

s.t. $Mx = d$,

$-A_{m+1}x \geq -\bar{L}$,

$-x_j \geq -\bar{x}_j$ for all $j \in W$,

$x \geq 0$.

If y solves LP-d then $y > 0$ since $d > 0$. Thus $A_{m+1}y > 0$.

The question that we are interested in is the following: <u>If</u> for some producible final demand vector, x^* is an optimal solution for the minimization problem, <u>then</u> is it the case that for all final demand vectors producible by $\{j|x_j^* > 0\}$, there exists an optimal solution for the minimization problem, such that the activities operated at a positive level at this optimal solution is a subset of $\{j|x_j^* > 0\}$?

Given a producible final demand vector d, the dual of LP-d denoted DLP-d is the following linear programming problem: Find $q \in \mathbb{R}^m$, an array of non-negative real numbers $\langle h_j | j \in W \rangle$ and a real number $\alpha \geq 0$ to solve:

Maximize $q^T d - \alpha\bar{L} - \sum_{j \in W} h_j \bar{x}_j$

s.t. $q^T M^j - \alpha A_{m+1,j} - h_j \leq wA_{m+1,j}$ for all $j \in W$,

$q^T M^j - \alpha A_{m+1,j} \leq wA_{m+1,j}$ for all $j \notin W$

Suppose there exists a producible final demand vector d^* and let x^* be an optimal solution for LP-d^*. By the Weak Duality Theorem for LP, there exists $q^* \in \mathbb{R}^m$, an array of non-negative real numbers $\langle h_j^* | j \in W \rangle$ and a real number $\alpha^* \geq 0$ such that

(i) $q^{*T} M^j - \alpha^* A_{m+1,j} - h_j^* \leq wA_{m+1,j}$ for all $j \in W$,

(ii) $q^{*T} M^j - \alpha^* A_{m+1,j} \quad \leq wA_{m+1,j}$ for all $j \notin W$,

(iii) $[q^{*T} M^j - \alpha^* A_{m+1,j} - h_j^* - wA_{m+1,j}]\, x_j^* = 0$ for all $j \in W$,

(iv) $[q^{*T} M^j - \alpha^* A_{m+1,j} - wA_{m+1,j}]\, x_j^* = 0$ for all $j \notin W$,

(v) $Mx^* = d^*$,

(vi) $\alpha^*[A_{m+1}x^* - \bar{L}] = 0$,

(vii) $x_j^* \leq \bar{x}_j$ for all $j \in W$,

(viii) $[x_j^* - \bar{x}_j]\, h_j^* = 0$ for all $j \in W$

(ix) $[q^{*T} d^* - \alpha^*\bar{L} - \sum_{j \in W} h_j^* \bar{x}_j] = wA_{m+1}x^*$.

Since by (v) $Mx^* = d^*$, (iii), (iv) and (vi) implies $[q^{*T} d^* - \alpha^*\bar{L} - \sum_{j \in W} h_j^* x_j^*] = wA_{m+1}x^*$.

$[q^{*T}d^{*}-\alpha^{*}\bar{L}-\sum_{j\in W} h_j^{*} x_j^{*}] = wA_{m+1}x^{*}$ combined with (viii) implies $[q^{*T}d^{*}-\alpha^{*}\bar{L}-\sum_{j\in W} h_j^{*} \bar{x}_j] = wA_{m+1}x^{*}$, which is (ix).

Thus, (iii), (iv), (v), (vi) and (viii) implies (ix).

Hence the required system of equations and inequalities are:

(i) $q^{*T}M^j - \alpha^{*}A_{m+1,j} - h_j^{*} \leq wA_{m+1,j}$ for all $j\in W$,

(ii) $q^{*T}M^j - \alpha^{*}A_{m+1,j} \leq wA_{m+1,j}$ for all $j\notin W$,

(iii) $[q^{*T}M^j - \alpha^{*}A_{m+1,j} - h_j^{*} - wA_{m+1,j}]\, x_j^{*} = 0$ for all $j\in W$,

(iv) $[q^{*T}M^j - \alpha^{*}A_{m+1,j} - wA_{m+1,j}]\, x_j^{*} = 0$ for all $j\notin W$,

(v) $Mx^{*} = d^{*}$,

(vi) $\alpha^{*}[A_{m+1}x^{*}-\bar{L}] = 0$,

(vii) $x_j^{*} \leq \bar{x}_j$ for all $j\in W$,

(viii) $[x_j^{*} - \bar{x}_j]\, h_j^{*} = 0$ for all $j\in W$

Note that $\{j|x_j^{*} > 0\} \neq \phi$, since $d^{*} > 0$, $\{j\in W|x_j^{*} > 0\} \subset \{j|q^{*T}M^j - \alpha^{*}a_{m+1,j} - h_j^{*} - wa_{m+1,j} = 0\}$, $\{j\notin W|x_j^{*} > 0\} \subset \{j\notin W|q^{*T}M^j - \alpha^{*}a_{m+1,j} - wa_{m+1,j} = 0\}$.

Let $J = \{j|x_j^{*} > 0\}$.

Let d be a final demand vector producible by J. Then clearly $d\in\{Mx\in\mathbb{R}_{+}^{m}\backslash\{0\}|x\in\mathbb{R}_{+}^{n}, x_j \leq \bar{x}_j$ for all $j\in W$, $A_{m+1}x \leq \bar{L}$, and $x_j = 0$ for all $j\notin J\}$.

Let x(d) solve the following linear programming problem:

Minimize $wA_{m+1}x$

s.t. $Mx = d$,

$-A_{m+1}x \geq -\bar{L}$,

$-x_j \geq -\bar{x}_j$ for all $j\in W$,

$x \geq 0$, $x_j = 0$, if $j\notin J$.

Since $J = \{j|x_j^{*} > 0\} = \{j\in W|x_j^{*} > 0\}\cup\{j\notin W|x_j^{*} > 0\}\subset\{j\in W|q^{*T}M^j - \alpha^{*}a_{m+1,j} - h_j^{*} - wa_{m+1,j} = 0\}\cup\{j\in W|q^{*T}M^j - \alpha^{*}a_{m+1,j} - wa_{m+1,j} = 0\}$ and $x_j(d) = 0$ for all $j\notin J$ it is clear that $[q^{*T}M^j - \alpha^{*}A_{m+1,j} - h_j^{*} - wA_{m+1,j}]x_j(d) = 0$ for all $j\in W$ and $[q^{*T}M^j - \alpha^{*}A_{m+1,j} - wA_{m+1,j}]x_j(d) = 0$ for all $j\notin W$.

Hence, the following system of equations and inequalities are satisfied:

(a) $Mx(d) = d$,

(b) $A_{m+1}x(d) \leq \bar{L}$

(c) $x_j(d) \leq \bar{x}_j$ for all $j\in W$

(d) $q^{*T}M^j - \alpha^* A_{m+1,j} - h_j^* \leq wA_{m+1,j}$ for all $j \in W$

(e) $q^{*T}M^j - \alpha^* A_{m+1,j} \leq wA_{m+1,j}$ for all $j \notin W$

(f) $[q^{*T}M^j - \alpha^* A_{m+1,j} - h_j^* - wA_{m+1,j}]x_j(d) = 0$, for all $j \in W$,

(g) $[q^{*T}M^j - \alpha^* A_{m+1,j} - wA_{m+1,j}]x_j(d) = 0$, for all $j \notin W$,

Thus x(d) satisfies all the constraints of LP-d and q^*, α^*, $<h_j^*|j \in W>$ solves all the constraints of the DLP-d. Further, $x_j(d) = 0$, if $j \notin J$

The value of the objective function of LP-d at x(d) is $wA_{m+1}x(d)$ and that of the dual DLP-d at q^*, α^*, $<h_j^*|j \in W>$ is $q^{*T}d - \alpha^* \bar{L} - \sum_{j \in W} h_j^* \bar{x}_j$

From (a) we get $q^{*T}d = q^{*T}Mx(d)$ and this combined with (f) and (g) gives us $[q^{*T}d - \alpha^* \bar{L} - \sum_{j \in W} h_j^* x_j(d)] = wA_{m+1}x(d)$.

At this point we invoke an assumption about activities operating up to "full capacity".

Assumption (about activities using their entire capacity): Suppose that $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$.

Now, $\sum_{j \in W} h_j^* x_j(d) = \sum_{\{j \in W | x_j^* < \bar{x}_j\}} h_j^* x_j(d) + \sum_{\{j \in W | x_j^* = \bar{x}_j\}} h_j^* x_j(d)$.

Clearly $\sum_{\{j \in W | x_j^* < \bar{x}_j\}} h_j^* x_j(d) = 0$, since $h_j^* = 0$ whenever $x_j^* < \bar{x}_j\}$

Thus, $\sum_{j \in W} h_j^* x_j(d) = \sum_{\{j \in W | x_j^* = \bar{x}_j\}} h_j^* x_j(d)$.

However, $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$.

Thus, $\sum_{j \in W} h_j^* x_j(d) = \sum_{\{j \in W | x_j^* = \bar{x}_j\}} h_j^* \bar{x}_j$.

Since $\sum_{\{j \in W | x_j^* < \bar{x}_j\}} h_j^* \bar{x}_j = 0$, we get $\sum_{j \in W} h_j^* \bar{x}_j$.

We already have, $[q^{*T}d - \alpha^* \bar{L} - \sum_{j \in W} h_j^* x_j(d)] = wA_{m+1}x(d)$.

Substituting $\sum_{j \in W} h_j^* \bar{x}_j$ for $\sum_{j \in W} h_j^* x_j(d)$ in the above equation gives $[q^{*T}d - \alpha^* \bar{L} - \sum_{j \in W} h_j^* \bar{x}_j] = wA_{m+1}x(d)$.

Thus, as is well known in the theory of linear programming, x(d) is an optimal solution for LP-d and so the answer to the question we have posed earlier is in the affirmative, <u>provided</u> $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$.

From (d) and (e) we get $q^{*T}M^j - \alpha^* A_{m+1,j} - h_j^* \leq wA_{m+1,j}$ for all $j \in W$ and $q^{*T}M^j - \alpha^* A_{m+1,j} \leq wA_{m+1,j}$ for all $j \notin W$

Thus for all $j \in \{1, \ldots, n\}$, there exists a non-negative real number ε_j such that $q^{*T}M^j - wA_{m+1,j} = \alpha^* A_{m+1,j} + \varepsilon_j$.

Clearly, $\varepsilon_j = h_j^*$ for all $j \in W$ satisfying $q^{*T}M^j - \alpha^* A_{m+1,j} - h_j^* - wA_{m+1,j} = 0$ and $\varepsilon_j = 0$ for all $j \notin W$ satisfying $q^{*T}M^j - \alpha^* A_{m+1,j} - wA_{m+1,j} = 0$.

Let $v \in \mathbb{R}^n_+$ be the row vector whose j^{th} coordinate is $\alpha^* A_{m+1,j} + \varepsilon_j$.

Since $q^{*T}M - wA_{m+1} = v$, v is profitable at wage rate w.

Hence if M is a weakly productive activity matrix, by Proposition 2 it follows that there exists an equilibrium price-vector p^* at the wage rate w and row-vector v.

Important Note: v depends on q^*, α^*, $<h_j^*|j \in W>$ which depends on J. Thus p^* depends on the producibility of 'd' by activities in J and on the assumption $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$.

Hence, as mentioned in the first section, there is a clear <u>dependence</u> of the **equilibrium price vector** <u>on</u> the **final demand vector**, unlike the conclusion of Sraffian economics.

This proves the following theorem, which is popularly known as the Non-Substitution Theorem.

Theorem 1: Given a CLAAM (M, $<\bar{x}_j|j \in W>$), suppose that for some producible final demand vector d^*, x^* is an optimal solution for LP-d^*. Let d be a final demand vector producible by $J = \{j | x_j^* > 0\}$. Let x(d) be an optimal solution for the linear programming problem LP-d <u>along with an additional constraint</u> "$x_j = 0$ for all $j \notin J$ (i.e., x is producible by J)":

Minimize $wA_{m+1}x$

s.t. $Mx = d^*$,

$A_{m+1}x \leq \bar{L}$,

$x_j \leq \bar{x}_j$ for all $j \in W$,

$x_j = 0$ for all $j \notin J$,

$x \geq 0$.

<u>If</u> $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$ (i.e., the capacities that are binding at x^* continue to remain binding at x(d)), <u>then</u> x(d) solves LP-d.

If in addition M is weakly productive, then there exists a price-vector p^* known as "efficiency price vector", an array of non-negative real numbers $<h_j^*|j \in W>$ and a real number $\alpha^* \geq 0$ - such that (i) $p^{*T}M^j - wA_{m+1,j} - h_j^* \leq \alpha^* A_{m+1,j}$ for all $j \in W$; (ii) $p^{*T}M^j - wA_{m+1,j} \leq \alpha^* A_{m+1,j}$ for all $j \notin W$; (iii) $p^{*T}M^j - wA_{m+1,j} - h_j^* = \alpha^* A_{m+1,j}$ for all $j \in W$ with $x_j^* > 0$; and (iv) $p^{*T}M^j - wA_{m+1,j} = \alpha^* A_{m+1,j}$ for all $j \notin W$ with $x_j^* > 0$.

Note: In the statement above, for each $j \in W$, h_j^* could be interpreted as the price of operating the j^{th} activity at unit level.

It is well known (see Lahiri (2022)) that if a linear programming problem- in this case LP-d^*- has an optimal solution, then it has a basic optimal solution, i.e. an optimal solution such that the columns of M corresponding to the co-ordinates of the optimal solution which have positive value are linearly independent.

Hence, if x^* is a basic optimal solution for LP-d^*, then the set of columns of M in $\{M^j | x_j^* > 0\}$ are linearly independent.

Thus, an alternative version of the Non-Substitution Theorem stated above (i.e. theorem 1) is one that begins by requiring that "x^* is a basic optimal solution for LP-d^*."

An immediate corollary if Theorem 1 is the following "compact" result valid only for (M,ϕ).

Corollary of Theorem 1: Given a CLAAM (M,ϕ), suppose that for some producible final demand vector d^*, x^* is an (a basic) optimal solution for LP-d^*. Let d be a final demand vector producible by $J = \{j | x_j^* > 0\}$. Let x(d) be an optimal solution for LP-d along with an additional constraint "x is producible by J". Then x(d) solves LP-d.

If in addition M is weakly productive, then there exists a price-vector p^* known as "efficiency price vector" and a real number $\alpha^* \geq 0$ -such that (i) $p^{*T}M^j - wA_{m+1,j} \leq \alpha^* A_{m+1,j}$ for all $j \in \{1,\ldots,n\}$; and (ii) $p^{*T}M^j - wA_{m+1,j} = \alpha^* A_{m+1,j}$ for all j with $x_j^* > 0$.

Note: In the proof of Theorem 1 presented in the form of a discussion prior to the statements of the two theorems, observe that, since x(d) must belong to $\mathbb{R}_+^n \backslash \{0\}$ and $A_{m+1} >> 0$, it must be the case that $wA_{m+1}x(d) > 0$.

Note: This paper generalizes and extends the framework of analysis as well as the results in an earlier paper by the author entitled "Production Analysis for Proper Activity Matrices" available at: https://drive.google.com/file/d/15E2SlIb5NeWBgtRI-O1gb1UA9CG6QhJc/view

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