Two-stage distributionally robust noncooperative games: Existence of Nash equilibria and its application to Cournot–Nash competition

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Abstract

Two-stage distributionally robust stochastic noncooperative games with continuous decision variables are studied. In such games, each player solves a two-stage distributionally robust optimization problem depending on the decisions of the other players. Existing studies in this area have been limited with strict assumptions, such as linear decision rules, and supposed that each player solves a two-stage linear distributionally robust optimization with a specifically structured ambiguity set. This limitation motivated us to generalize and analyze the game in a nonlinear case. The contributions of this study are (i) demonstrating the conditions for the existence of two-stage Nash equilibria under convexity and compactness assumptions, and (ii) consideration of a two-stage distributionally robust Cournot–Nash competition as an application, as well as an investigation into the conditions for the existence of market equilibria in an economic sense. We also report some results of numerical experiments to illustrate how distributional robustness affects the decision of each player in the Cournot–Nash competition.

Keywords — Two-stage distributionally robust Nash games; Stochastic Nash games; Noncooperative game; Cournot–Nash competition; Risk-averse equilibrium
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1 Introduction

A noncooperative game is a mathematical model for competitive situations in which players compete with each other to maximize their own profit. A Nash equilibrium is a tuple of the all players' strategies, by which the players have no incentive to change their strategy to gain more profit, which is one of the key concepts in the theory of noncooperative games. In other words, the Nash equilibrium simultaneously achieves the global optimality for the optimization problem of every player parametrized by the other rivals' strategies. The concept has been used in numerous fields, including microeconomics, management science, and computer science [8, 9, 12, 13, 15, 21, 22, 24].

According to Li et al. [23], methods for finding Nash equilibria are divided into two approaches. The first method is identifying some specific structures of the game, such as the supermodularity [45] or the property in which the profit of each player is characterized by a potential function, called a potential game [28]. The second approach is reformulating the game as a variational inequality [11, 34], which has been extensively studied and is efficient in finding the Nash equilibria of games with continuous decision variables in a uniform manner; to name a few, see [14, 16, 20, 21, 29, 33, 41]. To our best knowledge, the variational inequality reformulation approach was first proposed by Bensoussan [2].

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Noncooperative games in stochastic situations, which includes random variables in the optimization problem of each player, have also been extensively studied for both pure and mixed strategies. It is natural to consider stochastic noncooperative games in which each player makes multistage decisions in response to changes in the conditions. For multistage stochastic games played with in finite action spaces, many approaches have been extensively studied. However, in continuous cases, because multistage variational inequalities have not been well developed until quite recently, few studies on variational inequality approaches have been conducted.

The notion of multistage stochastic variational inequalities was first explicitly presented by Rockafellar and Wets [37]. Rockafellar and Sun [36] then proposed a progressive hedging algorithm for solving it. The method corresponds to the proximal point algorithm [35] for maximally monotone problems with a linear transformation. As an advantage of progressive hedging, its subproblem can be computed in parallel, which drastically reduces the computational time. Based on their developments, several researchers have extended the idea into the cases with continuously underlying random variables [4–6, 18, 19]. In [6, 18, 44, 49, 50], a variational inequality approach for two-stage stochastic noncooperative games was recently described.

Despite these various efforts on noncooperative games and variational inequalities under uncertainty, the ambiguity of the probability distributions have been ignored in both single- and multistage cases. Consider certain cases in which the available data may contain noise, and the number of sample data from an observation is small. An empirical distribution may be used and the above mentioned stochastic approaches can be applied. However, is the Nash equilibrium obtained through such approaches reliable? These questions have motivated researchers to consider distributionally robust stochastic noncooperative games, induced by the recent attention paid to distributionally robust optimization (DRO). Here, DRO aims to minimize the worst-case expected value of a measurable objective function from a set of probability distributions, called an *ambiguity set*. This model is supported by decision-making theory, which states that each player makes a decision based on the *maximin criteria*. The recent progress made in distributionally robust noncooperative games is shown in Table 1.

Table 1: Recent works on distributionally robust games

	Finite strategies (mixed strategies)[1]	Continuous pure strategies[2]
One-stage	Qu and Goh [32] Loizou [26, 27] Peng et al. [31]	Sun and Xu [43] Liu et al. [25] (Sun et al. [42][3])
Two-stage	———	Li et al. [23] (linear case) Chen et al. [6] (Cournot competition) **this paper (nonlinear case)**

One-stage models have been widely applied in both finite and continuous cases. To our best knowledge, Qu and Goh [32] were the first to consider the distributional robustness in noncooperative stochastic games. Its extensions were then discussed by Loizou [26, 27] and Peng et al. [31], both of which demonstrated the reformulation into a tractable optimization problem under specific-structured ambiguity sets. Sun and Xu [43] applied the framework to a continuous case. Liu et al. [25] then showed the conditions for the existence of Nash equilibrium in one-stage distributionally robust continuous games. They also demonstrated that some can be reduced to classic stochastic games in special cases with ambiguity sets.

In comparison with one-stage models, studies on two-stage models are still in their infancy. Li et al. [23] considered a linear case in which each player solves a two-stage distributionally

[1] We state that a mixed strategy with a finite strategy set is a probability assignment of taking from a set of finite pure strategy sets to a polyhedron.

[2] A case in which each player's pure strategy set is given by a subset of the Euclidean space as well as their decision variable takes continuous.

[3] Technical report.

robust linear stochastic programming with a Wiesemann–Kuhn–Sim-type ambiguity set [46]. They demonstrated that under the *linear decision rule* [40], an equilibrium of the game can be obtained by solving a deterministic conic variational inequality. Note that the linear decision rule is merely an assumption tailored to numerical tractability; thus, particularly in nonlinear cases, adopting the rule into the games may be more inaccurate than in linear cases because of the complexity of the decision-making. Chen et al. [6] discussed a two-stage distributionally robust Cournot–Nash competition based on an "ex-post" equilibrium concept corresponding to a distribution-free robust Nash equilibrium presented in [1]. However, an ex-post equilibrium may not exist depending on the ambiguity set even when the two-stage DRO of each player is "well-posed" in a certain sense, for example, convexity and compactness; hence, it may be occasionally an ill-posed problem when one considers high uncertainty situations.

Therefore, the motivation of this paper is to discuss a more general class of two-stage distributionally robust noncooperative games and give a more certain definition for the games based on the concept of Nash equilibria, which is unlike ex-post equilibrium. The contributions of this paper are summarized as follows:

- We consider a two-stage (nonlinear) distributionally robust noncooperative game. We propose a definition of an equilibrium concept based on Nash equilibrium under a general setting and show the existence of an equilibrium point under the continuity, compactness, and convexity of each player's optimization.
- As an application, we revisit the two-stage distributionally robust Cournot–Nash competition introduced in [6] and show the existence of an equilibrium based on the definition Nash equilibrium.
- We conduct a numerical experiment on the Cournot–Nash competition and investigate how distributional robustness affects the two-stage decisions of each player.

This paper is organized as follows. In Section 2, we introduce the model and define a two-stage distributionally robust Nash equilibrium. Then we present the conditions for the existence of the equilibria. In Section 3, we consider a reformulation of the game into a variational inequality for analysis and for the construction of solution methods. In Section 4, we introduce a two-stage distributionally robust Cournot–Nash competition as an application of the game and provide the conditions for the existence of equilibrium in an economic sense. In Section 5, we report the results of some numerical experiments conducted to illustrate how distributional robustness affects the decision-making of each player. In Section 6, we provide some concluding remarks.

Notations: We use the following notations throughout this paper: To avoid complexity in the notations, for vectors $x \in I\!R^n$ and $y \in I\!R^m$, $(x, y) \in I\!R^{n+m}$ denotes $(x^\top, y^\top)^\top$. When the vector $x \in I\!R^n$ consists of the subvectors $x_j \in I\!R^{n_j}$, $j = 1, \dots, N$, we occasionally denote $x = (x_j, x_{-j})$ to emphasize the j-th subvector x_j, where $x_{-j} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N) \in I\!R^{n_{-j}}$, $n := n_1 + \cdots + n_N$, and $n_{-j} := n - n_j$. For a continuously differentiable function $f \colon I\!R^n \times I\!R^m \to I\!R$, $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ denote partial gradients with respect to x and y, respectively. To addition, $I\!R^n_+ := \{x \in I\!R^n \mid x_i \geq 0,\ i = 1, \dots, n\}$ denotes the nonnegative orthant of $I\!R^n$. For two vectors $x \in I\!R^n$ and $y \in I\!R^n$, $x \perp y$ denotes $x^\top y = 0$. Finally, $[\cdot]_+$ represents $\max(0, \cdot)$.

2 Two-stage distributionally robust noncooperative games and the existence of Nash equilibrium

In this section, we introduce a distributionally robust two-stage stochastic noncooperative game. Then we present sufficient conditions under which the equilibrium point exists.

2.1 Model and definition of Nash equilibrium

Let $\xi\colon \Omega \to \Xi \subset I\!R^t$ be a random vector and $\mathscr{P}(\Xi) := \{P \mid P(\Xi) = 1,\ P(\cdot) \geq 0\}$ be a set of any probability measures supported over Ξ, which is equipped with the measurable space $(\Xi, \mathcal{B}(\Xi))$.

We describe the N-person two-stage stochastic noncooperative game considered in this paper. Hereafter, we use the following notations regarding player $j \in \{1, \ldots, N\}$:

- $x_j \in I\!R^{n_j}$, $y_j\colon \Xi \to I\!R^{m_j}$: first- (here-and-now) and second-stage (wait-and-see) strategies of player j, respectively;
- $X_j \subset I\!R^{n_j}$, $Y_j\colon X_j \times \Xi \rightrightarrows I\!R^{m_j}$: first- and second-stage strategy sets of player j, respectively;
- $\theta_j\colon I\!R^n \to I\!R$, $\gamma_j\colon I\!R^m \times I\!R^n \times \Xi \to I\!R$: first- and second-stage cost functions of player j, respectively;
- $\mathcal{Y}_j := \{y_j(\cdot) : y_j(\cdot) \in Y_j(x_j, \cdot) \quad \forall x_j \in X_j\}$: a set of all functions from Ξ to $Y_j(x_j, \xi) \subset I\!R^{m_j}$ for all $x_j \in X_j$;

Here, $n := \sum_{j=1}^N n_j$ and $m := \sum_{j=1}^N m_j$ are the sums of the dimensions for all players' strategy vectors at the first and second stages, respectively, and $X := \Pi_{j=1}^N X_j$ and $\mathcal{Y} := \Pi_{j=1}^N \mathcal{Y}_j$ are the Cartesian products of X_j and $\mathcal{Y}_j$, $j = 1, \ldots, N$.

We suppose that player j minimizes θ_j at the first stage and then minimizes γ_j at the second stage where $\xi \in \Xi$ is observed.

Player j solves the following optimization problem at the second stage:

$$\begin{aligned} Q_j(x_j, x_{-j}, \xi) := \min_{y_j(\xi) \in I\!R^{m_j}} \quad & \gamma_j(y_j(\xi), y_{-j}(\xi), x_j, x_{-j}, \xi) \\ \text{s.t.} \quad & y_j(\xi) \in Y_j(x_j, \xi), \end{aligned} \tag{1}$$

where $y_{-j}(\xi) \in I\!R^{m-m_j}$ indicates the other rival players' strategies. We call $Q_j(\cdot, \cdot, \xi)\colon\ I\!R^n \to I\!R$ a recourse function or an optimal value function at $\xi \in \Xi$. We also suppose that player j does not know which of the scenarios will occur when choosing x_j. Thus, the player tries to minimize the expected value $\mathbb{E}[Q_j(x_j, x_{-j}, \xi)]$ of the recourse function $Q_j(x_j, x_{-j}, \xi)$ before observing $\xi \in \Xi$ under a probability distribution.

However, our highest interest is for a case in which each player does not have strong confidence in the probability distribution (e.g., because of a lack of sample data to determine the distribution). Hence, we consider that they make their decisions based on the DRO framework, namely, the maximin criterion. That is, player j minimizes $\sup_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j, x_{-j}, \xi)]$, where $\mathscr{P}_j \subset \mathscr{P}$ denotes an ambiguity set or a collection of probability distributions from the observed data.

Consequently, the two-stage distributionally robust optimization (DRO) of player j in the first stage is formulated as

$$\begin{aligned} \min_{x_j \in I\!R^{n_j}} \quad & \Theta_j(x_j, x_{-j}) := \{\theta_j(x_j, x_{-j}) + \sup_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j, x_{-j}, \xi)]\} \\ \text{s.t.} \quad & x_j \in X_j, \end{aligned} \tag{2}$$

where $x_{-j} \in I\!R^{n-n_j}$ is a tuple of the other rival players' strategies.

Note that when $\mathscr{P}_j$ is a singleton, (2) reduces to a two-stage stochastic programming problem [30]. Moreover, when $\mathscr{P}_j = \mathscr{P}(\Xi)$, and Ξ is compact, the model (2) coincides with the two-stage (distribution-free) robust optimization since $\mathbb{E}_P[f(\xi)] \leq \max_{\xi \in \Xi} f(\xi)$ for any P and measurable functions f. Thus, the distributionally robust framework may also be regarded as a generalization of stochastic/robust approaches. In addition, Li et al. [23] deal with a case in which θ_j and γ_j are linear with respect to player j's decision variable, and X_j and $Y_j(x_j, \xi)$ are the sets of linear constraints, which means our model is a generalization of theirs.

We consider an equilibrium of the game consisting of (2), $j = 1, \ldots, N$.

Definition 2.1. *The point $(x^*, y^*(\cdot)) \in X \times \mathcal{Y}$ is the simultaneous strategy of the first and second stages, respectively, and is called a* two-stage distributionally robust Nash equilibrium *(TSDRNE) if and only if the following conditions hold for all $j \in \{1, \dots, N\}$:*

$$x_j^* \in \arg\min_{x_j \in X_j} \Theta_j(x_j, x_j^*), \tag{3}$$

$$y_j^*(\xi) \in \arg\min_{y_j(\xi) \in Y_j(x_j^*, \xi)} \gamma_j(y_j(\xi), y_{-j}^*(\xi), x_j^*, x_{-j}^*, \xi) \quad \forall \xi \in \Xi. \tag{4}$$

Note that a similar definition of equilibria is found in Zhang et al. [48, Definition 2.7]. However, under a first-stage condition (3), we consider the distributional robustness in Θ_j.

When two-stage DRO (2) is linear for all j, Li et al. [23] reformulate the two-stage distributionally robust noncooperative games into a deterministic variational inequality. The authors showed that the solution to the variational inequality satisfies (3) and (4), although they did not explicitly introduce the above definition.

2.2 Existence of two-stage Nash equilibrium

We provide some assumptions for the existence of a TSDRNE.

Assumption 2.2. *The following assertions hold for all $j \in \{1, \dots, N\}$:*

(a) *The function θ_j is continuous, and $\theta_j(\cdot, x_{-j})$ is convex for each fixed x_{-j};*

(b) *The feasible set $X_j \subset I\!R^{n_j}$ is compact and convex;*

(c) *The ambiguity set $\mathscr{P}_j$ is weakly compact[4];*

(d) *The function γ_j is continuous, and $\gamma_j(\cdot, y_{-j}(\xi), \cdot, x_{-j}, \xi)$ is jointly convex for each fixed x_{-j}, $y_{-j}(\xi)$ and $\xi \in \Xi$;*

(e) *$Y_j \colon X_j \times \Xi \rightrightarrows I\!R^{m_j}$ is continuous, and $Y_j(x_j, \xi)$ is nonempty (namely, is a relatively complete recourse), compact, and convex for every $(x_j, \xi) \in X_j \times \Xi$.*

Remark 2.1. *The convexity of $Q_j(\cdot, x_{-j}, \xi)$ is stated from Theorem 34 by Birge and Louveaux [3] when the following feasible set $Y_j(x_j, \xi)$ is convex for any $x_j \in X_j$ and $\xi \in \Xi$:*

$$Y_j(x_j, \xi) \coloneqq \left\{ y_j(\xi) \in I\!R^{m_j} : \begin{array}{l} g_j^i(y_j(\xi), x_j, \xi) \le 0, \;\; i = 1, \dots, r_j', \\ g_j^i(y_j(\xi), x_j, \xi) = 0, \;\; i = r_j' + 1, \dots, r_j. \end{array} \right\}.$$

where $g_j^i(\cdot, \cdot, \xi)$, $i = 1, \dots, r_j$, are continuous. The convexity of $Y_j(x_j, \xi)$ is guaranteed if the functions $g_j^i(\cdot, \cdot, \xi)$, $i = 1, \dots, r_j'$, are jointly convex and $g_j^i(\cdot, \cdot, \xi)$, $j = r_j' + 1, \dots, r_j$, are affine with respect to $(y_j(\xi), x_j)$.

For one-stage distributionally robust games, Liu et al. [25] do not assume the convexity of X_j but only its compactness. However, as we can see in the following lemma, the two-stage model also requires the convexity of X_j to ensure the convexity of the recourse function $Q_j(\cdot, x_{-j}, \xi)$.

Lemma 2.3. *Suppose that Assumption 2.2 holds. Then the recourse function $Q_j(\cdot, \cdot, \xi)$ is continuous, and $Q_j(\cdot, x_{-j}, \xi)$ is convex with respect to x_j for every fixed x_{-j} and $\xi \in \Xi$.*

Proof. It suffices to show the above assertion for a specific case in which player j's two-stage DRO is independent of the other rival players' strategies; hence, we omit label j and a tuple of rival players' strategies x_{-j} and $y_{-j}(\xi)$.

By Assumptions 2.2-(d) and (e), the continuity of the recourse function Q holds.

[4] We state a space $\mathscr{A}$ is weakly compact if and only if every sequence $\{P_i\} \subset \mathscr{A}$ contains a subsequence $\{P_{i'}\}$ and $P^* \in \mathscr{A}$ such that $P_{i'}$ weakly converges to P^*.

Next, we show the convexity of $Q(\cdot,\xi)$ for each fixed $\xi \in \Xi$. Let us define $\mathcal{S}(\xi) := Y(\xi) \times X$, where $Y(\xi) := \{y(\xi) \mid y(\xi) \in Y(x,\xi) \quad \forall x \in X\}$, and $\mathcal{S}(\xi)$ is convex for any $\xi \in \Xi$ by Assumptions 2.2-(b) and (e). Suppose that $y^1(\xi) \in Y(x^1,\xi)$ and $y^2(\xi) \in Y(x^2,\xi)$ are optimal solutions to the second stage problem for fixed $x^1 \in X$ and $x^2 \in X$, respectively. For any $\alpha \in (0,1)$, let $(y'(\xi),x') = \alpha(y^1(\xi),x^1) + (1-\alpha)(y^2(\xi),x^2)$, and thus $(y'(\xi),x') \in \mathcal{S}(\xi)$ by the convexity of $\mathcal{S}(\xi)$. It follows from the joint convexity of γ in Assumption 2.2-(d) that

$$Q(x',\xi) \le \gamma(y'(\xi),x',\xi) \le \alpha\gamma(y^1(\xi),x^1,\xi) + (1-\alpha)\gamma(y^2(\xi),x^2,\xi) = \alpha Q(x^1,\xi) + (1-\alpha)Q(x^2,\xi).$$

Therefore, we have completed the proof. □

Combining the continuity of Q_j by Lemma 2.3 and the relatively complete recourse (Assumption 2.2-(e)), $\mathbb{E}_P[Q_j(x_j,x_{-j},\xi)]$ is also continuous and bounded for any $x \in X$ and $P \in \mathscr{P}_j$. Thus, by the weak compactness of $\mathscr{P}_j$ (Assumption 2.2-(c)), there exists $P \in \mathscr{P}_j$ that achieves the maximum value of $\mathbb{E}_P[Q_j(x_j,x_{-j},\xi)]$ for any $x \in X$. Moreover, by the convexity of $Q_j(\cdot,x_{-j},\xi)$ from Lemma 2.3 and $\theta_j(\cdot,x_{-j})$ from Assumption 2.2-(a), we can easily show the convexity of $\Theta_j(\cdot,x_{-j})$, which is stated as follows.

Lemma 2.4. *Suppose that Assumption 2.2 holds. Then the objective function Θ_j in (2) is continuous and convex with respect to $x_j \in X_j$ for any x_{-j}.*

Before introducing the main assertion of this section, we give a key result that states the condition for the existence of Nash equilibria in a one-stage deterministic noncooperative game.

Proposition 2.5 (Rosen [39, Theorem 1]). *Consider a deterministic noncooperative game of N players in which player $j \in \{1,\dots,N\}$ solves the following optimization problem:*

$$\min_{x_j \in \bar{X}_j} \Theta_j(x_j,x_{-j}),$$

where $\bar{X}_j$ is nonempty and compact, and Θ_j is continuous in X and convex with respect to $x_j \in \bar{X}_j$ for every x_{-j}. Then there exists an equilibrium point $x^* \in \bar{X} := \bar{X}_1 \times \cdots \times \bar{X}_N$ such that

$$x_j^* \in \arg\min_{x_j \in \bar{X}_j} \Theta_j(x_j,x_{-j}^*) \qquad \forall j \in \{1,\dots,N\},$$

which is called a Nash equilibrium.

We now show the existence of TSDRNE points under Assumption 2.2.

Theorem 2.6. *Suppose that Assumption 2.2 holds. Then the two-stage distributionally robust Nash equilibrium $(x^*,y^*(\cdot)) \in X \times \mathcal{Y}$ exists.*

Proof. Lemma 2.4 states the continuity of Θ_j and convexity of $\Theta_j(\cdot,x_{-j})$ for each fixed x_{-j}. By Assumption 2.2-(b) and using Proposition 2.5, a Nash equilibrium $x^* \in X$ that satisfies (3) exists.

The existence of the second stage Nash equilibrium $y^*(\xi)$ can be likewise shown. By Assumptions 2.2-(d), (e), and using Proposition 2.5, there exists a Nash equilibrium $y^*(\xi)$, and the assertion holds for each fixed $\xi \in \Xi$, thus implying (4). Therefore, the proof is complete. □

3 Two-stage distributionally robust variational inequality under discrete probability distributions

This section presents a variational inequality reformulation for the condition of the TSDRNE stated in Definition 2.1.

Hereafter, we consider discrete probability cases, where the support set is given by $\Xi := \{\xi_1,\dots,\xi_K\}$, and the probability of ξ_k for player j is represented by $P_j(\xi_k)$. Let $P_j := (P_j(\xi_1),\dots,P_j(\xi_K)) \in$

$\mathbb{R}^K$ and $P := (P_1, \ldots, P_N) \in \mathbb{R}^{NK}$. We also denote $\Delta := \{p \in \mathbb{R}^K \mid \sum_{k=1}^K p_k = 1,\ p \geq 0\}$ as the polyhedron of the probabilities supported on Ξ, and $\mathscr{P} := \Pi_{j=1}^N \mathscr{P}_j$ is the Cartesian product of $\mathscr{P}_j$, $j = 1, \ldots, N$.

First, we give the definition of a two-stage distributionally robust variational inequality, which is inspired by the one-stage version in the technical paper [42].

Definition 3.1. *Let $\mathscr{P}_j \subset \Delta$, $j = 1, \ldots, N$ be convex ambiguity sets supported on Ξ. Suppose that $X \subset \mathbb{R}^n$ is a nonempty closed convex set and that $Y(\xi) \subset \mathbb{R}^m$ is also a nonempty closed convex set for each fixed $\xi \in \Xi$. A* two-stage distributionally robust variational inequality *(TSDRVI) is to find a pair $(x^*, y^*(\cdot)) \in X \times \mathcal{Y}$ and $P^* \in \mathscr{P}$ satisfying the following inclusions:*

$$0 \in \mathbb{E}_{P^*}[F(x^*, y^*(\xi), \xi)] + \mathcal{N}_X(x^*), \tag{5}$$

$$0 \in G(x^*, y^*(\xi), \xi) + \mathcal{N}_{Y(\xi)}(y^*(\xi)) \quad \forall \xi \in \Xi, \tag{6}$$

$$P_j^* \in \arg\max_{P \in \mathscr{P}_j} \mathbb{E}_P[f_j(x_j^*, x_{-j}^*, \xi)], \quad j = 1, \ldots, N, \tag{7}$$

where $F\colon \mathbb{R}^n \times \mathbb{R}^m \times \Xi \to \mathbb{R}^n$, $G\colon \mathbb{R}^n \times \mathbb{R}^m \times \Xi \to \mathbb{R}^m$, $f_j\colon \mathbb{R}^{n_j} \times \Xi \to \mathbb{R}$, and $\mathcal{N}_X(x) := \{z \in \mathbb{R}^n \mid \langle z, w - x\rangle \leq 0\ \forall w \in X\}$ denotes a normal cone of X at $x \in X$. The expectation operator $\mathbb{E}_P[F(x, y(\xi), \xi)]$ is defined as

$$\mathbb{E}_P[F(x, y(\xi), \xi)] := \left(\mathbb{E}_{P_1}[F_1(x, y(\xi), \xi)], \ldots, \mathbb{E}_{P_N}[F_N(x, y(\xi), \xi)]\right),$$

where $\mathbb{E}_{P_j}[F_j(x, y(\xi), \xi)] := \sum_k P_j(\xi_k) \cdot F_j(x, y(\xi_k), \xi_k)$ and $F_j := \mathbb{R}^n \times \mathbb{R}^m \times \Xi \to \mathbb{R}^{n_j}$.

When the ambiguity set $\mathscr{P}_j$ is a singleton for all j, the above TSDRVI reduces to two-stage stochastic variational inequalities [37, 44].

Note that for a variational inequality $0 \in \hat{F}(x) + \mathcal{N}_S(x)$, when X is given by the nonnegative orthant $\mathbb{R}^n_+$, the inclusion reduces to the complementarity $0 \leq x \perp \hat{F}(x) \geq 0$, which suggests that the class of variational inequalities includes complementarity problems.

Remark 3.1. *Chen et al. [6] considered a two-stage distributionally robust linear complementarity problem in the following form:*

$$0 \leq x \perp Ax + \mathbb{E}_P[B(\xi)y(\xi)] + q_1 \geq 0 \quad \forall P \in \mathscr{P}, \tag{8}$$

$$0 \leq y(\xi) \perp M(\xi)y(\xi) + N(\xi)x + q_2(\xi) \geq 0 \quad \textit{for } P\textit{– almost every } \xi \in \Xi, \tag{9}$$

where $A \in \mathbb{R}^{n\times n}, q_1 \in \mathbb{R}^n, B\colon \mathbb{R}^t \to \mathbb{R}^{n\times m}, M\colon \mathbb{R}^t \to \mathbb{R}^{m\times m}, N\colon \mathbb{R}^t \to \mathbb{R}^{n\times m}$ and $q_2\colon \mathbb{R}^t \to \mathbb{R}^m$ are continuous matrix/vector-valued mappings. When $X = \mathbb{R}^n_+$ and $Y(\xi) \equiv \mathbb{R}^m_+$ for all $\xi \in \Xi$ in (5) and (6), the difference between TSDRVI (5)–(7) and (8)–(9) is based on whether the solution $x \in \mathbb{R}^n_+$ to the first stage variational inequality (or the linear complementarity problem) depends on the probability distributions. In addition, we should emphasize that the solution to (8) and (9), called an 'ex-post' equilibrium, *is also the solution to TSDRVI (5)–(6). However, the converse does not hold in general. This suggests that the notion of TSDRVI (5)–(7) is weaker than that of the ex-post equilibrium formulation (8)-(9), which is shown in Appendix A.*

We now show the main assertion of this section.

Theorem 3.2. *Suppose that Assumption 2.2 holds and that $\mathscr{P}_j \subset \Delta$ is convex and compact for all j. The tuple $(x^*, y^*(\cdot))$ is a TSDRNE if and only if there exists $P^* \in \mathscr{P}$ such that $(x^*, y^*(\cdot))$ satisfies the following TSDRVI:*

$$0 \in F_\theta(x^*) + \mathbb{E}_{P^*}[v(x^*, \xi)] + \mathcal{N}_X(x^*), \tag{10}$$

$$0 \in G(x^*, y^*(\xi), \xi) + \mathcal{N}_{Y(x^*, \xi)}(y^*(\xi)) \quad \forall \xi \in \Xi, \tag{11}$$

$$P_j^* \in \arg\max_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j^*, x_{-j}^*, \xi)], \quad j = 1, \ldots, N, \tag{12}$$

where

$$F_\theta(x) \coloneqq [\nabla_{x_j}\theta_j(x_j, x_{-j})]_{j=1}^N,$$
$$v(x,\xi) \in \partial_{x_1} Q_1(x_1, x_{-1}, \xi) \times \cdots \times \partial_{x_N} Q_N(x_N, x_{-N}, \xi) \subset \mathbb{R}^n,$$
$$G(x, y(\xi), \xi) \coloneqq [\nabla_{y_j(\xi)}\gamma_j(y_j(\xi), y_{-j}(\xi), x_j, x_{-j}, \xi)]_{j=1}^N,$$
$$Y(x,\xi) \coloneqq \Pi_{j=1}^N Y_j(x_j, \xi).$$

Proof. We first show the 'only if' part. Let $(x^*, y^*(\cdot))$ be a TSDRNE. By the compactness of $\mathscr{P}_j$, for all $j \in \{1, \ldots, N\}$, there exists a probability vector $P_j^* \in \mathscr{P}_j$ that achieves the maximum of $\mathbb{E}_P[Q_j(x_j^*, x_{-j}^*, \xi)]$.

Now, we focus on the jth two-stage DRO, that is, problem (2) and the right-hand side of (1). Since $(x_j^*, y_j^*(\cdot))$ is the global optimal solution to (2), we have

$$0 \in \nabla_{x_j}\theta_j(x_j^*, x_{-j}^*) + \partial_{x_j}\mathbb{E}_{P_j^*}[Q_j(x_j^*, x_{-j}^*, \xi)] + \mathcal{N}_{X_j}(x_j^*), \tag{13}$$
$$0 \in \nabla_{y_j(\xi)}\gamma_j(y_j^*(\xi), y_{-j}^*(\xi), x_j^*, x_{-j}^*, \xi) + \mathcal{N}_{Y_j(x_j^*,\xi)}(y_j^*(\xi)) \quad \forall \xi \in \Xi, \tag{14}$$
$$P_j^* \in \arg\max_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j^*, x_{-j}^*, \xi)]. \tag{15}$$

Since $Q_j(\cdot, x_{-j}, \xi)$ is convex from Lemma 2.3, it is Clarke regular [7, Definition 2.3.4]. It then follows from [7, Corollary 3 (p.40)] that, for all $P_j \in \mathscr{P}_j$,

$$\partial_{x_j}\mathbb{E}_{P_j}[Q_j(x_j^*, x_{-j}^*, \xi)] = \mathbb{E}_{P_j}[\partial_{x_j} Q_j(x_j^*, x_{-j}^*, \xi)].$$

Hence, (13) implies that there exists $v_j(x_j^*, x_{-j}^*, \xi) \in \partial_{x_j} Q_j(x_j^*, x_{-j}^*, \xi)$ such that

$$0 \in \nabla_{x_j}\theta_j(x_j^*, x_{-j}^*) + \mathbb{E}_{P_j^*}[v_j(x_j^*, x_{-j}^*, \xi)] + \mathcal{N}_{X_j}(x_j^*). \tag{16}$$

It then follows from (16), (14), and (15) that (10)–(12) holds. This completes the proof of the 'only if' part.

Next, we show the 'if' part. Suppose that there exists $(x^*, y^*(\cdot))$ and $P^* \in \mathscr{P}$ that satisfy TSDRVI (10)–(12). In the first-stage variational inequality (10), by the definition of the normal cone, (10) implies (16) for all j. Since first-stage problem (2) is convex by Assumption 2.2-(b), (16) is the necessary and sufficient condition for the optimality of player j's first-stage optimization (2). Then, $x^* \in X$ satisfies condition (3) of the first-stage Nash equilibrium. Likewise, we can show that $y^*(\xi)$ satisfies condition (4) of the second-stage Nash equilibrium for every $\xi \in \Xi$. Therefore, $(x^*, y^*(\cdot))$ is a TSDRNE. The proof is completed. □

Unfortunately, solution methods for solving two-stage stochastic variational inequalities under a distributional ambiguity have yet to be established. However, a TSDRNE can be obtained using recent results on two-stage stochastic variational inequalities. For such an example, we will present the progressive hedging algorithm [36] in Section 5.1 for the numerical experiments.

4 Application to Cournot–Nash competition

In this section, we consider a two-stage distributionally robust Cournot–Nash competition in an oligopoly market and investigate the conditions for the existence of market equilibria in an economic sense.

First, let us distinguish conventional works from our study relating to two-stage Cournot–Nash competitions under uncertainty. The most similar two-stage distributionally robust Cournot–Nash competition was analyzed by Chen et al. [6] from the viewpoint of an ex-post equilibrium, i.e., the solution to (8) and (9) as shown in Remark 3.1. However, they do not provide the sufficient conditions for the existence of a solution satisfying (8) and (9). Also note that another similar two-stage stochastic Cournot–Nash competition under which the probability distribution is exactly known was developed by Zhang et al. [48] and Xu et al. [47]. The authors introduced a class of stochastic equilibrium problems with equilibrium constraints (SEPEC) to analyze the competition and find the market equilibria. However, solving SEPECs has some numerical difficulties, such as nonconvexity and nonmonotonicity, and hence it is difficult to find a global Nash equilibrium in general. In addition, an SEPEC approach requires the second-stage problem to have a unique solution for any given first-stage variables and $\xi \in \Xi$ in practice. As seen later, we do not need such uniqueness of the second-stage equilibrium for our model.

We now move on to our model. Consider an oligopoly market of N firms who compete in investing and supplying homogeneous products. In the first stage, firm j determines the upper capacity $x_j \geq 0$ of the product without certain information regarding the market demand in the future, and the investment cost of firm j for the capacity is defined by $\theta_j(x_j)$. Here, suppose that the future demand is only characterized by an inverse demand function $p(q(\xi), \xi)$, where $q(\xi) := \sum_{j=1}^{N} y_j(\xi)$ is the aggregate quantity of products in the market under scenario $\xi \in \Xi$. Firm $j \in \{1, \ldots, N\}$ tries to maximize the following worst-case expected profit from the ambiguity set $\mathscr{P}_j$:

$$\begin{aligned} \max_{x_j \in I\!R} \quad & \Theta_j(x_j) := \{\min_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j, \xi)] - \theta_j(x_j)\} \\ \text{s.t.} \quad & x_j \geq 0, \end{aligned} \tag{17}$$

where $Q_j(x_j, \xi)$ is the optimal value of the profit maximization under scenario $\xi \in \Xi$:

$$\begin{aligned} \max_{y_j(\xi) \in I\!R} \quad & p(q(\xi), \xi) y_j(\xi) - H_j(y_j(\xi), \xi) \\ \text{s.t.} \quad & 0 \leq y_j(\xi) \leq x_j, \end{aligned} \tag{18}$$

where $H_j(y_j(\xi), \xi)$ is a cost for supplying the product to the market.

Following Xu [47], we now make certain assumptions to ensure the convexity of the problems.

Assumption 4.1. *For all j, $\theta_j(x_j)$ and $H_j(y_j(\xi), \xi)$ are twice continuously differentiable with respect to x_j and $y_j(\xi)$, respectively, and*

$$\begin{aligned} & \theta_j'(x_j) \geq 0, \ \theta_j''(x_j) \geq 0 \quad \text{for} \quad x_j \geq 0, \\ & H_j'(y_j(\xi), \xi) \geq 0, \ H_j''(y_j(\xi), \xi) \geq 0 \quad \text{for} \quad y_j(\xi) \geq 0 \text{ and } \xi \in \Xi. \end{aligned}$$

Assumption 4.2. *The inverse demand function $p(q(\xi), \xi)$ satisfies the following conditions:*

- *$p(q(\xi), \xi)$ is twice continuously differentiable in $q(\xi)$ and $p_q'(q(\xi), \xi) < 0$ for $q(\xi) \geq 0$ and $\xi \in \Xi$;*
- *$p_q'(q(\xi), \xi) + q(\xi) p_{qq}''(q(\xi), \xi) \leq 0$ for $q(\xi) \geq 0$ and $\xi \in \Xi$.*

Under the above assumptions, Xu [47] established the following result.

Proposition 4.3 (Xu [47, Proposition 2.6])**.** *Suppose that Assumption 4.2 holds. Then the following assertions hold: For a fixed $\bar{q} \geq 0$,*

i. *$p_q'(q(\xi) + \bar{q}, \xi) + q p_{qq}''(q(\xi) + \bar{q}, \xi) \leq 0$ for $q(\xi) \geq 0$ and $\xi \in \Xi$;*

ii. *$q(\xi) p(q(\xi) + \bar{q}, \xi)$ is strictly concave in $q(\xi)$ for $q(\xi) \geq 0$ and $\xi \in \Xi$.*

Using the above results, it is easy to see that the first- and second-stage optimization problems of firm j are convex with respect to x_j and $y_j(\xi)$ for $\xi \in \Xi$, respectively (i.e., the objective functions and strategy sets at each stage are concave and convex, respectively).

Under convexity Assumptions 4.1 and 4.2, the necessary and sufficient condition for the optimality of problem (17) of firm j is written as

$$0 \in \theta_j'(x_j) - \mathbb{E}_{P_j}[\partial_{x_j} Q_j(x_j,\xi)] + \mathcal{N}_{[0,\infty)}(x_j), \tag{19}$$

$$\begin{aligned} 0 \le y_j(\xi) \perp H_j'(y_j(\xi),\xi) - p(q(\xi),\xi) \\ - p_q'(q(\xi),\xi))y_j(\xi) + \lambda_j(\xi) \ge 0 \quad \forall \xi \in \Xi, \end{aligned} \tag{20}$$

$$0 \le \lambda_j(\xi) \perp x_j - y_j(\xi) \ge 0 \quad \forall \xi \in \Xi, \tag{21}$$

$$P_j \in \arg\min_{P\in\mathscr{P}_j} \mathbb{E}_P[Q_j(x_j,\xi)], \tag{22}$$

where $\lambda_j(\xi)$ is the Lagrange multiplier for $y_j(\xi) \le x_j$. By the result of [6], $\partial_{x_j} Q_j(x_j,\xi)$ is calculated as

$$\partial_{x_j} Q_j(x_j,\xi) = \begin{cases} \lambda_j(\xi) & \text{if } x_j > 0, \\ \{\lambda_j(\xi) : \lambda_j(\xi) \ge [p(q(\xi),\xi) - H_j'(0,\xi)]_+\} & \text{if } x_j = 0. \end{cases}$$

Note that when $x_j > 0$, $\lambda_j(\xi)$ uniquely exists from the linear independence constraint qualification of problem (18).

Summarizing both cases finally yields the following two-stage distributionally robust variational inequality:

$$0 \le x_j \perp \theta_j'(x_j) - \mathbb{E}_P[\lambda_j(\xi)] \ge 0, \tag{23}$$

$$\begin{aligned} 0 \le y_j(\xi) \perp H_j'(y_j(\xi),\xi) - p(q(\xi),\xi) - \\ p_{y_j}'(q(\xi),\xi))y_j(\xi) + \lambda_j(\xi) \ge 0 \quad \forall \xi \in \Xi, \end{aligned} \tag{24}$$

$$0 \le \lambda_j(\xi) \perp x_j - y_j(\xi) \ge 0 \quad \forall \xi \in \Xi, \tag{25}$$

$$P_j \in \arg\min_{P\in\mathscr{P}_j} \mathbb{E}_P[Q_j(x_j,\xi)]. \tag{26}$$

Using the variational inequality reformulation above, we can state the following condition for the nontriviality of x_j in the competition.

Proposition 4.4. *For firm j, if the following inequality holds, then $x_j > 0$.*

$$\theta_j'(0) < \mathbb{E}_P[p(q_{-j}(\xi),\xi) - H_j'(0,\xi)] \quad \forall P \in \mathscr{P}_j, \tag{27}$$

where $q_{-j}(\xi) \coloneqq \sum_{j'\neq j}^N y_{j'}(\xi)$.

Proof. We show by contradiction: Assume that $x_j = 0$. The optimality condition for two-stage DRO of firm j is written as (23)–(26). By the assumption of $x_j = 0$, we have $y_j(\xi) \equiv 0$ for all $\xi \in \Xi$, and then we can reduce (23)–(25) to

$$\begin{aligned} \theta_j'(0) - \mathbb{E}_P[\lambda_j(\xi)] \ge 0, \\ \lambda_j(\xi) \ge [p(q_{-j}(\xi),\xi) - H_j'(0,\xi)]_+ \quad \forall \xi \in \Xi. \end{aligned} \tag{28}$$

It follows from the second inequality that

$$\mathbb{E}_P[\lambda_j(\xi)] \ge \mathbb{E}_P[p(q_{-j}(\xi),\xi) - H_j'(0,\xi)] \quad \forall P \in \mathscr{P}_j.$$

Hence, the above inequality, (28), and (27) yield

$$0 \le \theta_j'(0) - \mathbb{E}_P[\lambda_j(\xi)] \le \theta_j'(0) - \mathbb{E}_P[p(q_{-j}(\xi),\xi) - H_j'(0,\xi)] < 0 \quad \forall P \in \mathscr{P}_j.$$

This is a contradiction, and hence the proof is complete. □

Proposition 4.4 indicates that if the worst-case expected marginal profit of firm j at $x_j = 0$ ($y_j(\xi) \equiv 0$ for all $\xi \in \Xi$) is greater than the first-stage marginal cost, the firm has an incentive to invest at least a small number of products.

To ensure the existence of a TSDRNE in the Cournot–Nash competition, we need the following extra assumption.

Assumption 4.5. *For all j, there exists $\bar{x}_j \geq 0$ and $P_j \in \mathscr{P}_j$ such that*

$$\mathbb{E}_{P_j}[p(y_j(\xi), \xi)] < \theta_j'(x_j) \quad \text{for } x_j \geq \bar{x}_j \ (0 \leq y_j(\xi) \leq x_j \ \forall \xi \in \Xi), \tag{29}$$

where $P_j \in \arg\min_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j, \xi)]$.

In fact, Assumption 4.5 is not special when the law of increasing marginal costs holds in the market.

We now show the existence of the market equilibrium.

Theorem 4.6. *Suppose that Assumptions 4.1, 4.2, and 4.5 hold. Then a TSDRNE of the Cournot–Nash competition exists.*

Proof. Note that this game satisfies Assumption 2.2 except the compactness of the first-stage constraint set of each firm. If the set is compact, then a TSDRNE of the competition exists from Theorem 2.6. Thus, it suffices to show that there exists a finite number M_j such that

$$\sup_{x_j \geq 0} \Theta_j(x_j) = \max_{0 \leq x_j \leq M_j} \Theta_j(x_j).$$

By Assumptions 4.1 and 4.2, we have

$$\begin{aligned}[p_q'(q(\xi), \xi) y_j(\xi) + p(q(\xi), \xi) - H_j'(y_j(\xi), \xi)]_+ &\leq p(q(\xi), \xi) \\ &\leq p(y_j(\xi), \xi) \quad \forall \xi \in \Xi.\end{aligned}$$

Then the above inequalities also hold regarding their expected values under the probability distribution $P_j \in \arg\min_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j, \xi)]$.

Now we can take $\lambda_j(\xi) = [p_q'(q(\xi), \xi) y_j(\xi) + p(q(\xi), \xi) - H_j'(y_j(\xi), \xi)]_+$ for all $\xi \in \Xi$ by considering (23)–(25), and then it follows from Assumption 4.5 that

$$\mathbb{E}_{P_j}[\lambda_j(\xi)] \leq \mathbb{E}_{P_j}[p(y_j(\xi), \xi)] < \theta_j'(x_j) \quad \text{for } x_j \geq \bar{x}_j.$$

This implies that there exists $x_j \leq \bar{x}_j$ such that the first stage optimality (23) holds; that is,

$$\sup_{x_j \geq 0} \Theta_j(x_j) = \max_{0 \leq x_j \leq \bar{x}_j} \Theta_j(x_j).$$

We have thus completed the proof. □

Theorem 4.6 suggests that (even if firm j monopolized the market) if the marginal cost exceeds the expected marginal revenue (market price per one unit) under the worst-case probability for a certain $x_j \geq \bar{x}_j$, the firm has no incentive to invest more than $\bar{x}_j$; hence, an equilibrium point exists in the market.

In practice, the result of the existence of a TSDRNE can be easily obtained by assuming $x_j \leq M_j$ for a large number $M_j > 0$ in the first stage constraint of firm j. As a benefit of the absence of such a capacity limit, the first-stage complementarity condition does not require an additional Lagrange multiplier for the upper constraint $x_j \leq M_j$. In addition, Assumption 4.5 is meaningful for analyzing the detailed economic behavior of each firm.

5 Numerical experiments

In this section, we report some results of numerical experiments conducted to investigate the characteristics of the TSDRNE in the two-stage distributionally robust Cournot–Nash competition presented in Section 4. First, we consider a more specific case and provide a TSDRVI reformulation of the competition. We then provide a solution method for solving the TSDRVI. Finally, we report the results of the numerical experiments conducted.

Consider a duopoly market, i.e., $N = 2$. Recall that each firm competes with each other to maximize the worst-case expected profit: In the first stage, firm $j \in \{1, 2\}$ solves

$$\max_{x_j \geq 0} \quad \big\{ \min_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j, \xi)] - \theta_j(x_j) \big\}, \tag{30}$$

and the second-stage optimization is defined as

$$Q_j(x_j, \xi) \coloneqq \max_{0 \leq y_j(\xi) \leq x_j} \{ p(q(\xi), \xi) y_j(\xi) - H_j(y_j(\xi), \xi) \}. \tag{31}$$

In this experiment, the cost functions and inverse demand are given as follows:

$$\begin{aligned} \theta_j(x_j) &\coloneqq \frac{1}{2} a_j x_j^2 + b_j x_j + c_j \quad (a_j, b_j, c_j > 0), \\ H_j(y_j(\xi), \xi) &\coloneqq \frac{1}{2} \eta_j(\xi) y_j(\xi)^2 + \zeta_j(\xi) y_j(\xi) + s_j(\xi) \\ &\qquad\qquad\qquad\qquad (\eta_j(\xi), \zeta_j(\xi), s_j(\xi) > 0) \quad \forall \xi \in \Xi, \\ p(q(\xi), \xi) &\coloneqq \alpha(\xi) - \beta(\xi) q(\xi) \quad (\alpha(\xi) > \beta(\xi) > 0) \quad \forall \xi \in \Xi. \end{aligned}$$

Note that Assumptions 4.1, 4.2, and 4.5 hold for this case; hence, a TSDRNE of the Cournot–Nash competition exists in this competition.

Suppose that Ξ consists of K scenarios, i.e., $\Xi = \{\xi_1, \ldots, \xi_K\}$. Let the ambiguity set $\mathscr{P}_j$ of firm j be defined as follows:

$$\mathscr{P}_j \coloneqq \{ P \in \Delta \mid \mathbb{D}_{\mathrm{KL}}(P \| P_{0j}) \leq \rho_j \},$$

where $\rho_j \geq 0$, P_{0j} is a nominal (empirical) probability distribution of firm j, and $\mathbb{D}_{\mathrm{KL}}(\cdot \| \cdot)$ denotes the Kullback–Leibler (KL) divergence by

$$\mathbb{D}_{\mathrm{KL}}(P \| Q) \coloneqq \sum_{k=1}^{K} p_k \log \left(\frac{p_k}{q_k} \right),$$

where p_k and q_k are the probabilities when $\xi = \xi_k$ under the distributions P and Q, respectively, and also $\mathscr{P}_j$ is a convex set.

Since problems (30) and (31) are convex with respect to x_j and $y_j(\xi)$, respectively, the necessary and sufficient optimality conditions of firm j can be written as follows:

$$0 \leq x_j^* \perp a_j x_j^* + b_j - \mathbb{E}_{P_j^*}[\lambda_j^*(\xi)] \geq 0, \tag{32}$$

$$\begin{aligned} 0 \leq y_j^*(\xi) \perp (\eta_j(\xi) + 2\beta(\xi)) y_j^*(\xi) + \lambda_j^*(\xi) + \beta(\xi) \sum_{j' \neq j} y_{j'}^*(\xi) - \\ \alpha(\xi) + \zeta_j(\xi) \geq 0 \quad \forall \xi \in \Xi, \end{aligned} \tag{33}$$

$$0 \leq \lambda_j^*(\xi) \perp x_j^* - y_j^*(\xi) \geq 0 \quad \forall \xi \in \Xi, \tag{34}$$

$$P_j^* \in \arg\min_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j^*, \xi)]. \tag{35}$$

Gathering the systems (32)–(34) and (35) of all firms, the condition for the TSDRNE can be written as the following TSDRVI:

$$\begin{aligned} 0 \leq x^* \perp Ax^* - \mathbb{E}_{P^*}[\lambda^*(\xi)] + b \geq 0, \\ 0 \leq y^*(\xi) \perp \Pi(\xi) y^*(\xi) + \lambda^*(\xi) + r(\xi) \geq 0 \quad \forall \xi \in \Xi, \\ 0 \leq \lambda^*(\xi) \perp x^* - y^*(\xi) \geq 0 \quad \forall \xi \in \Xi, \end{aligned} \tag{36}$$

$$P_j^* \in \arg\min_{P \in \mathscr{P}_j} \mathbb{E}_P[Q_j(x_j, \xi)], \quad j = 1, \ldots, N. \tag{37}$$

where

$$A := \mathrm{diag}(a_1, a_2),\ b := (b_1, b_2)^\top,\ \Pi(\xi) := \mathrm{diag}(\eta_1(\xi), \eta_2(\xi)) + \beta(\xi)(I + \mathbf{1}\mathbf{1}^\top)$$
$$\mathbf{1} := (1,1)^\top \in I\!R^2,\ r(\xi) := (r_1(\xi), r_2(\xi)),\ r_j(\xi) := \zeta_j(\xi) - \alpha(\xi),\ j = 1, 2.$$

Note that since each firm's optimization problem and the ambiguity set $\mathscr{P}_j$ are convex, a solution of TSDRVI (36) and (37) is a TSDRNE of the Cournot–Nash competition by Theorem 3.2.

5.1 Solution method using progressive hedging

Here, an algorithm based on the progressive hedging is presented to solve TSDRVI (36) and (37).

The progressive hedging algorithm (PHA) for multistage stochastic variational inequalities was recently developed by Rockafellar and Sun [36] as an extension of [38] for solving multistage stochastic programming. The benefit of using the PHA is to reduce the computational complexity by solving the variational inequalities for each scenario $\xi \in \Xi$ in parallel.

Preserving the computational efficiency, our idea is to alternately solve (36) and (37) because if we fix a probability distribution, the two-stage stochastic variational inequality (36) can be solved by the PHA. The main loop of the proposed method is given in Algorithm 1, and its inner loop is shown in Algorithm 2, where the regularization parameter $\sigma > 0$ determines the performance of the algorithm and depends on numerical instances.

In practice, the sample data ξ used in the algorithm needs to follow a *reference probability distribution* which is an empirical distribution in a certain sense; hence, we assume that the ambiguity set consists of probability distributions that include the reference probability distribution. We may also suppose that the true probability distribution is included in $\mathscr{P}_j$.

Algorithm 1 Main loop: Solve TSDRVI

Input: $(x^0, y^0(\xi_1), \ldots, y^0(\xi_K), \lambda^0(\xi_1), \ldots, \lambda^0(\xi_K))$.
Output: $(x^*, y^*(\xi_1), \ldots, y^*(\xi_K), \lambda^*(\xi_1), \ldots, \lambda^*(\xi_K))$

1: Set $\nu = 0$, $x^\nu = x^0$, $y^\nu(\xi_k) = y^0(\xi_k)$, and $\lambda^\nu(\xi_k) = \lambda^0(\xi_k)$ for all k.

2: For each j, solve

$$P_j^{\nu+1} \in \arg\max_{P \in \mathscr{P}_j} \mathbb{E}_P[p(q^\nu(\xi), \xi) y_j^\nu(\xi) - H_j(y_j^\nu(\xi), \xi)].$$

3: Stop if $(x^\nu, y^\nu(\xi_1), \ldots, y^\nu(\xi_K), \lambda^\nu(\xi_1), \ldots, \lambda^\nu(\xi_K))$ and $P^{\nu+1}$ approximately satisfies (36) and (37).

4: Solve two-stage SVI (36) for a fixed $P^{\nu+1}$ using Algorithm 2 and obtain a solution $(x^{\nu+1}, y^{\nu+1}(\xi_1), \ldots, y^{\nu+1}(\xi_K), \lambda^{\nu+1}(\xi_1), \ldots, \lambda^{\nu+1}(\xi_K))$.

5: Set $\nu = \nu + 1$ and go to line 2.

In this experiment, the worst-case probability distribution in line 2 of Algorithm 1 is computed by the following formula [17] to reduce the computational burden:

$$P_j^{\nu+1}(\xi_k) := P_{0j}(\xi_k) \cdot \frac{g(y_j^\nu(\xi_k), y_{-j}^\nu(\xi_k), \alpha_j^{\nu+1})}{\mathbb{E}_{P_0}\left[g(y_j^\nu(\xi), y_{-j}^\nu(\xi), \alpha_j^{\nu+1})\right]} \quad \forall k,\ j = 1, \ldots, N,$$

where

$$\alpha_j^{\nu+1} := \arg\min_{\alpha \ge 0} \left\{\alpha \log \mathbb{E}_{P_0}[g(y_j^\nu(\xi), y_{-j}^\nu(\xi), \alpha)] + \alpha\rho_j\right\},$$
$$g(y_j(\xi), y_{-j}(\xi), \alpha) := \exp\left(\frac{p(q(\xi), \xi) y_j(\xi) - H_j(y_j(\xi), \xi)}{\alpha}\right).$$

We adopt the following stopping criterion for Algorithm 1:

$$\|\min(z, Mz + h)\|_2 \le \epsilon = 1.0 \times 10^{-6},$$

Algorithm 2 Progressive Hedging Algorithm

Input: $(x^{\nu,0}, y^{\nu,0}(\xi_1), \dots, y^{\nu,0}(\xi_K), \lambda^{\nu,0}(\xi_1), \dots, \lambda^{\nu,0}(\xi_K))$, $P^{\nu+1}$, and $w^0(\xi_1), \dots, w^0(\xi_K)$ such that $\sum_k w^0(\xi_k) = 0$.

Output: $(x^{\ell+1}, y^{\ell+1}(\xi_1), \dots, y^{\ell+1}(\xi_K), \lambda^{\ell+1}(\xi_1), \dots, \lambda^{\ell+1}(\xi_K))$

1: Set $\ell = 0$, $x^\ell(\xi_k) = x^{\nu,0}$, $y^\ell(\xi_k) = y^{\nu,0}(\xi_k)$ and $\lambda^\ell(\xi_k) = \lambda^{\nu,0}(\xi_k)$ for all k. Let $x^\ell := x^{\nu,0}$.

2: Stop if $(x^\ell, y^\ell(\xi_1), \dots, y^\ell(\xi_K), \lambda^\ell(\xi_1), \dots, \lambda^\ell(\xi_K))$ satisfies a stopping criterion.

3: For each scenario $\xi \in \Xi$, obtain a unique solution $(\hat{x}^\ell(\xi), \hat{y}^\ell(\xi), \hat{\lambda}^\ell(\xi))$ to

$$\begin{aligned}
0 \le x(\xi) &\perp Ax(\xi) - \lambda(\xi) + b + w^\ell(\xi) + \sigma(x(\xi) - x^\ell(\xi)) \ge 0\\
0 \le y(\xi) &\perp \Pi(\xi)y(\xi) + \lambda(\xi) + r(\xi) + \sigma(y(\xi) - y^\ell(\xi)) \ge 0\\
0 \le \lambda(\xi) &\perp x(\xi) - y(\xi) + \sigma(\lambda(\xi) - \lambda^\ell(\xi)) \ge 0
\end{aligned}$$

4: Let $\bar{x}_j^{\ell+1} := \sum_{k=1}^K P_j^{\nu+1}(\xi_k) \cdot \hat{x}^\ell(\xi_k)$ for $j = 1, 2$, and for all k, let

$$x^{\ell+1}(\xi_k) := x^{\ell+1} := \bar{x}^{\ell+1},\ y^{\ell+1}(\xi_k) := \hat{y}^\ell(\xi_k),\ \lambda^{\ell+1}(\xi_k) := \hat{\lambda}^\ell(\xi_k),$$
$$w^{\ell+1}(\xi_k) := w^\ell(\xi_k) + \sigma(\hat{x}^\ell(\xi_k) - \bar{x}^{\ell+1})$$

5: Set $\ell := \ell + 1$ and go to line 2.

where

$$M := \begin{bmatrix} A & B_1 & \dots & B_K \\ E & D_1 & & \\ \vdots & & \ddots & \\ E & & & D_K \end{bmatrix},\ h := [b^\top, \hat{h}_1^\top, \dots, \hat{h}_K^\top]^\top$$

$$B_k := [0,\ -\mathrm{diag}(P_1(\xi_k), P_2(\xi_k))],\ E := [0,\ I]^\top,\ D_k := \begin{bmatrix} \Pi(\xi_k) & I \\ -I & 0 \end{bmatrix},$$
$$\hat{h}_k := (r(\xi_k)^\top, 0^\top)^\top,\ z := (x, y(\xi_1), \dots, y(\xi_K), \lambda(\xi_1), \dots, \lambda(\xi_K)).$$

The tolerance of the inner iteration in Algorithm 2 is 10^{-8}, and the regularization parameter is set as $\sigma = 0.8$. Note that $\min(z, Mz + h) = 0$ is equivalent to two-stage stochastic variational inequality (36) for a fixed probability distribution.

5.2 Numerical results

We use the PATH5 solver [10] for MATLAB to obtain the solution to the variational inequalities in line 3 in Algorithm 2, the tolerance of which is 10^{-9}. Throughout this paper, we carry out the experiments on a computer with an Intel Xeon 2.10 GHz CPU, 128 GB of RAM, and 64-bit Windows 10 OS.

We use the following data for the model:

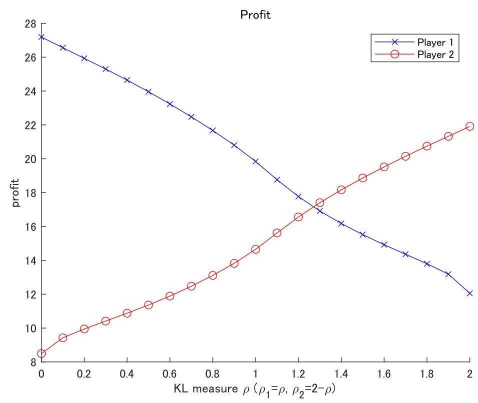
$$\begin{gathered}
K = 60, \Xi = [-1, 1]^3, a_1 = 0.0874, a_2 = 0.1767, b_1 = 1.7162, b_2 = 1.9021,\\
c_1 = 0.5212, c_2 = 0.8314, \alpha(\xi) = 20 + 5\xi_1, \beta(\xi) = 2 + \xi_2, \eta_1(\xi) = 0.2065 + 0.1\xi_1,\\
\eta_2(\xi) = 0.2700 + 0.1\xi_1, \zeta_1(\xi) = 0.5598 + 0.1\xi_1, \zeta_2(\xi) = 0.9748 + 0.1\xi_1,\\
s_1(\xi) = 0.1602 + 0.1\xi_1, s_2(\xi) = 0.1932 + 0.1\xi_1,\ \rho_1 = \rho \ge 0,\ \rho_2 = 2 - \rho \ge 0,
\end{gathered}$$

where $\xi \in I\!R^3$, $\xi_i := -1 + 2\xi_i^0$, $i = 1, 2, 3$, and the reference probability distribution of the random variable $\xi^0 \in [0, 1]^3$ is $\mathtt{BETA(3,3)}$.

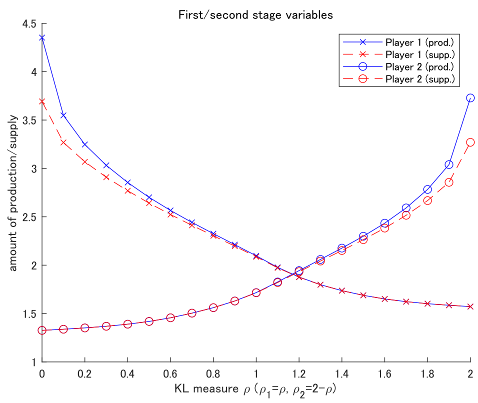
To avoid a sample dependence of ξ, we conducted 30 trials for each $\rho \in \{0, 0.1, 0.2, \ldots, 2.0\}$ by changing the sample data ξ during each trial and plotting the average results.

The numerical results are shown in Figures 1a–1d. Figures 1a and 1b represent the profit and strategy $(x_j^*, \bar{y}_j^*)$ for each player, respectively, where $\bar{y}_j^* := (y_j^*(\xi_1) + \cdots + y_j^*(\xi_K))/K$ denotes the average value of $y_j^*(\xi_k)$, $k = 1, \ldots, K$. These values decrease as ρ_j increases. It is noteworthy that the difference $x_j^* - \bar{y}_j^*$ also decreases as ρ_j increases; that is, when ρ_j is large, the difference $x_j^* - y_j^*(\xi)$ is zero for any scenario $\xi \in \Xi$. In addition, as Figure 1c indicates, the decrease of $x_j^* - \bar{y}_j^*$ also affects the slope of the curve for the average shadow price $\bar{\lambda}_j^* := (\lambda_j^*(\xi_1) + \cdots + \lambda_j^*(\xi_K))/K$. When the difference $x_j^* - \bar{y}_j^*$ is zero, and the difference $x_{-j}^* - \bar{y}_{-j}^*$ of the rival firm is positive, the curve of $\bar{\lambda}_j^*$ is decreasing, which means that firm j has a passive involvement in the market because the rival firm has much more market information than firm j. This also implies a monopolization by the rival firm. Note that under a catastrophe (e.g., economic crisis) in the market, the rival firm may incur significant losses, whereas firm j does not lose much in comparison.

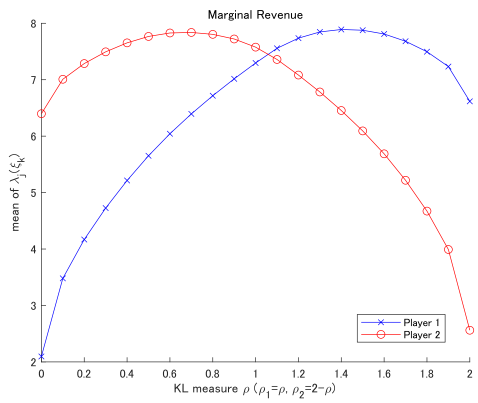
Figure 1d represents the curve of the maximum of the absolute directional derivative $|\delta^\top Q_j(x_j^*, \cdot)|$ for the expected value $\mathbb{E}_{P_j^*}[Q_j(x_j^*, \xi)]$ with respect to the perturbed probability $\delta \in I\!R^K$ subject to $\sum \delta_k = 0$ (because $P_j^* + \delta$ must be included in Δ under $\sum p_k^{*,j} = 1$). As the absolute value $|\delta^\top Q_j(x_j^*, \cdot)|$ decreases, it suggests that the solution x_j^* is robust regarding the perturbation of the probability distribution. Eventually, a small directional derivative indicates that the performance of the out-of-sample validation is less sensitive.



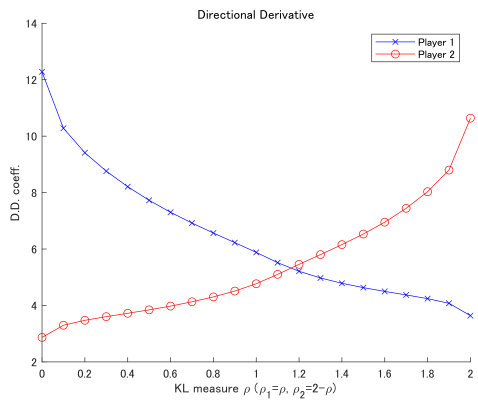
(a) Profit at TSDRNE



(b) Strategies



(c) Shadow price $\bar{\lambda}_j^*$



(d) Directional derivative

Figure 1: Results of numerical experiments in the Cournot–Nash competition ($N = 2$)

6 Conclusion

This paper discussed a class of nonlinear two-stage distributionally robust noncooperative games and demonstrated the existence of TSDRNE points under convexity, compactness, and continuity. We introduced a two-stage distributionally robust variational inequality to construct a solution method for finding an equilibrium point. We also considered a two-stage distributionally robust Cournot–Nash competition as an application and showed the existence of equilibria.

Two challenges still exist: 1) More efficient algorithms should be established to find a solution to TSDRVI, which also guarantees a global convergence. 2) The idea should be extended to a case in which the random variable ξ follows a continuous probability distribution, and the convergence of its discrete approximation methods should be analyzed.

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A Example in a one-stage stochastic game

As a case in which a distributionally robust Nash equilibrium exists whereas the ex-post equilibrium does not, consider the following example borrowed from [25].

Example A.1 (Boxed pig (for details, see Liu et al. [25, Example 2.1]))**.** *A large pig and a piglet are placed in a space with a lever at one end of the space and food dispenser at the other end. The pig that presses the lever must run to the other side to eat, and by the time it gets there, the other pig has eaten most, but not, all of the food. The large pig is dominant, and the piglet is subordinate. Therefore, the large pig is able to prevent the piglet from getting any food when both are at the food. When the large pig presses the lever, a disutility of* $\alpha = 6$ *units will be incurred for the large pig and* $\alpha = 2$ *units for the piglet (which can be interpreted as the energies to be consumed), and* ξ *units (a random variable taking integer values) of food will be released at the dispenser.*

The pigs have two choices, whether to press the lever or wait at the dispenser. Because the large pig dominates the game, if it gets to the dispenser first (wait at the dispenser) or at the same time (both press the lever and then run to the dispenser) as the piglet, it will receive the following amount of food:

$$p_d(\xi) := \begin{cases} \xi & \text{if } \xi \leq 9; \\ 9 + \log(\xi - 9) & \text{if } \xi \geq 10. \end{cases}$$

The piglet will receive the rest. Instead, if the piglet waits at the dispenser first, it will receive

$$p_s(\xi) := \begin{cases} \xi & \text{if } \xi \leq 4; \\ 4 + \log(\xi - 4) & \text{if } \xi \geq 5. \end{cases}$$

The game is summarized in Table 2.

Table 2: Boxed pigs

	(piglet) Pull the lever	(piglet) Wait
(big pig) Pull the lever	$(p_d(\xi) - 6, \xi - p_d(\xi) - 2)$	$(\xi - p_s(\xi) - 6, p_s(\xi))$
(big pig) Wait	$(p_d(\xi), \xi - p_d(\xi) - 2)$	$(0, 0)$

Now suppose that the random variable ξ follows the two potential distributions $P_1(\xi = 4) = 1/4$ and $P_1(\xi = 15) = 3/4$ or $P_2(\xi = 4) = 3/4$ and $P_2(\xi = 15) = 1/4$. The ambiguity set $\mathscr{P}$ of each pig is $\mathscr{P} = \{P_1, P_2\}$.

First, we consider a case in which both pigs initially know that ξ follows P_1. The expected utility under P_1 is shown in Table 3, where the stochastic Nash equilibrium is displayed in bold. The table indicates that, as the stochastic Nash equilibrium, either one of pigs pulls the lever,

Table 3: Stochastic Nash equilibrium under P_1 (**bold fonts**)

	(piglet) Pull the lever	(piglet) Wait
(big pig) Pull the lever	$(3.0938, 1.1562)$	$\mathbf{(0.4516, 5.7984)}$
(big pig) Wait	$\mathbf{(9.0938, 1.1562)}$	$(0, 0)$

and the other pig waits. Next, consider a case where neither pig knows the exact probability distribution. Suppose they play the game under the worst-case expected utilities. The table of the worst-case expected utilities over $\mathscr{P}$ is summarized in Table 4. As the distributionally robust Nash equilibrium, both pigs wait.

Table 4: Distributionally robust Nash equilibrium (**bold fonts**)

	(piglet) Pull the lever	(piglet) Wait
(big pig) Pull the lever	$(-0.3021, -0.9479)$	$(-3.8495, 4.5995)$
(big pig) Wait	$(5.6979, -0.9479)$	$(\mathbf{0}, \mathbf{0})$

These two results suggest that no ex-post equilibrium exists because the intersection between the sets of stochastic Nash equilibria under P_1 and the distributionally robust Nash equilibria is empty. Therefore, the concept of distributionally robust equilibria is weaker than that of ex-post equilibria.