

A momentum-based linearized augmented Lagrangian method for nonconvex constrained stochastic optimization

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Abstract

Nonconvex constrained stochastic optimization has emerged in many important application areas. Subject to general functional constraints it minimizes the sum of an expectation function and a nonsmooth regularizer. Main challenges arise due to the stochasticity in the random integrand and the possibly nonconvex functional constraints. To address these issues we propose a momentum-based linearized augmented Lagrangian method (MLALM). MLALM adopts a single-loop framework and incorporates a recursive momentum scheme to compute the stochastic gradient, which enables the construction of a stochastic approximation to the augmented Lagrangian function. We provide an analysis of global convergence of MLALM. Under mild conditions and with unbounded penalty parameters, we show that the sequences of average stationarity measure and constraint violations are convergent in expectation. Under a constraint qualification assumption the sequences of average constraint violation and complementary slackness measure converge to zero in expectation. We also explore properties of those related metrics when penalty parameters are bounded. Furthermore, we investigate oracle complexities of MLALM in terms of total number of stochastic gradient evaluations to find an ϵ -stationary point and an ϵ -KKT point when assuming the constraint qualification. Numerical experiments on two types of test problems reveal promising performances of the proposed algorithm.

Keywords: Nonconvex optimization, functional constraint, augmented Lagrangian function, stochastic gradient, momentum, global convergence, oracle complexity

Mathematics Subject Classification 2010: 90C30, 90C06, 65K05, 92C15

1 Introduction

In this paper, we consider the nonconvex constrained stochastic optimization problem

$$\begin{aligned} \min_{x \in X} \quad & \{f(x) \equiv \mathbb{E}_{\xi}[\mathbf{F}(x; \xi)]\} + h(x) \\ \text{s.t.} \quad & c_i(x) = 0, \quad i \in \mathcal{E}, \\ & c_i(x) \leq 0, \quad i \in \mathcal{I}, \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is a closed convex set, ξ is a random variable in the probability space Ξ , and independent of x . Here \mathcal{E} and \mathcal{I} are two finite sets of indices. For any fixed $\xi \in \Xi$, $\mathbf{F}(\cdot; \xi) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $c_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}, i \in \mathcal{E} \cup \mathcal{I}$, are continuously

differentiable and possibly nonconvex, and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is proper, lower-semicontinuous and convex. Without loss of generality we presume that both \mathcal{E} and \mathcal{I} are nonempty, the feasible set $\{x \in X : c_i(x) = 0, i \in \mathcal{E}; c_i(x) \leq 0, i \in \mathcal{I}\}$ is nonempty, and the objective function value of (1) over X is lower bounded by C^* . For problem (1), it can be expensive to compute the expectation or the distribution of ξ may not be expressed explicitly. Thus the exact function or gradient information of f can be hard to obtain. This type of problems widely appear in various application fields. For example, in deep learning, constraints are imposed on output of the deep neural networks [31] to enforce specific behaviors or properties, such as physics-constrained deep learning model [56], constraint-aware deep neural network compression [10], manifold regularized deep learning [38, 44]. Some recent study has also highlighted the advantages of incorporating various constraints when training deep neural networks [27, 36]. Other applications include, but not limited to, portfolio allocation [3, 43], two/multi-stage modeling [3, 43] and constrained maximum likelihood estimation [9, 17].

The past decade has witnessed great developments in nonconvex stochastic optimization. Since Ghadimi and Lan [18] proposed randomized SGD methods for unconstrained nonconvex optimization, a surge of works have emerged in this area of research. However, due to stochastic variances of approximate gradients, SGD methods often suffer from slow convergence [6]. To address this issue, several types of variance reduction techniques have been proposed. Related works include SAG [40], SAGA [15], SVRG [22], SARAH [32] and SPIDER [16]. Moreover, proximal variants aiming for stochastic composite optimization have also been studied [19, 51, 42, 33, 49]. Among those methods, SAG- and SAGA-type methods have high space requirements to store historical gradients at each sample point, while SVRG-, SARAH- and SPIDER-type methods require to compute a (nearly) accurate gradient at a checkpoint from time to time, which normally relies on large batch sizes. Recently, a stochastic recursive momentum method [14] attracts attention, in which only one sample is required to estimate the gradient at each iteration. Later a proximal variant was studied in [55] for nonconvex stochastic composite problems. Under the mean-squared smoothness condition, the aforementioned proximal algorithm can produce a stochastic ϵ -stationary point with the oracle complexity bounded by $\mathcal{O}(\epsilon^{-3})$, where the oracle complexity refers to the total number of stochastic gradient evaluations.

Nonconvex optimization with general functional constraints can be challenging since the feasibility to these constraints can be hard to maintain. Nonconvex constrained optimization in deterministic settings has been studied for decades [50]. Penalty methods and sequential quadratic programming (SQP) methods are two of most effective approaches for general constrained optimization. Penalty methods normally transform the original constrained problem into a sequence of unconstrained ones by penalizing the constraints into the objective in a term measuring the constraint violation. Among penalty methods, augmented Lagrangian (AL) methods attract much interest due to the fact that the AL function has more advantages in characterizing the optimality conditions for constrained problems and in designing effective algorithms. Nevertheless, classic penalty methods are normally double-loop algorithms, in which a penalty function needs to be (approximately) minimized in the inner-loop. Single-loop penalty methods with much simpler subproblems, such as $S\ell_1$ QP [50], linearized AL methods [47, 48], have thus been studied for constrained optimization. On the other hand, SQP methods try to compute search directions by solving a sequence of quadratic programming subproblems. Along with the developments of complexity theories, numerical methods for nonconvex constrained optimization with complexity analysis have been widely studied in the past ten years, including [7, 23, 24, 25, 39, 41, 54].

For general functional constrained optimization in stochastic settings, such as (1), the main concerns lie in that computing the exact gradient information of the expectation function can be expensive, sometimes even prohibitive, and that maintaining the feasibility to general constraints can be challenging. Proximal point methods [5, 4, 26] transfer problem (1) into a sequence of convex subproblems with proximal terms. These methods usually have multi-loop structure and need to call a subsolver in each inner-loop. For instance, the inexact constrained proximal point method with ConEx (ICPPC) in [4] transforms the original problem into a sequence of convex subproblems obtained after adding proximal terms and solves each subproblem with the solver ConEx, which is designed for convex functional constrained optimization. [46] studies penalty methods based on first- and zeroth-order stochastic approximations for equality constrained optimization, with each subproblem constructed based on ℓ_2 penalty function. Recently stochastic SQP methods have been studied in [2, 11, 12] for equality constrained stochastic optimization, with complexity analysis provided in [11]. Based on the linearized AL function, [53] studies a single-loop primal-dual stochastic gradient method (PDSG) for solving convex stochastic optimization problems. [21] extends PDSG and proposes a single-loop stochastic primal-dual (SPD) method for nonconvex problems with a large number of functional constraints.

1.1 Contributions

Our main contributions in this paper are summarized as follows.

1. We propose a Momentum-based Linearized Augmented Lagrangian Method (MLALM) for solving nonconvex constrained stochastic optimization problem (1). The presence of potentially nonconvex constraints poses challenges in finding feasible solutions. To cope with this issue, we adopt the idea of the linearized augmented Lagrangian (AL) function. This approach allows us to propose a single-loop algorithm framework and simplifies the subproblem at each iteration significantly, in contrast to double-loop algorithms like proximal point methods [4, 26]. The integration of the momentum technique within a single-loop algorithm framework is motivated by the prior work [21]. The SPD method proposed in [21] is based on the linearized AL function and aims for nonconvex constrained optimization with a large number of functional constraints. However, it requires large sampling sizes to compute stochastic gradients, resulting in relatively higher total oracle complexity to find an approximate solution, even when the initial iterate is feasible. To mitigate this issue, we employ a recursive momentum technique that only necessitates a small sampling size at each iteration, effectively controlling the variances of stochastic gradients.
2. We investigate the global convergence properties of MLALM. Our analysis reveals that, as the penalty parameter tends to infinity, the sequences of average stationarity measure in expectation converge to zero, and the average constraint violation sequence also exhibits convergence (refer to Theorem 1). Under a constraint qualification assumption (Assumption 5), we establish that the sequences of average constraint violation and average complementary slackness measure converge to zero (refer to Theorem 2). Additionally, we analyze the properties of MLALM when penalty parameters are bounded (refer to Theorem 3). In contrast to recent research on stochastic SQP methods for inequality constrained optimization, such as [28, 29, 30], which necessitate probabilistic conditions on the accuracy of gradient estimates, the related assumptions enforced in this paper can be considerably more relaxed (refer to Assumptions 3 and 4). Another closely related study, [13], investigates the convergence properties of an adaptive stochastic SQP algorithm for problems with deterministic equality and inequality constraints. However, the analysis in [13] relies on the occurrence of an event where the merit parameter sequence eventually becomes sufficiently small yet remains bounded away from zero.
3. We conduct an oracle complexity analysis for MLALM. Under certain conditions, through analyzing the measure of the output in terms of stationarity, constraint violations and complementary slackness, we show that the oracle complexities of MLALM to find an ϵ -stationary point (Definition 1) and to find an ϵ -KKT point (Definition 2) under the constraint qualification are both in order $\mathcal{O}(\epsilon^{-5})$. If the initial point is nearly feasible, previous orders can be reduced to $\mathcal{O}(\epsilon^{-4})$. The SPD method proposed in [21] achieves an ϵ -KKT point with an oracle complexity of $\mathcal{O}(\epsilon^{-6})$ (resp. $\mathcal{O}(\epsilon^{-5})$) without (resp. with) requiring an initial-feasibility condition. Both SPD and MLALM adopt the idea of using a linearized augmented Lagrangian function. However, the incorporation of momentum in MLALM enables us to achieve improved oracle complexities under the same problem settings and under the mean-squared smoothness assumption. The stochastic SQP method proposed in [11], which is designed for equality constrained optimization, relies on an adaptive strategy to update merit parameters and assumes prior knowledge of Lipschitz constants for the objective and constraint gradients. This assumption poses a challenge for the direct application of stochastic SQP to nonsmooth problems. Similarly, the ICPPC algorithm [4], designed for inequality constrained optimization, requires a strong feasibility assumption, depending on the availability of a strictly feasible point. The algorithm framework and theoretical analysis presented in [4, 5] and [11] are specifically tailored for problems with only inequality or equality constraints. In contrast, MLALM aims for more general problems. A more detailed comparison between MLALM, stochastic SQP (SSQP), and ICPPC is provided in Table 1.
4. We present the numerical performance analysis of the proposed algorithm MLALM on two problem classes: quadratically constrained nonconvex programs (QCNPs) and multi-class Neyman-Pearson classification problems (mNPCs). We first investigate the impact of the recursive momentum on QCNPs, to better understand how the introduction of momentum affects the algorithm's performance. We then compare MLALM with ICPPC [4] for QCNPs and with SPD [21] and ICPPC for mNPCs. Numerical results reveal that the use of momentum brings benefits and delivers competitive performance.

Algorithm	Problem	Stationarity measure	Assumptions	Comp.
SPD [21]	$\min_{x \in X} f(x) + h(x)$ s.t. $c_i(x) \leq 0, i \in \mathcal{I}$	$\mathbb{E}[\mathbf{d}(\nabla f(x) + \partial h(x) + \sum_{i \in \mathcal{I}} \lambda_i \nabla c_i(x), -\mathcal{N}_X(x))] \leq \epsilon,$ $\frac{1}{ \mathcal{I} } \mathbb{E}[\sum_{i \in \mathcal{I}} [c_i(x)]_+] \leq \epsilon,$ $\mathbb{E}[\sum_{i \in \mathcal{I}} \lambda_i c_i(x)] \leq \epsilon$	nonsingularity	$\mathcal{O}(\epsilon^{-6})$
				$\mathcal{O}(\epsilon^{-5})$ (if initial feasible)
SSQP [11]	$\min_{x \in \mathbb{R}^n} f(x)$ s.t. $c_i(x) = 0, i \in \mathcal{E}$	$\mathbb{E}[\ \nabla f(x) + \sum_{i \in \mathcal{E}} \lambda_i^{true} \nabla c_i(x)\] \leq \epsilon,$ $\mathbb{E}[\sqrt{\sum_{i \in \mathcal{E}} c_i(x) }] \leq \epsilon$	strong LICQ	$\tilde{\mathcal{O}}(\epsilon^{-4})$
				$\mathcal{O}(\epsilon^{-4})$ (if τ_{\min} known)
ICPPC [4]	$\min_{x \in X} f(x) + h(x)$ s.t. $c_i(x) + h_i(x) \leq 0, i \in \mathcal{I}$	$\mathbb{E}[\ \nabla f(\hat{x}) + \partial h(\hat{x}) + \sum_{i \in \mathcal{I}} \lambda_i (\nabla c_i(\hat{x}) + \partial h_i(\hat{x})), -\mathcal{N}_X(\hat{x})\] \leq \epsilon^2,$ and $\mathbb{E}[\sum_{i \in \mathcal{I}} \lambda_i c_i(\hat{x}) + h_i(\hat{x})] \leq \epsilon^2$	strong feasibility	$\mathcal{O}(\epsilon^{-4})$
MLALM (this paper)	$\min_{x \in X} f(x) + h(x)$ s.t. $c_i(x) = 0, i \in \mathcal{E}$ $c_i(x) \leq 0, i \in \mathcal{I}$	$\mathbb{E}[\mathbf{d}^2(\nabla f(x) + \partial h(x) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(x), -\mathcal{N}_X(x))] \leq \epsilon^2,$ $\mathbb{E}[\mathbf{d}^2(\nabla c_{\mathcal{E}}(x) c_{\mathcal{E}}(x) + \nabla c_{\mathcal{I}}(x) [c_{\mathcal{I}}(x)]_+, -\mathcal{N}_X(x))] \leq \epsilon^2$	mean-squared smoothness	$\mathcal{O}(\epsilon^{-5})$
				$\mathcal{O}(\epsilon^{-4})$ (if initial $\sqrt{\epsilon}$ -feasible)
		$\mathbb{E}[\mathbf{d}^2(\nabla f(x) + \partial h(x) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(x), -\mathcal{N}_X(x))] \leq \epsilon^2,$ $\mathbb{E}[\ c_{\mathcal{E}}(x)\ ^2 + \ [c_{\mathcal{I}}(x)]_+\ ^2] \leq \epsilon^2,$ $\mathbb{E}[\sum_{i \in \mathcal{I}} \lambda_i c_i(x)] \leq \epsilon$	mean-squared smoothness constraint qualification	$\mathcal{O}(\epsilon^{-5})$
				$\mathcal{O}(\epsilon^{-4})$ (if initial $\sqrt{\epsilon}$ -feasible)

Table 1: Comparison between algorithms for nonconvex constrained optimization, where $f(x) = \mathbb{E}_{\xi}[\mathbf{F}(x; \xi)]$, h and $h_i, i \in \mathcal{I}$ are convex but possibly nonsmooth, $\lambda_i, i \in \mathcal{I}$ are nonnegative, λ^{true} is a vector of Lagrange multipliers corresponding to x , τ_{\min} is the merit parameter threshold in SSQP, and “initial $\sqrt{\epsilon}$ -feasible” means the initial point satisfies $\|c_{\mathcal{E}}(x^1)\|^2 + \|[c_{\mathcal{I}}(x^1)]_+\|^2 \leq \epsilon$, while “initial feasible” means the initial point is feasible to (1). The “nonsingularity” condition refers to Assumption 3.1 in [21]. The “strong LICQ” condition for SSQP represents that the Jacobian of constraint functions have singular values that are lower bounded away from zero over a set containing all iterates for all realizations of the random variable. The “mean-squared smoothness” and “constraint qualification” condition for MLALM refer to Assumptions 3 and 5 in this paper, respectively.

1.2 Notation and preliminaries

We use $\|\cdot\|$ to denote the Euclidean norm of a vector without any specification. For brevity, we introduce $[k] := \{1, \dots, k\}$ for any positive integer k . For any $u \in \mathbb{R}$, we define its positive and negative parts as $[u]_+ := \max\{0, u\}$ and $[u]_- := \max\{0, -u\}$, respectively. Moreover, for any $u \in \mathbb{R}^n$, $[u]_+$ and $[u]_-$ are referred to as componentwise application of the operator $[\cdot]_+$ and $[\cdot]_-$, respectively. The gradient of f at x is denoted by $\nabla f(x)$. With a slight abuse of notation, we define $c_{\mathcal{E}} : \mathbb{R}^n \rightarrow \mathbb{R}^{|\mathcal{E}|}$ with components being $c_i(\cdot), i \in \mathcal{E}$, and $\nabla c_{\mathcal{E}} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times |\mathcal{E}|}$ with columns being $\nabla c_i(\cdot), i \in \mathcal{E}$. Notations $c_{\mathcal{I}}$ and $\nabla c_{\mathcal{I}}$ are defined in the same way. Given $X, Y \subseteq \mathbb{R}^n$, the distance between them is referred to $\mathbf{d}(X, Y) = \inf_{x \in X, y \in Y} \|x - y\|$. Furthermore, given random variables ξ and ζ , $\mathbb{E}_{\xi}[\cdot]$ represents the expectation with respect to ξ and $\mathbb{E}_{\xi}[\cdot | \zeta]$ represents the expectation conditioned on ζ . The inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$. The normal cone to a closed convex set X at a point $\bar{x} \in X$ is defined as

$$\mathcal{N}_X(\bar{x}) = \{v \mid \langle v, x - \bar{x} \rangle \leq 0, \forall x \in X\}.$$

And its dual cone is denoted by $\mathcal{N}_X^*(\bar{x})$. Let $h : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower-semicontinuous and convex. The set of subgradient of h at $x \in \text{dom}(h)$ is defined as

$$\partial h(x) = \{v \in \mathbb{R}^n \mid h(y) \geq h(x) + \langle v, y - x \rangle, \forall y \in \text{dom } h\}.$$

In general, finding a global or even a local minimizer for nonconvex constrained optimization can be NP-hard. Efforts are thus devoted to seeking more trackable solutions. Under certain constraint qualification, a local minimizer of (1) satisfies necessary conditions, e.g. KKT conditions. A point satisfying these conditions is called a KKT point. We assume in this paper that, there exist a KKT point $x^* \in X$ and a vector $\lambda^* \in \mathbb{R}^{|\mathcal{E} \cup \mathcal{I}|}$ with $\lambda_i^* \geq 0, i \in \mathcal{I}$, such that the KKT conditions are satisfied:

$$\mathbf{d}(\nabla f(x^*) + \partial h(x^*) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* \nabla c_i(x^*), -\mathcal{N}_X(x^*)) = 0; \quad c_{\mathcal{E}}(x^*) = \mathbf{0}, \quad c_{\mathcal{I}}(x^*) \leq \mathbf{0}; \quad \lambda_i c_i(x^*) = 0, i \in \mathcal{I}.$$

In practical computations, however, it is inevitable that the iteration may be trapped at an infeasible stationary point of the problem:

$$\min_{x \in X} \frac{1}{2} \|c_{\mathcal{E}}(x)\|^2 + \frac{1}{2} \|[c_{\mathcal{I}}(x)]_+\|^2. \quad (2)$$

From the optimality condition for (2), the following stationary holds:

$$\mathbf{d}(\nabla c_{\mathcal{E}}(x)c_{\mathcal{E}}(x) + \nabla c_{\mathcal{I}}(x)[c_{\mathcal{I}}(x)]_+, -\mathcal{N}_X(x)) = 0.$$

We next lay out assumptions that are used throughout the remainder of this paper.

Assumption 1 *Set X is closed and convex. Functions f and $c_i, i \in \mathcal{E} \cup \mathcal{I}$ are continuously differentiable over X with L -Lipschitz continuous gradients. Function h is proper, lower semicontinuous and convex over X . Moreover, the objective function value of (1) over X is lower bounded by C^* .*

Assumption 2 *There exist $C, G > 0$ such that for any $x \in X$,*

$$|c_i(x)| \leq C, \forall i \in \mathcal{E}; \quad c_i(x) \leq C, \forall i \in \mathcal{I}; \quad (3)$$

$$\|\nabla f(x)\| \leq G, \quad \|\partial h(x)\| \leq G, \quad \text{and} \quad \|\nabla c_i(x)\| \leq G, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. \quad (4)$$

Assumption 3 *$\mathbf{F}(\cdot; \xi)$ is continuously differentiable for each $\xi \in \Xi$ and satisfies*

$$\mathbb{E}_{\xi}[\|\nabla \mathbf{F}(u; \xi) - \nabla \mathbf{F}(v; \xi)\|^2] \leq L^2 \|u - v\|^2 \quad \forall u, v \in X.$$

Assumption 4 *There exists $\sigma > 0$ such that for any $x \in X$,*

$$\mathbb{E}_{\xi}[\nabla \mathbf{F}(x; \xi)] = \nabla f(x), \quad \mathbb{E}_{\xi}[\|\nabla \mathbf{F}(x; \xi) - \nabla f(x)\|^2] \leq \sigma^2,$$

Remark 1 *It is noteworthy that the boundedness in Assumption 2 holds naturally under Assumption 1 when X is compact, which is assumed in [4, 21]. Assumption 3 refers to the mean-squared smoothness condition, also known for L -average smoothness, is widely used in stochastic variance reduction-based methods [45, 55, 1]. It is slightly stronger than Lipschitz continuity of the expected value function f by Jensen's inequality. In [29, Assumption 3], a similar yet stronger assumption to Assumption 3 is made, where it requires \mathbf{F} be thrice differentiable and that $\nabla^2 \mathbf{F}(x, \xi)$ be uniformly bounded over a set containing all iterates and for all ξ . Besides, all constraint functions are assumed thrice continuously differentiable in [29].*

1.3 Outline

The rest of this paper is organized as follows. In Section 2 we introduce a momentum-based linearized augmented method for solving nonconvex constrained stochastic optimization problem (1). In Section 3 we present auxiliary lemmas that are required in subsequent sections. In Section 4 we investigate global convergence properties of the proposed algorithm. In Section 5 we establish oracle complexities to find an ϵ -stationary point and an ϵ -KKT point, respectively. In Section 6 we report some numerical experimental results and finally we give some conclusional remarks.

2 Momentum-based linearized augmented Lagrangian method

As is well-known, the augmented Lagrangian (AL) function plays a crucial role in characterizing optimality conditions for constrained optimization and is widely used in designing effective algorithms. The AL function associated with problem (1) is in the form as described in [37]:

$$\mathcal{L}_{\beta}(x, \lambda) := \phi_{\beta}(x, \lambda) + h(x), \quad (5)$$

where $\beta > 0$ is a penalty parameter, $\phi_{\beta}(x, \lambda) := f(x) + \Psi_{\beta}(x, \lambda)$ and

$$\Psi_{\beta}(x, \lambda) := \sum_{i \in \mathcal{E}} [\lambda_i c_i(x) + \frac{\beta}{2} c_i^2(x)] + \sum_{i \in \mathcal{I}} \psi_{\beta}(c_i(x), \lambda_i) \quad \text{with} \quad \psi_{\beta}(u, v) := \begin{cases} vu + \frac{\beta}{2} u^2 & \text{if } \beta u + v \geq 0, \\ -\frac{v^2}{2\beta} & \text{otherwise.} \end{cases}$$

It is easy to check that

$$\nabla_x \Psi_\beta(x, \lambda) = \sum_{i \in \mathcal{E}} (\lambda_i + \beta c_i(x)) \nabla c_i(x) + \sum_{i \in \mathcal{I}} [\lambda_i + \beta c_i(x)]_+ \nabla c_i(x). \quad (6)$$

Different from classical AL methods which try to minimize the AL function in the inner-loop, the linearized AL methods [47, 48, 54] construct a much simpler subproblem that minimizes an approximation to the AL function around current iterate x :

$$\min_{y \in X} \langle \nabla_x \phi_\beta(x, \lambda), y \rangle + \frac{1}{2\eta} \|y - x\|^2 + h(y),$$

where $\eta > 0$. However, due to the problem setting of (1), it is normally expensive sometimes even prohibitive to compute the exact gradient ∇f at an inquiry point $x \in X$. Under this circumstance, we can only get access to a stochastic gradient $\nabla \mathbf{F}(x; \xi)$ by randomly calling a sample ξ . As a result, we obtain a stochastic gradient $\nabla \Phi_\beta(x, \lambda; \xi)$, where

$$\Phi_\beta(x, \lambda; \xi) := \mathbf{F}(x; \xi) + \Psi_\beta(x, \lambda).$$

The SPD method [21] also adopts the linearized AL function to construct subproblems, but it requires large batch sizes when computing mini-batch stochastic gradients in order to derive desired iteration complexity. A natural way to reduce the total oracle complexity is to try adopting smaller batch size per iteration.

Before proceeding, let us consider the problem of minimizing a continuously differentiable function $f(x) = \mathbb{E}[\mathbf{F}(x; \xi)]$ over \mathbb{R}^n with $\xi \in \Xi$. Recall Nesterov's accelerated gradient approach, which reads

$$x^{t+1} = x^t - \eta_t s^t; \quad s^{t+1} = a_t s^t + b_t \nabla f(x^{t+1} - a_t s^t), \quad t \geq 1,$$

with $s^1 = \nabla f(x^1)$. Nevertheless, since the exact gradient of f cannot be accessed, we have to turn to its stochastic approximation by randomly picking one sample ξ^{t+1} :

$$\begin{aligned} s^{t+1} &= a_t s^t + b_t \nabla \mathbf{F}(x^{t+1} - a_t s^t; \xi^{t+1}) \\ &\approx a_t s^t + b_t \left[\left(1 + \frac{a_t}{\eta_t}\right) \nabla \mathbf{F}(x^{t+1}; \xi^{t+1}) - \frac{a_t}{\eta_t} \nabla \mathbf{F}(x^t; \xi^{t+1}) \right], \end{aligned}$$

where the above expression uses the linear Lagrange interpolating polynomial. Then, letting $a_t = 1 - \alpha_t$ and $b_t = \eta_t = \alpha_t$, we obtain the stochastic gradient estimation in the recursive momentum method [14] and its variant [55]:

$$s^{t+1} = \nabla \mathbf{F}(x^{t+1}; \xi^{t+1}) + (1 - \alpha_t)(s^t - \nabla \mathbf{F}(x^t; \xi^{t+1})), \quad t \geq 1.$$

As shown in [14, 55], under the mean-squared smoothness assumption momentum-based approaches can help to reduce the oracle complexity of SGD and proximal SGD methods. Motivated by this, we extend the idea to the general constrained optimization problem (1). Let $\{\beta_t\}$ be a sequence of penalty parameters for $t \geq 1$. We define

$$d^t = \begin{cases} \frac{1}{|\mathcal{J}_t|} \sum_{j \in \mathcal{J}_t} \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^t), & t = 1, \\ \frac{1}{|\mathcal{J}_t|} \sum_{j \in \mathcal{J}_t} \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^t) + (1 - \alpha_{t-1})(d^{t-1} - \frac{1}{|\mathcal{J}_t|} \sum_{j \in \mathcal{J}_t} \nabla_x \Phi_{\beta_{t-1}}(x^{t-1}, \lambda^{t-1}; \xi_j^t)), & t \geq 2, \end{cases} \quad (7)$$

where $\{\xi_j^t, j \in \mathcal{J}_t\}$ is a batch of samples selected randomly uniformly and independently from Ξ . Obviously, d^t is an approximation to $\nabla \phi_{\beta_t}(x^t, \lambda^t)$. The first term in the second line of (7) is a stochastic gradient estimated at x^t , while the second term is the difference between d^{t-1} and a stochastic gradient estimated at x^{t-1} . This correction aims to improve the accuracy of the stochastic approximation. We define the error

$$\varepsilon^t := d^t - \nabla_x \phi_{\beta_t}(x^t, \lambda^t), \quad t \geq 1.$$

When $\alpha_{t-1} = 1$, the expression for d^t reduces to the gradient approximation in the vanilla mini-batch SGD method. In this paper, we choose $\alpha_t \in (0, 1)$ for $t \geq 1$. To a certain extent, α_{t-1} can replace the batch \mathcal{J}_t to reduce variance, thereby reducing oracle complexity with a smaller batch. Based on the stochastic approximation d^t , as defined in (7), we propose the following scheme to update the primal variable:

$$x^{t+1} = \arg \min_{x \in X} \{ \langle d^t, x \rangle + h(x) + \frac{1}{2\eta_t} \|x - x^t\|^2 \}, \quad (8)$$

where $\eta_t > 0$, $t \geq 1$. To expect λ more trackable we propose to update λ^t through:

$$\lambda_i^{t+1} = \lambda_i^t + \begin{cases} \rho_t c_i(x^{t+1}), & i \in \mathcal{E}, \\ \rho_t \max\{-\frac{\lambda_i^t}{\beta_t}, c_i(x^{t+1})\}, & i \in \mathcal{I}, \end{cases} \quad (9)$$

where $\rho_t \in (0, \beta_t)$.

Algorithm 1 Momentum-based Linearized Augmented Lagrangian Method (MLALM)

Require: $x^1 \in \mathbb{R}^n$, $\lambda^1 \in \mathbb{R}^{|\mathcal{E} \cup \mathcal{I}|}$ with $\lambda_i^1 \geq 0, i \in \mathcal{I}$, a non-decreasing positive sequence $\{\beta_t\}$, and parameters $\{\eta_t > 0\}$, $\{\rho_t \in (0, \beta_t)\}$, $\{\alpha_t \in (0, 1)\}$.

1: **for** $t = 1, 2, \dots$ **do**
2: Calculate d^t through (7).
3: Calculate x^{t+1} through (8).
4: Calculate λ^{t+1} through (9).
5: **end for**

3 Auxiliary lemmas

In this section, we introduce auxiliary lemmas that will be useful in subsequent sections for global convergence and oracle complexity analysis. Let $\{x^t\}$ and $\{\lambda^t\}$ be generated by Algorithm 1.

Lemma 1 *For any $t \geq 1$, it holds that $\lambda_i^t \geq 0$, $i \in \mathcal{I}$.*

Proof. It is straightforward to obtain the conclusion by induction from $\lambda_i^1 \geq 0$, $\forall i \in \mathcal{I}$, $\rho_t \in (0, \beta_t)$ and (9). \square

Lemma 2 *Under Assumptions 1-2, it holds that for any $t \geq 1$,*

$$|\lambda_i^t| \leq |\lambda_i^1| + C \sum_{k=1}^{t-1} \rho_k, \quad \forall i \in \mathcal{E}; \quad \lambda_i^t \leq \lambda_i^1 + C \sum_{k=1}^{t-1} \rho_k, \quad \forall i \in \mathcal{I}, \quad (10)$$

and for any $t \in [T]$,

$$|\lambda_i^{t+1} - \lambda_i^t| \leq \rho_t \tilde{C}, \quad \forall i \in \mathcal{E} \cup \mathcal{I}, \quad (11)$$

where $\tilde{C} := \max(\frac{|\lambda_i^1| + C \sum_{k=1}^{\infty} \rho_k}{\beta_1}, C)$ and $\sum_{k=1}^0 \rho_k := 0$.

Proof. Firstly, by applying $\lambda^1 = \mathbf{0}$ and (9), we have that for any $t \geq 2$,

$$\begin{aligned} |\lambda_i^t| &\leq |\lambda_i^1| + \sum_{k=1}^{t-1} |\lambda_i^{k+1} - \lambda_i^k| \leq |\lambda_i^1| + \sum_{k=1}^{t-1} \rho_k |c_i(x^{k+1})| \leq |\lambda_i^1| + C \sum_{k=1}^{t-1} \rho_k, \quad \forall i \in \mathcal{E}, \\ \lambda_i^t &= \lambda_i^1 + \sum_{k=1}^{t-1} (\lambda_i^{k+1} - \lambda_i^k) \leq \lambda_i^1 + \sum_{\mathcal{K}} \rho_k c_i(x^{k+1}) \leq \lambda_i^1 + C \sum_{k=1}^{t-1} \rho_k, \quad \forall i \in \mathcal{I}, \end{aligned}$$

where $\mathcal{K} = \{k \in [t-1] \mid \lambda_i^{k+1} \geq \lambda_i^k\}$. We thus obtain (10). Subsequently, for any $i \in \mathcal{E}$, it is easy to obtain

$$|\lambda_i^{t+1} - \lambda_i^t| \leq \rho_t |c_i(x^{t+1})| \leq \rho_t \tilde{C}.$$

Then, for any $i \in \mathcal{I}$, it follows from (9), Lemma 1, Assumption 2 and (10) that

$$|\lambda_i^{t+1} - \lambda_i^t| = \rho_t \left| \max\left(-\frac{\lambda_i^t}{\beta_t}, c_i(x^{t+1})\right) \right| \leq \begin{cases} \rho_t C, & \text{if } c_i(x^{t+1}) \geq 0, \\ \frac{\rho_t \lambda_i^t}{\beta_t} \leq \rho_t \tilde{C}, & \text{otherwise,} \end{cases}$$

which yields (11). \square

Lemma 3 Under Assumptions 1-2, it holds that for any $t \geq 1$ and $\beta \geq \beta_1$,

$$|\psi_\beta(c_i(x^{t+1}), \lambda_i^{t+1}) - \psi_\beta(c_i(x^{t+1}), \lambda_i^t)| \leq \rho_t \tilde{C}^2, \quad i \in \mathcal{I}, \quad (12)$$

and

$$\|\nabla_x \Psi_\beta(x^{t+1}, \lambda^{t+1}) - \nabla_x \Psi_\beta(x^{t+1}, \lambda^t)\| \leq m \rho_t \tilde{C} G, \quad (13)$$

where $m := |\mathcal{E} \cup \mathcal{I}|$.

Proof. For any $t \in [T]$, by the definition of $\psi_\beta(u, v)$, we know that for any $i \in \mathcal{I}$,

$$\psi_\beta(c_i(x^{t+1}), \lambda_i) = \begin{cases} \lambda_i c_i(x^{t+1}) + \frac{\beta}{2} c_i^2(x^{t+1}), & \text{if } \beta c_i(x^{t+1}) + \lambda_i \geq 0, \\ -\frac{(\lambda_i)^2}{2\beta}, & \text{if } \beta c_i(x^{t+1}) + \lambda_i < 0. \end{cases}$$

Firstly, we consider the case when $\beta c_i(x^{t+1}) + \lambda_i^t < 0$. By Lemma 1 and (9), we have $\lambda_i^{t+1} \leq \lambda_i^t$. Thus $\beta c_i(x^{t+1}) + \lambda_i^{t+1} < 0$. Hence, the following relations hold true:

$$\begin{aligned} |\psi_\beta(c_i(x^{t+1}), \lambda_i^{t+1}) - \psi_\beta(c_i(x^{t+1}), \lambda_i^t)| &= \frac{(\lambda_i^t)^2 - (\lambda_i^{t+1})^2}{2\beta} = \frac{\lambda_i^t + \lambda_i^{t+1}}{2\beta} |\lambda_i^{t+1} - \lambda_i^t| \\ &\leq \frac{\lambda_i^1 + C \sum_{k=1}^t \rho_k}{\beta_1} |\lambda_i^{t+1} - \lambda_i^t| \\ &\leq \rho_t \tilde{C}^2. \end{aligned}$$

Secondly, when $\beta c_i(x^{t+1}) + \lambda_i^t \geq 0$, we have $-\frac{\lambda_i^t}{\beta} \leq c_i(x^{t+1}) \leq C$, then (12) is derived obviously if $\beta c_i(x^{t+1}) + \lambda_i^{t+1} \geq 0$. If $\beta c_i(x^{t+1}) + \lambda_i^{t+1} < 0$, since $\psi_\beta(u, v)$ is monotonically decreasing in $v \geq 0$ when $u < 0$, it follows that

$$\begin{aligned} |\psi_\beta(c_i(x^{t+1}), \lambda_i^{t+1}) - \psi_\beta(c_i(x^{t+1}), \lambda_i^t)| &= -\frac{(\lambda_i^{t+1})^2}{2\beta} - \lambda_i^t c_i(x^{t+1}) - \frac{\beta}{2} c_i^2(x^{t+1}) \\ &\leq \lambda_i^{t+1} c_i(x^{t+1}) + \frac{\beta}{2} c_i^2(x^{t+1}) - \lambda_i^t c_i(x^{t+1}) - \frac{\beta}{2} c_i^2(x^{t+1}) \\ &= -c_i(x^{t+1}) |\lambda_i^{t+1} - \lambda_i^t| \\ &\leq \frac{\lambda_i^t}{\beta} |\lambda_i^{t+1} - \lambda_i^t| \leq \rho_t \tilde{C}^2 \end{aligned}$$

which yields (12). In addition, (6), together with (11), indicates that for any $t \geq 1$,

$$\begin{aligned} &\|\nabla_x \Psi_\beta(x^{t+1}, \lambda^{t+1}) - \nabla_x \Psi_\beta(x^{t+1}, \lambda^t)\| \\ &\leq \left\| \sum_{i \in \mathcal{E}} (\lambda_i^{t+1} - \lambda_i^t) \nabla c_i(x^{t+1}) \right\| + \left\| \sum_{i \in \mathcal{I}} ([\beta c_i(x^{t+1}) + \lambda_i^{t+1}]_+ - [\beta c_i(x^{t+1}) + \lambda_i^t]_+) \nabla c_i(x^{t+1}) \right\| \\ &\leq G \sum_{i \in \mathcal{E}} |\lambda_i^{t+1} - \lambda_i^t| + G \sum_{i \in \mathcal{I}} |[\beta c_i(x^{t+1}) + \lambda_i^{t+1}]_+ - [\beta c_i(x^{t+1}) + \lambda_i^t]_+| \\ &\leq G \sum_{i \in \mathcal{E} \cup \mathcal{I}} |\lambda_i^{t+1} - \lambda_i^t| \leq m \rho_t \tilde{C} G \end{aligned}$$

which is exactly (13). □

The lemma below characterizes the smoothness of $\phi_\beta(x, \lambda)$ with respect to x for fixed λ .

Lemma 4 Under Assumptions 1-2, it holds that for any $u, v \in X$, $t \geq 1$ and $\beta \geq \beta_1$,

$$\|\nabla_x \phi_\beta(u, \lambda^t) - \nabla_x \phi_\beta(v, \lambda^t)\| \leq L_\beta \|u - v\|. \quad (14)$$

Furthermore, if Assumption 3 holds as well, then

$$\mathbb{E}_\xi [\|\nabla_x \Phi_\beta(u, \lambda^t; \xi) - \nabla_x \Phi_\beta(v, \lambda^t; \xi)\|^2] \leq L_\beta^2 \|u - v\|^2, \quad (15)$$

where $L_\beta := \beta \tilde{L}$ with $\tilde{L} := \frac{L + mCL \sum_{k=1}^\infty \rho_k}{\beta_1} + mG^2 + mCL$.

Proof. It follows from Assumptions 1-2, (6) and (10) that for any $u, v \in X$,

$$\begin{aligned}
& \|\nabla_x \Psi_\beta(u, \lambda^t) - \nabla_x \Psi_\beta(v, \lambda^t)\| \\
& \leq \sum_{i \in \mathcal{E}} \|(\beta c_i(u) + \lambda_i^t) \nabla c_i(u) - (\beta c_i(v) + \lambda_i^t) \nabla c_i(v)\| \\
& \quad + \sum_{i \in \mathcal{I}} \|[\beta c_i(u) + \lambda_i^t]_+ \nabla c_i(u) - [\beta c_i(v) + \lambda_i^t]_+ \nabla c_i(v)\| \\
& = \sum_{i \in \mathcal{E}} \|[(\beta c_i(u) + \lambda_i^t) - (\beta c_i(v) + \lambda_i^t)] \nabla c_i(u) + (\beta c_i(v) + \lambda_i^t) (\nabla c_i(u) - \nabla c_i(v))\| \\
& \quad + \sum_{i \in \mathcal{I}} \|[[\beta c_i(u) + \lambda_i^t]_+ - [\beta c_i(v) + \lambda_i^t]_+] \nabla c_i(u) + [\beta c_i(v) + \lambda_i^t]_+ (\nabla c_i(u) - \nabla c_i(v))\| \\
& \leq \sum_{i \in \mathcal{E}} [\beta |c_i(u) - c_i(v)| \|\nabla c_i(u)\| + (\beta c_i(v) + \lambda_i^t) L \|u - v\|] \\
& \quad + \sum_{i \in \mathcal{I}} [\beta |c_i(u) - c_i(v)| \|\nabla c_i(u)\| + [\beta c_i(v) + \lambda_i^t]_+ L \|u - v\|] \\
& \leq \sum_{i \in \mathcal{E} \cup \mathcal{I}} [\beta G^2 \|u - v\| + L(\beta C + |\lambda_i^t|) \|u - v\|] \\
& \leq m(\beta G^2 + CL(\beta + \sum_{k=1}^{t-1} \rho_k)) \|u - v\|.
\end{aligned}$$

Then, Assumptions 1 and 2 indicate

$$\|\nabla_x \phi_\beta(u, \lambda^t) - \nabla_x \phi_\beta(v, \lambda^t)\| \leq \|\nabla f(u) - \nabla f(v)\| + \|\nabla_x \Psi_\beta(u, \lambda^t) - \nabla_x \Psi_\beta(v, \lambda^t)\| \leq L_\beta \|u - v\|,$$

where the last inequality is due to $\beta \geq \beta_1$. Analogously, it holds from Assumption 3 that

$$\begin{aligned}
& \mathbb{E}_\xi [\|\nabla_x \Phi_\beta(u, \lambda^t; \xi) - \nabla_x \Phi_\beta(v, \lambda^t; \xi)\|^2] \\
& \leq \mathbb{E}_\xi [\|\nabla \mathbf{F}(u; \xi) - \nabla \mathbf{F}(v; \xi)\|^2] + 2\mathbb{E}_\xi [\|\nabla \mathbf{F}(u; \xi) - \nabla \mathbf{F}(v; \xi)\| \|\nabla_x \Psi_\beta(u, \lambda^t) - \nabla_x \Psi_\beta(v, \lambda^t)\|] \\
& \quad + \|\nabla_x \Psi_\beta(u, \lambda^t) - \nabla_x \Psi_\beta(v, \lambda^t)\|^2 \\
& \leq L^2 \|u - v\|^2 + 2mL(\beta G^2 + CL(\beta + \sum_{k=1}^{t-1} \rho_k)) \|u - v\|^2 + (m(\beta G^2 + CL(\beta + \sum_{k=1}^{t-1} \rho_k)))^2 \|u - v\|^2
\end{aligned}$$

which yields (15). □

Lemma 5 *Under Assumptions 1-2, it holds that*

$$\mathbf{d}^2(\nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1})) \leq 4(m\rho_t \tilde{C}G)^2 + 4\|\varepsilon^t\|^2 + 4(L_{\beta_t}^2 + \frac{1}{\eta_t^2}) \|x^{t+1} - x^t\|^2, \quad (16)$$

where $L_{\beta_t} := \beta_t \tilde{L}$.

Proof. Optimality conditions for (8) imply that for any $t \geq 1$,

$$\mathbf{d}(d^t + \partial h(x^{t+1}) + \frac{1}{\eta_t}(x^{t+1} - x^t), -\mathcal{N}_X(x^{t+1})) = 0.$$

Then from (13) and (14) we obtain

$$\begin{aligned}
& \mathbf{d}^2(\nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1})) \\
& = \mathbf{d}^2(d^t + \partial h(x^{t+1}) + \frac{1}{\eta_t}(x^{t+1} - x^t) + \nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) - d^t - \frac{1}{\eta_t}(x^{t+1} - x^t), -\mathcal{N}_X(x^{t+1}))
\end{aligned}$$

$$\begin{aligned}
&\leq \|\nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) - d^t - \frac{1}{\eta_t}(x^{t+1} - x^t)\|^2 \\
&= \|\nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) - \nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^t) + \nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^t) - \nabla_x \phi_{\beta_t}(x^t, \lambda^t) + \nabla_x \phi_{\beta_t}(x^t, \lambda^t) - d^t \\
&\quad - \frac{1}{\eta_t}(x^{t+1} - x^t)\|^2 \\
&\leq 4(m\rho_t \tilde{C}G)^2 + 4L_{\beta_t}^2 \|x^{t+1} - x^t\|^2 + 4\|\varepsilon^t\|^2 + \frac{4}{\eta_t^2} \|x^{t+1} - x^t\|^2,
\end{aligned}$$

which completes the proof. \square

Lemma 6 *Under Assumptions 1-2, the following relation holds true:*

$$\Psi_{\beta_{t+1}}(x, \lambda) \leq \Psi_{\beta_t}(x, \lambda) + \frac{\beta_{t+1} - \beta_t}{2} \sum_{i \in \mathcal{E} \cup \mathcal{I}} c_i^2(x). \quad (17)$$

Furthermore, we have

$$\left(\frac{1}{2\eta_t} - \frac{L_{\beta_t}}{2}\right) \|x^{t+1} - x^t\|^2 \leq \mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \frac{\beta_{t+1} - \beta_t}{2} mC^2 + \frac{\eta_t}{2} \|\varepsilon^t\|^2 + m\rho_t \tilde{C}^2. \quad (18)$$

Proof. The key to prove (17) is to verify that for any $i \in \mathcal{I}$,

$$\psi_{\beta_{t+1}}(c_i(x), \lambda_i) \leq \psi_{\beta_t}(c_i(x), \lambda_i) + \frac{\beta_{t+1} - \beta_t}{2} c_i^2(x), \quad \forall i \in \mathcal{I}.$$

According to the definition of ψ_β and $\beta_{t+1} \geq \beta_t$, one has

$$\psi_{\beta_{t+1}}(c_i(x), \lambda_i) - \psi_{\beta_t}(c_i(x), \lambda_i) = \begin{cases} \frac{\beta_{t+1} - \beta_t}{2} c_i^2(x), & \beta_{t+1} c_i(x) + \lambda_i \geq 0, \beta_t c_i(x) + \lambda_i \geq 0, \\ -\frac{\lambda_i^2}{2\beta_{t+1}} - \lambda_i c_i(x) - \frac{\beta_t}{2} c_i^2(x), & \beta_{t+1} c_i(x) + \lambda_i < 0, \beta_t c_i(x) + \lambda_i \geq 0, \\ \frac{\lambda_i^2}{2\beta_t} - \frac{\lambda_i^2}{2\beta_{t+1}}, & \beta_{t+1} c_i(x) + \lambda_i < 0, \beta_t c_i(x) + \lambda_i < 0. \end{cases}$$

For the second case, $\beta_{t+1} c_i(x) + \lambda_i < 0$, we have

$$-\frac{\lambda_i^2}{2\beta_{t+1}} - \lambda_i c_i(x) - \frac{\beta_t}{2} c_i^2(x) \leq \lambda_i c_i(x) + \frac{\beta_{t+1}}{2} c_i^2(x) - \lambda_i c_i(x) - \frac{\beta_t}{2} c_i^2(x) = \frac{\beta_{t+1} - \beta_t}{2} c_i^2(x).$$

Further, if $\beta_t c_i(x) + \lambda_i < 0$, then $\lambda_i^2 < \beta_t \beta_{t+1} c_i^2(x)$. It holds that

$$\frac{\lambda_i^2}{2\beta_t} - \frac{\lambda_i^2}{2\beta_{t+1}} \leq \frac{\beta_{t+1} - \beta_t}{2} c_i^2(x).$$

Thus, (17) can be derived. Moreover, Assumption 2 together with (11) and (12) implies that

$$\begin{aligned}
\mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t) &= \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^{t+1}) + \sum_{i \in \mathcal{E}} [(\lambda_i^t c_i(x^{t+1}) + \frac{\beta_t}{2} c_i^2(x^{t+1})) - (\lambda_i^{t+1} c_i(x^{t+1}) + \frac{\beta_t}{2} c_i^2(x^{t+1}))] \\
&\quad + \sum_{i \in \mathcal{I}} [\psi_{\beta_t}(c_i(x^{t+1}), \lambda_i^t) - \psi_{\beta_t}(c_i(x^{t+1}), \lambda_i^{t+1})] \\
&\geq \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^{t+1}) - \sum_{i \in \mathcal{E}} |c_i(x^{t+1})| |\lambda_i^t - \lambda_i^{t+1}| - \sum_{i \in \mathcal{I}} \rho_t \tilde{C}^2 \\
&\geq \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^{t+1}) - m\rho_t \tilde{C}^2.
\end{aligned} \quad (19)$$

Note that by optimality conditions for (8), there exists a vector $u \in \partial h(x^{t+1})$ such that

$$\langle d^t + u + \frac{1}{\eta_t}(x^{t+1} - x^t), x - x^{t+1} \rangle \geq 0 \quad \forall x \in X.$$

Then by the convexity of h and setting $x = x^t$, we obtain

$$h(x^{t+1}) - h(x^t) \leq \langle u, x^{t+1} - x^t \rangle \leq -\langle d^t + \frac{1}{\eta_t}(x^{t+1} - x^t), x^{t+1} - x^t \rangle. \quad (20)$$

Thus it indicates

$$\begin{aligned} & \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t) - \mathcal{L}_{\beta_t}(x^t, \lambda^t) \\ &= \phi_{\beta_t}(x^{t+1}, \lambda^t) - \phi_{\beta_t}(x^t, \lambda^t) + h(x^{t+1}) - h(x^t) \\ &\leq \langle \nabla_x \phi_{\beta_t}(x^t, \lambda^t), x^{t+1} - x^t \rangle + \frac{L_{\beta_t}}{2} \|x^{t+1} - x^t\|^2 + h(x^{t+1}) - h(x^t) \\ &\leq \langle d^t - \varepsilon^t, x^{t+1} - x^t \rangle - \langle d^t + \frac{1}{\eta_t}(x^{t+1} - x^t), x^{t+1} - x^t \rangle + \frac{L_{\beta_t}}{2} \|x^{t+1} - x^t\|^2 \\ &= -\langle \varepsilon^t, x^{t+1} - x^t \rangle + (\frac{L_{\beta_t}}{2} - \frac{1}{\eta_t}) \|x^{t+1} - x^t\|^2 \\ &\leq \frac{\eta_t}{2} \|\varepsilon^t\|^2 + \frac{1}{2\eta_t} \|x^{t+1} - x^t\|^2 + (\frac{L_{\beta_t}}{2} - \frac{1}{\eta_t}) \|x^{t+1} - x^t\|^2 \\ &\leq \frac{\eta_t}{2} \|\varepsilon^t\|^2 + (\frac{L_{\beta_t}}{2} - \frac{1}{2\eta_t}) \|x^{t+1} - x^t\|^2, \end{aligned} \quad (21)$$

where the first inequality follows from (14), the second inequality comes from (20), and the third inequality is due to Young's inequality. Thus it together with (19) yields that

$$\begin{aligned} \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^{t+1}) - \mathcal{L}_{\beta_t}(x^t, \lambda^t) &= \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^{t+1}) - \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t) + \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t) - \mathcal{L}_{\beta_t}(x^t, \lambda^t) \\ &\leq \frac{\eta_t}{2} \|\varepsilon^t\|^2 + (\frac{L_{\beta_t}}{2} - \frac{1}{2\eta_t}) \|x^{t+1} - x^t\|^2 + m\rho_t \tilde{C}^2. \end{aligned} \quad (22)$$

Then rearranging the terms and together with (17) derives (18). \square

From now on and for simplicity, we introduce $\lambda^t \in \mathbb{R}^{|\mathcal{E} \cup \mathcal{I}|}$, $t \geq 1$, defined componentwise by

$$\tilde{\lambda}_i^t = \begin{cases} \beta_{t-1} c_i(x^t) + \lambda_i^t, & i \in \mathcal{E}, \\ [\beta_{t-1} c_i(x^t) + \lambda_i^t]_+, & i \in \mathcal{I}. \end{cases} \quad (23)$$

Lemma 7 *Under Assumptions 1-2, it holds that for any $T \geq 1$,*

$$\begin{aligned} & \frac{1}{T} \sum_{t=1}^T \mathbf{d}^2(\nabla c_{\mathcal{E}}(x^{t+1}) c_{\mathcal{E}}(x^{t+1}) + \nabla c_{\mathcal{I}}(x^{t+1}) [c_{\mathcal{I}}(x^{t+1})]_+, -\mathcal{N}_X(x^{t+1})) \\ &\leq \frac{4}{\beta_1^2} \frac{1}{T} \sum_{t=1}^T \mathbf{d}^2(\nabla f(x^{t+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{t+1} \nabla c_i(x^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1})) \\ &\quad + \frac{4(2 + m^2(\|\lambda_1\| + C \sum_{k=1}^T \rho_k)^2) G^2}{T} \sum_{t=1}^T \frac{1}{\beta_t^2}. \end{aligned}$$

Proof. It is apparent that there exists $v^{t+1} \in \partial h(x^{t+1})$ such that

$$\begin{aligned} & \mathbf{d}(\nabla f(x^{t+1}) + v^{t+1} + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{t+1} \nabla c_i(x^{t+1}), -\mathcal{N}_X(x^{t+1})) \\ &= \mathbf{d}(\nabla f(x^{t+1}) + \partial h(x^{t+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{t+1} \nabla c_i(x^{t+1}), -\mathcal{N}_X(x^{t+1})). \end{aligned}$$

Then for any $t \geq 1$, one has

$$\begin{aligned}
& \mathbf{d}^2(\nabla c_{\mathcal{E}}(x^{t+1})c_{\mathcal{E}}(x^{t+1}) + \nabla c_{\mathcal{I}}(x^{t+1})[c_{\mathcal{I}}(x^{t+1})]_+, -\mathcal{N}_X(x^{t+1})) \\
& \leq \frac{1}{\beta_t^2} \mathbf{d}^2(\sum_{i \in \mathcal{E}} \beta_t c_i(x^{t+1}) \nabla c_i(x^{t+1}) + \sum_{i \in \mathcal{I}} [\beta_t c_i(x^{t+1})]_+ \nabla c_i(x^{t+1}), -\mathcal{N}_X(x^{t+1})) \\
& \leq \frac{4}{\beta_t^2} [\mathbf{d}^2(\nabla f(x^{t+1}) + v^{t+1} + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{t+1} \nabla c_i(x^{t+1}), -\mathcal{N}_X(x^{t+1})) + \|\nabla f(x^{t+1})\|^2 \\
& \quad + \|v^{t+1}\|^2 + (\sum_{i \in \mathcal{E} \cup \mathcal{I}} |\lambda_i^{t+1}| \|\nabla c_i(x^{t+1})\|)^2] \\
& \leq \frac{4}{\beta_t^2} [\mathbf{d}^2(\nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1})) + (2 + m^2(\|\lambda^1\| + C \sum_{k=1}^t \rho_k)^2) G^2]. \tag{24}
\end{aligned}$$

Summing up the above inequality over $t = 1, \dots, T$ and dividing it by T yields the conclusion from $\beta_t \geq \beta_1$ for any $t \geq 1$. \square

The lemma below provides a recursive bound on the error ε^t . For notation simplicity, we denote in the following that:

$$\xi^t = \{\xi_j^t, j \in \mathcal{J}_t\}, \quad \xi^{[t]} = \{\xi^1, \dots, \xi^t\}, \quad t \geq 1.$$

Lemma 8 *Under Assumptions 1-4, it holds that for any $t \geq 1$,*

$$\begin{aligned}
\mathbb{E}_{\xi^{[t+1]}} [\|\varepsilon^{t+1}\|^2] & \leq (1 - \alpha_t)^2 \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] + \frac{1}{|\mathcal{J}_{t+1}|} (2\alpha_t^2 \sigma^2 + 8(1 - \alpha_t)^2 (\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2 \\
& \quad + 8(1 - \alpha_t)^2 (m \rho_t \tilde{C} G)^2 + 4(1 - \alpha_t)^2 L_{\beta_t}^2 \mathbb{E}_{\xi^{[t]}} [\|x^{t+1} - x^t\|^2]). \tag{25}
\end{aligned}$$

Proof. Recall that

$$\begin{aligned}
\varepsilon^{t+1} & = d^{t+1} - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) \\
& = \frac{1}{|\mathcal{J}_{t+1}|} \sum_{j \in \mathcal{J}_{t+1}} \nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) \\
& \quad + (1 - \alpha_t)(d^t - \frac{1}{|\mathcal{J}_{t+1}|} \sum_{j \in \mathcal{J}_{t+1}} \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})) \\
& = \frac{1}{|\mathcal{J}_{t+1}|} \sum_{j \in \mathcal{J}_{t+1}} \nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) \\
& \quad + (1 - \alpha_t)\varepsilon^t + (1 - \alpha_t)(\nabla_x \phi_{\beta_t}(x^t, \lambda^t) - \frac{1}{|\mathcal{J}_{t+1}|} \sum_{j \in \mathcal{J}_{t+1}} \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})). \tag{26}
\end{aligned}$$

Since $\varepsilon^t, x^t, \lambda^t, x^{t+1}$ and λ^{t+1} are independent of ξ^{t+1} , taking expectation with respect to ξ^{t+1} conditioned on $\xi^{[t]}$ yields

$$\begin{aligned}
\mathbb{E}_{\xi^{t+1}} [\langle \frac{1}{|\mathcal{J}_{t+1}|} \sum_{j \in \mathcal{J}_{t+1}} \nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}), \varepsilon^t \rangle \mid \xi^{[t]}] & = 0, \\
\mathbb{E}_{\xi^{t+1}} [\langle \frac{1}{|\mathcal{J}_{t+1}|} \sum_{j \in \mathcal{J}_{t+1}} \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}) - \nabla_x \phi_{\beta_t}(x^t, \lambda^t), \varepsilon^t \rangle \mid \xi^{[t]}] & = 0.
\end{aligned}$$

Hence, squaring both sides of (26) and then taking expectation with respect to ξ^{t+1} conditioned on $\xi^{[t]}$, we have

$$\begin{aligned}
& \mathbb{E}_{\xi^{t+1}}[\|\varepsilon^{t+1}\|^2 \mid \xi^{[t]}] \\
& \leq (1 - \alpha_t)^2 \|\varepsilon^t\|^2 + \mathbb{E}_{\xi^{t+1}}\left[\left\|\frac{1}{|\mathcal{J}_{t+1}|} \sum_{j \in \mathcal{J}_{t+1}} (\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) \right.\right. \\
& \quad \left.\left. + (1 - \alpha_t)(\nabla_x \phi_{\beta_t}(x^t, \lambda^t) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}))\right\|^2 \mid \xi^{[t]}\right] \\
& = (1 - \alpha_t)^2 \|\varepsilon^t\|^2 + \mathbb{E}_{\xi^{t+1}}\left[\frac{1}{|\mathcal{J}_{t+1}|^2} \sum_{j \in \mathcal{J}_{t+1}} \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) \right. \\
& \quad \left. + (1 - \alpha_t)(\nabla_x \phi_{\beta_t}(x^t, \lambda^t) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}))\|^2 \mid \xi^{[t]}\right]. \tag{27}
\end{aligned}$$

Let us focus on the second term in R.H.S. of (27). Note that for any $j \in \mathcal{J}_{t+1}$,

$$\begin{aligned}
& \mathbb{E}_{\xi^{t+1}}[\|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + (1 - \alpha_t)(\nabla_x \phi_{\beta_t}(x^t, \lambda^t) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}))\|^2 \mid \xi^{[t]}] \\
& = \mathbb{E}_{\xi^{t+1}}[\|\alpha_t(\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1})) + (1 - \alpha_t)(\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) \\
& \quad - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \nabla_x \phi_{\beta_t}(x^t, \lambda^t) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}))\|^2 \mid \xi^{[t]}] \\
& \leq \mathbb{E}_{\xi^{t+1}}[2\alpha_t^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1})\|^2 + 2(1 - \alpha_t)^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) \\
& \quad - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \nabla_x \phi_{\beta_t}(x^t, \lambda^t) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})\|^2 \mid \xi^{[t]}] \\
& = \mathbb{E}_{\xi^{t+1}}[2\alpha_t^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1})\|^2 \\
& \quad + 2(1 - \alpha_t)^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})\|^2 \\
& \quad - 4(1 - \alpha_t)^2 \langle \nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}), \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \phi_{\beta_t}(x^t, \lambda^t) \rangle \\
& \quad + 2(1 - \alpha_t)^2 \|\nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \phi_{\beta_t}(x^t, \lambda^t)\|^2 \mid \xi^{[t]}] \\
& = \mathbb{E}_{\xi^{t+1}}[2\alpha_t^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1})\|^2 \\
& \quad + 2(1 - \alpha_t)^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})\|^2 \\
& \quad - 2(1 - \alpha_t)^2 \|\nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \phi_{\beta_t}(x^t, \lambda^t)\|^2 \mid \xi^{[t]}] \\
& \leq \mathbb{E}_{\xi^{t+1}}[2\alpha_t^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1})\|^2 \\
& \quad + 2(1 - \alpha_t)^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})\|^2 \mid \xi^{[t]}] \\
& \leq 2\alpha_t^2 \sigma^2 + \mathbb{E}_{\xi^{t+1}}[4(1 - \alpha_t)^2 \|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})\|^2 \\
& \quad + 4(1 - \alpha_t)^2 \|\nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})\|^2 \mid \xi^{[t]}],
\end{aligned}$$

where the last equality follows from

$$\mathbb{E}_{\xi^{t+1}}[\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1}) \mid \xi^{[t]}] = \nabla_x \phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \phi_{\beta_t}(x^t, \lambda^t).$$

Hence, we obtain the following relation:

$$\begin{aligned}
& \mathbb{E}_{\xi^{t+1}} [\|\varepsilon^{t+1}\|^2 \mid \xi^{[t]}] \\
& \leq (1 - \alpha_t)^2 \|\varepsilon^t\|^2 + \frac{2\alpha_t^2 \sigma^2}{|\mathcal{J}_{t+1}|} + \frac{4(1 - \alpha_t)^2}{|\mathcal{J}_{t+1}|^2} \sum_{j \in \mathcal{J}_{t+1}} \mathbb{E}_{\xi^{t+1}} [\|\nabla_x \Phi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^{t+1}, \lambda^t; \xi_j^{t+1})\|^2 \mid \xi^{[t]}] \\
& \quad + \frac{4(1 - \alpha_t)^2}{|\mathcal{J}_{t+1}|^2} \sum_{j \in \mathcal{J}_{t+1}} \mathbb{E}_{\xi^{t+1}} [\|\nabla_x \Phi_{\beta_t}(x^{t+1}, \lambda^t; \xi_j^{t+1}) - \nabla_x \Phi_{\beta_t}(x^t, \lambda^t; \xi_j^{t+1})\|^2 \mid \xi^{[t]}] \\
& \leq (1 - \alpha_t)^2 \|\varepsilon^t\|^2 + \frac{2\alpha_t^2 \sigma^2}{|\mathcal{J}_{t+1}|} + \frac{4(1 - \alpha_t)^2}{|\mathcal{J}_{t+1}|} \|\nabla_x \Psi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \Psi_{\beta_t}(x^{t+1}, \lambda^t)\|^2 \\
& \quad + \frac{4(1 - \alpha_t)^2}{|\mathcal{J}_{t+1}|} L_{\beta_t}^2 \|x^{t+1} - x^t\|^2, \tag{28}
\end{aligned}$$

where the second inequality follows from (15) and the definition of $\Phi_{\beta_t}(x, \lambda; \xi)$. To derive the conclusion, the remainder is to estimate the third term in R.H.S. of (28). As a matter of fact, due to (13) and

$$\begin{aligned}
& \|\nabla_x \Psi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \Psi_{\beta_t}(x^{t+1}, \lambda^{t+1})\| \\
& = \left\| \sum_{i \in \mathcal{E}} (\beta_{t+1} - \beta_t) c_i(x^{t+1}) \nabla c_i(x^{t+1}) + \sum_{i \in \mathcal{I}} ([\lambda_i^{t+1} + \beta_{t+1} c_i(x^{t+1})]_+ - [\lambda_i^{t+1} + \beta_t c_i(x^{t+1})]_+) \nabla c_i(x^{t+1}) \right\| \\
& \leq (\beta_{t+1} - \beta_t) \sum_{i \in \mathcal{E} \cup \mathcal{I}} |c_i(x^{t+1})| \|\nabla c_i(x^{t+1})\| \leq (\beta_{t+1} - \beta_t) mCG,
\end{aligned}$$

we obtain

$$\begin{aligned}
& \|\nabla_x \Psi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \Psi_{\beta_t}(x^{t+1}, \lambda^t)\|^2 \\
& \leq 2\|\nabla_x \Psi_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) - \nabla_x \Psi_{\beta_t}(x^{t+1}, \lambda^{t+1})\|^2 + 2\|\nabla_x \Psi_{\beta_t}(x^{t+1}, \lambda^{t+1}) - \nabla_x \Psi_{\beta_t}(x^{t+1}, \lambda^t)\|^2 \\
& \leq 2(\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2 + 2m^2 \rho_t^2 \tilde{C}^2 G^2.
\end{aligned}$$

Plugging above inequality into (28) yields the conclusion. \square

Interestingly, parameter α_t and batch \mathcal{J}_{t+1} are somewhat intertwined in our approach. More specifically, when $\alpha_t = 1$, the gradient estimate d^t turns to the vanilla mini-batch SGD approximation, where the batch size is normally chosen large enough to reduce stochastic variances. However, in this paper, we focus on the case where $0 < \alpha_t < 1$. In this scenario, the current error ε^t is controlled by previous error ε^{t-1} and corrections accumulated in past iterations. We can prove that, under appropriate parameter settings, the term $\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2]$ tends to zero even without the use of batches \mathcal{J}_t (refer to (36)). However, it is worthy to note that without using appropriate batches, we cannot guarantee the boundedness of $\mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2]$, which is crucial for the global convergence analysis (refer to (44)).

4 Global convergence analysis

In this section, we conduct a global convergence analysis for MLALM. To this end, we assume that the parameters used in Algorithm 1 satisfy the conditions below:

$$\eta_t L_{\beta_t} < \frac{1}{2}, \quad \frac{4\eta_t^2 L_{\beta_t}^2}{1 - \eta_t L_{\beta_t}} \leq \alpha_t < 1, \quad \frac{\eta_t L_{\beta_t}^2}{1 - \eta_t L_{\beta_t}} \leq \tilde{L}, \tag{29}$$

where \tilde{L} is introduced in Lemma 4.

Lemma 9 Suppose that Assumptions 1-4 and (29) hold. Then for any $T \geq 1$ the following is true:

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \alpha_t \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] &\leq \frac{(1 + 2 \sum_{t=1}^T \alpha_t^2) \sigma^2}{T} + \frac{8m^2 C^2 G^2}{T} \sum_{t=1}^T \frac{(\beta_{t+1} - \beta_t)^2}{|\mathcal{J}_{t+1}|} + \frac{8m^2 \tilde{C}^2 G^2}{T} \sum_{t=1}^T \rho_t^2 \\ &\quad + \frac{8\tilde{L}}{T} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + \frac{mC^2}{2} \sum_{t=1}^T \frac{\beta_{t+1} - \beta_t}{|\mathcal{J}_{t+1}|} + 2m\tilde{C}^2 \sum_{t=1}^T \rho_t), \end{aligned} \quad (30)$$

where C^* is the lower bound of the objective function of (1) over X .

Proof. From Lemma 6 it follows that

$$\|x^{t+1} - x^t\|^2 \leq \frac{2\eta_t}{1 - \eta_t L_{\beta_t}} (\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \frac{\beta_{t+1} - \beta_t}{2} mC^2 + \frac{\eta_t}{2} \|\varepsilon^t\|^2 + m\rho_t \tilde{C}^2). \quad (31)$$

We then substitute the above relation into (25) indicating

$$\begin{aligned} &\mathbb{E}_{\xi^{[t+1]}} [\|\varepsilon^{t+1}\|^2] \\ &\leq (1 - \alpha_t)^2 \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] + \frac{1}{|\mathcal{J}_{t+1}|} (2\alpha_t^2 \sigma^2 + 8(1 - \alpha_t)^2 (\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2 + 8(1 - \alpha_t)^2 (m\rho_t \tilde{C}G)^2 \\ &\quad + 4(1 - \alpha_t)^2 L_{\beta_t}^2 \mathbb{E}_{\xi^{[t]}} [\|x^{t+1} - x^t\|^2]) \\ &\leq (1 - \alpha_t)^2 (1 + \frac{4\eta_t^2 L_{\beta_t}^2}{|\mathcal{J}_{t+1}|(1 - \eta_t L_{\beta_t})}) \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] \\ &\quad + \frac{1}{|\mathcal{J}_{t+1}|} (2\alpha_t^2 \sigma^2 + 8(1 - \alpha_t)^2 (\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2 + 8(1 - \alpha_t)^2 (m\rho_t \tilde{C}G)^2) \\ &\quad + \frac{8(1 - \alpha_t)^2 \eta_t L_{\beta_t}^2}{|\mathcal{J}_{t+1}|(1 - \eta_t L_{\beta_t})} \mathbb{E}_{\xi^{[t]}} [\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \frac{\beta_{t+1} - \beta_t}{2} mC^2 + m\rho_t \tilde{C}^2] \\ &\leq (1 - \alpha_t) \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] + \frac{1}{|\mathcal{J}_{t+1}|} (2\alpha_t^2 \sigma^2 + 8(1 - \alpha_t)^2 (\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2 + 8(1 - \alpha_t)^2 (m\rho_t \tilde{C}G)^2) \\ &\quad + \frac{8\eta_t L_{\beta_t}^2}{|\mathcal{J}_{t+1}|(1 - \eta_t L_{\beta_t})} \mathbb{E}_{\xi^{[t]}} [\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \frac{\beta_{t+1} - \beta_t}{2} mC^2 + m\rho_t \tilde{C}^2], \end{aligned}$$

where the last inequality is due to the condition $\frac{4\eta_t^2 L_{\beta_t}^2}{|\mathcal{J}_{t+1}|(1 - \eta_t L_{\beta_t})} \leq \alpha_t < 1$. By summing the above inequality over $t = 1, \dots, T$ and using $\mathbb{E}_{\xi^1} [\|\varepsilon^1\|^2] \leq \sigma^2$, we obtain from $\frac{\eta_t L_{\beta_t}^2}{1 - \eta_t L_{\beta_t}} \leq \tilde{L}$ that

$$\begin{aligned} \sum_{t=1}^T \alpha_t \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] &\leq (1 + 2 \sum_{t=1}^T \alpha_t^2) \sigma^2 + 8m^2 C^2 G^2 \sum_{t=1}^T \frac{(\beta_{t+1} - \beta_t)^2}{|\mathcal{J}_{t+1}|} + 8m^2 \tilde{C}^2 G^2 \sum_{t=1}^T \frac{\rho_t^2}{|\mathcal{J}_{t+1}|} \\ &\quad + 8\tilde{L} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - \mathbb{E}_{\xi^{[T]}} [\mathcal{L}_{\beta_{T+1}}(x^{T+1}, \lambda^{T+1})] + \frac{mC^2}{2} \sum_{t=1}^T \frac{\beta_{t+1} - \beta_t}{|\mathcal{J}_{t+1}|} + m\tilde{C}^2 \sum_{t=1}^T \frac{\rho_t}{|\mathcal{J}_{t+1}|}). \end{aligned} \quad (32)$$

Moreover, we can upper bound $\mathcal{L}_{\beta_1}(x^1, \lambda^1) - \mathbb{E}_{\xi^{[T]}} [\mathcal{L}_{\beta_{T+1}}(x^{T+1}, \lambda^{T+1})]$ by

$$\mathcal{L}_{\beta_1}(x^1, \lambda^1) - \mathbb{E}_{\xi^{[T]}} [\mathcal{L}_{\beta_{T+1}}(x^{T+1}, \lambda^{T+1})]$$

$$\begin{aligned}
&= \mathcal{L}_{\beta_1}(x^1, \lambda^1) - \mathbb{E}_{\xi^{[T]}}[f(x^{T+1}) + h(x^{T+1}) + \sum_{i \in \mathcal{E}} [\lambda_i^{T+1} c_i(x^{T+1}) + \frac{\beta_t}{2} c_i^2(x^{T+1})] + \sum_{i \in \mathcal{I}} \psi_{\beta_t}(c_i(x^{T+1}), \lambda_i^{T+1})] \\
&\leq \mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* - \mathbb{E}_{\xi^{[T]}}[\sum_{i \in \mathcal{E}} [\lambda_i^{T+1} c_i(x^{T+1}) + \frac{\beta_t}{2} c_i^2(x^{T+1})] + \sum_{i \in \mathcal{I}} \psi_{\beta_t}(c_i(x^{T+1}), \lambda_i^{T+1})] \\
&\leq \mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + \mathbb{E}_{\xi^{[T]}}[\sum_{i \in \mathcal{E}} |\lambda_i^{T+1} c_i(x^{T+1})| + \sum_{i \in \mathcal{I}} \frac{(\lambda_i^{T+1})^2}{2\beta_t}] \\
&\leq \mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C(\sum_{i \in \mathcal{E}} (|\lambda_i^1| + C \sum_{t=1}^T \rho_t) + \frac{1}{2\beta_1} \sum_{i \in \mathcal{I}} (\lambda_i^1 + C \sum_{t=1}^T \rho_t)^2) \\
&\leq \mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + m\tilde{C}^2 \sum_{t=1}^T \rho_t, \tag{33}
\end{aligned}$$

where the second inequality comes from $\psi_{\beta_t}(u, v) \geq -\frac{v^2}{2\beta_t}$, and the third inequality holds due to (10). Plugging the above inequality into (32) and dividing the whole inequality by T yield the desired result. \square

Lemma 10 *Under the conditions of Lemma 9, suppose that $\{\alpha_t\}$ and $\{\eta_t\}$ are non-increasing sequences. Then it holds that for any $T \geq 1$,*

$$\begin{aligned}
&\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\mathbf{d}^2(\nabla f(x^{t+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{t+1} c_i(x^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1}))] \\
&\leq \frac{4m^2 \tilde{C}^2 G^2}{T} (1 + \frac{28}{\alpha_T}) \sum_{t=1}^T \rho_t^2 + \frac{14(1 + 2 \sum_{t=1}^T \alpha_t^2) \sigma^2}{\alpha_T T} + \frac{112m^2 C^2 G^2}{\alpha_T T} \sum_{t=1}^T (\beta_{t+1} - \beta_t)^2 \\
&\quad + \frac{1}{T} (\frac{112\tilde{L}}{\alpha_T} + \frac{20}{\eta_T}) (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + \frac{m\beta_{T+1}C^2}{2} + 2m\tilde{C}^2 \sum_{t=1}^T \rho_t). \tag{34}
\end{aligned}$$

Proof. By using the definition of ϕ_β and summing up (16) for $t = 1, \dots, T$, we can subsequently divide the resulting expression by T to obtain

$$\begin{aligned}
&\frac{1}{T} \sum_{t=1}^T \mathbf{d}^2(\nabla f(x^{t+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{t+1} c_i(x^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1})) \\
&\leq \frac{4m^2 \tilde{C}^2 G^2}{T} \sum_{t=1}^T \rho_t^2 + \frac{4}{T} \sum_{t=1}^T \|\varepsilon^t\|^2 + \frac{4}{T} \sum_{t=1}^T (L_{\beta_t}^2 + \frac{1}{\eta_t^2}) \|x^{t+1} - x^t\|^2. \tag{35}
\end{aligned}$$

On the one hand, since $\{\alpha_t\}$ is non-increasing and $|\mathcal{J}_t| \geq 1$, by Lemma 9 the second term in R.H.S. of (35) can be upper bounded by

$$\begin{aligned}
&\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\|\varepsilon^t\|^2] \leq \frac{(1 + 2 \sum_{t=1}^T \alpha_t^2) \sigma^2}{\alpha_T T} + \frac{8m^2 C^2 G^2}{\alpha_T T} \sum_{t=1}^T (\beta_{t+1} - \beta_t)^2 + \frac{8m^2 \tilde{C}^2 G^2}{\alpha_T T} \sum_{t=1}^T \rho_t^2 \\
&\quad + \frac{8\tilde{L}}{\alpha_T T} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + \frac{m\beta_{T+1}C^2}{2} + 2m\tilde{C}^2 \sum_{t=1}^T \rho_t). \tag{36}
\end{aligned}$$

On the other hand, it follows from (31) that

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T (L_{\beta_t}^2 + \frac{1}{\eta_t^2}) \|x^{t+1} - x^t\|^2 \\
& \leq \frac{1}{T} \sum_{t=1}^T \frac{2(1 + \eta_t^2 L_{\beta_t}^2)}{\eta_t(1 - \eta_t L_{\beta_t})} (\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \frac{\beta_{t+1} - \beta_t}{2} mC^2 + \frac{\eta_t}{2} \|\varepsilon^t\|^2 + m\rho_t \tilde{C}^2) \\
& \leq \frac{5}{\eta_T T} \sum_{t=1}^T (\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \frac{\beta_{t+1} - \beta_t}{2} mC^2 + m\rho_t \tilde{C}^2) + \frac{5}{2T} \sum_{t=1}^T \|\varepsilon^t\|^2 \\
& \leq \frac{5}{\eta_T T} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + \frac{m\beta_{T+1}C^2}{2} + 2m\tilde{C}^2 \sum_{t=1}^T \rho_t) + \frac{5}{2T} \sum_{t=1}^T \|\varepsilon^t\|^2, \tag{37}
\end{aligned}$$

where the second inequality follows from the fact that $\frac{1+u^2}{1-u} < \frac{5}{2}$ when $0 < u < \frac{1}{2}$ and $\eta_T \leq \eta_t$, the last comes from (33). Therefore, we obtain the conclusion. \square

4.1 Unbounded penalty parameters

In this subsection, we detect the global convergence of MLALM with unbounded penalty parameters. The theorem below shows global convergence properties of the sequences of average stationarity measure and average constraint violation in expectation, respectively.

Theorem 1 *Suppose that Assumptions 1-4 hold, and the parameters used in Algorithm 1 satisfy $\beta_t = \beta_0 t^\iota$ and*

$$\rho_t = \frac{\rho}{t^\theta}, \quad \eta_t = \frac{\eta}{t^\iota L_{\beta_t}}, \quad \alpha_t = \frac{4\alpha\eta^2}{t^\iota(t^\iota - \eta)}, \quad t \geq 1, \tag{38}$$

where $\rho, \beta_0, \iota > 0$, $\eta \in (0, \frac{\sqrt{17}-1}{8})$, $\theta \in (1, \infty)$, $\iota \in (0, \frac{1}{2})$ and $\alpha \in [1, \frac{1-\eta}{4\eta^2})$ are given constants. Then the followings are true:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\mathbf{d}^2(\nabla f(x^{t+1}) + \partial h(x^{t+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{t+1} \nabla c_i(x^{t+1}), -\mathcal{N}_X(x^{t+1}))] = 0, \tag{39}$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\mathbf{d}^2(\nabla c_{\mathcal{E}}(x^{t+1}) c_{\mathcal{E}}(x^{t+1}) + \nabla c_{\mathcal{I}}(x^{t+1}) [c_{\mathcal{I}}(x^{t+1})]_+, -\mathcal{N}_X(x^{t+1}))] = 0. \tag{40}$$

Furthermore, suppose that $|\mathcal{J}_t| = t^q$ with $q > 1$ for any $t \geq 1$, and there exists $C_u > 0$ such that $\mathbb{E}[f(x^t) + h(x^t)] \leq C_u$ for any $t \geq 1$. Then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\|c_{\mathcal{E}}(x^t)\|^2 + \|[c_{\mathcal{I}}(x^t)]_+\|^2] \text{ exists and is finite.} \tag{41}$$

Proof. We can verify that the settings defined in (38) satisfy the required conditions specified in (29). It is noteworthy that the values of $\alpha \in [1, \frac{1-\eta}{4\eta^2})$ are derived from the inequality $\frac{4\eta_t^2 L_{\beta_t}^2}{1 - \eta_t L_{\beta_t}} \leq \alpha_t \leq 1$. Additionally, the range for $\eta \in (0, \frac{\sqrt{17}-1}{8})$ is determined based on the condition $\frac{1-\eta}{4\eta^2} > 1$. From the upper bounds as shown in (34) and Lemma 7, the key to prove (39) and (40) is to derive

$$\frac{\sum_{t=1}^T \rho_t^2}{\alpha_T T} \rightarrow 0, \quad \frac{\sum_{t=1}^T \alpha_t^2}{\alpha_T T} \rightarrow 0, \quad \frac{\sum_{t=1}^T (\beta_{t+1} - \beta_t)^2}{\alpha_T T} \rightarrow 0,$$

$$\frac{1}{T} (\frac{1}{\alpha_T} + \frac{1}{\eta_T}) (\beta_{T+1} + \sum_{t=1}^T \rho_t) \rightarrow 0 \quad \text{and} \quad \frac{1}{T} \sum_{t=1}^T \frac{1}{\beta_t^2} \rightarrow 0,$$

as T increases to infinity. These can be achieved under parameter settings of the theorem and the fact that

$$\sum_{t=1}^T (\beta_{t+1} - \beta_t)^2 = \beta_0^2 \sum_{t=1}^T ((t+1)^\iota - t^\iota)^2 \leq \beta_0^2 \sum_{t=1}^T t^{2(\iota-1)}$$

which is upper bounded due to $\iota < \frac{1}{2}$.

We next prove (41). Recall that it is derived in (21) that

$$\mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t) - \mathcal{L}_{\beta_t}(x^t, \lambda^t) \leq \frac{\eta_t}{2} \|\varepsilon^t\|^2 + \left(\frac{L_{\beta_t}}{2} - \frac{1}{2\eta_t}\right) \|x^{t+1} - x^t\|^2$$

which yields

$$\|x^{t+1} - x^t\|^2 \leq \frac{2\eta_t}{1 - \eta_t L_{\beta_t}} (\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t) + \frac{\eta_t}{2} \|\varepsilon^t\|^2).$$

Plugging the above inequality into (25) implies

$$\begin{aligned} \mathbb{E}_{\xi^{[t+1]}} [\|\varepsilon^{t+1}\|^2] &\leq (1 - \alpha_t)^2 \left(1 + \frac{4\eta_t^2 L_{\beta_t}^2}{1 - \eta_t L_{\beta_t}}\right) \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] + (1 - \alpha_t)^2 \frac{8\eta_t L_{\beta_t}^2}{1 - \eta_t L_{\beta_t}} (\mathbb{E}_{\xi^{[t]}} [\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t)]) \\ &\quad + \frac{1}{|\mathcal{J}_{t+1}|} (2\alpha_t^2 \sigma^2 + 8(1 - \alpha_t)^2 (\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2 + 8(1 - \alpha_t)^2 (m\rho_t \tilde{C}G)^2) \\ &\leq \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] + \frac{8\eta_t L_{\beta_t}^2}{1 - \eta_t L_{\beta_t}} (\mathbb{E}_{\xi^{[t]}} [\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t)]) \\ &\quad + \frac{1}{|\mathcal{J}_{t+1}|} (2\alpha_t^2 \sigma^2 + 8(\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2 + 8(m\rho_t \tilde{C}G)^2). \end{aligned}$$

Moreover, dividing both sides of the above inequality by β_t , rearranging the terms and taking the expectation lead to

$$\begin{aligned} A_t &:= \mathbb{E}_{\xi^{[t+1]}} \left[\frac{1}{\beta_t} \mathcal{L}_{\beta_t}(x^{t+1}, \lambda^t) + \frac{1 - \eta_t L_{\beta_t}}{8\eta_t \beta_t L_{\beta_t}^2} \|\varepsilon^{t+1}\|^2 \right] \\ &\leq \mathbb{E}_{\xi^{[t]}} \left[\frac{1}{\beta_t} \mathcal{L}_{\beta_t}(x^t, \lambda^t) + \frac{1 - \eta_t L_{\beta_t}}{8\eta_t \beta_t L_{\beta_t}^2} (\|\varepsilon^t\|^2 + \frac{2\alpha_t^2 \sigma^2}{|\mathcal{J}_{t+1}|} + \frac{8(\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2}{|\mathcal{J}_{t+1}|} + \frac{8(m\rho_t \tilde{C}G)^2}{|\mathcal{J}_{t+1}|}) \right] =: B_t. \end{aligned} \tag{42}$$

Thus, $\{\frac{1}{T} \sum_{p=1}^T \sum_{t=1}^p (A_t - B_t)\}_{T \geq 1}$ is non-increasing sequence as T increases to infinity. Recalling the definition of \mathcal{L}_β , it is obvious that

$$\begin{aligned} &\frac{1}{T} \sum_{p=1}^T \sum_{t=1}^p (A_t - B_t) \\ &= \frac{1}{T} \sum_{p=1}^T \sum_{t=1}^p \frac{\mathbb{E}_{\xi^{[t+1]}} [f(x^{t+1}) + h(x^{t+1}) - f(x^t) - h(x^t)]}{\beta_t} \\ &\quad + \frac{1}{T} \sum_{p=1}^T \mathbb{E}_{\xi^{[p]}} \left[\sum_{t=1}^p \frac{\Psi_{\beta_t}(x^{t+1}, \lambda^t) - \Psi_{\beta_t}(x^t, \lambda^t)}{\beta_t} - \frac{1}{2} (\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2) \right] \\ &\quad + \frac{1}{2T} \sum_{p=1}^T \mathbb{E}_{\xi^{[p]}} [\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2] \end{aligned}$$

$$+ \frac{1}{T} \sum_{p=1}^T \sum_{t=1}^p \frac{1 - \eta_t L_{\beta_t}}{8\eta_t \beta_t L_{\beta_t}^2} (\mathbb{E}_{\xi^{[t+1]}} [\|\varepsilon^{t+1}\|^2 - \|\varepsilon^t\|^2] - \frac{2\alpha_t^2 \sigma^2}{|\mathcal{J}_{t+1}|} - \frac{8(\beta_{t+1} - \beta_t)^2 m^2 C^2 G^2}{|\mathcal{J}_{t+1}|} - \frac{8(m\rho_t \tilde{C}G)^2}{|\mathcal{J}_{t+1}|}). \quad (43)$$

Firstly, under assumptions of this theorem, we can infer that the following inequality holds for any $p \geq 1$:

$$|\sum_{t=1}^p \mathbb{E}[f(x^{t+1}) + h(x^{t+1}) - f(x^t) - h(x^t)]| = |\mathbb{E}[f(x^{p+1}) + h(x^{p+1})] - f(x^1) - h(x^1)| \leq C_u + C^*.$$

Then, due to the fact that the sequence $\{\beta_t\}_{t \geq 1}$ is monotonically increasing and $\lim_{t \rightarrow \infty} \frac{1}{\beta_t} = 0$, by utilizing Dirichlet's Test we deduce that

$$\lim_{p \rightarrow \infty} \sum_{t=1}^p \frac{\mathbb{E}[f(x^{t+1}) + h(x^{t+1}) - f(x^t) - h(x^t)]}{\beta_t} \text{ exists.}$$

Therefore, the arithmetic mean (the first item on R.H.S. of (43)) converges as $T \rightarrow \infty$. Secondly, by Lemma A.1 we are able to prove that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{p=1}^T \sum_{t=1}^p \frac{\mathbb{E}[\Psi_{\beta_t}(x^{t+1}, \lambda^t) - \Psi_{\beta_t}(x^t, \lambda^t)]}{\beta_t} - \frac{1}{2} \mathbb{E}[\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{T}}(x^{p+1})]_+\|^2] \text{ exists.}$$

Thirdly, for the last item in R.H.S. of (43), it implies from (27) that for any $t \geq 1$,

$$\mathbb{E}_{\xi^{[t+1]}} [\|\varepsilon^{t+1}\|^2] \leq \mathbb{E}_{\xi^{[t]}} [\|\varepsilon^t\|^2] + \frac{4\sigma^2}{|\mathcal{J}_{t+1}|} \leq \mathbb{E}_{\xi^1} [\|\varepsilon^1\|^2] + \sum_{k=1}^t \frac{4\sigma^2}{|\mathcal{J}_{k+1}|} = \sigma^2 + \sum_{k=1}^t \frac{4\sigma^2}{|\mathcal{J}_{k+1}|}. \quad (44)$$

Then it derives that

$$\begin{aligned} & |\sum_{t=1}^p (\mathbb{E}[\|\varepsilon^{t+1}\|^2 - \|\varepsilon^t\|^2] - \frac{2\alpha_t^2 \sigma^2}{|\mathcal{J}_{t+1}|} - \frac{8m^2 \tilde{C}^2 G^2 \rho_t^2}{|\mathcal{J}_{t+1}|} - \frac{8m^2 C^2 G^2 (\beta_{t+1} - \beta_t)^2}{|\mathcal{J}_{t+1}|})| \\ &= |\mathbb{E}[\|\varepsilon^{p+1}\|^2] - \|\varepsilon^1\|^2 - \sum_{t=1}^p (\frac{2\alpha_t^2 \sigma^2}{|\mathcal{J}_{t+1}|} + \frac{8m^2 \tilde{C}^2 G^2 \rho_t^2}{|\mathcal{J}_{t+1}|} + \frac{8m^2 C^2 G^2 (\beta_{t+1} - \beta_t)^2}{|\mathcal{J}_{t+1}|})| \end{aligned}$$

is bounded following the settings of $\alpha_t, \rho_t, |\mathcal{J}_{t+1}|$ and

$$(\beta_{t+1} - \beta_t)^2 = \beta_0^2 ((t+1)^\iota - t^\iota)^2 \leq \beta_0^2 t^{2(\iota-1)},$$

where the inequality holds by $(t+1)^\iota - t^\iota \leq t^{\iota-1}$ for any $t \geq 1$. At the meanwhile, we can verify that $\{\frac{1-\eta_t L_{\beta_t}}{8\eta_t \beta_t L_{\beta_t}^2}\}_{t \geq 1}$ is monotonically decreasing and converging to zero. Therefore, the last term of (43) is also convergent by applying Dirichlet's test. In summary, our analysis has demonstrated that the first, second, and fourth terms in R.H.S. of (43) converge to finite values as T increases to infinity. Consequently, the sequence $\{\frac{1}{T} \sum_{p=1}^T \sum_{t=1}^p (A_t - B_t)\}_{T \geq 1}$ is uniformly lower bounded by a finite value, and since it is non-increasing, it must converge as T tends to infinity. Therefore, the third term in R.H.S. of (43), which represents the average of constraint violations across all previous T iterates, also converges to a finite value as T approaches infinity. This completes the proof of (41). \square

Like in most penalty methods for nonconvex constrained optimization, the iterates generated by MLALM may get trapped around an infeasible stationary point, when no constraint qualification is assumed. Hence, to further analyze feasibility and the complementary slackness of iterates, it is necessary to impose a constraint qualification. Variants of constraint qualification have been widely used in the literature for nonconvex constrained optimization. Given that MLALM operates as a stochastic approximation method, it becomes necessary to analyze its theoretical performance in an average sense by considering the average of relevant stationarity measures over all previous iterates. However, it is important to note that the iterates produced by MLALM can be infeasible during the algorithmic process. Therefore, it is crucial to establish a broader region beyond the feasible region where a constraint qualification holds. Drawing motivation from [20] and [34, 35, 52], we introduce the following assumption regarding a constraint qualification.

Assumption 5 *There exist positive constants δ and Z such that for any $t \geq 1$, there exists $z^t \in -\mathcal{N}_X^*(x^t)$ with $\|z^t\| \leq Z$ solving*

$$\begin{aligned} \delta \cdot \text{sgn}(c_i(x^t)) + \nabla c_i(x^t)^T z^t &= 0, & i \in \mathcal{E} : c_i(x^t) \neq 0; \\ \delta + \nabla c_i(x^t)^T z^t &\leq 0, & i \in \mathcal{I} : c_i(x^t) > 0. \end{aligned} \quad (45)$$

Remark 2 *The constraint qualification assumed in Assumption 5 can be regarded as an extended variant of MFCQ which was originally proposed for smooth nonconvex constrained optimization [50]. In the case of infeasible methods, especially stochastic approximation methods for nonconvex constrained optimization, a constraint qualification (or nonsingularity condition) is often imposed on infeasible iterates in the literature. The necessity of a nonsingularity condition is evident in works such as [23, 25, 39], where it is utilized to analyze the complexity of penalty methods for nonconvex constrained optimization. Assumption 5 is weaker than the strong LICQ in [2, 13], where it assumes that $\nabla c(x)^T$ has full row rank and its singular values are uniformly lower bounded away from zero over a convex and compact set containing all iterates. When such assumption is imposed on all iterates $\{x^t\}_{t \geq 1}$ and $X = \mathbb{R}^n$, (45) is solvable with $\{z^t\}_{t \geq 1}$ being uniformly bounded. Compared with the constraint qualification assumed in [29, Assumption 4] for smooth deterministic constrained stochastic optimization, we replace $c_i(x^t)$ with $\delta \cdot \text{sgn}(c_i(x^t))$ for infeasible equality constraints and with δ for infeasible inequality constraints in the linear system (45). With this modification, we are able to prove the average of expected Lagrange multiplier vectors $\tilde{\lambda}^t, t \geq 1$, as defined in (23), is upper bounded in the lemma below.*

Lemma 11 *Under the conditions and parameter settings of Theorem 1 with $\iota \in (0, \frac{1}{3}]$, suppose that Assumption 5 holds, then $\{\mathbb{E}_{\xi^{[T]}}[\frac{1}{T} \sum_{t=1}^T \|\tilde{\lambda}^t\|]\}_{T \geq 1}$ is uniformly bounded.*

Proof. The detailed proof is presented in Appendix B. □

The following theorem provides a characterization of global convergence, in which the average of expected constraint violation and complementary slackness converge to zero, respectively.

Theorem 2 *Under the same conditions and parameter settings of Theorem 1 with $\iota \in (0, \frac{1}{3}]$, suppose that Assumption 5 holds, then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\|c_{\mathcal{E}}(x^{t+1})\|^{\frac{1}{2}} + \|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{1}{2}}] = 0, \quad (46)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\sum_{i \in \mathcal{I}} (\tilde{\lambda}_i^{t+1} |c_i(x^{t+1})|)^{\frac{1}{4}}] = 0. \quad (47)$$

Proof. As demonstrated in Lemma 11, there exists a constant $\tilde{\Lambda} > 0$ such that

$$\mathbb{E}_{\xi^{[T]}} [\frac{1}{T} \sum_{t=1}^T \|\tilde{\lambda}^{t+1}\|] \leq \tilde{\Lambda} \quad \forall T \geq 1.$$

Recall that as shown in Lemma 2, there exists $\Lambda > 0$ such that $\|\lambda^t\| \leq \Lambda, t \geq 1$. Then by the definition of $\tilde{\lambda}^t$, we obtain

$$\mathbb{E}_{\xi^{[T]}} [\frac{1}{T} \sum_{t=1}^T \beta_t (\|c_{\mathcal{E}}(x^{t+1})\| + \|[c_{\mathcal{I}}(x^{t+1})]_+\|)] \leq \tilde{\Lambda} + \Lambda \quad \forall T \geq 1. \quad (48)$$

Therefore, the following relation holds true:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [(\|c_{\mathcal{E}}(x^{t+1})\| + \|[c_{\mathcal{I}}(x^{t+1})]_+\|)^{\frac{1}{2}}]$$

$$\begin{aligned}
&= \mathbb{E}_{\xi^{[T]}} \left[\frac{1}{T} \sum_{t=1}^T \beta_t^{-\frac{1}{2}} \beta_t^{\frac{1}{2}} (\|c_{\mathcal{E}}(x^{t+1})\| + \|[c_{\mathcal{I}}(x^{t+1})]_+\|) \right] \\
&\leq \left(\frac{1}{T} \sum_{t=1}^T \beta_t^{-1} \right)^{\frac{1}{2}} \mathbb{E}_{\xi^{[T]}} \left[\left(\frac{1}{T} \sum_{t=1}^T \beta_t (\|c_{\mathcal{E}}(x^{t+1})\| + \|[c_{\mathcal{I}}(x^{t+1})]_+\|) \right)^{\frac{1}{2}} \right] \leq \left(\frac{1}{T} \sum_{t=1}^T \beta_t^{-1} \right)^{\frac{1}{2}} \sqrt{\bar{\Lambda} + \Lambda}
\end{aligned} \tag{49}$$

which indicates (46) from $a^{\frac{1}{2}} + b^{\frac{1}{2}} \leq \sqrt{2}(a+b)^{\frac{1}{2}}$ and the setting of β_t . The remainder of the proof is to show (47). Note that if $c_i(x^{t+1}) \leq -\frac{\lambda_i^{t+1}}{\beta_t}$, $\tilde{\lambda}_i^{t+1} = 0$. Then by using notations

$$\mathcal{I}_1 := \{i \in \mathcal{I} : c_i(x^{t+1}) > 0\}, \quad \mathcal{I}_2 := \{i \in \mathcal{I} : -\frac{\lambda_i^t}{\beta_t} \leq c_i(x^{t+1}) \leq 0\}, \tag{50}$$

we derive

$$\begin{aligned}
\sum_{i \in \mathcal{I}} (\tilde{\lambda}_i^{t+1} |c_i(x^{t+1})|)^{\frac{1}{4}} &= \sum_{i \in \mathcal{I}_1} (\beta_t (c_i(x^{t+1}))^2 + \lambda_i^{t+1} c_i(x^{t+1}))^{\frac{1}{4}} + \sum_{i \in \mathcal{I}_2} (-\lambda_i^{t+1} c_i(x^{t+1}) - \beta_t (c_i(x^{t+1}))^2)^{\frac{1}{4}} \\
&\leq \sum_{i \in \mathcal{I}_1} (\beta_t^{\frac{1}{4}} c_i^{\frac{1}{2}}(x^{t+1}) + (\lambda_i^{t+1} c_i(x^{t+1}))^{\frac{1}{4}}) + \sum_{i \in \mathcal{I}_2} (-\lambda_i^{t+1} c_i(x^{t+1}))^{\frac{1}{4}} \\
&\leq \sum_{i \in \mathcal{I}_1} \beta_t^{\frac{1}{4}} c_i^{\frac{1}{2}}(x^{t+1}) + \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} (\lambda_i^{t+1} |c_i(x^{t+1})|)^{\frac{1}{4}} \\
&\leq \beta_t^{\frac{1}{4}} \sum_{i \in \mathcal{I}_1} c_i^{\frac{1}{2}}(x^{t+1}) + \Lambda^{\frac{1}{4}} \sum_{i \in \mathcal{I}_1} c_i^{\frac{1}{4}}(x^{t+1}) + \Lambda^{\frac{1}{2}} |\mathcal{I}_2| \beta_t^{-\frac{1}{4}} \\
&\leq |\mathcal{I}_1|^{\frac{3}{4}} \beta_t^{\frac{1}{4}} \|c_{\mathcal{I}_1}(x^{t+1})\|^{\frac{1}{2}} + |\mathcal{I}_1|^{\frac{7}{8}} \Lambda^{\frac{1}{4}} \|c_{\mathcal{I}}(x^{t+1})\|^{\frac{1}{4}} + \Lambda^{\frac{1}{2}} |\mathcal{I}_2| \beta_t^{-\frac{1}{4}} \\
&\leq |\mathcal{I}|^{\frac{3}{4}} \beta_t^{\frac{1}{4}} \|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{1}{2}} + |\mathcal{I}|^{\frac{7}{8}} \Lambda^{\frac{1}{4}} \|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{1}{4}} + \Lambda^{\frac{1}{2}} |\mathcal{I}| \beta_t^{-\frac{1}{4}},
\end{aligned} \tag{51}$$

where the last inequality holds by Jensen's inequality. Then taking expectation with respect to $\xi^{[t]}$ on both sides of (51) and sum-averaging over the first T iterations yield

$$\begin{aligned}
&\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} \left[\sum_{i \in \mathcal{I}} (\tilde{\lambda}_i^{t+1} |c_i(x^{t+1})|)^{\frac{1}{4}} \right] \\
&\leq \frac{1}{T} \sum_{t=1}^T (\mathbb{E}_{\xi^{[t]}} [|\mathcal{I}|^{\frac{3}{4}} \beta_t^{\frac{1}{4}} \|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{1}{2}} + |\mathcal{I}|^{\frac{7}{8}} \Lambda^{\frac{1}{4}} \|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{1}{4}} + \Lambda^{\frac{1}{2}} |\mathcal{I}| \beta_t^{-\frac{1}{4}}]) \\
&\leq |\mathcal{I}|^{\frac{3}{4}} \left(\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\beta_t \|[c_{\mathcal{I}}(x^{t+1})]_+\|] \right)^{\frac{1}{2}} \left(\frac{1}{T} \sum_{t=1}^T \beta_t^{-\frac{1}{2}} \right)^{\frac{1}{2}} + |\mathcal{I}|^{\frac{7}{8}} \Lambda^{\frac{1}{4}} \left(\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{1}{2}}] \right)^{\frac{1}{2}} \\
&\quad + \frac{|\mathcal{I}| \Lambda^{\frac{1}{2}}}{T} \sum_{t=1}^T \beta_t^{-\frac{1}{4}},
\end{aligned}$$

where the second equality comes from Cauchy-Schwarz inequality and $(\mathbb{E}[u])^2 \leq \mathbb{E}[u^2]$ for a random variable $u > 0$. Then the desired result is concluded by (48) and (49). \square

4.2 Bounded penalty parameters

In this subsection, we analyze convergence properties of MLALM when the penalty parameters are bounded. Without loss of generality, we assume that $\beta_t \equiv \beta$ for all $t \geq 1$.

Theorem 3 *Suppose that Assumptions 1-4 hold, and the parameters used in Algorithm 1 satisfy (38) and $\beta_t \equiv \beta$, $t \geq 1$. Then (39) holds and there exists a positive constant δ_1 (independent of β) such that*

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\mathbf{d}^2(\nabla c_{\mathcal{E}}(x^{t+1})c_{\mathcal{E}}(x^{t+1}) + \nabla c_{\mathcal{I}}(x^{t+1})[c_{\mathcal{I}}(x^{t+1})]_+, -\mathcal{N}_X(x^{t+1}))] \leq \delta_1 \beta^{-2}. \quad (52)$$

Furthermore, assume that $|\mathcal{J}_t| = t^q$, $t \geq 1$, with $q > 1$, and there exists $C_u > 0$ such that $\mathbb{E}[f(x^t) + h(x^t)] \leq C_u$ for any $t \geq 1$. Then there exists a positive constant δ_2 (independent of β) such that for all sufficiently large T ,

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\|c_{\mathcal{E}}(x^t)\|^2 + \|[c_{\mathcal{I}}(x^t)]_+\|^2] \leq C_{vio}^1 + \delta_2 \beta^{-1}, \quad (53)$$

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} \left[\sum_{i \in \mathcal{I}} (\tilde{\lambda}_i^{t+1})^{\frac{1}{2}} |c_i(x^{t+1})| \right] \leq |\mathcal{I}|^{\frac{1}{4}} (\beta^{\frac{1}{2}} (C_{vio}^1 + \delta_2 \beta^{-1})^{3/4} + \Lambda^{\frac{1}{2}} (C_{vio}^1 + \delta_2 \beta^{-1})^{1/2} + \beta^{-1} \Lambda^{3/2}), \quad (54)$$

where $C_{vio}^1 = \|c_{\mathcal{E}}(x^1)\|^2 + \|[c_{\mathcal{I}}(x^1)]_+\|^2$.

Proof. In analogy to Theorem 1, by analyzing the upper bound in (34) and following the parameter setting, we can obtain (39). Lemma 7 can be applied to derive (52).

To prove (53), it suffices to analyze the bound of (43) when $\beta_t \equiv \beta$ for all $t \geq 1$. Following the analysis to (43) in previous theorem and by Lemma A.2, we can obtain

$$\begin{aligned} 0 &\geq \frac{1}{T} \sum_{p=1}^T \sum_{t=1}^p (A_t - B_t) = \frac{1}{2T} \sum_{p=1}^T \mathbb{E}_{\xi^{[p]}} [\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2] \\ &\quad - \frac{1}{2} (\|c_{\mathcal{E}}(x^1)\|^2 + \|[c_{\mathcal{I}}(x^1)]_+\|^2) + \mathcal{O}(\beta^{-1}). \end{aligned}$$

Due to the non-increasing property of $A_t - B_t$ we derive (53).

To prove (54), it follows from the definitions of \mathcal{I}_1 and \mathcal{I}_2 in (50) that

$$\begin{aligned} \sum_{i \in \mathcal{I}} (\tilde{\lambda}_i^{t+1})^{\frac{1}{2}} |c_i(x^{t+1})| &= \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} ([\lambda_i^{t+1} + \beta c_i(x^{t+1})]_+)^{\frac{1}{2}} |c_i(x^{t+1})| \\ &\leq \beta^{\frac{1}{2}} \sum_{i \in \mathcal{I}_1} (c_i(x^{t+1}))^{\frac{3}{2}} + \sum_{i \in \mathcal{I}_1} (\lambda_i^{t+1})^{\frac{1}{2}} c_i(x^{t+1}) + \sum_{i \in \mathcal{I}_2} (\lambda_i^{t+1})^{\frac{1}{2}} |c_i(x^{t+1})| \\ &\leq |\mathcal{I}_1|^{\frac{1}{4}} (\beta^{\frac{1}{2}} \|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{3}{2}} + \Lambda^{\frac{1}{2}} \|[c_{\mathcal{I}}(x^{t+1})]_+\|) + |\mathcal{I}_2|^{\frac{1}{4}} \beta^{-1} \Lambda^{\frac{3}{2}} \\ &\leq |\mathcal{I}|^{\frac{1}{4}} (\beta^{\frac{1}{2}} \|[c_{\mathcal{I}}(x^{t+1})]_+\|^{\frac{3}{2}} + \Lambda^{\frac{1}{2}} \|[c_{\mathcal{I}}(x^{t+1})]_+\| + \beta^{-1} \Lambda^{\frac{3}{2}}). \end{aligned}$$

Therefore, taking the expectation of the above formula and the average over $t = 1, \dots, T$ yields

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} \left[\sum_{i \in \mathcal{I}} (\tilde{\lambda}_i^{t+1})^{\frac{1}{2}} |c_i(x^{t+1})| \right] \\
& \leq |\mathcal{I}|^{\frac{1}{4}} \left(\frac{\beta^{\frac{1}{2}}}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\| [c_{\mathcal{I}}(x^{t+1})]_+ \|^{\frac{3}{2}}] + \frac{\Lambda^{\frac{1}{2}}}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\| [c_{\mathcal{I}}(x^{t+1})]_+ \|] + \beta^{-1} \Lambda^{\frac{3}{2}} \right) \\
& \leq |\mathcal{I}|^{\frac{1}{4}} \left(\frac{\beta^{\frac{1}{2}}}{T} \sum_{t=1}^T (\mathbb{E}_{\xi^{[t]}} [\| [c_{\mathcal{I}}(x^{t+1})]_+ \|^2])^{\frac{3}{4}} + \frac{\Lambda^{\frac{1}{2}}}{T} \sum_{t=1}^T (\mathbb{E}_{\xi^{[t]}} [\| [c_{\mathcal{I}}(x^{t+1})]_+ \|^2])^{\frac{1}{2}} + \beta^{-1} \Lambda^{\frac{3}{2}} \right) \tag{55} \\
& \leq |\mathcal{I}|^{\frac{1}{4}} \left(\beta^{\frac{1}{2}} \left(\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\| [c_{\mathcal{I}}(x^{t+1})]_+ \|^2] \right)^{\frac{3}{4}} + \Lambda^{\frac{1}{2}} \left(\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}} [\| [c_{\mathcal{I}}(x^{t+1})]_+ \|^2] \right)^{\frac{1}{2}} + \beta^{-1} \Lambda^{\frac{3}{2}} \right) \\
& \leq |\mathcal{I}|^{\frac{1}{4}} \left(\beta^{\frac{1}{2}} (C_{vio}^1 + \delta_2 \beta^{-1})^{\frac{3}{4}} + \Lambda^{\frac{1}{2}} (C_{vio}^1 + \delta_2 \beta^{-1})^{\frac{1}{2}} + \beta^{-1} \Lambda^{\frac{3}{2}} \right),
\end{aligned}$$

where the second and third inequalities come from Jensen's inequality. \square

Remark 3 *It is noteworthy that Theorem 3 does not assume Assumption 5. However, if we do impose this assumption, we can actually obtain stronger results, which can be seen in Subsection 5.2.*

5 Oracle complexity analysis

In order to investigate oracle complexities of MLALM, we limit the maximum number of iterations to a fixed integer, denoted by T with a little abuse of notation, and set the batch size

$$|\mathcal{J}_t| = 1, \quad t = 1, \dots, T.$$

We randomly select an iterate x^{R+1} as the output, where R follows a uniform distribution over $\{1, \dots, T\}$. To characterize the output of the algorithm, we define two types of approximate solutions for problem (1), and we will analyze oracle complexities of MLALM to find those solutions accordingly.

Definition 1 *Given $\epsilon > 0$, we call $x \in X$ an ϵ -stationary point of (1), if there exists $\lambda \in \mathbb{R}^{|\mathcal{E} \cup \mathcal{I}|}$ with $\lambda_i \geq 0$, $i \in \mathcal{I}$, such that*

$$\mathbb{E}[\mathbf{d}^2(\nabla f(x) + \partial h(x) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(x), -\mathcal{N}_X(x))] \leq \epsilon^2, \tag{56}$$

$$\mathbb{E}[\mathbf{d}^2(\nabla c_{\mathcal{E}}(x) c_{\mathcal{E}}(x) + \nabla c_{\mathcal{I}}(x) [c_{\mathcal{I}}(x)]_+, -\mathcal{N}_X(x))] \leq \epsilon^2. \tag{57}$$

Definition 2 *Given $\epsilon > 0$, we call $x \in X$ an ϵ -KKT point of (1), if there exists $\lambda \in \mathbb{R}^{|\mathcal{E} \cup \mathcal{I}|}$ with $\lambda_i \geq 0$, $i \in \mathcal{I}$, such that (56) holds and*

$$\mathbb{E}[\|c_{\mathcal{E}}(x)\|^2 + \|c_{\mathcal{I}}(x)_+\|^2] \leq \epsilon^2, \quad \mathbb{E}[\sum_{i \in \mathcal{I}} \lambda_i |c_i(x)|] \leq \epsilon.$$

5.1 Towards an ϵ -stationary point

In this subsection, we will analyze the complexity of MLALM for finding an ϵ -stationary point of (1). Towards this end, we need to first estimate the stationarity measure, i.e., L.H.S. of (56), at x^{R+1} .

Lemma 12 Under Assumptions 1-4 and (29), set $\rho_t \equiv \frac{\rho}{T}$ and positive parameters $\beta_t \equiv \beta_1$, $\eta_t \equiv \eta_1$, $\alpha_t \equiv \alpha_1$, $t \geq 1$, then it holds that with $\tilde{\lambda}$ defined through (23),

$$\begin{aligned} & \mathbb{E}_{R;\xi^{[T]}}[\mathbf{d}^2(\nabla f(x^{R+1}) + \partial h(x^{R+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{R+1} \nabla c_i(x^{R+1}), -\mathcal{N}_X(x^{R+1}))] \\ & \leq \frac{4m^2 \rho^2 \tilde{C}^2 G^2}{T^2} (1 + \frac{28}{\alpha_1}) + \frac{1}{T} (\frac{112\tilde{L}}{\alpha_1} + \frac{20}{\eta_1}) (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + 2m\rho\tilde{C}^2) + \frac{14(1 + 2T\alpha_1^2)\sigma^2}{\alpha_1 T}. \end{aligned} \quad (58)$$

Proof. Similar to the proof of Lemma 10, we first give an upper bound on the stationarity measure as shown in (56). By taking expectation on both sides of (16) and average over $t = 1, \dots, T$, we obtain

$$\begin{aligned} & \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\mathbf{d}^2(\nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1}))] \\ & \leq \frac{4(m\rho\tilde{C}G)^2}{T^2} + \frac{4}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\|\varepsilon^t\|^2] + \frac{4}{T} \sum_{t=1}^T (L_{\beta_t}^2 + \frac{1}{\eta_t^2}) \mathbb{E}_{\xi^{[t]}}[\|x^{t+1} - x^t\|^2]. \end{aligned} \quad (59)$$

From Lemma 9 it follows that

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\|\varepsilon^t\|^2] \leq \frac{(1 + 2T\alpha_1^2)\sigma^2}{\alpha_1 T} + \frac{8m^2 \rho^2 \tilde{C}^2 G^2}{\alpha_1 T^2} + \frac{8\tilde{L}}{\alpha_1 T} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + 2m\rho\tilde{C}^2). \quad (60)$$

For the last term in R.H.S. of (59), it is easy to attain from (31) that

$$\begin{aligned} & \frac{1}{T} \sum_{t=1}^T (L_{\beta_t}^2 + \frac{1}{\eta_t^2}) \mathbb{E}_{\xi^{[t]}}[\|x^{t+1} - x^t\|^2] \\ & \leq \frac{1}{T} \sum_{t=1}^T \frac{2(1 + \eta_t^2 L_{\beta_t}^2)}{\eta_t(1 - \eta_t L_{\beta_t})} \mathbb{E}_{\xi^{[t]}}[(\mathcal{L}_{\beta_t}(x^t, \lambda^t) - \mathcal{L}_{\beta_{t+1}}(x^{t+1}, \lambda^{t+1}) + \frac{\eta_t}{2} \|\varepsilon^t\|^2 + m\rho_t \tilde{C}^2)] \\ & \leq \frac{5}{\eta_1 T} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + 2m\rho\tilde{C}^2) + \frac{5}{2T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\|\varepsilon^t\|^2], \end{aligned} \quad (61)$$

where the last inequality is due to $\frac{1+u^2}{1-u} < \frac{5}{2}$ for $0 < u < \frac{1}{2}$ and $\eta_t = \eta_1$ for all $t \in [T]$. Then, plugging (61) and (60) into (59) together with the definition of $\tilde{\lambda}$ in (23), we obtain

$$\begin{aligned} & \mathbb{E}_{R;\xi^{[T]}}[\mathbf{d}^2(\nabla f(x^{R+1}) + \partial h(x^{R+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{R+1} \nabla c_i(x^{R+1}), -\mathcal{N}_X(x^{R+1}))] \\ & = \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\mathbf{d}^2(\nabla_x \phi_{\beta_t}(x^{t+1}, \lambda^{t+1}) + \partial h(x^{t+1}), -\mathcal{N}_X(x^{t+1}))] \\ & \leq \frac{4m^2 \rho^2 \tilde{C}^2 G^2}{T^2} + \frac{20}{\eta_1 T} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + 2m\rho\tilde{C}^2) + \frac{14}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[t]}}[\|\varepsilon^t\|^2] \\ & \leq \frac{4m^2 \rho^2 \tilde{C}^2 G^2}{T^2} (1 + \frac{28}{\alpha_1}) + \frac{1}{T} (\frac{112\tilde{L}}{\alpha_1} + \frac{20}{\eta_1}) (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + 2m\rho\tilde{C}^2) + \frac{14(1 + 2T\alpha_1^2)\sigma^2}{\alpha_1 T} \end{aligned}$$

which yields the conclusion. \square

As in general it may be intractable to find a feasible solution for a nonconvex constrained optimization problem. Hence, we need to characterize the infeasible stationarity measure.

Lemma 13 *Under the conditions of Lemma 12, it holds that*

$$\begin{aligned} & \mathbb{E}_{R;\xi^{[T]}}[\mathbf{d}^2(\nabla c_{\mathcal{E}}(x^{R+1})c_{\mathcal{E}}(x^{R+1}) + \nabla c_{\mathcal{I}}(x^{R+1})[c_{\mathcal{I}}(x^{R+1})]_+, -\mathcal{N}_X(x^{R+1})))] \\ & \leq \frac{4}{\beta_1^2}((2 + m^2 \rho^2 C^2)G^2 + \frac{4m^2 \rho^2 \tilde{C}^2 G^2}{T^2}(1 + \frac{28}{\alpha_1}) \\ & \quad + \frac{1}{T}(\frac{112\tilde{L}}{\alpha_1} + \frac{20}{\eta_1})(\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + C\|\lambda^1\|_1 + 2m\rho\tilde{C}^2) + \frac{14(1 + 2\sum_{t=1}^T \alpha_1^2)\sigma^2}{\alpha_1 T}). \end{aligned}$$

Proof. By applying (24) and taking expectation with respect to R and $\xi^{[T]}$ on both sides, we obtain the conclusion from the setting of β_t . \square

Selecting appropriate parameters is crucial as the ultimate oracle complexities depend on these choices. To ensure (29), we set the parameters as follows:

$$\beta_t = \beta_0 T^\tau, \quad \eta_t = \frac{\eta}{L_{\beta_t} T^\iota}, \quad \alpha_t = \frac{4\alpha\eta^2}{T^\iota(T^\iota - \eta)}, \quad (62)$$

where $\beta_0, \tau, \iota > 0$, $\eta \in (0, \frac{\sqrt{17}-1}{8})$, $\alpha \in [1, \frac{1-\eta}{4\eta^2})$ are given constants that are independent of T . Then the upper bounds shown in Lemmas 12 is of order

$$\mathcal{O}(T^{\tau+2\iota-1}, T^{2\tau+\iota-1}, T^{-2\iota}). \quad (63)$$

Obviously, when $\tau = \iota = \frac{1}{5}$, above order can reach the lowest order. We summarize above analysis into the following theorem.

Theorem 4 *Under Assumptions 1-4, suppose that*

$$\rho_t \equiv \frac{\rho}{T}, \quad \beta_t = T^{1/5}, \quad \eta_t \equiv \frac{\eta}{L_{\beta_t} T^{1/5}}, \quad \alpha_t \equiv \frac{4\alpha\eta^2}{T^{1/5}(T^{1/5} - \eta)}, \quad \forall t \in [T]$$

with $\rho \in (0, T^{6/5}]$, $\eta \in (0, \frac{\sqrt{17}-1}{8})$, $\alpha \in [1, \frac{1-\eta}{4\eta^2})$ are constants independent of T . Then it holds that with $\tilde{\lambda}$ defined through (23),

$$\mathbb{E}_{R;\xi^{[T]}}[\mathbf{d}^2(\nabla f(x^{R+1}) + \partial h(x^{R+1}) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \tilde{\lambda}_i^{R+1} \nabla c_i(x^{R+1}), -\mathcal{N}_X(x^{R+1})))] = \mathcal{O}(T^{-2/5}), \quad (64)$$

$$\mathbb{E}_{R;\xi^{[T]}}[\mathbf{d}^2(\nabla c_{\mathcal{E}}(x^{R+1})c_{\mathcal{E}}(x^{R+1}) + \nabla c_{\mathcal{I}}(x^{R+1})[c_{\mathcal{I}}(x^{R+1})]_+, -\mathcal{N}_X(x^{R+1})))] = \mathcal{O}(T^{-2/5}). \quad (65)$$

Consequently, the oracle complexity of MLALM to reach an ϵ -stationary point of (1) is of order $\mathcal{O}(\epsilon^{-5})$.

Proof. It is straightforward to obtain (64) by Lemmas 12 together with the analysis to (63). Under the parameter setting (62), the upper bound in Lemma 13 is of order $\mathcal{O}(T^{2\iota-\tau-1}, T^{\iota-1}, T^{-2\tau})$. Then (65) can be derived from $\tau = \iota = \frac{1}{5}$. Hence, to achieve an ϵ -stationary point of (1), the maximum number of iterations T should be of order $\mathcal{O}(\epsilon^{-5})$. Since the computation of the stochastic gradient only requires one sampling per iteration, as shown in (7), the total number of calls to stochastic oracles is in the order of $\mathcal{O}(T)$, which is of order $\mathcal{O}(\epsilon^{-5})$. \square

It is important to note that Lemmas 12 and 13 demonstrate that the term $\mathcal{L}_{\beta_1}(x^1, \lambda^1)$ explicitly appears in the upper bounds. Then the term $\frac{\beta_1}{2}(\|c_{\mathcal{E}}(x^1)\|^2 + \|[c_{\mathcal{I}}(x^1)]_+\|^2)$ directly affects these upper bounds. In particular, we need to consider the impact of potentially large values of β_1 on the complexity order derived in (63). However, when the initial point is sufficiently close to the feasible region, such that the aforementioned term is of order $\mathcal{O}(1)$, the influence of large β_1 on the complexity order can be reduced. In this case, under the parameter setting (62) the upper bound in Lemma 12 is in order

$$\mathcal{O}(T^{2\iota-1}, T^{\tau+\iota-1}, T^{-2\iota}).$$

To achieve the lowest order $\mathcal{O}(T^{-\frac{1}{2}})$ we can choose $\tau = \iota = \frac{1}{4}$. We can also obtain that the upper bound in Lemma 13 is in order $\mathcal{O}(T^{-\frac{1}{2}})$. We thus derive the following corollary, with the proof omitted.

Corollary 1 Under Assumptions 1-4, suppose that

$$\rho_t \equiv \frac{\rho}{T}, \quad \beta_t = T^{1/4}, \quad \eta_t \equiv \frac{\eta}{L_{\beta_t} T^{1/4}}, \quad \alpha_t \equiv \frac{4\alpha\eta^2}{T^{1/4}(T^{1/4} - \eta)}, \quad \forall t \in [T]$$

with $\rho \in (0, T^{5/4}]$, $\eta \in (0, \frac{\sqrt{17}-1}{8})$, $\alpha \in [1, \frac{1-\eta}{4\eta^2})$ being constants independent of T . If $\|c_{\mathcal{E}}(x^1)\|^2 + \|[c_{\mathcal{I}}(x^1)]_+\|^2 = \mathcal{O}(\epsilon)$, the oracle complexity to reach an ϵ -stationary point of (1) is of order $\mathcal{O}(\epsilon^{-4})$.

5.2 Towards an ϵ -KKT point

In this subsection, we will analyze the oracle complexity of MLALM towards an ϵ -KKT point of (1). With the help of Assumption 5 we can obtain the following lemma characterizing the near-feasibility of x^{R+1} and the complementary slackness in expectation.

Theorem 5 Suppose that Assumption 5 and the conditions of Theorem 4 hold, then

$$\mathbb{E}_{R;\xi^{[T]}}[\|c_{\mathcal{E}}(x^{R+1})\|^2 + \|[c_{\mathcal{I}}(x^{R+1})]_+\|^2] = \mathcal{O}(T^{-2/5}), \quad (66)$$

$$\mathbb{E}_{R;\xi^{[T]}}[\sum_{i \in \mathcal{I}} \tilde{\lambda}_i^{R+1} |c_i(x^{R+1})|] = \mathcal{O}(T^{-1/5}). \quad (67)$$

Consequently, the oracle complexity of MLALM to reach an ϵ -KKT point of (1) is of order $\mathcal{O}(\epsilon^{-5})$.

Proof. Under Assumption 5, it is easy to obtain

$$\begin{aligned} c_i(x^t)^T \nabla c_i(x^t)^T z^t &= -\delta |c_i(x^t)|, \quad i \in \mathcal{E} : c_i(x^t) \neq 0, \\ [c_i(x^t)]_+^T \nabla c_i(x^t)^T z^t &\leq -\delta [c_i(x^t)]_+, \quad i \in \mathcal{I} : c_i(x^t) > 0, \end{aligned}$$

which further yields

$$c_{\mathcal{E}}(x^t)^T \nabla c_{\mathcal{E}}(x^t)^T z + [c_{\mathcal{I}}(x^t)]_+^T \nabla c_{\mathcal{I}}(x^t)^T z \leq -\delta (\|c_{\mathcal{E}}(x^t)\|_1 + \|[c_{\mathcal{I}}(x^t)]_+\|_1).$$

Moreover, since $v^T z \leq 0$ for all $v \in \mathcal{N}_X(x^t)$, we have

$$c_{\mathcal{E}}(x^t)^T \nabla c_{\mathcal{E}}(x^t)^T z + [c_{\mathcal{I}}(x^t)]_+^T \nabla c_{\mathcal{I}}(x^t)^T z^t + v^T z \leq -\delta (\|c_{\mathcal{E}}(x^t)\|_1 + \|[c_{\mathcal{I}}(x^t)]_+\|_1).$$

Hence, it holds that for any $v \in \mathcal{N}_X(x^t)$,

$$\begin{aligned} \delta (\|c_{\mathcal{E}}(x^t)\|_1 + \|[c_{\mathcal{I}}(x^t)]_+\|_1) &\leq |c_{\mathcal{E}}(x^t)^T \nabla c_{\mathcal{E}}(x^t)^T z^t + [c_{\mathcal{I}}(x^t)]_+^T \nabla c_{\mathcal{I}}(x^t)^T z^t + v^T z^t| \\ &\leq \|\nabla c_{\mathcal{E}}(x^t) c_{\mathcal{E}}(x^t) + \nabla c_{\mathcal{I}}(x^t) [c_{\mathcal{I}}(x^t)]_+ + v\| \|z^t\|. \end{aligned}$$

Due to $\|z^t\| \leq Z$ and the arbitrariness of $v \in \mathcal{N}_X(x^t)$, we obtain

$$\frac{\delta}{Z} (\|c_{\mathcal{E}}(x^t)\|_1 + \|[c_{\mathcal{I}}(x^t)]_+\|_1) \leq \mathbf{d}(\nabla c_{\mathcal{E}}(x^t) c_{\mathcal{E}}(x^t) + \nabla c_{\mathcal{I}}(x^t) [c_{\mathcal{I}}(x^t)]_+, -\mathcal{N}_X(x^t)).$$

Therefore, (66) can be derived from (64).

In light of $|c_i(x^{t+1})| = [c_i(x^{t+1})]_+ + [c_i(x^{t+1})]_-$ for $i \in \mathcal{I}$ and using the notations

$$\mathcal{I}_1 := \{i \mid c_i(x^{t+1}) \geq 0\}, \quad \mathcal{I}_2 := \{i \mid -\frac{\lambda_i^{t+1}}{\beta_t} \leq c_i(x^{t+1}) < 0\},$$

we can derive the following relations

$$\sum_{i \in \mathcal{I}} \tilde{\lambda}_i^{t+1} |c_i(x^{t+1})| = \sum_{i \in \mathcal{I}_1} [\beta_t c_i^2(x^{t+1}) + \lambda_i^{t+1} c_i(x^{t+1})] + \sum_{i \in \mathcal{I}_2} [-c_i(x^{t+1}) (\beta_t c_i(x^{t+1}) + \lambda_i^{t+1})]$$

$$\begin{aligned}
&\leq \sum_{i \in \mathcal{I}_1} [\beta_t c_i^2(x^{t+1}) + \lambda_i^{t+1} c_i(x^{t+1})] + \sum_{i \in \mathcal{I}_2} \frac{|\lambda_i^{t+1}|^2}{\beta_t} \\
&\leq \beta_t \sum_{i \in \mathcal{I}_1} c_i^2(x^{t+1}) + \rho C \sum_{i \in \mathcal{I}_1} c_i(x^{t+1}) + \frac{m\rho^2 C^2}{\beta_t},
\end{aligned} \tag{68}$$

where the first inequality comes from $-u(bu + a) \leq \frac{a^2}{b}, \forall u \in \mathbb{R}$ with $b > 0$, and the second inequality is due to $|\lambda_i^{t+1}| \leq \lambda_i^1 + \rho C, \forall i \in \mathcal{I}$ by (10) and $\rho_t = \frac{\rho}{T}$. Then taking expectation with respect to R and $\xi^{[T]}$ on both sides of (68) yields

$$\begin{aligned}
\mathbb{E}_{R; \xi^{[T]}} \left[\sum_{i \in \mathcal{I}} \tilde{\lambda}_i^{R+1} |c_i(x^{R+1})| \right] &= \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[T]}} \left[\sum_{i \in \mathcal{I}} \tilde{\lambda}_i^{t+1} |c_i(x^{t+1})| \right] \\
&\leq \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi^{[T]}} \left[\beta_t \sum_{i \in \mathcal{I}_1} c_i^2(x^{t+1}) + \rho C \sum_{i \in \mathcal{I}_1} c_i(x^{t+1}) \right] + \frac{m\rho^2 C^2}{\beta_t} \\
&= \beta_1 \mathbb{E}_{R; \xi^{[T]}} [\| [c_{\mathcal{I}}(x^{R+1})]_+ \|^2] + \rho C |\mathcal{I}| \mathbb{E}_{R; \xi^{[T]}} [\| [c_{\mathcal{I}}(x^{R+1})]_+ \|] + \frac{m\rho^2 C^2}{\beta_1} \\
&= \mathcal{O}(T^{-1/5}),
\end{aligned}$$

where the third equality comes from (66) and $(\mathbb{E}[u])^2 \leq \mathbb{E}[u^2]$ for a random variable $u \in \mathbb{R}$.

Consequently, the oracle complexity of MLALM to reach an ϵ -KKT point of (1) is $\mathcal{O}(T)$ which is in order $\mathcal{O}(\epsilon^{-5})$. \square

In analogy to Corollary 1, when the initial point is sufficiently close to the feasible region of (1), it is straightforward to obtain the following results.

Corollary 2 *Suppose that Assumption 5 and the conditions of Corollary 1 hold, then the oracle complexity of MLALM to reach an ϵ -KKT point of (1) is in order $\mathcal{O}(\epsilon^{-4})$.*

6 Numerical Simulations

In this section, we conduct a series of numerical experiments to evaluate the practical performance of MLALM, as outlined in Algorithm 1. All experiments were performed using Matlab 2021b on a 64-bit Linux machine equipped with a 4.90 GHz Intel Core i7-12700K CPU and 32GB of memory.

6.1 Quadratically Constrained Nonconvex Program

In this subsection, we test the proposed method on solving quadratically constrained nonconvex programs [21]:

$$\begin{aligned}
\min_{x \in X} \quad & f(x) = \frac{1}{N} \sum_{i=1}^N \log(1 + \frac{1}{2} \|H_i x - c_i\|^2) \\
\text{s.t.} \quad & f_j(x) = \frac{1}{2} x^T Q_j x + a_j^T x \leq b_j, \quad j = 1, \dots, M,
\end{aligned} \tag{69}$$

where $X = [-10, 10]^n$. For each $i \in [N]$, we generate $H_i \in \mathbb{R}^{p \times n}$ randomly with elements independently following the standard Gaussian distribution. For each $j \in [M]$, $Q_j \in \mathbb{R}^{n \times n}$ is sum of a random matrix and a diagonal matrix with elements randomly selected from $U[-1, 1]$, and $a_j \sim$ is randomly generated following $U[0.1, 1.1]^n$. Then we generate a random point $x_* \sim U(0, 1)^n$ and set $c_i = H_i x_*, i \in [N]$, $b_j = \frac{1}{2} x_*^T Q_j x_* + a_j^T x_*, j \in [M]$. It is worth noting that x_* is feasible for (69) and has an objective value of $f(x_*) = 0$. Therefore, x_* is the optimal solution for problem (69). In this subsection, we present the average performance over 10 independent runs of the algorithm in each scenario.

To assess the impact of the parameter α_t on the numerical performance of (7), we implement experiments by solving (69). We select α_t from the set $\{0, 0.2, 0.4, 0.6, 0.8, 1\}$. In all experiments, we set $n = M = 50$, $p = 5$, $N = 1000$, and initialize the algorithm with $x^1 = \mathbf{0}$. The maximum number of iterations is set to $T = 2000$, and the penalty parameter is chosen as $\beta_t = T^{1/4}$. Additionally, we set $\rho_t = T^{1/4}$ and consider two step sizes, $\eta_t \in \{0.15/T^{1/4}, 0.17/T^{1/4}\}$. To obtain reliable results, in Figures 1 and 2 the left subplot presents the trend of average objective function values at all previous iterates, while the right one illustrates the average of constraint violation $\sum_{j=1}^M [f_j(x) - b_j]_+$ over past iterates. We observe that, for a given step size η_t , the best-performing value of α_t tends to lie within an intermediate range, rather than being the largest or smallest option. Additionally, by comparing the optimal α_t values depicted in Figures 1 and 2, we notice that as the step size η_t increases, the optimal α_t value also increases. This aligns with the positive correlation between α_t and η_t , as indicated in the parameter settings of Theorem 1. They highlight the importance of selecting an appropriate α_t value within an intermediate range to achieve optimal performance in terms of both the objective function value and the constraint violation.

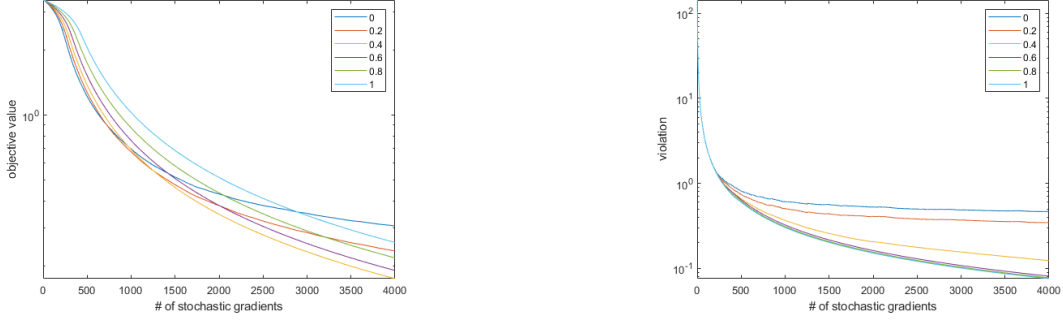


Figure 1: The impact of α_t on MLALM with $\eta_t = 0.15/T^{1/4}$

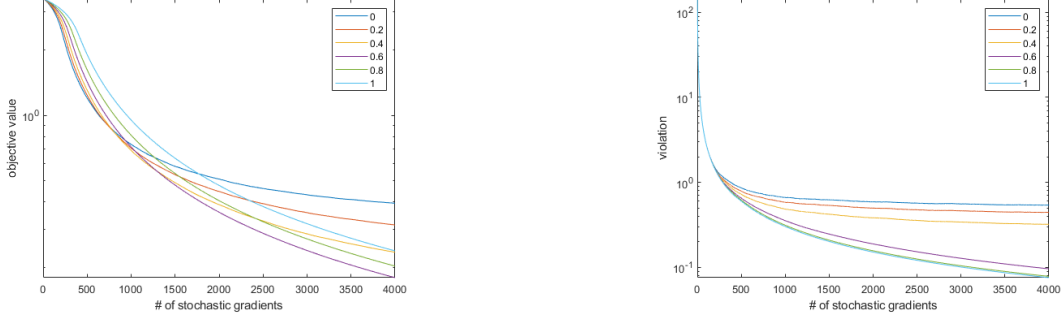


Figure 2: The impact of α_t on MLALM with $\eta_t = 0.17/T^{1/4}$

We next compare the performance of MLALM with that of ICPPC [4] for solving problem (69). We adopt the following experimental settings. For both algorithms, we set $n \in \{50, 100\}$, $p = 5$, $N = 1000$, and $M \in \{50, 100\}$. The initial point is chosen as $x^1 = \mathbf{0}$, and the maximum number of iterations is set to $T = 2000$. For ICPPC, we set $t_0 = 2$, $\mathcal{M} = 0.1M$. All other parameters for ICPPC are set according to the requirements specified in [4]. It is important to note that for ICPPC we set the maximum number of inner iterations as 2, based on its favorable performance observed in numerical tests. Regarding MLALM, we set $\eta_t = 0.03/T^{1/4}$, $\alpha_t = 0.5$, $\beta_t = T^{1/4}$, and $\rho_t = 10$. Figure 3 illustrates the performance of both algorithms in terms of objective function values and constraint violations on QCNP problems under different scenarios. All reported results are the average values obtained from 10 independent runs of each algorithm. The observations from the figures indicate that, within the same number of stochastic gradient evaluations, MLALM reduces the constraint violations much faster throughout the algorithm's progress, although the speed to reduce the objective function value by both algorithms is comparable.

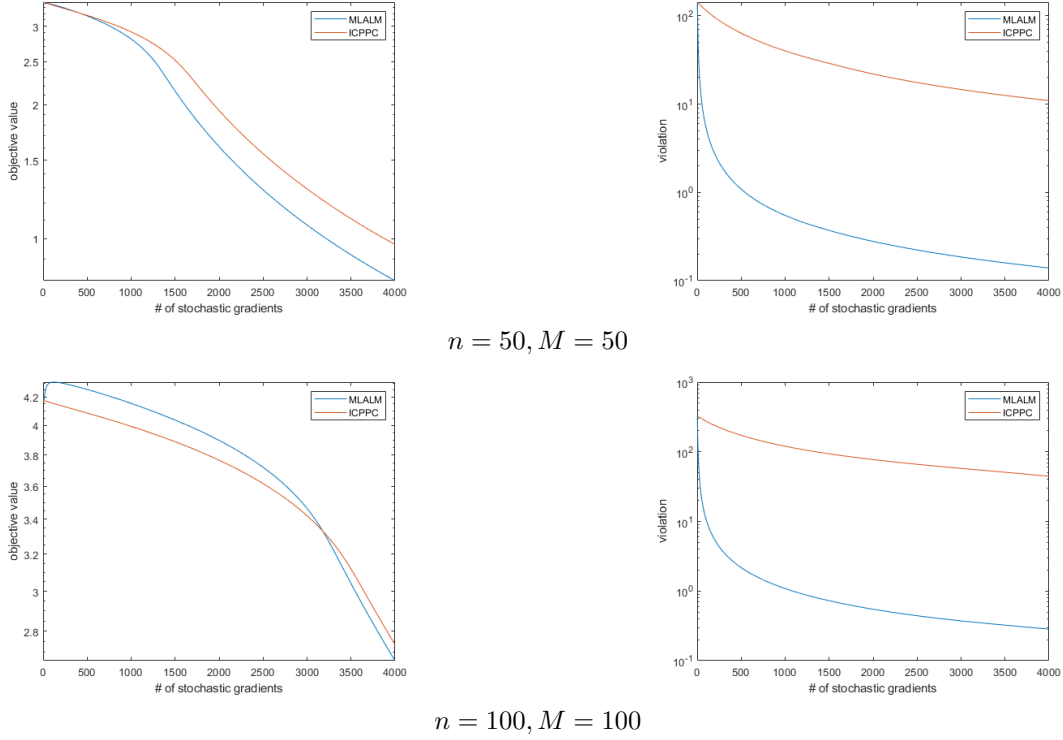


Figure 3: Comparison between MLALM and ICPPC for solving QCNP problems

6.2 Multi-class Neyman-Pearson classification problems

In this subsection, we consider multi-class Neyman-Pearson classification (mNPC) problems [25]. The mNPC problem focuses on the task of learning K models x_k , where $k \in [K]$, in order to predict the class of a potential data point ξ by selecting the model that maximizes the inner product $x_k^T \xi$. More precisely, the optimization problem aims to minimize the loss associated with one specific class while controlling the loss values for the remaining classes. The problem formulation can be expressed as follows:

$$\begin{aligned}
 \min_{\|x_k\| \leq \lambda, k \in [K]} \quad & \frac{1}{|\mathcal{D}_1|} \sum_{l>1} \sum_{\xi \in \mathcal{D}_1} h(x_1^T \xi - x_l^T \xi) \\
 \text{s.t.} \quad & \frac{1}{|\mathcal{D}_k|} \sum_{l \neq k} \sum_{\xi \in \mathcal{D}_k} h(x_k^T \xi - x_l^T \xi) \leq \gamma_k, \quad k = 2, \dots, K,
 \end{aligned} \tag{70}$$

where $h(z) = 1/(1 + e^z)$ is the loss function and \mathcal{D}_k represents the training data of the k -th class. We use two datasets from LibSVM [8]: *covtype* ($K = 7$) and *mnist* ($K = 10$). Besides, we set $\gamma_k = 0.5(K - 1)$ and $\lambda = 0.3$.

We now compare MLALM with SPD [21] and ICPPC [4]. For these three algorithms, we set the maximum number of stochastic gradient computations is 4000. For MLALM, we set parameter $\alpha_t = 0.7$ for both datasets. Other parameters are set same for both MLALM and SPD. Specifically, for *covtype* we set $\eta_t = 0.01/t^{1/4}$, $\beta_t = 5T^{1/4}$, $\rho_t = 0.67$, while for *mnist* we set $\eta_t = 0.005/t^{1/4}$, $\beta_t = T^{1/4}$, $\rho_t = 0.0067$. It is worth noting that the number of stochastic gradients calculated by SPD at t th iteration is $t^{1/4}$. Figures 4 and 5 present the performances of MLALM and SPD on the respective datasets for solving mNPC problems. The parameters of the ICPPC algorithm for the two datasets are selected as follows: For *covtype*, we set $\theta_t = 0.67$, $\tau_t = 2.5$ and $\eta_t = 2.6 \times 10^{-4}$ in subproblem solver ConEx; for *mnist*, we set $\theta_t = 0.67$, $\tau_t = 2.5$ and $\eta_t = 0.003$. The inner-iteration number is set to 2. All reported results are the average values obtained from 5 independent runs of each algorithm. From the figures we observe that for the *covtype* dataset, MLALM demonstrates a faster reduction in the objective function value compared to the other algorithms, while ICPPC maintains a lower level of constraint violations. Regarding the *mnist*

dataset, MLALM exhibits superior performance in both the objective function value and the constraint violations. Furthermore, it is worth mentioning that MLALM outperforms SPD significantly, indicating that the incorporation of momentum brings notable benefits to the numerical performance.

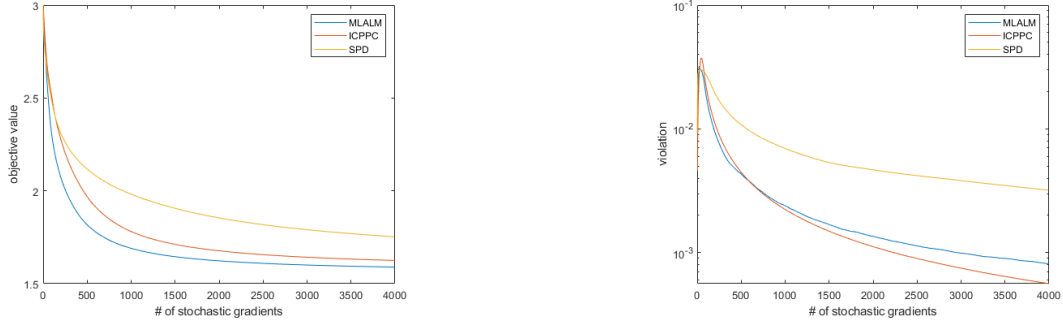


Figure 4: Comparison between MLALM, ICPPC and SPD on dataset *covtype*

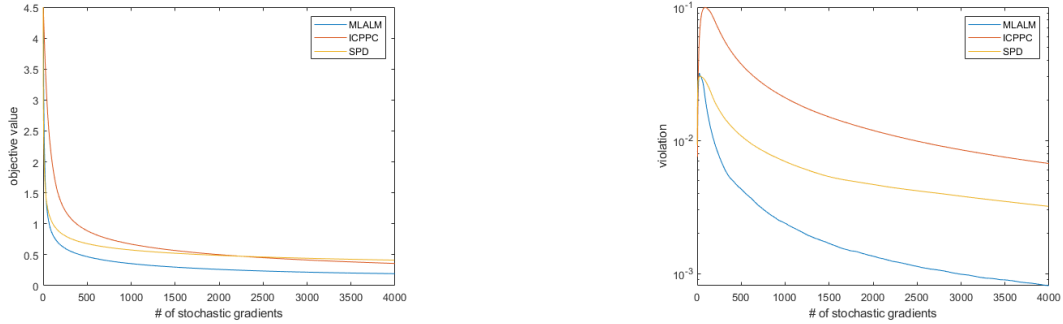


Figure 5: Comparison between MLALM, ICPPC and SPD on dataset *mnist*

7 Conclusion

In this paper, we present MLALM, a single-loop linearized augmented Lagrangian method for nonconvex optimization problems with an expectation function in the objective and deterministic functional constraints. To address potentially nonconvex constraints, we leverage the linearized augmented Lagrangian function to construct a stochastic approximation, enabling updates to the primal variable. Inspired by recent developments in momentum-type methods for unconstrained optimization in the literature, we introduce a momentum step to compute the stochastic gradient at each iteration. We conduct a comprehensive analysis of the global convergence properties of the proposed algorithm. Our analysis reveals that, under appropriate parameter settings and with unbounded increasing penalty parameters, the sequence of average stationarity measure in expectation converges to zero. Additionally, the sequence of average constraint violations exhibits convergence in expectation. Moreover, under a constraint qualification assumption, both the sequences of average expected constraint violations and complementary slackness measures converge to zero. We also investigate the properties of MLALM when penalty parameters are bounded. Furthermore, we analyze the oracle complexity of the algorithm in achieving an ϵ -stationary point and an ϵ -KKT point. Specifically, the oracle complexity to reach an ϵ -stationary point is of the order $\mathcal{O}(\epsilon^{-5})$. Under the constraint qualification assumption, the proposed algorithm can reach an ϵ -KKT point with oracle complexity bounded by $\mathcal{O}(\epsilon^{-5})$. When the initial point is nearly feasible within a certain accuracy, the complexity orders can be improved to $\mathcal{O}(\epsilon^{-4})$ accordingly. Finally, we implement numerical experiments on two types of test problems to evaluate the performance of our algorithm. The experimental results demonstrate promising performance and validate the effectiveness of our proposed approach.

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Appendix

A. Auxiliary lemmas

Lemma A.1 *Under the same conditions of Theorem 1, it holds that*

$$\lim_{p \rightarrow \infty} \sum_{t=1}^p \frac{\Psi_{\beta_t}(x^{t+1}, \lambda^t) - \Psi_{\beta_t}(x^t, \lambda^t)}{\beta_t} - \frac{1}{2}(\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2) \text{ exists.}$$

Proof. By the definition of Ψ_{β_t} , we have

$$\begin{aligned} & \sum_{t=1}^p \frac{\Psi_{\beta_t}(x^{t+1}, \lambda^t) - \Psi_{\beta_t}(x^t, \lambda^t)}{\beta_t} - \frac{1}{2}(\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2) \\ &= \sum_{t=1}^p \sum_{i \in \mathcal{E}} \left(\frac{\lambda_i^t}{\beta_t} c_i(x^{t+1}) + \frac{1}{2} c_i^2(x^{t+1}) - \frac{\lambda_i^t}{\beta_t} c_i(x^t) - \frac{1}{2} c_i^2(x^t) \right) - \frac{1}{2} \|c_{\mathcal{E}}(x^{p+1})\|^2 \\ & \quad + \sum_{t=1}^p \sum_{i \in \mathcal{I}} \frac{\psi_{\beta_t}(c_i(x^{t+1}), \lambda_i^t) - \psi_{\beta_t}(c_i(x^t), \lambda_i^t)}{\beta_t} - \frac{1}{2} \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2. \end{aligned} \quad (71)$$

We next consider the terms in (71) related with equality constraints and inequality constraints in the following part (a) and (b) separately.

(a) Note that

$$\begin{aligned} & \sum_{t=1}^p \sum_{i \in \mathcal{E}} \left(\frac{\lambda_i^t}{\beta_t} c_i(x^{t+1}) + \frac{1}{2} c_i^2(x^{t+1}) - \frac{\lambda_i^t}{\beta_t} c_i(x^t) - \frac{1}{2} c_i^2(x^t) \right) - \frac{1}{2} \|c_{\mathcal{E}}(x^{p+1})\|^2 \\ &= \frac{1}{2} (\|c_{\mathcal{E}}(x^{p+1})\|^2 - \|c_{\mathcal{E}}(x^1)\|^2) + \sum_{i \in \mathcal{E}} \sum_{t=1}^p \frac{\lambda_i^t}{\beta_t} (c_i(x^{t+1}) - c_i(x^t)) - \frac{1}{2} \|c_{\mathcal{E}}(x^{p+1})\|^2 \\ &= -\frac{1}{2} \|c_{\mathcal{E}}(x^1)\|^2 + \sum_{i \in \mathcal{E}} \sum_{t=1}^p \frac{\lambda_i^t}{\beta_t} (c_i(x^{t+1}) - c_i(x^t)). \end{aligned} \quad (72)$$

For the second term in above equation:

$$\sum_{i \in \mathcal{E}} \sum_{t=1}^p \frac{\lambda_i^t}{\beta_t} (c_i(x^{t+1}) - c_i(x^t)) = \sum_{i \in \mathcal{E}} \frac{\lambda_i^p}{\beta_p} c_i(x^{p+1}) + \sum_{i \in \mathcal{E}} \sum_{t=1}^{p-1} \left(\frac{\lambda_i^t}{\beta_t} - \frac{\lambda_i^{t+1}}{\beta_{t+1}} \right) c_i(x^{t+1}), \quad (73)$$

it is noteworthy that for any $i \in \mathcal{E}$, $\{\frac{\lambda_i^p}{\beta_p} c_i(x^{p+1})\}_{p \geq 1}$ converges to zero by the boundedness of λ^t and $c_i(x^t)$ for all $i \in \mathcal{E}$, $t \geq 1$, and

$$\begin{aligned} \sum_{t=1}^{p-1} \left(\frac{\lambda_i^t}{\beta_t} - \frac{\lambda_i^{t+1}}{\beta_{t+1}} \right) c_i(x^{t+1}) &= \sum_{t=1}^{p-1} \left(\frac{\lambda_i^t}{\beta_t} - \frac{\lambda_i^{t+1}}{\beta_t} + \frac{\lambda_i^{t+1}}{\beta_t} - \frac{\lambda_i^{t+1}}{\beta_{t+1}} \right) c_i(x^{t+1}) \\ &= \sum_{t=1}^{p-1} \frac{-\rho_t c_i^2(x^{t+1})}{\beta_t} + \sum_{t=1}^{p-1} \left(\frac{1}{\beta_t} - \frac{1}{\beta_{t+1}} \right) \lambda_i^{t+1} c_i(x^{t+1}) \\ &= \sum_{t=1}^{p-1} \frac{-\rho_t c_i^2(x^{t+1})}{\beta_t} + \sum_{t=1}^{p-1} \frac{(t+1)^{\frac{t}{2}}}{\beta_0 t^t} \frac{1}{(t+1)^{\frac{t}{2}}} (1 - (\frac{t}{t+1})^t) \lambda_i^{t+1} c_i(x^{t+1}). \end{aligned} \quad (74)$$

Here, $|\sum_{t=1}^{p-1} -\rho_t c_i^2(x^{t+1})|$ is bounded following the setting of ρ_t . Since $\iota < 1$, one has $1 - (\frac{t}{t+1})^\iota < \frac{1}{t+1}$, then $|\sum_{t=1}^{p-1} \frac{1}{(t+1)^{\frac{\iota}{2}}} (1 - (\frac{t}{t+1})^\iota) \lambda_i^{t+1} c_i(x^{t+1})|$ is bounded due to the boundedness of λ^t , $t \geq 1$. Then owing to the fact that both sequences $\{\frac{1}{\beta_t}\}$ and $\{\frac{(t+1)^{\frac{\iota}{2}}}{\beta_0 t^\iota}\}_{t \geq 1}$ are monotonically decreasing and converging to zero, it follows from Dirichlet's Test that $\sum_{t=1}^{p-1} (\frac{\lambda_i^t}{\beta_t} - \frac{\lambda_i^{t+1}}{\beta_{t+1}}) c_i(x^{t+1})$ converges as $p \rightarrow \infty$, so does (73). Therefore, it is straightforward to obtain the convergence of L.H.S. of (72) as $p \rightarrow \infty$.

(b) We now consider the remaining part in (71) related with inequality constraints. Our goal is to prove its convergence as $p \rightarrow \infty$. Towards this end, for any $i \in \mathcal{I}$ we consider the following index sets:

$$\begin{aligned}\mathcal{T}_1 &= \{t \in [p-1] : \beta_t c_i(x^{t+1}) + \lambda_i^t \geq 0, \beta_{t+1} c_i(x^{t+1}) + \lambda_i^{t+1} \geq 0\}; \\ \mathcal{T}_2 &= \{t \in [p-1] : \beta_t c_i(x^{t+1}) + \lambda_i^t \geq 0, \beta_{t+1} c_i(x^{t+1}) + \lambda_i^{t+1} < 0\}; \\ \mathcal{T}_3 &= \{t \in [p-1] : \beta_t c_i(x^{t+1}) + \lambda_i^t < 0, \beta_{t+1} c_i(x^{t+1}) + \lambda_i^{t+1} < 0\}; \\ \mathcal{T}_4 &= \{t \in [p-1] : \beta_t c_i(x^{t+1}) + \lambda_i^t < 0, \beta_{t+1} c_i(x^{t+1}) + \lambda_i^{t+1} \geq 0\}.\end{aligned}\tag{75}$$

Here we omit their dependence upon i . Obviously, $\cup_{i \in [4]} \mathcal{T}_i = [p-1]$. One can verify that $\mathcal{T}_4 = \emptyset$. This is because whenever $\beta_t c_i(x^{t+1}) + \lambda_i^t < 0$, $c_i(x^{t+1}) < 0$ and $\lambda^{t+1} = \lambda^t - \frac{\rho_t \lambda^t}{\beta_t} < \lambda^t$ by (9), thus there must have $\beta_{t+1} c_i(x^{t+1}) + \lambda^{t+1} < \beta_t c_i(x^{t+1}) + \lambda^t < 0$. Then, for any $i \in \mathcal{I}$, the following relations hold from the definition of ψ_β :

$$\begin{aligned}& \sum_{t=1}^p \frac{\psi_{\beta_t}(c_i(x^{t+1}), \lambda_i^t) - \psi_{\beta_t}(c_i(x^t), \lambda_i^t)}{\beta_t} - \frac{1}{2}([c_i(x^{p+1})]_+)^2 \\ &= \sum_{t=1}^p \frac{\psi_{\beta_t}(c_i(x^{t+1}), \lambda_i^t)}{\beta_t} - \sum_{t=0}^{p-1} \frac{\psi_{\beta_{t+1}}(c_i(x^{t+1}), \lambda_i^{t+1})}{\beta_{t+1}} - \frac{1}{2}([c_i(x^{p+1})]_+)^2 \\ &= \frac{\psi_{\beta_p}(c_i(x^{p+1}), \lambda_i^p)}{\beta_p} - \frac{\psi_{\beta_1}(c_i(x^1), \lambda_i^1)}{\beta_1} + \sum_{t=1}^{p-1} \left(\frac{\psi_{\beta_t}(c_i(x^{t+1}), \lambda_i^t)}{\beta_t} - \frac{\psi_{\beta_{t+1}}(c_i(x^{t+1}), \lambda_i^{t+1})}{\beta_{t+1}} \right) - \frac{1}{2}([c_i(x^{p+1})]_+)^2 \\ &= \frac{\lambda_i^p \max\{c_i(x^{p+1}), -\frac{\lambda_i^p}{\beta_p}\}}{\beta_p} + \frac{(\max\{c_i(x^{p+1}), -\frac{\lambda_i^p}{\beta_p}\})^2}{2} - \frac{1}{2}([c_i(x^{p+1})]_+)^2 - \frac{\psi_{\beta_1}(c_i(x^1), \lambda_i^1)}{\beta_1} \\ &+ \sum_{t \in \mathcal{T}_1} \left(\frac{\lambda_i^t}{\beta_t} - \frac{\lambda_i^{t+1}}{\beta_{t+1}} \right) c_i(x^{t+1}) + \sum_{t \in \mathcal{T}_2} \left(\frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} + \frac{\lambda_i^t}{\beta_t} c_i(x^{t+1}) + \frac{1}{2} \|c_i(x^{t+1})\|^2 \right) + \sum_{t \in \mathcal{T}_3} \left(\frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} - \frac{(\lambda_i^t)^2}{2\beta_t^2} \right).\end{aligned}\tag{76}$$

Without loss of generality, we assume that as p approaches infinity, \mathcal{T}_1 , \mathcal{T}_2 , and \mathcal{T}_3 are infinite sets. It is easy to see that the sum of the first three terms in R.H.S. of (76) converges to zero as p approaches infinity. Therefore, to demonstrate the convergence of L.H.S. of (76), it suffices to prove that the terms pertaining to sets \mathcal{T}_1 , \mathcal{T}_2 , and \mathcal{T}_3 converge as $p \rightarrow \infty$.

(b1) Our first claim is that $\sum_{t \in \mathcal{T}_1} (\frac{\lambda_i^t}{\beta_t} - \frac{\lambda_i^{t+1}}{\beta_{t+1}}) c_i(x^{t+1})$ converges as p approaches infinity. This claim can be shown by following the same analysis as in the previous proof for (74).

(b2) For $t \in \mathcal{T}_2$, $\frac{\lambda_i^{t+1}}{\beta_{t+1}} < -c_i(x^{t+1}) \leq \frac{\lambda_i^t}{\beta_t}$. It indicates

$$\begin{aligned}\frac{(\lambda_i^{t+1})^2}{\beta_{t+1}^2} - \frac{(\lambda_i^t)^2}{\beta_t^2} &\leq \frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} + \frac{\lambda_i^t}{\beta_t} c_i(x^{t+1}) + \frac{1}{2} \|c_i(x^{t+1})\|^2 \leq \frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} - \frac{\lambda_i^t \lambda_i^{t+1}}{\beta_t \beta_{t+1}} + \frac{(\lambda_i^t)^2}{2\beta_t^2} \\ &= \frac{1}{2} \left(\frac{\lambda_i^{t+1}}{\beta_{t+1}} - \frac{\lambda_i^t}{\beta_t} \right)^2.\end{aligned}\tag{77}$$

Then we multiply both sides of the above inequality by $\frac{\beta_{t+1}}{t^{\frac{1}{2}}}$ and subsequently sum up the resulting inequality over \mathcal{T}_2 , obtaining

$$\sum_{t \in \mathcal{T}_2} \frac{(\lambda_i^{t+1})^2}{\beta_{t+1} t^{\frac{1}{2}}} - \frac{\beta_{t+1} (\lambda_i^t)^2}{\beta_t^2 t^{\frac{1}{2}}} \leq \sum_{t \in \mathcal{T}_2} \frac{\beta_{t+1}}{t^{\frac{1}{2}}} \left(\frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} + \frac{\lambda_i^t}{\beta_t} c_i(x^{t+1}) + \frac{1}{2} \|c_i(x^{t+1})\|^2 \right) \leq \sum_{t \in \mathcal{T}_2} \frac{\beta_{t+1}}{2t^{\frac{1}{2}}} \left(\frac{\lambda_i^{t+1}}{\beta_{t+1}} - \frac{\lambda_i^t}{\beta_t} \right)^2. \quad (78)$$

By Lemma 2 and the setting of ρ_t , there exists $\Lambda > 0$ such that $\|\lambda^t\| \leq \Lambda$ for any $t \geq 1$. And for any $t \in \mathcal{T}_2$, $\lambda_i^t \geq \lambda_i^{t+1} = \lambda_i^t + \rho_t c_i(x^{t+1}) \geq (1 - \frac{\rho_t}{\beta_t}) \lambda_i^t$, thus $\frac{\lambda_i^{t+1}}{\beta_{t+1}} < \frac{\lambda_i^t}{\beta_t}$. Then due to $(1 + \frac{1}{t})^{2\iota} < 1 + \frac{1}{t}$ by $\iota < \frac{1}{2}$, it can be implied that

$$\begin{aligned} 0 &> \sum_{t \in \mathcal{T}_2} \frac{(\lambda_i^{t+1})^2}{\beta_{t+1} t^{\frac{1}{2}}} - \frac{\beta_{t+1} (\lambda_i^t)^2}{\beta_t^2 t^{\frac{1}{2}}} \geq \sum_{t \in \mathcal{T}_2} \left(\frac{(1 - \frac{\rho_t}{\beta_t})^2}{\beta_{t+1} t^{\frac{1}{2}}} - \frac{\beta_{t+1}}{\beta_t^2 t^{\frac{1}{2}}} \right) \Lambda^2 \\ &= \frac{1}{\beta_0} \sum_{t \in \mathcal{T}_2} \frac{1}{t^{\frac{1}{2}} (t+1)^\iota} \left((1 - \frac{\rho_t}{\beta_t})^2 - \frac{(t+1)^{2\iota}}{t^{2\iota}} \right) \Lambda^2 \\ &\geq \frac{1}{\beta_0} \sum_{t \in \mathcal{T}_2} \frac{1}{t^{\frac{1}{2}} (t+1)^\iota} \left(1 - \frac{2\rho_t}{\beta_t} - 1 - \frac{1}{t} \right) \Lambda^2 \\ &\geq \frac{1}{\beta_0} \sum_{t \in \mathcal{T}_2} \frac{1}{t^{\frac{1}{2}} (t+1)^\iota} \left(-\frac{2\rho}{\beta_0 t^{\theta+\iota}} - \frac{1}{t} \right) \Lambda^2 \\ &= \frac{1}{\beta_0} \sum_{t \in \mathcal{T}_2} \frac{1}{t^{1+\frac{1}{2}} (t+1)^\iota} \left(-\frac{2\rho}{\beta_0 t^{\theta+\iota-1}} - 1 \right) \Lambda^2, \end{aligned} \quad (79)$$

which is lower bounded by a finite value, since $\theta + \iota > 1$. On the other hand, it follows from $\beta_{t+1} \geq \beta_t \geq \beta_1$, $\|\lambda^t\| \leq \Lambda$ and $\lambda_i^{t+1} \geq (1 - \frac{\rho_t}{\beta_t}) \lambda_i^t$ that

$$\begin{aligned} \sum_{t \in \mathcal{T}_2} \frac{\beta_{t+1}}{2t^{\frac{1}{2}}} \left(\frac{\lambda_i^{t+1}}{\beta_{t+1}} - \frac{\lambda_i^t}{\beta_t} \right)^2 &\leq \sum_{t \in \mathcal{T}_2} \frac{\beta_{t+1}}{2t^{\frac{1}{2}}} \left(\frac{(1 - \frac{\rho_t}{\beta_t})}{\beta_{t+1}} \lambda_i^t - \frac{\lambda_i^t}{\beta_t} \right)^2 \\ &\leq \sum_{t \in \mathcal{T}_2} \frac{1}{2\beta_{t+1} t^{\frac{1}{2}}} \left((1 - \frac{\rho_t}{\beta_t}) - \frac{\beta_{t+1}}{\beta_t} \right)^2 \Lambda^2 \\ &= \sum_{t \in \mathcal{T}_2} \frac{1}{2\beta_{t+1} t^{\frac{1}{2}}} \left((1 - \frac{\rho}{\beta_0 t^{\theta+\iota}}) - (1 + \frac{1}{t})^\iota \right)^2 \Lambda^2 \\ &\leq \sum_{t \in \mathcal{T}_2} \frac{1}{2\beta_{t+1} t^{\frac{1}{2}}} \left(\frac{\rho}{\beta_0 t^{\theta+\iota}} + \frac{1}{t} \right)^2 \Lambda^2, \end{aligned} \quad (80)$$

which is upper bounded by a finite value since $\theta > 1$. Therefore, by (78), $\sum_{t \in \mathcal{T}_2} \frac{\beta_{t+1}}{t^{\frac{1}{2}}} \left(\frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} + \frac{\lambda_i^t}{\beta_t} c_i(x^{t+1}) + \frac{1}{2} \|c_i(x^{t+1})\|^2 \right)$ is bounded, then from $\frac{t^{\frac{1}{2}}}{\beta_{t+1}} \rightarrow 0$ as $t \rightarrow \infty$ we derive that $\sum_{t \in \mathcal{T}_2} \left(\frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} + \frac{\lambda_i^t}{\beta_t} c_i(x^{t+1}) + \frac{1}{2} \|c_i(x^{t+1})\|^2 \right)$ converges as $p \rightarrow \infty$.

(b3) For $t \in \mathcal{T}_3$, by (9) we have $\lambda_i^{t+1} = (1 - \frac{\rho_t}{\beta_t}) \lambda_i^t$. Then it indicates from the monotonically increasing property of $\{\beta_t\}$ that $\frac{\lambda_i^{t+1}}{\beta_{t+1}} \leq \frac{\lambda_i^t}{\beta_t}$, and from (79) that $\sum_{t \in \mathcal{T}_3} \frac{\beta_{t+1}}{t^{\frac{1}{2}}} \left(\frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} - \frac{(\lambda_i^t)^2}{2\beta_t^2} \right)$ is bounded. Furthermore, since $\{\frac{t^{\frac{1}{2}}}{\beta_{t+1}}\}_{t \geq 1}$ is a monotonically decreasing sequence converging to zero, then $\sum_{t \in \mathcal{T}_3} \left(\frac{(\lambda_i^{t+1})^2}{2\beta_{t+1}^2} - \frac{(\lambda_i^t)^2}{2\beta_t^2} \right)$ converges as $p \rightarrow \infty$. \square

Lemma A.2 *Under the same conditions of Theorem 3, it holds that*

$$|\sum_{t=1}^p \frac{\Psi_\beta(x^{t+1}, \lambda^t) - \Psi_\beta(x^t, \lambda^t)}{\beta} - \frac{1}{2}(\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2)| = C_{vio}^1 + \mathcal{O}(\beta^{-1}),$$

where $C_{vio}^1 = \frac{1}{2}\|c_{\mathcal{E}}(x^1)\|^2 + \frac{1}{2}\|[c_{\mathcal{I}}(x^1)]_+\|^2$.

Proof. Similar to Lemma A.1, to prove the conclusion it suffices to provide estimate bounds on terms in (71) with $\beta_t \equiv \beta$ for all $t \geq 1$, that is

$$\begin{aligned} & \sum_{t=1}^p \frac{\Psi_\beta(x^{t+1}, \lambda^t) - \Psi_\beta(x^t, \lambda^t)}{\beta} - \frac{1}{2}(\|c_{\mathcal{E}}(x^{p+1})\|^2 + \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2) \\ &= \sum_{t=1}^p \sum_{i \in \mathcal{E}} \left(\frac{\lambda_i^t}{\beta} c_i(x^{t+1}) + \frac{1}{2} c_i^2(x^{t+1}) - \frac{\lambda_i^t}{\beta} c_i(x^t) - \frac{1}{2} c_i^2(x^t) \right) - \frac{1}{2} \|c_{\mathcal{E}}(x^{p+1})\|^2 \\ & \quad + \sum_{t=1}^p \sum_{i \in \mathcal{I}} \frac{\psi_\beta(c_i(x^{t+1}), \lambda_i^t) - \psi_\beta(c_i(x^t), \lambda_i^t)}{\beta} - \frac{1}{2} \|[c_{\mathcal{I}}(x^{p+1})]_+\|^2. \end{aligned} \quad (81)$$

Since the analysis is essentially the same as Lemma A.1 except replacing β_t with constant β , we simply state the main ideas here.

(a) Following (72)-(73), we obtain

$$\sum_{t=1}^p \sum_{i \in \mathcal{E}} \left(\frac{\lambda_i^t}{\beta} c_i(x^{t+1}) + \frac{1}{2} c_i^2(x^{t+1}) - \frac{\lambda_i^t}{\beta} c_i(x^t) - \frac{1}{2} c_i^2(x^t) \right) - \frac{1}{2} \|c_{\mathcal{E}}(x^{p+1})\|^2 = -\frac{1}{2} \|c_{\mathcal{E}}(x^1)\|^2 + \mathcal{O}(\beta^{-1}). \quad (82)$$

(b) With a slight abuse of notation, corresponding to inequality constraints we still use \mathcal{T}_i , $i = 1, \dots, 4$ as defined in (75), except $\beta_t \equiv \beta$ in this scenario. It is easy to check from the definition of \mathcal{T}_1 together with (11) and the setting of ρ_t that for any $i \in \mathcal{I}$,

$$\begin{aligned} & \left| \sum_{t \in \mathcal{T}_1} \left(\frac{\lambda_i^t}{\beta} - \frac{\lambda_i^{t+1}}{\beta} \right) c_i(x^{t+1}) \right| \leq \beta^{-1} \sum_{t \in \mathcal{T}_1} |\lambda_i^t - \lambda_i^{t+1}| |c_i(x^{t+1})| \\ & \leq \beta^{-1} \sum_{t \in \mathcal{T}_1} |\lambda_i^t - \lambda_i^{t+1}| \cdot \max\left\{ \frac{\lambda_i^t}{\beta}, \frac{\lambda_i^{t+1}}{\beta}, C \right\} = \mathcal{O}(\beta^{-1}). \end{aligned}$$

From (77), Lemma 2 and the setting of ρ_t , it follows that for any $i \in \mathcal{I}$,

$$\begin{aligned} & \left| \sum_{t \in \mathcal{T}_2} \left(\frac{(\lambda_i^{t+1})^2}{2\beta^2} + \frac{\lambda_i^t}{\beta} c_i(x^{t+1}) + \frac{1}{2} \|c_i(x^{t+1})\|^2 \right) \right| = \mathcal{O}(\beta^{-2}), \\ & \left| \sum_{t \in \mathcal{T}_3} \left(\frac{(\lambda_i^{t+1})^2}{2\beta^2} - \frac{(\lambda_i^t)^2}{2\beta^2} \right) \right| = \mathcal{O}(\beta^{-2}). \end{aligned}$$

Hence, we obtain from (76) that

$$\sum_{t=1}^p \sum_{i \in \mathcal{I}} \frac{\psi_\beta(c_i(x^{t+1}), \lambda_i^t) - \psi_\beta(c_i(x^t), \lambda_i^t)}{\beta_t} - \frac{1}{2} ([c_i(x^{p+1})]_+)^2 = -\frac{1}{2} \|[c_{\mathcal{I}}(x^1)]_+\|^2 + \mathcal{O}(\beta^{-1}).$$

Consequently, the conclusion of this lemma can be derived. \square

B. Proof of Lemma 11

Let us prove the conclusion by contradiction. For notation simplicity, we define $a_k := \mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k \|\hat{\lambda}^t\|]$, $k \geq 1$ with

$$\hat{\lambda}_i^t := \begin{cases} \beta_t c_i(x^t) + \lambda_i^t, & i \in \mathcal{E}, \\ [\beta_t c_i(x^t) + \lambda_i^t]_+, & i \in \mathcal{I}. \end{cases}$$

We assume that $\limsup_{k \rightarrow \infty} a_k = +\infty$, that is, for any $M > 0$, there exists an infinite subsequence $\{a_k\}_{k \in \mathcal{K}}$ such that $a_k \geq M$, for all $k \in \mathcal{K}$. In view of the first-order optimality condition of the subproblem (8), there exists $u^{t+1} \in \partial h(x^{t+1})$ such that

$$\langle d^t + u^{t+1} + \frac{1}{\eta_t}(x^{t+1} - x^t), x - x^{t+1} \rangle \geq 0, \quad \forall x \in X; \forall t \geq 1.$$

Subsequently, dividing both sides of the above inequality by a_k and recalling $d^t = \nabla f(x^t) + \nabla c(x^t)\hat{\lambda}^t + \varepsilon^t$ we have for any $x \in X$, $t > 0$, $k \in \mathcal{K}$,

$$\frac{1}{a_k} \langle \nabla f(x^t) + \nabla c(x^t)\hat{\lambda}^t + \varepsilon^t + u^{t+1} + \frac{1}{\eta_t}(x^{t+1} - x^t), x - x^{t+1} \rangle \geq 0.$$

Equivalently, there exists $v^{t+1} \in \mathcal{N}_X(x^{t+1})$ such that

$$\frac{\nabla f(x^t)}{a_k} + \frac{\nabla c(x^t)\hat{\lambda}^t}{a_k} + \frac{\varepsilon^t}{a_k} + \frac{u^{t+1}}{a_k} + \frac{x^{t+1} - x^t}{\eta_t a_k} + v^{t+1} = 0, \quad \forall t > 0; \forall k \in \mathcal{K}. \quad (83)$$

Following Assumption 5, for any $t \geq 1$, there exists $z^{t+1} \in -\mathcal{N}_X^*(x^{t+1})$ with $\|z^{t+1}\| \leq Z$ such that

$$\begin{aligned} \nabla c_i(x^{t+1})^T z^{t+1} &= -\delta \cdot \text{sgn}(c_i(x^{t+1})), \quad i \in \mathcal{E} : c_i(x^{t+1}) \neq 0; \\ \nabla c_i(x^{t+1})^T z^{t+1} &\leq -\delta, \quad i \in \mathcal{I} : c_i(x^{t+1}) > 0. \end{aligned} \quad (84)$$

Then taking inner product with z^{t+1} on both sides, summing (83) over $[k]$, dividing the result by k , and concurrently taking the expectation yield

$$\begin{aligned} 0 &= \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (z^{t+1})^T \left(\frac{\nabla f(x^t)}{a_k} + \frac{\nabla c(x^t)\hat{\lambda}^t}{a_k} + \frac{\varepsilon^t}{a_k} + \frac{u^{t+1}}{a_k} + \frac{x^{t+1} - x^t}{\eta_t a_k} + v^{t+1} \right) \right] \\ &= \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (z^{t+1})^T \left(\frac{\nabla f(x^t)}{a_k} + \frac{\varepsilon^t}{a_k} + \frac{u^{t+1}}{a_k} + \frac{(\nabla c(x^t) - \nabla c(x^{t+1}))\hat{\lambda}^t}{a_k} + \frac{x^{t+1} - x^t}{\eta_t a_k} \right) \right] \\ &\quad + \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (z^{t+1})^T \left(\frac{\nabla c(x^{t+1})\hat{\lambda}^t}{a_k} + v^{t+1} \right) \right]. \end{aligned} \quad (85)$$

It is easy to prove that for all $k \in \mathcal{K}$, one has

$$\left| \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \frac{(z^{t+1})^T \nabla f(x^t)}{a_k} \right] \right| \leq \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \|z^{t+1}\| \|\nabla f(x^t)\| \right] \leq \frac{ZG}{M},$$

and

$$\left| \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \frac{(z^{t+1})^T u^{t+1}}{a_k} \right] \right| \leq \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \|z^{t+1}\| \|u^{t+1}\| \right] \leq \frac{ZG}{M}.$$

Besides, we can also verify that

$$\begin{aligned}
|\mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k \frac{(z^{t+1})^T \varepsilon^t}{a_k}]| &\leq \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k \|z^{t+1}\| \|\varepsilon^t\|] \\
&\leq \frac{Z}{a_k} \cdot \mathbb{E}_{\xi^{[k]}}[\sqrt{\frac{1}{k} \sum_{t=1}^k \|\varepsilon^t\|^2}] \\
&\leq \frac{Z}{a_k} \cdot \sqrt{\mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k \|\varepsilon^t\|^2]}.
\end{aligned}$$

With the setting of α_t , β_t and ρ_t in (38), we have

$$\begin{aligned}
\sum_{t=1}^k \alpha_t^2 &\leq \sum_{t=1}^k \frac{64\alpha^2\eta^4}{t^{4\iota}} \leq \begin{cases} \frac{64\alpha^2\eta^4}{1-4\iota}(k^{1-4\iota} - 1) + 64\alpha^2\eta^4, & \iota \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{3}], \\ 64\alpha^2\eta^4 \ln k + 64\alpha^2\eta^4, & \iota = \frac{1}{4}, \end{cases} \\
\sum_{t=1}^k (\beta_{t+1} - \beta_t)^2 &\leq \sum_{t=1}^k \beta_0^2 t^{2\iota-2}, \quad \sum_{t=1}^k \rho_t^2 \leq \sum_{t=1}^k \rho_0^2 t^{-2\theta}, \quad \sum_{t=1}^k \rho_t \leq \sum_{t=1}^k \rho_0 t^{-\theta}.
\end{aligned}$$

Since $\iota \leq \frac{1}{3}$, $\sum_{t=1}^k (\beta_{t+1} - \beta_t)^2$, $\sum_{t=1}^k \rho_t^2$, $\sum_{t=1}^k \rho_t$ are upper bounded by a positive constant, and $\sum_{t=1}^k \alpha_t^2$ is upper bounded by $\mathcal{O}(k^{1-4\iota} + \ln k)$. Therefore, from (36), there exists a positive constant C_{eps} such that

$$\mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k \|\varepsilon^t\|^2] = \frac{1}{k} \sum_{t=1}^k \mathbb{E}_{\xi^{[t]}}[\|\varepsilon^t\|^2] \leq \frac{C_{eps}^2}{4} k^{2\iota-1} (k^{1-4\iota} + \ln k + k^\iota + 1) \leq C_{eps}^2. \quad (86)$$

Hence, we obtain

$$|\mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k \frac{(z^{t+1})^T \varepsilon^t}{a_k}]| \leq \frac{Z C_{eps}}{M}.$$

Furthermore, the following relations hold:

$$\begin{aligned}
&|\mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k (z^{t+1})^T (\frac{(\nabla c(x^t) - \nabla c(x^{t+1})) \hat{\lambda}^t}{a_k} + \frac{x^{t+1} - x^t}{\eta_t a_k})]| \\
&\leq \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k (\|z^{t+1}\| \|\nabla c(x^t) - \nabla c(x^{t+1})\|_{\text{F}} \|\hat{\lambda}^t\| + \frac{\|z^{t+1}\| \|x^{t+1} - x^t\|}{\eta_t})] \\
&\leq \frac{Z}{a_k} \cdot \mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k (C_\lambda L_{\beta_t} \|\nabla c(x^t) - \nabla c(x^{t+1})\|_{\text{F}} + \frac{\|x^{t+1} - x^t\|}{\eta_t})] \\
&\leq \frac{Z}{a_k} \cdot \mathbb{E}_{\xi^{[k]}}[\frac{1}{k} \sum_{t=1}^k (\sqrt{m} L C_\lambda L_{\beta_t} \|x^{t+1} - x^t\| + \frac{\|x^{t+1} - x^t\|}{\eta_t})] \\
&\leq \frac{Z}{a_k} \cdot \mathbb{E}_{\xi^{[k]}}[\frac{\sqrt{m} L C_\lambda + 1}{k} \sum_{t=1}^k (L_{\beta_t} + \frac{1}{\eta_t}) \|x^{t+1} - x^t\|] \\
&\leq \frac{Z}{a_k} \cdot \mathbb{E}_{\xi^{[k]}}[(\sqrt{m} L C_\lambda + 1) \sqrt{\frac{1}{k} \sum_{t=1}^k (L_{\beta_t} + \frac{1}{\eta_t})^2 \|x^{t+1} - x^t\|^2}]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{Z}{a_k} \cdot \sqrt{2}(\sqrt{m}LC_\lambda + 1) \sqrt{\mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (L_{\beta_t}^2 + \frac{1}{\eta_t^2}) \|x^{t+1} - x^t\|^2 \right]} \\
&\leq \frac{Z}{a_k} \cdot \sqrt{2}(\sqrt{m}LC_\lambda + 1) C_{sum} \leq \frac{\sqrt{2}Z(\sqrt{m}LC_\lambda + 1)C_{sum}}{M},
\end{aligned}$$

where the second inequality is due to the fact that there exists C_λ independent of β_t such that $\|\hat{\lambda}^t\| \leq C_\lambda L_{\beta_t}$ and C_{sum} denotes the upper bound derived by (37) and (86), i.e., there is a $C_{sum} > 0$ such that

$$\begin{aligned}
&\mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (L_{\beta_t}^2 + \frac{1}{\eta_t^2}) \|x^{t+1} - x^t\|^2 \right] \\
&\leq \frac{5}{\eta_k k} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + \frac{m\beta_{k+1}C^2}{2} + 2m\tilde{C}^2 \sum_{t=1}^k \rho_t) + \mathbb{E}_{\xi^{[k]}} \left[\frac{5}{2k} \sum_{t=1}^k \|\varepsilon^t\|^2 \right] \\
&\leq \frac{5}{\eta_k k} (\mathcal{L}_{\beta_1}(x^1, \lambda^1) - C^* + \frac{m\beta_{k+1}C^2}{2} + 2m\tilde{C}^2 \sum_{t=1}^k \rho_t) + \frac{5}{2} C_{eps}^2 \leq C_{sum}^2,
\end{aligned}$$

owing to the setting of η_t , β_t , ρ_t in (38) and $\iota \leq \frac{1}{3}$. Therefore, it follows from (85) that

$$\mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (z^{t+1})^T \left(\frac{\nabla c(x^{t+1}) \hat{\lambda}^t}{a_k} + v^{t+1} \right) \right] \geq - \frac{Z(2G + C_{eps} + \sqrt{2}(\sqrt{m}LC_\lambda + 1)C_{sum})}{M} =: \frac{C_{lower}}{M}.$$

We next introduce notations:

$$\begin{aligned}
\mathcal{E}_1 &:= \{i \in \mathcal{E} : c_i(x^{t+1}) \neq 0\}, \quad \mathcal{E}_2 := \{i \in \mathcal{E} : c_i(x^{t+1}) = 0\}, \quad \mathcal{I}_1 := \{i \in \mathcal{I} : c_i(x^{t+1}) > 0\}, \\
\mathcal{I}_2 &:= \{i \in \mathcal{I} : -\frac{\lambda_i^t}{\beta_t} \leq c_i(x^{t+1}) \leq 0\}, \quad \mathcal{I}_3 := \{i \in \mathcal{I} : c_i(x^{t+1}) < -\frac{\lambda_i^t}{\beta_t}\}.
\end{aligned} \tag{87}$$

For the sake of simplicity, and without causing any potential confusion, we omit the index t in the notations as defined in (87). For any $i \in \mathcal{E}_1$, it follows from $|\lambda_i^t| \geq |\hat{\lambda}_i^t| - \beta_t |c_i(x^t)|$ and $\beta_t |c_i(x^t)| - \beta_t |c_i(x^{t+1})| \leq \beta_t |c_i(x^t) - c_i(x^{t+1})|$ that

$$\begin{aligned}
(z^{t+1})^T \nabla c_i(x^{t+1}) &= -\delta \cdot \text{sgn}(c_i(x^{t+1})) \hat{\lambda}_i^t \\
&= -\delta \cdot \text{sgn}(c_i(x^{t+1})) (\lambda_i^t + \beta_t c_i(x^{t+1}) - \beta_t c_i(x^{t+1}) + \beta_t c(x^t)) \\
&\leq -\delta |\lambda_i^t| + 2\delta |\lambda_i^t| - \delta \beta_t |c_i(x^{t+1})| + \delta \beta_t |c_i(x^{t+1}) - c_i(x^t)| \\
&\leq -\delta |\hat{\lambda}_i^t| + \delta \beta_t |c_i(x^t)| + 2\delta |\lambda_i^t| - \delta \beta_t |c_i(x^{t+1})| + \delta \beta_t |c_i(x^{t+1}) - c_i(x^t)| \\
&\leq -\delta |\hat{\lambda}_i^t| + 2\delta |\lambda_i^t| + 2\delta \beta_t |c_i(x^{t+1}) - c_i(x^t)| \\
&\leq -\delta |\hat{\lambda}_i^t| + 2\delta |\lambda_i^t| + 2\delta G \beta_t \|x^{t+1} - x^t\|,
\end{aligned}$$

then we obtain from (84) together with $\tilde{L} \geq mCL + mG^2$ and $(z^{t+1})^T v^{t+1} \leq 0$ that

$$\begin{aligned}
&\mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (z^{t+1})^T \left(\frac{\nabla c(x^{t+1}) \hat{\lambda}^t}{a_k} + v^{t+1} \right) \right] \\
&\leq \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (-\delta \|\hat{\lambda}_{\mathcal{E}_1}^t\|_1 + 2\delta \|\lambda_{\mathcal{E}_1}^t\|_1 + 2\delta |\mathcal{E}_1| G \beta_t \|x^{t+1} - x^t\|) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (-\delta \|\hat{\lambda}_{\mathcal{I}_1}^t\|_1 + (z^{t+1})^\top \nabla c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^{t+1}) \hat{\lambda}_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}^t) \right] \\
& \leq \frac{1}{a_k} \cdot (-\delta \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \|\hat{\lambda}^t\|_1 \right] + \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (2\delta \|\lambda_{\mathcal{E}_1}^t\|_1 + \delta \|\hat{\lambda}_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}^t\|_1 + (z^{t+1})^\top \nabla c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^{t+1}) \hat{\lambda}_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}^t) \right]) \\
& \quad + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k 2\delta |\mathcal{E}_1| G \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{2\delta |\mathcal{E}| \Lambda}{a_k} + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (\delta + ZG) \|\hat{\lambda}_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}^t\|_1 + 2\delta |\mathcal{E}_1| G \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{2\delta |\mathcal{E}| \Lambda}{a_k} + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (\delta + ZG) (\|\lambda_{\mathcal{E}_2}^t + \beta_t c_{\mathcal{E}_2}(x^t)\|_1 + \|\lambda_{\mathcal{I}_2 \cup \mathcal{I}_3}^t + \beta_t c_{\mathcal{I}_2 \cup \mathcal{I}_3}(x^t)\|_1) \right] \\
& \quad + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k 2\delta G^{-1} \tilde{L} \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{2\delta |\mathcal{E}| \Lambda}{a_k} + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (\delta + ZG) \|\lambda_{\mathcal{E}_2}^t + \beta_t c_{\mathcal{E}_2}(x^t) - \beta_t c_{\mathcal{E}_2}(x^{t+1})\|_1 \right] \\
& \quad + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (\delta + ZG) \|\lambda_{\mathcal{I}_2 \cup \mathcal{I}_3}^t + \beta_t c_{\mathcal{I}_2 \cup \mathcal{I}_3}(x^t) - \beta_t c_{\mathcal{I}_2 \cup \mathcal{I}_3}(x^{t+1})\|_1 \right] \\
& \quad + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k 2\delta G^{-1} \tilde{L} \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{2\delta |\mathcal{E}| \Lambda}{a_k} + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (\delta + ZG) \|\lambda_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}^t + \beta_t c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^t) - \beta_t c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^{t+1})\|_1 \right] \\
& \quad + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k 2\delta G^{-1} \tilde{L} \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{2\delta |\mathcal{E}| \Lambda}{a_k} + \frac{\delta + ZG}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k (\|\lambda_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}^t\|_1 + \beta_t \|c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^t) - c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^{t+1})\|_1) \right] \\
& \quad + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k 2\delta G^{-1} \tilde{L} \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{m(3\delta + ZG)\Lambda}{a_k} + \frac{\delta + ZG}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \beta_t \|c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^t) - c_{\mathcal{E}_2 \cup \mathcal{I}_2 \cup \mathcal{I}_3}(x^{t+1})\|_1 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k 2\delta G^{-1} \tilde{L} \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{m(3\delta + ZG)\Lambda}{a_k} + \frac{(3\delta + ZG)G^{-1}}{a_k} \cdot \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \tilde{L} \beta_t \|x^{t+1} - x^t\| \right] \\
& \leq -\delta + \frac{m(3\delta + ZG)\Lambda}{a_k} + \frac{(3\delta + ZG)G^{-1}}{a_k} \cdot (\mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k L_{\beta_t}^2 \|x^{t+1} - x^t\|^2 \right])^{\frac{1}{2}} \\
& \leq -\delta + \frac{(3\delta + ZG)(m\Lambda + G^{-1}C_{sum})}{a_k} \leq -\delta + \frac{(3\delta + ZG)(m\Lambda + G^{-1}C_{sum})}{M} =: -\delta + \frac{C_{upper}}{M},
\end{aligned}$$

where $\Lambda > 0$ is a constant such that $\|\lambda^t\| \leq \Lambda$ for any $t \geq 1$ by Lemma 2 and the setting of ρ_t , and the 5-th inequality is due to $c_{\mathcal{E}_2}(x^{t+1}) = 0$ and $c_{\mathcal{I}_2 \cup \mathcal{I}_3}(x^{t+1}) \leq 0$. Summarizing above analysis we obtain $M \leq \delta^{-1}(C_{upper} - C_{lower})$. However, this contradicts the arbitrariness of M . Thus we derive that $\limsup_{k \rightarrow \infty} a_k < +\infty$.

Now we aim to bound $\mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \|\tilde{\lambda}^t\| \right]$. According to the definitions of $\tilde{\lambda}$ and $\hat{\lambda}$, we have

$$\begin{aligned}
\|\tilde{\lambda}^t\| & \leq \|\lambda_{\mathcal{E}}^t + \beta_{t-1}c_{\mathcal{E}}(x^t)\| + \|[\lambda_{\mathcal{I}}^t + \beta_{t-1}c_{\mathcal{I}}(x^t)]_+\| \\
& = \|\lambda_{\mathcal{E}}^t + \beta_{t-1}c_{\mathcal{E}}(x^t) - \beta_t c_{\mathcal{E}}(x^t) + \beta_t c_{\mathcal{E}}(x^t)\| + \|[\lambda_{\mathcal{I}}^t + \beta_{t-1}c_{\mathcal{I}}(x^t) - \beta_t c_{\mathcal{I}}(x^t) + \beta_t c_{\mathcal{I}}(x^t)]_+\| \\
& \leq \|\hat{\lambda}_{\mathcal{E}}^t\| + \|\beta_{t-1}c_{\mathcal{E}}(x^t) - \beta_t c_{\mathcal{E}}(x^t)\| + \|\hat{\lambda}_{\mathcal{I}}^t\| + \|[\beta_{t-1}c_{\mathcal{I}}(x^t) - \beta_t c_{\mathcal{I}}(x^t)]_+\| \\
& \leq 2\|\hat{\lambda}^t\| + mC(\beta_t - \beta_{t-1}).
\end{aligned}$$

Therefore, with the setting of β_t , one has

$$\limsup_{k \rightarrow \infty} \mathbb{E}_{\xi^{[k]}} \left[\frac{1}{k} \sum_{t=1}^k \|\tilde{\lambda}^t\| \right] \leq 2 \limsup_{k \rightarrow \infty} a_k + mC \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=1}^k (\beta_t - \beta_{t-1}) < \infty.$$

The proof is completed.