

Large independent sets in Markov random graphs

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Abstract

Computing the maximum size of an independent set in a graph is a famously hard combinatorial problem that has been well-studied for various classes of graphs. When it comes to random graphs, only the classical binomial random graph $G_{n,p}$ has been analysed and shown to have largest independent sets of size $\Theta(\log n)$ w.h.p. This classical model does not capture any dependency structure between edges that can appear in real-world networks. We initiate study in this direction by defining random graphs $G_{n,p}^r$ whose existence of edges is determined by a Markov process that is also governed by a decay parameter $r \in (0, 1]$. We prove that w.h.p. $G_{n,p}^r$ has independent sets of size $(\frac{1-r}{2+\varepsilon}) \frac{n}{\log n}$ for arbitrary $\varepsilon > 0$, which implies an asymptotic lower bound of $\Omega(\pi(n))$ where $\pi(n)$ is the prime-counting function. This is derived using bounds on the terms of a harmonic series, Turán bound on stability number, and a concentration analysis for a certain sequence of dependent Bernoulli variables that may also be of independent interest. Since $G_{n,p}^r$ collapses to $G_{n,p}$ when there is no decay, it follows that having even the slightest bit of dependency (any $r < 1$) in the random graph construction leads to the presence of large independent sets and thus our random model has a phase transition at its boundary value of $r = 1$. For the maximal independent set output by a greedy algorithm, we deduce that it has a performance ratio of at most $1 + \frac{\log n}{(1-r)}$ w.h.p. when the lowest degree vertex is picked at each iteration, and also show that under any other permutation of vertices the algorithm outputs a set of size $\Omega(n^{1/1+\tau})$, where $\tau = 1/(1-r)$, and hence has a performance ratio of $O(n^{\frac{1}{2-\tau}})$.

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1 Introduction

An independent set in a graph $G = (V, E)$ is a subset of V such that no two vertices in this subset have an edge between them. The maximum cardinality of an independent set in G is called the stability number $\alpha(G)$, and this is a difficult combinatorial problem that is strongly NP-hard to compute exactly and also to approximate within factor arbitrarily close to $|V|$. There is vast amount of literature on approximating this number for general and special graphs, in theory and also computationally through optimization algorithms [cf. [Bom+99](#), [GGV09](#), [Reb+11](#), [GSW22](#)]. Graphs generated through some randomisation technique are natural candidates for analysis of many structural graph properties. The most basic class of random graphs is the classical binomial random graph, also called the Erdős-Rényi-Gilbert random graph. This is denoted by $G_{n,p}$ and is a n -vertex graph where each edge has a given fixed probability p of being present in the graph.¹ There is rich history on bounding $\alpha(G_{n,p})$ asymptotically and it is known that the

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¹To be precise, the random graph model proposed by Erdős and Rényi fixes the number of edges instead of fixing the edge probability, but it is well-known that the two definitions are equivalent.

largest independent sets in $G_{n,p}$ are of size $\log n$ with high probability (w.h.p.) [BE76, Mat76, Fri90]. Bounds have also been computed by solving convex optimization problems such as those from the Lovász ϑ -function and the Lovász-Schrijver lift-and-project operator [Gia+09, GL17] and the latter produce a tight relaxation w.h.p. after being applied for $\Theta(\log n)$ rounds on $G_{n,1/2}$ [FK03]. The analysis of $\alpha(G_{n,p})$ has spawned interest in the chromatic number $\chi(G_{n,p})$, which is related as per the relation $\chi(G)\alpha(G) \geq n$ for any n -vertex graph G , and its concentration [cf. McD90, CPS08, Hec21]. Many other properties of $G_{n,p}$ have also been studied in great detail; see recent texts such as the monograph [FK15] and [Bré17, chap. 10].

We continue this line of work on asymptotic analysis of $\alpha(G)$ but initiate it on a new class of random graphs for which $G_{n,p}$ is a boundary condition. Random graph models have been used for analyzing topological structures of networks, such as social network, citation network, etc. [NWS02, Bar16, Fos+18], and independent sets present some important properties of these networks. Typical network models have non-uniform edge probability and nonzero correlations between edges. To the best of our knowledge, there is no literature on analyzing the stability number of a random graph with some dependency structure. Hence, this paper bridges this gap and also creates a new random graph model that can be further scrutinised for many of its properties as has been done for the classical model.

The edges in our random graph are generated dynamically using a Markov process. Given n, p and a decay parameter $r \in (0, 1]$, starting from the singleton graph $(\{v_1\}, \emptyset)$, a graph $G_{n,p}^r$ having n vertices is generated in $n - 1$ iterations where at each iteration $t \geq 2$, the vertex v_t is added to the graph and edges (v_i, v_t) for $1 \leq i \leq t - 1$ are added as per a Bernoulli r.v. X_i^t . The success probability of X_i^t is equal to p for $i = 1$ and for $i \geq 2$, it is independent of the values of $\{X_1^t, \dots, X_{i-2}^t\}$ and is equal to the success probability of X_{i-1}^t when $X_{i-1}^t = 0$, and is reduced by a factor r when $X_{i-1}^t = 1$.

Definition 1. For any $p, r \in (0, 1]$, a *Markov random graph* $G_{n,p}^r$ is a random graph on n vertices $\{v_1, \dots, v_n\}$ wherein the edge probabilities are defined as follows :

1. for $2 \leq i \leq n$, edge (v_1, v_i) exists with probability equal to p ,
2. for $2 \leq j < i \leq n$, probability of edge (v_j, v_i) depends on whether edge (v_{j-1}, v_i) is present or not, and is independent of all other edges, and we have the following dependency structure on their conditional probabilities,

$$\begin{aligned} \mathbb{P} \{ \text{edge } (v_{j+1}, v_i) \text{ exists} \mid \text{edge } (v_j, v_i) \text{ does not exist} \} &= \mathbb{P} \{ \text{edge } (v_j, v_i) \text{ exists} \} \\ \mathbb{P} \{ \text{edge } (v_{j+1}, v_i) \text{ exists} \mid \text{edge } (v_j, v_i) \text{ exists} \} &= r \mathbb{P} \{ \text{edge } (v_j, v_i) \text{ exists} \}. \end{aligned}$$

We think of the parameter p as the *initial probability* since it governs the edge probabilities between the first vertex and other vertices, and the parameter r as the *decay parameter* since it decreases the edge probability by this factor whenever a previous edge (think of the vertices sorted from $1, \dots, n$ and edges between them) is present. Taking $r = 1$ makes all edges have equal probability p and so $G_{n,p}^1$ is isomorphic to $G_{n,p}$, thereby making our Markov random graph a generalization of the Erdős-Rényi-Gilbert model.

Notation. The random graphs $G_{n,p}^r$ form an obvious finite probability space. Random variables in this paper are generally some function of the random graph $G_{n,p}^r$. We suppress ω from a r.v. $\xi(\omega)$ and write it simply as ξ . Commonly appearing random variables are the stability number $\alpha(G_{n,p}^r)$, vertex degree which is $\deg(v_i)$, average degree $d(G_{n,p}^r)$, edge (v_j, v_i) Bernoulli r.v. X_j^i . We use \mathbb{E} to take expected value and $\mathbb{P} \{ \}$ for probability measure, where the corresponding probability space is obvious from context. When a sequence of random variables $\{\xi_n\}$

on the same probability space converges to another random variable ξ , we denote it as $\xi_n \xrightarrow{p} \xi$, and when ξ is a constant we sometimes say that ξ_n concentrates to ξ . A property \mathcal{P}_n is said to be with high probability (w.h.p.) if $\mathbb{P}\{\mathcal{P}_n \text{ is true}\} \rightarrow 1$ as $n \rightarrow \infty$. Logarithm to the base e (natural log) is denoted by \log . Since the constant $1 - r$ will appear many times in this paper, we denote

$$\gamma := 1 - r.$$

1.1 Main Results and Outline

We assume throughout that $r < 1$, and so our results do not apply to the binomial random graph. Our first main result is an asymptotic lower bound on $\alpha(G_{n,p}^r)$ in terms of the prime-counting function $\pi(n)$, which is defined as the number of primes less than or equal to n .

Theorem 1.1. *For every $\varepsilon > 0$, we have w.h.p. that*

$$\alpha(G_{n,p}^r) \geq \frac{\gamma}{2 + \varepsilon} \frac{n}{\log n},$$

and consequently, $\alpha(G_{n,p}^r) \geq \frac{\gamma}{2(2\varepsilon + 1)} \pi(n)$ w.h.p.

A key ingredient of this proof in §4 is establishing a concentration result about the average vertex degree $d(G_{n,p}^r)$.

Theorem 1.2. $d(G_{n,p}^r)/\log n$ concentrates to $\frac{2}{\gamma}$.

The proof for this requires analysing the sequence of dependent Bernoulli r.v.'s corresponding to each vertex v_i . This analysis is carried out in §7, but we use [Theorem 7.1](#) and [Corollary 7.3](#) corresponding to it when proving our above concentration result.

Another lower bound can be obtained from the maximal independent set produced by a greedy algorithm. Recall that a greedy algorithm sorts the vertices in some order and then iteratively adds them to the output set if the addition retains the independent set property for the subset. In §5 we lower bound the size of the maximal independent set obtained by running a greedy algorithm on the permutation of vertices from v_1 to v_n .

Proposition 1.3. *Our greedy algorithm outputs w.h.p. an independent set of size $\Omega\left(n^{\frac{\gamma}{\gamma+1}}\right)$.*

For upper bound on the size of independent sets in $G_{n,p}^r$, we prove in §6 a nontrivial constant $c < 1$ that bounds $\alpha(G_{n,p}^r) \leq cn$.

Theorem 1.4. *We have w.h.p. that $\alpha(G_{n,p}^r) \leq \left(e^{-r} + \frac{r}{10}\right) n$.*

This leaves open the question of whether our lower bound of $n/\log n$ is tight.

Question 1. Does there exist a function $f(n) = \Omega(n/\log n)$ such that $f(n) = o(n)$ and $\alpha(G_{n,p}^r) = \Theta(f(n))$ w.h.p.?

A crucial first step in our entire analysis is to establish the edge probabilities, i.e., success probability for Bernoulli r.v. corresponding to each edge. This is not a straightforward task, unlike the binomial graph $G_{n,p}$ for which this probability is readily available as the input parameter p . Although we don't derive an exact expression for the edge probabilities, we derive tight lower and upper bounds on it in §3. This is done by analyzing in §2 the rate at which terms in the recurrence formula $f_a : x \in \mathbb{R} \mapsto x(1 - ax)$ grow. This recurrence appears due to the Markov process that generates our random graph.

1.2 Some Consequences

For any n -vertex graph G with m edges, the average degree $d(G)$ is equal to $2m/n$ and we have $\delta(G) \leq d(G) \leq \Delta(G)$. Hence, [Theorem 1.2](#) implies that $G_{n,p}^r$ has w.h.p. approximately $\Theta(n \log n)$ many edges, and there exist vertices with degrees $O(\log n)$ and those with degrees $\Omega(\log n)$, where the constants hiding in this notation are linear in r .

Corollary 1.5. *For every $\varepsilon > 0$, we have w.h.p. that $\delta(G_{n,p}^r) \leq \left(\frac{2}{\gamma} + \varepsilon\right) \log n$ and $\Delta(G_{n,p}^r) \geq \left(\frac{2}{\gamma} - \varepsilon\right) \log n$.*

The chromatic number $\chi(G)$ of any n -vertex graph G is lower-bounded by $\alpha(G)$ through the relation $\chi(G) \geq n/\alpha(G)$, which means that an upper bound on $\alpha(G)$ provides a lower bound on $\chi(G)$. Since our upper bound on $\alpha(G_{n,p}^r)$ is $O(n)$, this basic relation does not tell us anything useful. Instead, we use the Turán bound on chromatic number to deduce a lower bound on $\chi(G_{n,p}^r)$ in terms of the prime counting function $\pi(n)$. We also note a lower bound on the edge-chromatic number χ' .

Corollary 1.6. *For every $\varepsilon > 0$, we have w.h.p. that*

$$\chi(G_{n,p}^r) \geq \frac{\pi(n)}{\pi(n) + \varepsilon - \frac{2}{\gamma}}, \quad \text{and} \quad \chi'(G_{n,p}^r) \geq \left(\frac{2}{\gamma} - \varepsilon\right) \log n.$$

Proof. The lower bound on χ' is immediate from [Corollary 1.5](#) and the fact that $\chi'(G) \geq \Delta(G)$ for any graph G . The bound on χ is using the bound from Turan-type arguments [[CHS03](#), Theorem 8] which says that $\chi(G) \geq n^2/(n^2 - 2m)$ for any graph G with n vertices and m edges. Since $d(G) = 2m/n$ makes this bound equal to $n/(n - d(G))$, we obtain that

$$\chi(G_{n,p}^r) \geq \frac{n}{n - d(G_{n,p}^r)} = \frac{1}{1 - \frac{d(G_{n,p}^r)/\log n}{n/\log n}} \geq \frac{1}{1 - \frac{2/\gamma - \varepsilon}{\pi(n)}} \quad \text{w.h.p. } \forall \varepsilon > 0,$$

where in the last inequality we have used the concentration from [Theorem 1.2](#) and the fact that $\pi(n) \geq n/\log n$ for $n \geq 17$. \square

Although this lower bound on $\chi(G_{n,p}^r)$ is not a constant, it is clear that it is a weak bound and so there remains scope for improvement on it.

Another question related to chromatic number that has raised interest for the binomial random graph $G_{n,p}$ is that of its Hadwiger number $h(G_{n,p})$, which for a graph G is the largest value of t such that G contains a K_t -minor. This is related to the chromatic number $\chi(G)$ because of the famous Hadwiger's conjecture that $\chi(G) \leq h(G)$ for any graph G . Erde et al. [[EKK21](#)] showed recently that $h(G_{n,p}) = \Omega(\sqrt{k})$ when $p = (1 + \varepsilon)/k$, where k is a lower bound on $\delta(G_{n,p})$. Although an analysis of the Hadwiger number of $G_{n,p}^r$ requires separate attention that is beyond the scope of this paper, we note some observations here. Recall that the line graph $L(G)$ of a simple graph G is obtained by vertex-edge duality.

Corollary 1.7. *For every $\varepsilon > 0$, w.h.p. there exists a K_t -minor in $G_{n,p}^r$ for $t = \Omega_\varepsilon\left(\frac{\log n}{\sqrt{\log \log n}}\right)$ and a K_s -minor in $L(G_{n,p}^r)$ for $s = \Omega_\varepsilon(\log n)$, where Ω_ε hides a constant in terms of ε ,*

Proof. A graph G with average degree $d(G) \geq k$ has $h(G) = \Omega(k/\sqrt{\log k})$ [cf. [EKK21](#), Corollary 2.2]. This bound is a convex function of k that is increasing for $k \geq 2$. Since the average degree concentration in [Theorem 1.2](#) tells us that $d(G_{n,p}^r) \geq \left(\frac{2}{\gamma} - \varepsilon\right) \log n$ w.h.p., it follows that

$G_{n,p}^r$ has a complete minor of size at least $\Omega_\varepsilon\left(\frac{\log n}{\sqrt{\log \log n}}\right)$. For line graphs of simple graphs, it is well-known that Hadwiger's conjecture is true as a direct consequence of Vizing's theorem which says that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ and because $\chi(L(G)) = \chi'(G)$. Hence, we have $h(L(G_{n,p}^r)) \geq \chi(L(G_{n,p}^r)) = \chi'(G_{n,p}^r) \geq \Delta(G_{n,p}^r) \geq \left(\frac{2}{\gamma} - \varepsilon\right) \log n$, where the last inequality is from [Corollary 1.5](#). \square

[Proposition 1.3](#) gives a weaker lower bound on $\alpha(G_{n,p}^r)$ than that from [Theorem 1.1](#). Since the construction of our random graph is such that one can intuitively see that the vertex degrees are in the descending order from v_1 to v_n w.h.p., the greedy algorithm we consider gives a worst-case performance bound on the greedy algorithm that one typically uses to approximate $\alpha(G_{n,p}^r)$ by sorting vertices from smallest to largest degree (in our case that would be v_n to v_1). This is noted formally in the next result which compares the performance ratios of the two methods. The performance ratio of an algorithm A that outputs an independent set of size $\alpha_A(G)$ for any graph G (and hence provides a lower bound on $\alpha(G)$) is denoted by $\rho_A(n)$ and defined as the maximum ratio $\alpha(G)/\alpha_A(G)$ over all n -vertex graphs G . Let \mathbf{Gr} denote our greedy algorithm which iteratively adds vertices in the order v_1 to v_n and $\mathbf{Gr-min}$ denote the algorithm which picks the smallest degree vertex at each iteration.

Corollary 1.8. *W.h.p., $\rho_{\mathbf{Gr}} = O(n^{\frac{1}{1+\gamma}})$ and $\rho_{\mathbf{Gr-min}} \leq 1 + \left(\frac{1}{\gamma} + \varepsilon\right) \log n$ for $\varepsilon > 0$.*

Proof. The first bound is directly from combining [Proposition 1.3](#) and [Theorem 1.4](#). The second bound uses the general result of Halldórsson and Radhakrishnan [[HR97](#), Corollary 4] which says that $\rho_{\mathbf{Gr-min}} \leq (d(G) + 2)/2$, and the upper bound on $d(G_{n,p}^r)$ from [Theorem 1.2](#). \square

1.3 Discussion

The stability number of Erdős-Rényi-Gilbert random model $G_{n,p}$ has been well-studied. If $p \in (0, 1)$ is fixed, then Matula [[Mat76](#)] has shown that w.h.p. the stability number $\alpha(G_{n,p}) \approx 2 \log_q(n)$, where the logarithm base is $q = 1/(1-p)$. The graph $G_{n,p}$ with fixed p is usually considered as a dense random graph. For sparse random graphs with $np = d$ fixed, Frieze [[Fri90](#)] showed that

$$\left| \alpha(G_{n,p}) - \frac{2n}{d} (\log d - \log \log d - \log 2 + 1) \right| \leq \frac{\varepsilon n}{d}$$

holds w.h.p. for all $d \geq d_\varepsilon$ where d_ε is a constant depending on the chosen $\varepsilon > 0$. In contrast, our lower bound of $\Omega(n/\log n)$ from [Theorem 1.1](#) indicates that there are much larger independent sets in our random graph $G_{n,p}^r$ when p and r are fixed. Since all of our analysis heavily depends on $r < 1$, our results don't generalise those for $G_{n,p}$, thus implying that a phase transition occurs in our random graph model at the boundary value $r = 1$.

The $\log n$ term in our main results comes from bounding the partial sum of a harmonic series that arises due to the Markovian dependence between the edge probabilities. The lower bound in [Theorem 1.1](#) makes use of Turán's lower bound on $\alpha(G)$ for any graph G . The crucial part of our analysis is the average degree concentration in [Theorem 1.2](#). Our proof for this theorem uses Chebyshev's inequality. Due to the absence of independence structure between the r.v.s, we cannot apply Chernoff- or Hoeffding-type inequalities, and use of martingale tail inequalities also does not help. For fixed p , this concentration result shows $G_{n,p}^r$ to be more sparse than $G_{n,p}$ in terms of the number of edges. Intuitively, a denser graph has smaller stability number and our result indeed complies with this intuition,

$$\alpha(G_{n,p}^r) = \Omega\left(\frac{n}{\log n}\right) = \Omega\left(\log_{\frac{1}{1-p}}(n)\right) = \Omega(\alpha(G_{n,p})).$$

Theorem 1.2 relies on the asymptotic result of the sequence $\frac{S_n}{\log n}$ in **Theorem 7.1** and this result does not trivially follow from the literature. We know of two studies [JKQ, LQY] that show S_n/n converges to constants under the assumption that $\mathbb{P}\{Y_{n+1} = 1 \mid Y_1, \dots, Y_n\} = \theta_n + g_n(S_n)$ where g_n is a linear function and θ_n is a constant. Our work makes a different assumption where the conditional probability does not depend on the partial sum S_n . Furthermore, the variables we consider form a nonhomogenous Markov chain and to the best of our knowledge such a sequence has not been analysed before, although we mention that partial sums of a Bernoulli sequence with a homogenous Markov chain has been studied before [Edw60, Wan81]. Another powerful method to deal with a sequence of variables is martingale analysis. It is easy to see $\{S_n\}_{n \geq 1}$ is a super-martingale because $\mathbb{E}[S_{n+1} \mid Y_1, \dots, Y_n] = \mathbb{E}[Y_{n+1} \mid Y_n] + S_n > 0$. Since $\{S_n\}_{n \geq 1}$ is non-decreasing, Doob's inequality collapses to Chebyshev's inequality $\mathbb{P}\{S_n > C\} \leq \mathbb{E}S_n/C$. So it appears that a martingale technique would not improve our analysis.

2 Growth Rates in a Recurrence Formula

This section establishes some technical results on the following recursion formula that will be used throughout this paper,

$$x_{n+1} = f_a(x_n) \quad \text{for } n \geq 1, \quad \text{where } f_a : x \mapsto x(1 - ax), \quad \text{for } a \in (0, 1], \quad (1)$$

and it is assumed that $x_1 \in (0, 1)$. We begin by observing that the function $f_a : x \mapsto x(1 - ax)$ used to generate this recurrence has the following properties from elementary calculus.

Lemma 2.1. *f_a is a concave quadratic that*

1. *is increasing on $(-\infty, a/2]$ and decreasing on $(a/2, \infty)$,*
2. *has a maximum value of $a^2/4$,*
3. *does not have any nonzero fixed point,*
4. *is positive-valued over $(0, 1)$,*
5. *satisfies $f_a(x) < x$ for $x > 0$,*
6. *satisfies $af_a(x) = f_1(ax)$.*
7. *satisfies $f_a(x) < f_b(x)$ for $x > 0$ if $a > b$.*

Proof. Rewrite $f_a = x(1 - ax) = -(x - a/2)^2 + a^2/4$ and then the first 3 claims are obvious. Observe that $f_a(0) = 0$ and $f_a(1) = 1 - a \geq 0$. The first claim implies $f_a(x) > \min\{f_a(0), f_a(1)\} = 0$ for all $x \in (0, 1)$ and therefore, the fourth claim is true. Next, for all $x > 0$, the fifth claim is equivalent to $1 - ax < 1$ which is essentially $a > 0$ followed from our assumption on a . The sixth claim is no more than a simple computation $af_a(x) = ax(1 - ax) = f_1(ax)$. To prove the seventh claim, we evaluate $f_a(x) - f_b(x) = (b - a)x^2$ which is negative if $a > b$ and $x > 0$. \square

The terms in the recurrence are lower and upper bounded by constant multiples of $1/n$.

Lemma 2.2. *Recurrence (1) generates a decreasing sequence with each term x_n , for $n \geq 2$, bounded as*

$$\frac{1}{a}\ell_n(x_1) \leq x_n < \min \left\{ \frac{a^2}{4}, \frac{1}{a} \cdot \frac{1}{n - 1 + \frac{1}{x_1}} \right\}, \quad \text{where } \ell_n(x_1) := \begin{cases} \frac{1}{(\sqrt{n+1})^2}, & n \geq \left(\frac{1}{f_1(x_1)} - 1\right)^2 \\ \frac{f_1(x_1)}{n}, & \text{otherwise.} \end{cases} \quad (2)$$

Proof. We immediately have from the fifth and second claims in [Lemma 2.1](#) that (1) generates a decreasing sequence over $(0, a^2/4]$. The sixth claim in [Lemma 2.1](#) implies that the sequence $\{y_n\}$ formed by the recurrence $y_n = f_1(y_{n-1})$ is related to $\{x_n\}$ by $y_n = ax_n$. Therefore, to obtain the other bounds on x_n it suffices to bound with $a = 1$ and then scale the results by dividing with a . Henceforth, assume $a = 1$.

We prove by induction that $x_n < \frac{1}{n-1+\frac{1}{x_1}}$. The base case of $n = 2$ holds because $x_2 = f_1(x_1) = x_1(1-x_1) < x_1/(1+x_1) = 1/(1+1/x_1)$, where the inequality is from $1-x_1^2 < 1$. Suppose $x_n < 1/(n+\phi)$ for some $n \geq 2$, where for convenience of notation we let $\phi = \frac{1}{x_1} - 1$. Note that $\phi > 0$ because $x_1 \in (0, 1)$, and hence $1/(n+\phi) \in (0, 1/2)$ and $1/(n+\phi) < 1/n$ for $n \geq 2$. Monotonicity of f_1 over $(0, 1/2)$ from the first claim in [Lemma 2.1](#) and our induction hypothesis imply that $f_1(1/(n+\phi)) > f_1(x_n) = x_{n+1}$. Since $f_1(1/(n+\phi)) = (n+\phi-1)/(n+\phi)^2 < 1/(n+1+\phi)$ and $x_{n+1} = f_1(x_n)$, we have arrived at $x_{n+1} < 1/(n+1+\phi)$, which completes the induction for the upper bound.

For the lower bound, define two functions $\phi(\lambda) := \max\{\frac{\lambda}{\lambda-1}, \frac{1}{f_1(x_1)}\}$ and $g_n(\lambda) := \phi(\lambda) + n\lambda$, with domain $\lambda > 1$. We argue, by induction, that $1/g_n(\lambda)$ is a parametric lower bound on x_n . Denote $\phi := \phi(\lambda)$. The base case is $n = 2$. From recurrence formula $x_2 = f_1(x_1)$ and $\phi \geq 1/f_1(x_1)$, we get $x_2 \geq 1/\phi$ and thus $x_2 > 1/(\phi + 2\lambda)$ since $\lambda > 0$. For the inductive step, assume $x_n \geq 1/(\phi + \lambda n)$ is true. [Lemma 2.2](#) says that $x_n \leq 1/4$ for all $n \geq 2$ and the first claim in [Lemma 2.1](#) asserts $f_1(x)$ is monotone increasing on $(0, 1/4]$. $x_{n+1} = f_1(x_n) \geq f_1\left(\frac{1}{\phi + \lambda n}\right) = \frac{(\phi + \lambda n) - 1}{(\phi + \lambda n)^2}$. Since $(\phi + n\lambda - 1)(\phi + (n+1)\lambda) = (\phi + n\lambda)^2 + \lambda(\phi + n\lambda) - (\phi + n\lambda + \lambda)$ and $\lambda(\phi + n\lambda) - (\phi + n\lambda + \lambda) = (\phi + n\lambda)(\lambda - 1) - \lambda > 0$ where the inequality is because $\phi + n\lambda > \phi = \lambda/(\lambda - 1)$, it follows that $f_1\left(\frac{1}{\phi + \lambda n}\right) > 1/(\phi + (n+1)\lambda)$. This leads to $x_{n+1} > 1/(\phi + (n+1)\lambda) = \frac{1}{\phi + \lambda(n+1)}$.

Now we show that our claimed lower bound on x_n is a lower bound on the supremum of $1/g_n(\lambda)$ over $(1, \infty)$, or equivalently an upper bound on $\inf_{\lambda > 1} g_n(\lambda)$. Denote $\lambda^* = 1/(1 - f_1(x_1))$. This value is such that $\lambda^*/(\lambda^* - 1) = 1/f_1(x_1)$. Since $\lambda/\lambda - 1$ is a decreasing function of λ , we have that the function ϕ has non-differentiability at λ^* and g_n can be written as

$$g_n(\lambda) = \begin{cases} \frac{\lambda}{\lambda-1} + n\lambda, & 1 < \lambda \leq \lambda^* \\ \frac{1}{f_1(x_1)} + n\lambda, & \lambda \geq \lambda^*. \end{cases}$$

Therefore,

$$\inf_{\lambda > 1} g_n(\lambda) = \min \left\{ \inf_{1 < \lambda \leq \lambda^*} \frac{\lambda}{\lambda-1} + n\lambda, \inf_{\lambda \geq \lambda^*} \frac{1}{f_1(x_1)} + n\lambda \right\}, \quad (3)$$

and let us evaluate the two infimums separately. The second one is obviously equal to $\frac{1}{f_1(x_1)} + n\frac{1}{1-f_1(x_1)}$. For the first infimum, it is easy to verify that the function to be minimised is convex in λ , and the first derivative is $n - 1/(\lambda - 1)^2$, so that the stationary point is at $\tilde{\lambda} = 1 + 1/\sqrt{n}$. Hence, the minimum value of this function over the real line is its value at $\tilde{\lambda}$, which is equal to $(\sqrt{n} + 1)^2$. We are interested in the minimum over the interval $(1, \lambda^*]$. The function has a vertical asymptote $\lambda = 1$ and its value at the other endpoint is $\frac{1}{f_1(x_1)} + \frac{n}{1-f_1(x_1)}$. It is easy to verify that $(\sqrt{n} + 1)^2 \leq \frac{1}{f_1(x_1)} + \frac{n}{1-f_1(x_1)}$ if and only if $((\sqrt{n} + 1)f_1(x_1) - 1)^2 \geq 0$, which is obviously true. Therefore, the minimum over $(1, \lambda^*]$ is $(\sqrt{n} + 1)^2$ if $\tilde{\lambda} \leq \lambda^*$, otherwise the minimum is $\frac{1}{f_1(x_1)} + \frac{n}{1-f_1(x_1)}$. The second value can be upper bounded as follows,

$$\frac{1}{f_1(x_1)} + n\frac{1}{1-f_1(x_1)} \leq \frac{1}{f_1(x_1)} + n\frac{1}{f_1(x_1)(1-\frac{1}{4})} = \left(1 + \frac{n}{3}\right) \frac{1}{f_1(x_1)} < \frac{n}{f_1(x_1)},$$

where the first inequality is by applying the following fact that is simple to verify by cross-multiplying denominators,

$$0 < \xi_1 \leq \xi_2 < 1 \implies \frac{1}{1 - \xi_1} \leq \frac{\xi_2}{\xi_1(1 - \xi_2)},$$

to $\xi_1 = f_1(x_1)$ and $\xi_2 = 1/4$ (this upper bound on $x_2 = f_1(x_1)$ is from [Lemma 2.2](#)), and the second inequality is because $n > 1 + n/3$ for $n \geq 2$. Thus, the infimum in (3) is upper bounded by $1/(\sqrt{n} + 1)^2$ if $\tilde{\lambda} \leq \lambda^*$, otherwise the bound is $n/f_1(x_1)$.

It remains to simplify the condition $\tilde{\lambda} \leq \lambda^*$, which becomes $1 + 1/\sqrt{n} \leq 1/(1 - f_1(x_1))$. This is equivalent to $f_1(x_1) \geq 1/(1 + \sqrt{n})$, which rearranges to $n \geq (\frac{1}{f_1(x_1)} - 1)^2$. \square

Another result that we will need is lower and upper bounds on the sum of terms in the recurrence for large enough n . To prove the lower bound, let us recall that the n^{th} harmonic number is

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

which is the partial sum of the harmonic series $\sum_i \frac{1}{i}$. It is obvious from the Riemann approximation of an integral that

$$H_n \leq \log(n + 1) + 1.$$

A well-known fact about the harmonic number is that it exhibits a logarithmic growth rate, which can be derived using the Euler-Maclaurin expansion formula.

Lemma 2.3 ([\[BW71\]](#)). $H_n = \gamma + \log n + \frac{1}{2n} - R_n$, where $0 \leq R_n \leq \frac{1}{8n^2}$ and $\gamma \approx 0.57721$ is the Euler constant.

We can now argue that the partial sums in the recurrence are $\Theta(\log n)$.

Lemma 2.4. Denote $f_1 := f_1(x_1)$ and $\eta(x_1) := 2(1 - f_1) \log f_1 - \left(\frac{5}{2} + \log 2\right) f_1$. For $n \geq (\frac{1}{f_1} - 1)^2$, we have

$$ax_1 + \eta(x_1) + \frac{2}{1 + \sqrt{n}} + \log n \leq a \sum_{i=1}^n x_i \leq ax_1 + \log(1 + (n - 1)x_1).$$

Proof. For the upper bound, we have

$$a \sum_{i=1}^n x_i \leq ax_1 + \sum_{i=2}^n \frac{1}{i - 1 + \frac{1}{x_1}} \leq ax_1 + \int_{\frac{1}{x_1}}^{\frac{1}{x_1} + n - 1} \frac{1}{t} dt = ax_1 + \log(1 + (n - 1)x_1),$$

where the first inequality is using the upper bound in [Lemma 2.2](#) and the second inequality is due to the summation being the right Riemann sum of the decreasing function $t \mapsto 1/(t + \frac{1}{x_1})$ over the interval $[0, n - 1]$.

For the lower bound, let us denote $n^* := (\frac{1}{f_1} - 1)^2$ for convenience. Using the lower bound

from [Lemma 2.2](#) for each term in the sequence, the partial sum can be lower bounded as

$$\begin{aligned}
a \sum_{i=1}^n x_i &\geq ax_1 + \sum_{i=2}^n \ell_i(x_1) = ax_1 + \sum_{i=2}^{n^*-1} \frac{f_1(x_1)}{i} + \sum_{i=n^*}^n \frac{1}{(\sqrt{i}+1)^2} \\
&= ax_1 + f_1(x_1)(H_{n^*-1} - 1) + \sum_{i=n^*}^n \frac{1}{(\sqrt{i}+1)^2} \\
&\geq ax_1 + f_1(x_1)(H_{n^*-1} - 1) + \int_{n^*}^n \frac{1}{(\sqrt{t}+1)^2} dt \\
&= ax_1 + f_1(x_1)(H_{n^*-1} - 1) + 2 \left[\frac{1}{1+\sqrt{t}} + \log(1+\sqrt{t}) \right]_{n^*}^n \\
&\geq ax_1 + f_1(x_1)(H_{n^*-1} - 1) - \frac{2}{1+\sqrt{n^*}} - 2 \log(1+\sqrt{n^*}) \\
&\quad + \frac{2}{1+\sqrt{n}} + \log n,
\end{aligned}$$

where the first equality is by definition of ℓ_i in [\(2\)](#), the second equality is the definition of harmonic number H_n , the second inequality is due to the summation being the left Riemann sum of the decreasing function $t \mapsto 1/(1+\sqrt{t})^2$ over the interval $[n^*, n]$, and the last inequality is from the fact that $2 \log(1+\sqrt{t}) = \log(1+t+2\sqrt{t}) > \log t$. Now we simplify the terms in the middle involving n^* and argue that they are lower bounded by $\eta(x_1)$. Since $n^* := (\frac{1}{f_1} - 1)^2$, we have $1+\sqrt{n^*} = 1/f_1$ and $\log(1+\sqrt{n^*}) = -\log f_1$. Therefore, the middle terms depending on n^* are

$$f_1(H_{n^*-1} - 1) - 2f_1 + 2 \log f_1 = f_1(H_{n^*-1} - 3) + 2 \log f_1,$$

and we have to argue that this is lower bounded by $\eta(x_1)$. By [Lemma 2.3](#), the n^{th} harmonic number can be lower-bounded as

$$H_n \geq \gamma + \log n + \frac{1}{2n} - \frac{1}{8n^2} \implies H_n - 3 \geq \gamma + \log n + \frac{1}{2n} - \frac{1}{8n^2} - 3 \geq \log n - \frac{5}{2},$$

where the last inequality is because $\gamma > 1/2$ and $1/2n - 1/8n^2 > 0$. Substituting this lower bound into the middle terms depending on n^* gives us the lower bound

$$f_1 \left(\log(n^* - 1) - \frac{5}{2} \right) + 2 \log f_1.$$

Since $n^* - 1 = (1/f_1 - 2)/f_1$, we have $\log(n^* - 1) = \log(1/f_1 - 2) - \log f_1$, and so the lower bound on the middle terms becomes

$$\begin{aligned}
(2 - f_1) \log f_1 + f_1 \left(\log \left(\frac{1}{f_1} - 2 \right) - \frac{5}{2} \right) &= (2 - f_1) \log f_1 + f_1 \left(\log(1 - 2f_1) - \log f_1 - \frac{5}{2} \right) \\
&\geq 2(1 - f_1) \log f_1 - f_1 \left(\log 2 + \frac{5}{2} \right) =: \eta(x_1),
\end{aligned}$$

where the inequality is because $\log(1 - 2f_1) \geq -\log 2$ due to $f_1 \leq 1/4$ from [Lemma 2.1](#). \square

3 Estimating Edge Probabilities

In any random graph model, each edge can be associated with a Bernoulli random variable (r.v.) which is equal to 1 if and only if that edge is present in the random graph. The success

probability for this r.v. depends on the construction of the random graph model; for the binomial random graph $\mathcal{G}_{n,p}$ by Erdős-Rényi-Gilbert, this probability is equal to the parameter p , which implies that any two Bernoulli r.v.'s are iid. This significantly helps the analysis of these graphs for their stability number, concentration of chromatic number and many other graph properties. The same is not true for arbitrary edges in our Markov random graph $G_{n,p}^r$ since we have a dependency structure between the edges. Denote the Bernoulli r.v. X_j^i by

$$X_j^i = 1 \iff \text{edge } (v_j, v_i) \text{ is present in graph } G_{n,p}^r, \quad 1 \leq j < i \leq n.$$

and its success probability by

$$p_j^i := \mathbb{P} \left\{ X_j^i = 1 \right\}, \quad 1 \leq j < i \leq n.$$

For each j , the edges $\{(v_j, v_i) : j < i\}$ have their corresponding Bernoulli variables equal in distribution and also independent of each other.

Lemma 3.1. X_k^i and X_k^j are iid random variables for $k < \min\{i, j\}$ and $i \neq j$.

For every vertex i , there is a corresponding Markov chain formed by the sequence of Bernoulli r.v.'s from edges to vertices

Lemma 3.2. For every $i \geq 2$, the sequence $\{X_1^i, \dots, X_{i-1}^i\}$ forms a non-homogenous Markov chain with transition matrix from X_k^i to X_{k+1}^i , for $1 \leq k \leq i-2$, given by

$$\mathbf{P}_k^i = \begin{bmatrix} 1 - p_k^i & p_k^i \\ \gamma p_k^i & r p_k^i \end{bmatrix}$$

Since all edges to v_1 occur with the same probability p , we have $p_1^i = p$ for all i . For the other edges, the success probabilities for the Markov chain $\{X_1^i, \dots, X_{i-1}^i\}$, namely the sequence $\{p_1^i, \dots, p_{i-1}^i\}$, satisfies a recursion.

Lemma 3.3. For $2 \leq j \leq i-1$, we have

$$p_j^i = p_{j-1}^i \left[1 - \gamma p_{j-1}^i \right],$$

and consequently,

$$\frac{\ell_j(\gamma p)}{\gamma} \leq p_j^i < \min \left\{ p_{j-1}^i, \frac{1}{\gamma j} \right\}.$$

Proof. This follows from the law of total probability,

$$\begin{aligned} \mathbb{P} \left\{ X_j^i = 1 \right\} &= \mathbb{P} \left\{ X_j^i = 1 \mid X_{j-1}^i = 1 \right\} \mathbb{P} \left\{ X_{j-1}^i = 1 \right\} + \mathbb{P} \left\{ X_j^i = 1 \mid X_{j-1}^i = 0 \right\} \mathbb{P} \left\{ X_{j-1}^i = 0 \right\}, \\ &= r \mathbb{P} \left\{ X_{j-1}^i = 1 \right\} \mathbb{P} \left\{ X_{j-1}^i = 1 \right\} + \mathbb{P} \left\{ X_{j-1}^i = 1 \right\} \mathbb{P} \left\{ X_{j-1}^i = 0 \right\}, \\ &= r(p_{j-1}^i)^2 + p_{j-1}^i(1 - p_{j-1}^i), \\ &= p_{j-1}^i \left[1 - \gamma p_{j-1}^i \right], \end{aligned}$$

where the second equality is from the conditional probabilities in [Definition 1](#). Thus, for every i , the sequence $\{p_1^i, p_2^i, \dots, p_{i-1}^i\}$ obeys the recursion $x_{k+1} = x_k(1 - \gamma x_k)$ starting with $x_1 = p_1^i = p$. This recursion is the same as the one analysed earlier in [\(1\)](#) and therefore, the claimed bounds follow from applying [Lemma 2.2](#) with $a = \gamma$, and relaxing the upper bound of $1/\gamma(j-1+1/p)$ to $1/\gamma j$ because $1/p \geq 1$. \square

The transition matrix in [Lemma 3.2](#) for the Markov chain $\{X_1^i, X_2^i, \dots, X_{i-1}^i\}$ gives conditional probabilities for consecutive variables. For two arbitrary variables, the conditional probability can be bounded as follows.

Lemma 3.4. *For $1 \leq k < j < i$,*

$$1 - p_{j-1}^i \leq \mathbb{P}\{X_j^i = 0 \mid X_k^i = 0\} \leq \gamma p_{j-1}^i, \quad r p_{j-1}^i \leq \mathbb{P}\{X_j^i = 1 \mid X_k^i = 1\} \leq p_{j-1}^i.$$

Proof. Fix any $i \geq 2$. For ease of readability, we suppress the superscript i in the probabilities. The Markovian property gives us $\mathbb{P}\{X_j^i = 0 \mid X_k^i = 0\} = (\mathbf{P}_k \mathbf{P}_{k+1} \cdots \mathbf{P}_{j-1})_{11}$. Since \mathbf{P}_k is a stochastic matrix for all k , the product matrix $\mathbf{M} := \mathbf{P}_k \mathbf{P}_{k+1} \cdots \mathbf{P}_{j-1}$ is also stochastic, which implies that $(\mathbf{M} \mathbf{P}_{j-1})_{11} = (1 - p_{j-1})a_{11} + (\gamma p_{j-1})(1 - a_{11})$, thereby leading to $1 - p_{j-1} \leq (\mathbf{M} \mathbf{P}_{j-1})_{11} \leq \gamma p_{j-1}$. A similar argument gives bounds on the other conditional probability after observing that $\mathbb{P}\{X_j^i = 1 \mid X_k^i = 1\} = (\mathbf{M} \mathbf{P}_{j-1})_{22}$. \square

We now use these bounds on edge probabilities to estimate the probability that a subset of vertices is independent. The first step towards this is to recognise that this probability can be expressed as a product of probabilities for a vertex to be disconnected from a subset.

Lemma 3.5. *For any subset $S := \{v_{j_1}, v_{j_2}, \dots, v_{j_k}\}$ of k vertices in $G_{n,p}^r$, we have that*

$$\mathbb{P}\{S \text{ is independent}\} = \prod_{i=2}^k \mathbb{P}\{v_{j_i} \text{ is disconnected from } \{v_{j_1}, \dots, v_{j_{i-1}}\}\}.$$

Proof. Wlog, assume the vertices in S are sorted such that $j_1 < j_2 < \dots < j_k$. Since a subset is independent if and only if it is pairwise disconnected, it is clear that the desired probability is equal to the joint probability $\mathbb{P}\{X_{j_i}^{j_r} = 0 : r > i\}$. Partition these edge variables based on their upper index into disjoint sets $B_{j_1}, B_{j_2}, \dots, B_{j_k}$ where $B_{j_r} = \{X_{j_1}^{j_r}, X_{j_2}^{j_r}, \dots, X_{j_{r-1}}^{j_r}\}$. Note that $B_{j_1} = \emptyset$. From [Lemma 3.1](#), two different edge variables are dependent if and only if they have the same upper index and their lower indices are two consecutive numbers. Therefore, all variables inside B_{j_r} are independent from these in B_{j_l} with $l \neq r$ because their upper indices j_r and j_l are not the same. This independence implies that the probability can be written as the product (over $i = 2, \dots, k$) of the probability that all variables in B_{j_i} are equal to 0. The event that all variables in B_{j_i} equal to 0 is equivalent to saying that v_{j_i} is disconnected from $\{v_{j_1}, \dots, v_{j_{i-1}}\}$. \square

Now we use bounds on edge probabilities to bound the probability of disconnection for a vertex from a subset. Our bounds are independent from the indices of vertices in the subset and are solely determined by the size of the subset.

Lemma 3.6. *For any subset $A \subseteq \{v_1, \dots, v_k\}$ of size m , for all $i \geq k$,*

$$(1 - p) \prod_{j=1}^{m-1} (1 - p_j^i) \leq \mathbb{P}\{v_i \text{ is disconnected from } A\} \leq \prod_{j=i-m}^{i-1} (\gamma p_j^i).$$

Proof. Fix $i \geq k$ and let $D_i(m)$ denote the event that v_i is disconnected from A . We argue the lower bound first.

$$\mathbb{P}\{D_i(m)\} = \mathbb{P}\{X_{j_1}^i = 0, \dots, X_{j_m}^i = 0\} = \mathbb{P}\{X_{j_1}^i = 0\} \prod_{k=2}^m \mathbb{P}\{X_{j_k}^i = 0 \mid X_{j_{k-1}}^i = 0\}.$$

By [Lemma 3.4](#), $\mathbb{P}\{X_{j_k}^i = 0 \mid X_{j_{k-1}}^i = 0\}$ is bounded below by $1 - p_{j_{k-1}}$, where we suppress the superscript i on probabilities for ease of readability. Therefore,

$$\mathbb{P}\{D_i(m)\} \geq \mathbb{P}\{X_{j_1}^i = 0\} \prod_{k=2}^m (1 - p_{j_{k-1}}) = (1 - p_{j_1}) \prod_{k=2}^m (1 - p_{j_{k-1}})$$

We know from [Lemma 3.3](#) that the sequence $\{p_j\}$ is decreasing, and hence the sequence $\{1 - p_j\}$ is increasing. Immediately, we have $1 - p_{j_1} \geq 1 - p$ since $p = p_1$ is the largest value among all p_j . Note that $\prod_{k=2}^m (1 - p_{j_{k-1}})$ contains $m - 1$ distinct values $p_{j_1-1}, p_{j_2-1}, \dots, p_{j_{m-1}}$ where $j_m > \dots > j_1 \geq 1$. Clearly, $j_k \geq k$ for all $1 \leq k \leq m - 1$. Therefore, $p_{j_{k-1}} \leq p_{k-1}$ for all i and then $\prod_{k=2}^m (1 - p_{j_{k-1}}) \geq \prod_{j=1}^{m-1} (1 - p_j)$. Consequently, $(1 - p_{j_1}) \prod_{k=2}^m (1 - p_{j_{k-1}}) \geq (1 - p) \prod_{j=1}^{m-1} (1 - p_j)$ which is our desired lower bound.

The upper bound can be obtained similarly.

$$\begin{aligned} \mathbb{P}\{D_i(m)\} &= \mathbb{P}\{X_{j_1}^i = 0, \dots, X_{j_m}^i = 0\} = \mathbb{P}\{X_{j_1}^i = 0\} \prod_{k=2}^m \mathbb{P}\{X_{j_k}^i = 0 \mid X_{j_{k-1}}^i = 0\} \\ &\leq \mathbb{P}\{X_{j_1}^i = 0\} \prod_{k=2}^m (\gamma p_{j_{k-1}}) \\ &\leq \prod_{j=i-m}^{i-1} (\gamma p_j). \end{aligned}$$

The first inequality is obtained using [Lemma 3.4](#). The last inequality is because $\mathbb{P}\{X_{j_1}^i = 0\} \leq 1$ and we claim that the product term is upper bounded by $\prod_{j=i-m}^{i-1} (\gamma p_j)$. To argue this claim, we use the assumption that A only contains vertices with index less than i . Hence, $j_1 < j_2 < \dots < j_m \leq i - 1$ and $j_k \leq i - 1 - (m - k)$ for all $1 \leq k \leq m - 1$, which implies that $p_{j_{k-1}} \geq p_{i-2-m+k} \geq p_{i-1-m+k}$. Therefore, $\prod_{k=2}^m (\gamma p_{j_{k-1}}) \leq \prod_{j=i-m}^{i-1} (\gamma p_j)$. \square

Putting together the previous two lemmata leads to the following upper bound for an independent set.

Proposition 3.7. *The probability P_k that a subset of k vertices in $G_{n,p}^r$ is an independent set satisfies the following bound when $n - k + 1 \geq (1/\tau - 1)^2$, for $\tau := f_1(\gamma p) = \gamma p(1 - \gamma p)$,*

$$P_k \leq \prod_{i=1}^{k-1} \left(1 - \frac{r}{\gamma} \frac{1}{(\sqrt{n-i} + 1)^2} \right)^i.$$

Proof. We use the expression from [Lemma 3.5](#). For each term in the product, applying the upper bound from [Lemma 3.6](#) relaxes it as follows,

$$\mathbb{P}\{v_{j_i} \text{ is disconnected from } \{v_{j_1}, \dots, v_{j_{i-1}}\}\} \leq \prod_{k=j_i-(i-1)}^{j_i-1} (\gamma p_k), \quad (4)$$

where for the sake of brevity, we have suppressed the superscript j_i on the probabilities. Since $j_1 < j_2 < \dots < j_k \leq n$, we have $j_i \leq n - (k - i)$. [Lemma 3.3](#) tells us that $\{p_k\}$ is a decreasing

sequence. Therefore, $p_{j_i-t} \geq p_{n-(k-i)-t}$ for all $t \in \{1, 2, \dots, i-1\}$ and now we can further relax (4) as

$$\mathbb{P} \{v_{j_i} \text{ is disconnected from } \{v_{j_1}, \dots, v_{j_{i-1}}\}\} \leq \prod_{k=j_i-(i-1)}^{j_i-1} (\gamma p_k) \leq \prod_{t=k-(i-1)}^{k-1} (\gamma p_{n-t})$$

Applying this inequality to the expression from Lemma 3.5,

$$P_k \leq \prod_{i=2}^k \prod_{t=k-i+1}^{k-1} (\gamma p_{n-t}) = \prod_{i=1}^{k-1} (1 - r p_{n-i})^i.$$

We use the lower bound $p_{n-i} \geq \psi_{p,r}(n-i)/\gamma$ obtained from Lemma 3.3 to get rid of p_{n-i} .

$$\mathbb{P} \{subset \text{ is independent}\} \leq \prod_{i=1}^{k-1} (1 - r q_{n-i})^i \leq \prod_{i=1}^{k-1} \left(1 - \frac{r}{\gamma} \psi_{p,r}(n-i)\right)^i$$

Since $\psi_{p,r}(n-i) := \ell_{n-i}(\gamma p)$, and $\ell_n(x) = 1/(\sqrt{n} + 1)^2$ if $n \geq 1/(f_1(x) + 1)^2$, we obtain an explicit upper bound for sufficiently large $n - k$. \square

Remark 1. It is natural to ask about a lower bound on the probability P_k from Proposition 3.7. Using the expression from Lemma 3.5, the lower bound from Lemma 3.6, and the upper bound on each p_j^i from Lemma 3.3 yields a lower bound of a higher power of $1 - p$, and so we do not obtain a very good estimate that goes to 1 for high values of n .

4 Lower Bounds from Vertex Degrees

Our main goal here is to prove Theorem 1.1. A key part of this proof is analyzing the average vertex degree in this graph. Let us formally define vertex degree and average degree in $G_{n,p}^r$. The degree of vertex v_i is a random variable denoting the number of edges incident on v_i , given by the expression

$$\deg(v_i) = \sum_{j=1}^{i-1} X_j^i + \sum_{j=i+1}^n X_i^j.$$

The average degree $d(G_{n,p}^r)$ is the average vertex degree across the entire graph,

$$d(G_{n,p}^r) = \frac{1}{n} \sum_{i=1}^n \deg(v_i)$$

4.1 Concentration of average degree

Our main tool in lower bounding $\alpha(G_{n,p}^r)$ is the following concentration result about average degree, which can also be equivalently stated as saying that $d(G_{n,p}^r)/\log n \xrightarrow{p} 2\gamma$. To establish this concentration, we use a key technical result about a dependent Bernoulli sequence; the analysis of this sequence is presented later in §7.

Proof of Theorem 1.2. The sum of all edges is

$$\sum_{i=1}^n d(v_i) = \sum_{i=1}^n \left(\sum_{j=1}^{i-1} X_j^{(i)} + \sum_{k=i+1}^n X_i^{(k)} \right) = 2 \sum_{i=1}^n \sum_{j=1}^{i-1} X_j^{(i)} = 2 \sum_{i=1}^n S_{i-1}^{(i)} = 2 \sum_{i=1}^{n-1} S_i^{(i+1)}$$

where $S_i^{(i+1)} \sim S_i$ (meaning they follow the same distribution) the sum of first i variables in the Markov process $\{Y_j\}_{j \geq 1}$. The average degree of the graph is

$$d(G_{n,p}^r) = \frac{1}{n} \sum_{i=1}^n d(v_i) = \frac{2}{n} \sum_{i=1}^{n-1} S_i^{(i+1)}$$

Note that each random variable in the Markov process $\{Y_i\}_{i \geq 1}$ is nonnegative, so

$$0 \leq_{st} S_1 \leq_{st} S_2 \leq_{st} \cdots \leq_{st} S_n$$

where we denote stochastic dominance with \leq_{st} ($X \leq_{st} Y$ means that $P\{X \geq c\} \leq P\{Y \geq c\}$ for all real c) and where 0 denotes the constant random variable with probability 1 to be 0. Since $S_i^{(i+1)} \sim S_i$, we obtain $S_i^{(i+1)} \leq_{st} S_n$ for all $i \leq n$. Furthermore,

$$d(G_{n,p}^r) = \frac{2}{n} \sum_{i=1}^{n-1} S_i^{(i+1)} \leq_{st} \frac{2(n-1)}{n} S_n \leq_{st} 2S_n \quad (5)$$

From [Corollary 7.3](#) which is proved later in this paper, we have that for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \left| \frac{S_n}{\log n} - \frac{1}{\gamma} \right| \leq \frac{\varepsilon}{\gamma} \right\} = 1 \quad (6)$$

and then (5) yields the following upper bound,

$$\mathbb{P} \left\{ \frac{d(G_{n,p}^r)}{\log n} \geq \frac{2 + \varepsilon_0}{\gamma} \right\} \leq \mathbb{P} \left\{ \frac{2S_n}{\log n} \geq \frac{2 + \varepsilon_0}{\gamma} \right\} \leq \mathbb{P} \left\{ \left| \frac{S_n}{\log n} - \frac{1}{\gamma} \right| \geq \frac{\varepsilon_0}{2\gamma} \right\}. \quad (7)$$

Moreover,

$$\mathbb{P} \left\{ \left| \frac{S_n}{\log n} - \frac{1}{\gamma} \right| \geq \frac{\varepsilon_0}{2\gamma} \right\} \leq 1 - \mathbb{P} \left\{ \left| \frac{S_n}{\log n} - \frac{1}{\gamma} \right| \leq \frac{\varepsilon_0}{3\gamma} \right\}$$

If we pick $\varepsilon := \varepsilon_0/3$ in (6) and take limit of both sides, the right hand side of the above goes to 0 and so is the left. Together with (7), we have for all $\varepsilon_0 > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{d(G_{n,p}^r)}{\log n} \geq \frac{2 + \varepsilon_0}{\gamma} \right\} = 0 \quad \implies \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{d(G_{n,p}^r)}{\log n} \leq \frac{2(1 + \varepsilon_0)}{\gamma} \right\} = 1. \quad (8)$$

Therefore, $d(G_{n,p}^r)$ is asymptotically bounded above by $2(1 + \varepsilon_0)/\gamma \log n$. For deriving the lower bound, we first mention that for $n_c := \lfloor cn \rfloor \in \mathbb{Z}_{\geq 0}$ with $c \in (0, 1)$,

$$d(G_{n,p}^r) = \frac{2}{n} \sum_{i=1}^{n-1} S_i^{(i+1)} \geq_{st} \frac{2}{n} \sum_{i=n_c}^{n-1} S_i^{(i+1)} \geq_{st} \frac{2(n - n_c)}{n} S_{n_c} \geq_{st} 2(1 - c)S_{n_c}.$$

For the lower bound of $\frac{d(G_{n,p}^r)}{\log n}$, let's now fix $\varepsilon > 0$. From (6),

$$\forall \varepsilon > 0, \quad \exists N_\varepsilon \in \mathbb{Z}_{\geq 0}, \quad \mathbb{P} \left\{ \left| \frac{S_n}{\log n} - \frac{1}{\gamma} \right| \leq \frac{\varepsilon}{2\gamma} \right\} \geq 1, \quad \forall n \geq N_\varepsilon$$

Let

$$c := \frac{\varepsilon - 2\varepsilon^2 + 2\varepsilon^3}{2 - \varepsilon} \quad (9)$$

Note that $c \in (0, 1)$ is well-defined when $\varepsilon \in (0, 1)$. In addition, c is monotone increasing in terms of ε . Define

$$M_\varepsilon := \max \left\{ \exp \left\{ -\frac{1 + \varepsilon^2}{\varepsilon^2} \log c \right\}, \frac{N_\varepsilon}{c} \right\}.$$

If $n \geq M_\varepsilon$, then $n \geq N_\varepsilon/c$ and so $n_c = \lfloor cn \rfloor \geq N_\varepsilon$. Moreover,

$$\frac{\log n}{\log n_c} = \frac{\log n}{\log n + \log c - 1} \leq 1 + \varepsilon^2. \quad (10)$$

From $d(G_{n,p}^r) \geq_{st} 2(1-c)S_{n_c}$,

$$\mathbb{P} \left\{ \frac{d(G_{n,p}^r)}{\log n} \geq \frac{2(1-\varepsilon)}{\gamma} \right\} \geq \mathbb{P} \left\{ \frac{2(1-c)S_{n_c}}{\log n} \geq \frac{2(1-\varepsilon)}{\gamma} \right\} \geq \mathbb{P} \left\{ \frac{S_{n_c}}{\log n_c} \geq \frac{(1+\varepsilon^2)(1-\varepsilon)}{(1-c)\gamma} \right\}$$

where the last inequality comes from $(1+\varepsilon^2)S_{n_c}/\log n \geq S_{n_c}/\log n_c$ which is a straightforward result of (10). Substitute (9) and simplify the above term, we have

$$\begin{aligned} \mathbb{P} \left\{ \frac{S_{n_c}}{\log n_c} \geq \frac{(1+\varepsilon^2)(1-\varepsilon)}{(1-c)\gamma} \right\} &= \mathbb{P} \left\{ \frac{S_{n_c}}{\log n_c} \geq \frac{2-\varepsilon}{2\gamma} \right\} \\ &\geq \mathbb{P} \left\{ \left| \frac{S_{n_c}}{\log n_c} - \frac{1}{\gamma} \right| \leq \frac{\varepsilon}{2\gamma} \right\} \geq 1 - \varepsilon. \end{aligned}$$

So far, we have shown that, for all $\varepsilon > 0$, there exists $M_\varepsilon \in \mathbb{Z}_{\geq 0}$ such that

$$\mathbb{P} \left\{ \frac{d(G_{n,p}^r)}{\log n} \geq \frac{2(1-\varepsilon)}{\gamma} \right\} \geq 1 - \varepsilon, \quad \forall n \geq M_\varepsilon.$$

Consequently, $\frac{d(G_{n,p}^r)}{\log n} \geq \frac{2(1-\varepsilon)}{\gamma}$ w.h.p.. Together with the upper bound, we conclude that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \left| \frac{d(G_{n,p}^r)}{\log n} - \frac{2}{\gamma} \right| \leq \frac{\varepsilon}{\gamma} \right\} = 1, \quad \forall \varepsilon > 0. \quad \square$$

4.2 Proof of Theorem 1.1

We will use the following lower bound on stability number of general graphs that arises from applying Turán's theorem, which says that any graph not having a K_{t+1} subgraph cannot have more edges than the Turán graph corresponding to n and t .

Lemma 4.1 ([CHS03, Theorems 10 and 11]). *For any n -vertex graph G having m edges, the stability number $\alpha(G)$ can be lower bounded as follows,*

$$\alpha(G) \geq \max \left\{ \frac{n^2}{n+2m}, \frac{2n-m}{3} \right\}.$$

The first bound is equivalent to $n/(1+d(G))$, where $d(G) = 2m/n$ is the average degree of G , and this bound has been noted before, for e.g. by Griggs [Gri83]. It is dominated by the second bound if and only if $n/2 < m < n$.

We are now ready to argue that

$$\alpha(G_{n,p}^r) \geq \frac{\gamma}{2+\varepsilon} \cdot \frac{n}{\log n} \quad \text{w.h.p.}$$

For fixed $\varepsilon > \varepsilon_0 > 0$, we observe if n is sufficiently large then

$$\mathbb{P} \left\{ \alpha(G_{n,p}^r) \geq \frac{n}{\frac{2+\varepsilon}{\gamma} \log n} \right\} \geq \mathbb{P} \left\{ \alpha(G_{n,p}^r) \geq \frac{n}{\frac{2+\varepsilon_0}{\gamma} \log n + 1} \right\} \geq \mathbb{P} \left\{ \frac{d(G_{n,p}^r)}{\log n} \leq \frac{2+\varepsilon_0}{\gamma} \right\}, \quad (11)$$

where the first inequality follows from $\frac{2+\varepsilon}{\gamma} \log n > \frac{2+\varepsilon_0}{\gamma} \log n + 1$ for $n > e^{\frac{\gamma}{\varepsilon-\varepsilon_0}}$, and the second inequality has used the first bound in [Lemma 4.1](#) which tells us that $\alpha(G_{n,p}^r) \geq n/(1+d(G_{n,p}^r))$. After taking the limit throughout in (11), the right-most probability goes to 1 by [Theorem 1.2](#), and so the left-most term also has limit 1, which yields the lower bound of $\Omega(n/\log n)$ on $\alpha(G_{n,p}^r)$. The bound in terms of the prime-counting function $\pi(n)$ is an immediate corollary of the following well-known Prime Number Theorem which says that $\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\log n} = 1$. For every $\varepsilon > 0$, replacing $n/\log n$ in the leftmost event in (11) with its lower bound $\pi(n)/(1+\varepsilon)$ implies that w.h.p.

$$\alpha(G_{n,p}^r) \geq \frac{\gamma}{2+\varepsilon} \frac{\pi(n)}{1+\varepsilon}.$$

Since the denominator $\varepsilon^2 + 3\varepsilon + 2$ is less than $4\varepsilon + 2$, our proof is complete.

5 Greedy algorithm

Now we analyse the performance of a greedy algorithm for approximating (lower bounding) $\alpha(G_{n,p}^r)$. This algorithm was introduced near the statement of [Proposition 1.3](#) in [§1.1](#). Our algorithm sorts vertices in the order v_1 to v_n , and starting with an empty set iteratively builds an independent set in n iterations where at iteration i the vertex v_i is added to the independent set if this addition preserves the new set to also be independent. The definition of our random graph is such that one can intuitively see that the vertex degrees are in the descending order from v_1 to v_n w.h.p. Hence, our simple greedy algorithm can be regarded as the opposite of what one would use as a greedy algorithm for approximating $\alpha(G_{n,p}^r)$ wherein the vertices would be sorted in increasing order of their degrees (v_n to v_1 for $G_{n,p}^r$). Thus, lower bounding the performance of our algorithm tells us (intuitively speaking) the worst we can do in any greedy algorithm for approximating $\alpha(G_{n,p}^r)$.

We prove here the asymptotic lower bound guarantee that was stated earlier in [Proposition 1.3](#). Let $\alpha_G(G_{n,p}^r)$ be the output of our greedy algorithm.

Proof of [Proposition 1.3](#). We adopt a similar strategy as Grimmett and McDiarmid [[GM75](#)] for $G_{n,p}$. Let STAB be the independent set created during the iterations of the greedy algorithm. We analyze the number of iterations needed to increase the size of STAB by 1. Denote R_{m+1} to be the number of iterations needed to add a new vertex into STAB since when $|\text{STAB}| = m$. Observe that R_{m+1} follows a geometric distribution whose parameter is $\mathbb{P}\{v_i \text{ is disconnected from STAB}\}$. Suppose $\text{STAB} = \{v_{j_1}, \dots, v_{j_m}\}$, then

$$\mathbb{E}R_{m+1} = \frac{1}{\mathbb{P}\{v_i \text{ is disconnected from STAB}\}}, \quad \forall i > j_m$$

We need to estimate the probability of v_i can be added into STAB i.e. v_i is disconnected from STAB. By [Lemma 3.6](#),

$$\frac{1}{\prod_{j=i-m}^{i-1}(\gamma p_j)} \leq \mathbb{E}R_{m+1} \leq \frac{1}{(1-p) \prod_{j=1}^{m-1}(1-p_j)}$$

Note that there exists a positive integer w such that $p_i \leq w/i$ for all i . More precisely, we can take $\tau = \lceil 1/\gamma \rceil$ by [Lemma 3.3](#). Denote $\gamma = (1-p) \prod_{j=1}^{\tau}(1-p_j)$. Then we can further relax the upper bound above when $m > \tau$.

$$\mathbb{E}R_{m+1} \leq \frac{1}{\gamma \prod_{j=\tau+1}^m(1-\frac{\tau}{j})} = \frac{1}{\gamma} \frac{m!}{(m-\tau)! \tau!} = \frac{1}{\gamma} \binom{m}{\tau} \leq \frac{1}{\gamma} \left(\frac{me}{\tau}\right)^{\tau}.$$

If k is large enough, then

$$\mathbb{E} \sum_{m=1}^k R_m \leq \sum_{m=1}^{\tau} \mathbb{E}R_m + \sum_{m=\tau+1}^k \frac{1}{\gamma} \left(\frac{me}{\tau}\right)^{\tau} = O\left(k^{\tau+1}\right)$$

Note that $\sum_{m=1}^{\tau} \mathbb{E}R_m$ is a constant and $\sum_{m=\tau+1}^k \frac{1}{\gamma} \left(\frac{me}{\tau}\right)^{\tau}$ is bounded above by $\frac{k}{\gamma} \left(\frac{ke}{\tau}\right)^{\tau} = O\left(k^{\tau+1}\right)$. For the variance, since R_m follows geometric distribution,

$$\begin{aligned} \text{Var} \sum_{m=1}^k R_m &\leq \sum_{m=1}^k \mathbb{E}R_m^2 \leq 2 \sum_{m=1}^k \mathbb{E}R_m^2 \\ &\leq \sum_{m=1}^w \mathbb{E}R_m + \sum_{m=d+1}^k \frac{1}{\gamma^2} \left(\frac{m}{\tau}\right)^{2\tau} \\ &= O\left(k^{2\tau+1}\right) \end{aligned}$$

For convenience of notation, we denote $\sum_{m=1}^k R_m$ as Z_k . Let

$$k := k(n) := \left(\left(\frac{\tau}{e}\right)^{\tau} \cdot \frac{\gamma}{3} \cdot n \right)^{\frac{1}{\tau+1}} = O\left(n^{\frac{1}{\tau+1}}\right)$$

Then, we have $\mathbb{E}Z_k = O(n)$ and $\text{Var} Z_k = O\left(n^{\frac{2\tau+1}{\tau+1}}\right)$. In particular, $\mathbb{E}Z_k < \frac{n}{2}$ for sufficiently large n , which implies that

$$\mathbb{P}\{Z_k \geq n\} \leq \mathbb{P}\left\{Z_k \geq \mathbb{E}Z_k + \frac{n}{2}\right\} \leq \mathbb{P}\left\{|Z_k - \mathbb{E}Z_k| \geq \frac{n}{2}\right\}.$$

Therefore, when $n \rightarrow \infty$, we get that

$$\mathbb{P}\{Z_k \geq n\} \leq \mathbb{P}\left\{|Z_k - \mathbb{E}Z_k| \geq \frac{n}{2}\right\} \leq \frac{4 \text{Var} Z_k}{n^2} = O\left(n^{-\frac{1}{\tau+1}}\right).$$

Hence, Z_k , the number of iterations needed to obtain a independent set of size k , is at most n w.h.p. when $n \rightarrow \infty$ and $k = n^{1/(\tau+1)}$. Conversely, if our algorithm has n iterations, then the size of independent set is at least k w.h.p., i.e. the greedy stability number of $G_{n,p}^r$ is $\Omega\left(n^{1/(\tau+1)}\right)$. \square

Obviously, $\alpha_G(G_{n,p}^r)$ gives another lower bound on $\alpha(G_{n,p}^r)$. However, this lower bound is much weaker than $\Omega(n/\log(n))$ in [Theorem 1.1](#).

6 Upper Bound

Our goal here is to prove the upper bound on $\alpha(G_{n,p}^r)$ from [Theorem 1.4](#). We use the first moment argument, which has also been used for the binomial random graph model [[Mat76](#), [Fri90](#), [FK15](#)]. More precisely, let $H_{k,n}$ be the number of independent sets of order k in $G_{n,p}^r$. We want to find the minimum k such that $\mathbb{P}\{H_{k,n} > 0\} \rightarrow 0$ as $n \rightarrow \infty$. The first step towards this is to estimate the probability that a subset of vertices forms an independent set.

6.1 Proof of [Theorem 1.4](#)

The function

$$f_r(c) := c(1 - \log c) + \frac{r}{\gamma} (c + \log(1 - c)) = \frac{c}{\gamma} - c \log c + \frac{r}{\gamma} \log(1 - c).$$

appears in our analysis and we will need to use negative values of this function over the positive part of its domain, which lies within $(0, 1)$. For this purpose we claim that there is a unique positive root beyond which f_r is negative-valued, and although it is difficult to characterise this root explicitly, we provide an upper bound on it in terms of r .

Claim 1. *For every $r \in (0, 1)$ the function f_r has a unique positive root c^* and $f_r(c) < 0$ for all $c \in (c^*, 1)$. Furthermore, $c^* < e^{-r} + 0.1r$.*

We argue an upper bound on $\alpha(G_{n,p}^r)$ that is arbitrarily close to the root c^* of the function f_r . In particular, we prove that for every $\varepsilon > 0$, w.h.p. $\alpha(G_{n,p}^r) \leq (c^* + \varepsilon)n$. Since [Claim 1](#) tells us that $c^* < e^{-r} + 0.1r$, it follows that $\alpha(G_{n,p}^r) \leq (e^{-r} + 0.1r)n$.

Let $H_{k,n}$ be the number of independent sets of order k in $G_{n,p}^r$. We want to find a minimum $k := k(n)$ such that $\mathbb{P}\{H_{k,n} > 0\} \rightarrow 0$ as $n \rightarrow \infty$. Then, with such a $k(n)$, we have w.h.p. $\alpha(G_{n,p}^r) \leq k(n)$. Observe that $\mathbb{E}H_{k,n} = \sum_{i=1}^n i \mathbb{P}\{H_{k,n} = i\} \geq \sum_{i=1}^n \mathbb{P}\{H_{k,n} = i\} = \mathbb{P}\{H_{k,n} > 0\}$. It is sufficient to show that $\mathbb{E}H_{k,n} \rightarrow 0$ w.h.p.

Observe that $\mathbb{E}H_{k,n}$ can be written as the sum of probabilities of all possible subsets of size k to be independent set,

$$\mathbb{E}H_{k,n} = \sum_{\substack{A \subseteq \{v_1, v_2, \dots, v_n\}: \\ |A|=k}} \mathbb{P}\{A \text{ is independent set}\}$$

Using the upper bound from [Proposition 3.7](#) for the probability in the summand, it follows that when $n - k$ is large enough,

$$\mathbb{E}H_{k,n} \leq \binom{n}{k} \prod_{i=1}^{k-1} \left(1 - \frac{r}{\gamma (\sqrt{a-i} + 1)^2}\right)^i. \quad (12)$$

Let k be an upper bound on $\alpha(G_{n,p}^r)$. Since $G_{n,p}^r$ is a n -vertex graph, we can assume that $k = cn$ for some constant c which is to be determined. We claim that

Claim 2. *We have*

$$\prod_{i=1}^k \left(1 - \frac{h}{(\sqrt{a-i} + 1)^2}\right)^i \leq \exp\{h(\log(1-c) + c)n + O(n)\}.$$

Before proving this claim, let us argue why it finishes our proof of this theorem. When k is linear in terms of n , it is well-known that binomial coefficient $\binom{n}{k}$ is bounded above by $(ne/k)^k$. Equation (2) applied to (12) implies that

$$\mathbb{E}H_{k,n} \leq \left(\frac{ne}{k}\right)^k \prod_{i=1}^{k-1} \left(1 - \frac{r}{\gamma(\sqrt{a-i}+1)^2}\right)^i = \exp\{nf_r(c) + O(n)\}.$$

If $f_r(c) < 0$, then $\lim_{n \rightarrow \infty} \mathbb{E}H_{k,n} = 0$ which is what we want for $k = cn$ to be an upper bound on the stability number. For the tightest bound, we take the smallest value of c with $f_r(c) < 0$.

Claim 1 shows that $f_r(c) < 0$ for all $c \in (c^*, 1)$ where c^* is the unique positive root of f_r . Therefore, taking $c = c^* + \varepsilon$ yields the desired upper bound of $c^* + \varepsilon$.

It remains to prove the two claims used in the above proof.

6.1.1 Proof of Claim 1

It is clear that the domain of f_r is in $(-\infty, 1)$. Let us first prove uniqueness of the root in $(0, 1)$. We have

$$\frac{\partial}{\partial c} f_r(c) = \frac{r}{\gamma} \left(1 - \frac{1}{1-c}\right) - \log c, \quad \frac{\partial^2}{\partial c^2} f_r(c) = -\frac{1}{c} - \frac{r}{\gamma} \cdot \frac{1}{(1-c)^2}.$$

As the second derivative $\frac{\partial^2}{\partial c^2} f_r(c)$ is always negative, $f_r(c)$ is concave for fixed r . Therefore, there are at most 2 roots. Next, we observe that

$$\lim_{c \rightarrow 0} f_r(c) = 0, \quad \lim_{c \rightarrow 0} f_r'(c) = \infty, \quad \lim_{c \rightarrow 1} f_r(c) = -\infty$$

The boundary value of $f_r(c)$ at $c = 0$ is 0. However, $c = 0$ is not in the domain of $f_r(c)$ and so there is at most 1 root for $f_r(c)$. We now need to show such root actually exists between 0 and 1. Notice that the slope near $c = 0$ is positive and there must exist $a(r) \in (0, 1)$ such that $f_r(a(r)) > 0$ for all $r \in (0, 1)$. Since the either boundary value at $c = 1$ is $-\infty$, the continuity of $f_r(c)$ implies the existence of a root $c^* \in (a(r), 1)$. In particular, $f_r(c) < 0$ for all $c \in (c^*, 1)$ and $f_r(c) > 0$ for all $c \in (0, c^*)$.

Although there is no explicit formula for the root c^* , we can find some upper bound function $c(r)$ such that $1 > c(r) > c^*$ for all $r \in (0, 1)$. Consider $c(r) = e^{-r} + \frac{r}{10}$ and denote the function $g(r)$ by

$$g(r) := f_r(c(r)) = \left(1 - \frac{1}{1 - e^{-r} - \frac{r}{10}}\right) r - \log\left(e^{-r} + \frac{r}{10}\right).$$

We need to show $g(r) < 0$ for all $r \in (0, 1)$. It is clear that the boundary values of $g(r)$ are $g(0) = 0$ and $g(1) \approx -0.1197$. Since $g(r)$ is continuous and both boundary values are non-positive, it is sufficient to prove $g(r)$ has no root within interval $(0, 1)$. Suppose not, then $\exists r^* \in (0, 1)$ such that

$$\left(1 - \frac{1}{1 - e^{-r^*} - \frac{r^*}{10}}\right) r^* = \log\left(e^{-r^*} + \frac{r^*}{10}\right)$$

Observe that $e^{-r} + r/10$ is monotone decreasing for $r \in (0, 1)$. Therefore, for $r \in (0, 1)$, the right hand side $\log(e^{-r} + r/10)$ is also monotone decreasing while the left hand side

$\left(1 - \frac{1}{1 - e^{-r} - \frac{r}{10}}\right) \cdot r$ is monotone increasing. Furthermore, they both attain 0 at $r = 0$.

$$\log\left(e^{-r} + \frac{r}{10}\right)\Big|_{x=0} = 0 = \left(1 - \frac{1}{1 - e^{-r} - \frac{r}{10}}\right) r \Big|_{x=0}$$

Since one is decreasing and other is increasing, there is no $r^* \in (0, 1)$ making them equal.

6.1.2 Proof of Claim 2

Since $1 + x \leq e^x$, we have

$$\prod_{i=1}^k \left(1 - \frac{h}{a-i}\right)^i \leq \exp\left\{-h \cdot \sum_{i=1}^k \frac{i}{(\sqrt{a-i}+1)^2}\right\} \quad (13)$$

The function $x/(\sqrt{a-x}+1)^2$ is monotone increasing for $0 \leq x \leq a$. We can estimate the inner summation in (13) by evaluating an integral

$$\begin{aligned} \sum_{i=1}^k \frac{i}{(\sqrt{a-i}+1)^2} &\geq \int_0^{cn} \frac{x}{(\sqrt{a-x}+1)^2} dx \\ &= -cn + 6\sqrt{n} - 6 + \frac{(4c-6)n+6}{1+\sqrt{(1-c)n}} + 2(n-3) \log\left(\frac{1+\sqrt{n}}{1+\sqrt{(1-c)n}}\right) \end{aligned} \quad (14)$$

When $n \rightarrow \infty$, the lower bound above is dominated by $-cn$ and $2n \log\left(\frac{1+\sqrt{n}}{1+\sqrt{(1-c)n}}\right)$. For the second term, its asymptotic value can be obtained from the Taylor expansion of $\log x$, i.e.

$$\log\left(\frac{1+\sqrt{n}}{1+\sqrt{(1-c)n}}\right) = \log\left(\frac{1}{\sqrt{1-c}} + O\left(\frac{1}{\sqrt{n}}\right)\right) = -\frac{1}{2} \log(1-c) + O\left(\frac{1}{\sqrt{n}}\right).$$

By using big-O notation, (14) can be written as

$$\sum_{i=1}^k \frac{i}{(\sqrt{a-i}+1)^2} \geq -(\log(1-c) + c)n + O(n).$$

Therefore, we can apply this bound to (13) to get

$$\prod_{i=1}^k \left(1 - \frac{h}{a-i}\right)^i \leq \exp\{h(\log(1-c) + c)n + O(n)\},$$

which finishes our proof for this claim.

7 Concentration of a Bernoulli Sequence

In this section, we establish a theorem about convergence of a dependent Bernoulli sequence, which is fundamental to our analysis of the average vertex degree in [Theorem 1.2](#). We also present several consequences of our theorem, in particular, a generalisation in [Corollary 7.4](#)

wherein the marginal probabilities of the sequence are sandwiched between two converging sequences.

Consider a sequence of Bernoulli r.v.'s $\{Y_n\}$, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that are not assumed to be iid. Denote the marginal probabilities and partial sum by

$$p_n := \mathbb{P}\{Y_n = 1\} \quad n \geq 1, \quad S_n := Y_1 + Y_2 + \cdots + Y_n.$$

We analyse the ratio of this partial sum to $\log n$, and so for convenience let us denote

$$\mathcal{Y}_n := \frac{S_n}{\log n}.$$

Our main result of this section is to show that when the probability sequence $\{p_n\}$ satisfies the recurrence formula (1) with a equal to some constant β , then \mathcal{Y}_n concentrates to $1/\beta$. Furthermore, we also establish that the mean of \mathcal{Y}_n , which is equal to $\mathbb{E}S_n/\log n$, converges to $1/\beta$ and the distribution of \mathcal{Y}_n grows at a quadratic rate in the limit.

Theorem 7.1. *Suppose the sequence $\{p_n\}$ is such that there is some $\beta \in (0, 1]$ for which $p_n = f_\beta(p_{n-1})$ for all $n \geq 2$. Then, $\mathbb{E}\mathcal{Y}_n \rightarrow 1/\beta$ and $\mathcal{Y}_n \xrightarrow{p} 1/\beta$. Furthermore, the distribution of \mathcal{Y}_n grows at a quadratic rate in the limit as follows,*

$$\frac{1}{1-\theta} \geq \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{\mathcal{Y}_n}{\mathbb{E}\mathcal{Y}_n} \geq 1 - \theta \right\} \geq \theta^2, \quad \theta \in [0, 1).$$

This theorem is proved in the next few sections. We discuss here some of its consequences. First, we have that the partial sum is dominated by arbitrary powers of n .

Corollary 7.2. *Under the conditions of Theorem 7.1, w.h.p. $S_n = O(n^\varepsilon)$ for every $\varepsilon > 0$.*

Proof. Fix any $\varepsilon > 0$. Note that $\log n = O(n^\varepsilon)$, and so there exists a constant C_ε and integer N_ε such that $\log n \leq C_\varepsilon n^\varepsilon$ for all $n \geq N_\varepsilon$. Therefore, for all $n \geq N_\varepsilon$,

$$\mathbb{P} \left\{ \frac{S_n}{C_\varepsilon n^\varepsilon} \geq 1 + \frac{1}{\beta} \right\} \leq \mathbb{P} \left\{ \frac{S_n}{C_\varepsilon n^\varepsilon} \geq \varepsilon + \frac{1}{\beta} \right\} \leq \mathbb{P} \left\{ \frac{S_n}{\log n} \geq \varepsilon + \frac{1}{\beta} \right\} \leq \mathbb{P} \left\{ \left| \frac{S_n}{\log n} - \frac{1}{\beta} \right| \geq \varepsilon \right\},$$

where the first inequality is due to $\varepsilon + 1/\beta < 1 + 1/\beta$, and the last inequality is due to $|S_n/\log n - 1/\beta| \geq S_n/\log n - 1/\beta$. The concentration of $S_n/\log n$ to $1/\beta$ in Theorem 7.1 means that the rightmost probability converges to 0, and then the above chain implies that the leftmost probability also goes to 0, which leads to $S_n = O(n^\varepsilon)$ w.h.p. \square

The second consequence of our theorem is that it subsumes the case where the stochastic process $\{Y_n\}$ forms a non-homogenous Markov chain whose transition matrix obeys a decay property.

Corollary 7.3. *Suppose that $\{Y_n\}$ is a Markov chain of Bernoulli r.v.'s, so that for every $n \geq 1$ we have*

$$\mathbb{P}\{Y_{n+1} = y_{n+1} \mid Y_1 = y_1, \dots, Y_n = y_n\} = \mathbb{P}\{Y_{n+1} = y_{n+1} \mid Y_n = y_n\},$$

and also suppose that there is some $\beta \in [0, 1)$ such that

$$\mathbb{P}\{Y_{n+1} = 1 \mid Y_n = 0\} = \mathbb{P}\{Y_n = 1\}, \quad \mathbb{P}\{Y_{n+1} = 1 \mid Y_n = 1\} = (1 - \beta)\mathbb{P}\{Y_n = 1\}.$$

Then, $\mathbb{E}\mathcal{Y}_n \rightarrow 1/\beta$ and $\mathcal{Y}_n \xrightarrow{p} 1/\beta$.

Proof. Using arguments analogous to those already provided in [Lemma 3.3](#) in the context of the Markov chain $\{X_j^i\}_{j < i}$ for each vertex v_i in the random graph $G_{n,p}^r$, it follows that $p_n = f_\beta(p_{n-1})$, and so this becomes a special case of [Theorem 7.1](#). \square

This makes [Theorem 7.1](#) relevant to our Markov random graph $G_{n,p}^r$ by taking $\beta = \gamma$, and we use this convergence result to prove concentration of the average degree of this graph in [Theorem 1.2](#).

A third consequence of our main theorem, which may be of independent interest and is stated below and proved later in [§7.4](#), is that we show convergence of expectation for the sequence $\{\mathcal{Y}_n\}$ under a weaker assumption, namely, when the probability sequence $\{p_n\}$ is sandwiched between two sequences $\{f_{a_n}(p_{n-1})\}$ and $\{f_{b_n}(p_{n-1})\}$ that are generated using two converging sequences $\{a_n\}$ and $\{b_n\}$.

Corollary 7.4. *Suppose there exist two converging sequences $\{a_n\}, \{b_n\} \subset (0, 1]$ having the same limit $\beta > 0$ and such that the corresponding sequences $\{f_{a_n}\}$ and $\{f_{b_n}\}$ bound the marginal probabilities of $\{Y_n\}$ as $f_{a_n}(p_{n-1}) \leq p_n \leq f_{b_n}(p_{n-1})$ for all $n \geq 2$. Then, $\mathbb{E}\mathcal{Y}_n \rightarrow 1/\beta$.*

Note that the weaker assumption made in this corollary also means that we do not guarantee convergence in probability, unlike as in [Corollary 7.3](#).

Let us also make some remarks about the claims in [Theorem 7.1](#).

Remark 2. For a general random sequence, convergence of expected value and convergence in probability do not imply each other, but when the sequence is uniformly integrable then the latter implies the former [cf. [Han12](#)]. This indicates that if we can show that the sequence $\{\mathcal{Y}_n\}$ is uniformly integrable then establishing $\mathcal{Y}_n \xrightarrow{p} 1/\beta$ gives us $\mathbb{E}\mathcal{Y}_n \rightarrow 1/\beta$. However, we do not use this implication in our proof since we do not think that $\{\mathcal{Y}_n\}$ is uniformly integrable. Instead, our proof first establishes convergence of $\mathbb{E}\mathcal{Y}_n$ and uses it to argue convergence in probability.

Remark 3. For a random sequence $\{\mathcal{X}_n\}$, there are two notions when talking about the expected values converging. The first is that of convergence *of* expectation where there is another r.v. X such that $\mathbb{E}\mathcal{X}_n \rightarrow \mathbb{E}X$ (if $\mathbb{E}\mathcal{X}_n$ converges to a finite number c then one could simply define X to be equal to c a.s.). The second notion is that of convergence *in* expectation where $\mathbb{E}|\mathcal{X}_n - X| \rightarrow 0$ for some r.v. X . Markov's inequality gives us that the latter implies convergence in probability, and it is well-known that the reverse implication is true if and only if $\{\mathcal{X}_n\}$ is uniformly integrable. Since we do not think that uniform integrability holds for our sequence $\{\mathcal{Y}_n\}$, we cannot use [Theorem 7.1](#) to say that $\{\mathcal{Y}_n\}$ converges in expectation to $1/\beta$.

7.1 Convergence of Expectation

We prove here the first claim of [Theorem 7.1](#) that $\mathbb{E}\mathcal{Y}_n$ converges to $1/\beta$. This convergence is a step towards establishing convergence in probability in the next section.

Since $\mathbb{E}\mathcal{Y}_n = \mathbb{E}S_n / \log n$, establishing convergence of expectation to $1/\beta$ is equivalent to showing that $\beta \mathbb{E}S_n / \log n$ converges to 1. The definition of S_n implies that $\mathbb{E}S_n = \sum_{i=1}^n \mathbb{E}Y_i = \sum_{i=1}^n p_i$. Since the sequence $\{p_i\}$ follows the recursion $p_i = f_\beta(p_{i-1})$, we can use the bounding analysis of the recurrence formula that was done earlier in this paper. In particular, applying [Lemma 2.4](#) with $p_i = x_i$ and $a = \beta$ gives us the bounds

$$\beta p_1 + \eta(p_1) + \frac{2}{1 + \sqrt{n}} + \log n \leq \beta \mathbb{E}S_n \leq \beta p_1 + \log(1 + (n-1)p_1). \quad (15)$$

Divide throughout by $\log n$ to get

$$1 + \frac{\beta p_1 + \eta(p_1) + \frac{2}{1+\sqrt{n}}}{\log n} \leq \beta \mathbb{E} \mathcal{Y}_n \leq \frac{\beta p_1 + \log(1 + (n-1)p_1)}{\log n}.$$

We argue that the above lower and upper bounds have limit (as $n \rightarrow \infty$) equal to 1, and then the squeeze theorem implies that $\mathbb{E} \mathcal{Y}_n$ converges to $1/\beta$. It is easy to see the limit of the lower bound because $2/(1 + \sqrt{n}) \leq 2$, and β , p_1 , and $\eta(p_1)$ are all constants. Now consider the upper bound. This can be written as

$$\frac{\beta p_1 + \log(1 + (n-1)p_1)}{\log n} = 1 + \frac{\beta p_1 + \log(1 + (n-1)p_1) - \log n}{\log n} = 1 + \frac{\beta p_1 + \log\left(p_1 + \frac{1-p_1}{n}\right)}{\log n}$$

Denote $g(n) := \beta p_1 + \log\left(p_1 + \frac{1-p_1}{n}\right)$. Since $g(n)$ is decreasing in n , it is clear that $\lim_{m \rightarrow \infty} g(m) \leq g(n) \leq g(1)$ for every positive integer n . The limit of g is equal to $\beta p_1 + \log p_1$, and so we have $g(n) \in \beta p_1 + [\log p_1, 0]$. Since these bounds on $g(n)$ are all constants and independent of n , we get that the limit of $g(n)/\log n$ is equal to 0 by the squeeze theorem. This means that the upper bound on $\beta \mathbb{E} \mathcal{Y}_n$, which in our notation is $1 + g(n)/\log n$, has limit equal to 1.

7.2 Convergence in Probability

This section proves the second claim of [Theorem 7.1](#) that \mathcal{Y}_n concentrates to $1/\beta$. We first argue an upper bound on the second moment of S_n , and then use this bound in conjunction with two key technical results on sequences of random variables to finally deduce our claim on the concentration of \mathcal{Y}_n .

7.2.1 Second Moment of Partial Sum

The definition $S_n = Y_1 + Y_2 + \dots + Y_n$ implies that

$$S_n^2 = \sum_{i=1}^n Y_i^2 + 2 \sum_{i < j} Y_i Y_j = \sum_{i=1}^n Y_i + 2 \sum_{i < j} Y_i Y_j,$$

where the second equality is due to each Y_i being a Bernoulli r.v. Therefore, linearity of expectation and $\mathbb{E} S_n = \sum_{i=1}^n \mathbb{E} Y_i$ means that the second moment of S_n is

$$\mathbb{E} S_n^2 = \mathbb{E} S_n + 2 \sum_{i < j} \mathbb{E}(Y_i Y_j). \quad (16)$$

The next lemma upper bounds the expected value of each product term $Y_i Y_j$.

Lemma 7.5. *For $n \geq 2$,*

$$\mathbb{E} S_n^2 \leq \frac{2r}{\gamma} \mathbb{E} S_n + \left(\frac{\log n}{\gamma} \right)^2.$$

Proof. If $i \leq j - 2$, the joint probability $\mathbb{P}\{Y_i Y_j = 1\}$ is at most $p_i p_{j-1}$. Indeed, by [Lemma 3.4](#),

$$\mathbb{P}\{Y_i Y_j = 1\} = \mathbb{P}\{Y_i = 1, Y_j = 1\} \mathbb{P}\{Y_j = 1 \mid Y_i = 1\} \mathbb{P}\{Y_i = 1\} \leq p_i p_{j-1}$$

Furthermore, for two dependent variables Y_{j-1} and Y_j ,

$$\mathbb{P}\{Y_{j-1} Y_j = 0\} = \mathbb{P}\{Y_j = 0\} + \mathbb{P}\{Y_{j-1} = 0\} \mathbb{P}\{Y_j = 1 \mid Y_{j-1} = 0\} = 1 - p_j + p_{j-1}(1 - p_{j-1})$$

The recurrence relation implies $p_j = p_{j-1}(\gamma r p_{j-1})$. Thus,

$$\mathbb{P}\{Y_i Y_j = 1\} \leq \begin{cases} p_i p_{j-1}, & i \leq j-2 \\ \left(\frac{1}{\gamma} - 1\right)(p_{j-1} - p_j), & i = j-1 \end{cases}$$

We are ready to compute an upper bound for $\sum_{i < j} \mathbb{E}Y_i Y_j$.

$$\begin{aligned} \sum_{i < j} \mathbb{E}Y_i Y_j &= \sum_{j=2}^n \sum_{i=1}^{j-1} \mathbb{E}Y_i Y_j = \sum_{j=2}^n \mathbb{E}Y_{j-1} Y_j + \sum_{i=3}^n \sum_{i=1}^{j-2} \mathbb{E}Y_i Y_j \\ &\leq \left(\frac{1}{\gamma} - 1\right) \sum_{i=2}^n (p_{j-1} - p_j) + \sum_{j=3}^n \sum_{i=1}^{j-2} p_i p_{j-1} \\ &\leq \left(\frac{1}{\gamma} - 1\right) (p_1 - p_n) + \sum_{j=3}^n \sum_{i=1}^{j-2} p_i p_{j-1}. \end{aligned}$$

Note that $\mathbb{E}S_n = \sum_{j=1}^n \mathbb{E}X_j = \sum_{j=1}^n p_j$. Then (16) can be written as

$$\mathbb{E}S_n^2 = \mathbb{E}S_n + \frac{2r}{\gamma} (p_1 - p_n) + 2 \sum_{j=3}^n \sum_{i=1}^{j-2} p_i p_{j-1}.$$

From Lemma 3.3, γp_i has an upper bound $1/i$, and so

$$\sum_{j=3}^n \sum_{i=1}^{j-2} p_i p_{j-1} \leq \frac{1}{\gamma} \sum_{j=3}^n p_{j-1} \sum_{i=1}^{j-2} \frac{1}{i}.$$

The partial sum of harmonic series $\sum_{i=1}^{j-2} \frac{1}{i} \leq \log(j-1) + 1$. Therefore,

$$\sum_{j=3}^n \sum_{i=1}^{j-2} p_i p_{j-1} \leq \frac{1}{\gamma} \sum_{j=3}^n p_{j-1} + \frac{1}{\gamma} \sum_{j=3}^n \log(j-1) p_{j-1} = \frac{\mathbb{E}S_n - p_1 - p_2}{\gamma} + \frac{1}{\gamma} \sum_{j=3}^n \log(j-1) p_{j-1}.$$

Then, we reuse the bound $p_j \leq 1/(\gamma j)$ for $j \in \mathbb{N}$ from Lemma 3.3,

$$\frac{1}{\gamma} \sum_{j=2}^{n-1} (\log j) p_j \leq \frac{1}{\gamma^2} \sum_{j=2}^{n-1} \frac{\log j}{j} \leq \frac{1}{\gamma^2} \int_1^n \frac{\log t}{t} dt = \frac{1}{2} \left(\frac{\log n}{\gamma} \right)^2.$$

Combining the two inequalities above, we obtain

$$\sum_{j=3}^n \sum_{i=1}^{j-2} p_i p_{j-1} \leq \frac{\mathbb{E}S_n - p_1 - p_2}{\gamma} + \frac{1}{2} \left(\frac{\log n}{\gamma} \right)^2.$$

It follows that

$$\begin{aligned} \mathbb{E}S_n^2 &\leq \frac{2r}{\gamma} \mathbb{E}S_n - 2p_1 - \frac{2}{\gamma} p_2 - \frac{2r}{\gamma} p_n + \left(\frac{\log n}{\gamma} \right)^2 \\ &\leq \frac{2r}{\gamma} \mathbb{E}S_n + \left(\frac{\log n}{\gamma} \right)^2, \end{aligned}$$

which is the desired result. \square

7.2.2 Two Lemmata on Random Sequences

Let $\{\mathcal{X}_n\}$ be a sequence of random variables all of which are defined on the same probability space. The following result about the terms in the sequence concentrating to their mean is a straightforward consequence of a classic concentration inequality.

Lemma 7.6. *Suppose that $\mathcal{X}_n \geq 0$ a.s. for all n , and the mean $\mu_n \in (0, \infty)$ and variance σ_n^2 of \mathcal{X}_n obey $\sigma_n^2 = o(\mu_n^2)$. Then, \mathcal{X}_n/μ_n concentrates to 1.*

Proof. Chebyshev's inequality applied to \mathcal{X}_n tells us that $\mathbb{P}\{|\mathcal{X}_n - \mu_n| \geq k\sigma_n\} \leq 1/k^2$ for any $k > 0$. Taking $k = \varepsilon\mu_n/\sigma_n$ for arbitrary $\varepsilon > 0$ gives us $\mathbb{P}\{|\mathcal{X}_n - \mu_n| \geq \varepsilon\mu_n\} \leq \sigma_n^2/(\varepsilon^2\mu_n^2)$. The assumption of μ_n being positive makes the probability equal to $\mathbb{P}\{|\mathcal{X}_n/\mu_n - 1| \geq \varepsilon\}$. Now taking limit on both sides leads to

$$\lim_{n \rightarrow \infty} \mathbb{P}\left\{\left|\frac{\mathcal{X}_n}{\mu_n} - 1\right| \geq \varepsilon\right\} \leq \lim_{n \rightarrow \infty} \frac{\sigma_n^2}{\varepsilon^2\mu_n^2}.$$

Since $\sigma_n^2 = o(\mu_n^2)$, the right-hand side goes to zero, which implies that the left-hand side is also zero and therefore, $\mathcal{X}_n/\mu_n \xrightarrow{p} 1$. \square

The second useful result is a special case of Slutsky's theorem, or more generally the fact that convergence in probability is preserved under multiplication [cf. [Cin11](#), Theorem III.3.6].

Lemma 7.7. *Let $\{\mathcal{Z}_n\}$ be a converging sequence of positive reals with limit τ and such that $\mathcal{X}_n/\mathcal{Z}_n \xrightarrow{p} \lambda$ for some constant $\lambda > 0$. Then, $\mathcal{X}_n \xrightarrow{p} \lambda\tau$.*

Note that the sequence $\{\mathcal{Y}_n\}$ is of positive reals and hence it trivially converges in probability.

7.2.3 Deducing the Concentration Result

Claim 3. $\lim_{n \rightarrow \infty} \frac{\mathbb{E}S_n^2}{(\mathbb{E}S_n)^2} = 1$.

Proof of Claim. Since the variance of S_n is equal to $\mathbb{E}S_n^2 - (\mathbb{E}S_n)^2$ and this variance is nonnegative, we know that the ratio $\mathbb{E}S_n^2/(\mathbb{E}S_n)^2$ is at least 1. So, it remains to show that the limit is upper bounded by 1. For this proof, we only need a weaker lower bound for $\mathbb{E}S_n$ in [\(15\)](#) by dropping out $\frac{2}{1+\sqrt{n}}$ to get

$$\mathbb{E}S_n \geq p_1 + \frac{1}{\beta} (\eta(p_1) + \log n),$$

which implies that

$$(\mathbb{E}S_n)^2 \geq p_1^2 + \frac{2p_1}{\beta} (\eta(p_1) + \log n) + \frac{1}{\beta^2} (\eta(p_1) + \log n)^2.$$

Combining this lower bound with the upper bound on $\mathbb{E}S_n^2$ from [Lemma 7.5](#) leads to

$$\frac{\mathbb{E}S_n^2}{(\mathbb{E}S_n)^2} \leq \frac{2r}{\gamma\mathbb{E}S_n} + \frac{\left(\frac{\log n}{\gamma}\right)^2}{p_1^2 + \frac{2p_1}{\gamma} (\eta(w) + \log n) + \frac{1}{\gamma^2} (\eta(w) + \log n)^2}.$$

Note that $\lim_{n \rightarrow \infty} \mathbb{E}S_n = \infty$ and thus the first term goes to 0. The limit of second term is dominated by coefficients of $(\log n)^2$. Then

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}S_n^2}{(\mathbb{E}S_n)^2} \leq \lim_{m \rightarrow \infty} \frac{\left(\frac{\log n}{\gamma}\right)^2}{\frac{1}{\gamma^2}(\log n)^2} = 1,$$

which completes the proof of this claim. \square

This implies that the variance of S_n is equal to $o((\mathbb{E}S_n)^2)$. Therefore, by [Lemma 7.6](#), $S_n/\mathbb{E}S_n$ concentrates to 1. Equivalently, $\mathcal{Y}_n/\mathbb{E}\mathcal{Y}_n$ concentrates to 1. From the convergence of expectation result in [§7.1](#) we know that $\mathbb{E}\mathcal{Y}_n$ converges to $1/\beta$. Then, applying [Lemma 7.7](#) with $\mathcal{X}_n = \mathcal{Y}_n$ and $\mathcal{Z}_n = \mathbb{E}S_n/\log n$ yields the desired result that \mathcal{Y}_n concentrates to $1/\beta$.

7.3 Distribution Function

The distribution of $\mathcal{Y}_n/\mathbb{E}\mathcal{Y}_n$ is the same as that of $S_n/\mathbb{E}S_n$. Since $S_n \geq 0$ a.s., Markov's inequality immediately provides the upper bound $\mathbb{P}\left\{\frac{S_n}{\mathbb{E}S_n} \geq 1 - \theta\right\} \leq 1/(1 - \theta)$, which then also holds in the limit as $n \rightarrow \infty$. For the lower bound, we use the Paley–Zygmund concentration inequality which gives us

$$\mathbb{P}\{S_n \geq (1 - \theta)\mathbb{E}S_n\} \geq \theta^2 \frac{(\mathbb{E}S_n)^2}{\mathbb{E}S_n^2}.$$

Taking $n \rightarrow \infty$, [Claim 3](#) yields $\lim_{n \rightarrow \infty} \mathbb{P}\{S_n \geq (1 - \theta)\mathbb{E}S_n\} \geq \theta^2$.

7.4 Proof of Corollary 7.4

Choose any $\varepsilon > 0$ such that $\varepsilon < \min\{1 - \beta, \beta/3\}$. Since both $\{a_n\}$ and $\{b_n\}$ converge to β , there exists an integer N such that for every $n \geq N$ we have

$$\beta - \varepsilon < a_n < \beta + \varepsilon \quad \text{and} \quad \beta - \varepsilon < b_n < \beta + \varepsilon.$$

Let $\beta^+ = \beta + \varepsilon$ and $\beta^- = \beta - \varepsilon$. From the seventh property of f_a in [Lemma 2.1](#),

$$f_{\beta^+}(p_n) \leq f_{a_n}(p_n) \leq p_{n+1} \leq f_{b_n}(p_n) \leq f_{\beta^-}(p_n), \quad \forall n \geq N. \quad (17)$$

Construct two new sequences $\{l_i\}_{i \geq N}$ and $\{u_i\}_{i \geq N}$ in which $l_N = u_N = p_N$ and subsequent elements for $n \geq N + 1$ are generated by the recurrence formulas as follows.

$$l_n = f_{\beta^+}(l_{n-1}), \quad u_n = f_{\beta^-}(u_{n-1}).$$

We want to show $l_n \leq p_n \leq u_n$ for all $n \geq N$. Note that when $n = N$, it's obviously true by our construction. We now prove the inequality for $n \geq N + 1$ by induction. The base case $l_{N+1} \leq p_{N+1} \leq u_{N+1}$ follows directly from (17). Suppose $l_n \leq p_n \leq u_n$ for some $n \geq N + 1$. By (17),

$$u_n \leq f_{\beta^-}(u_{n-1}) \leq \max_{x \in (0,1)} f_{\beta^-}(x) = \frac{1}{4\beta^-} \leq \frac{1}{2\beta^+}, \quad \forall n \geq N + 1$$

where the last inequality follows from our choice of ε . As a result, $l_n, p_n, u_n \in (0, 1/4\beta^+]$. The first property in [Lemma 2.1](#) shows f_{β^+} and f_{β^-} are both monotone increasing on $(0, 1/2\beta^+]$. Therefore,

$$l_{n+1} = f_{\beta^+}(l_n) \leq f_{\beta^+}(p_n) \leq p_{n+1} \leq f_{\beta^-}(p_n) \leq f_{\beta^-}(u_n) = u_{n+1}$$

We extend $\{l_n\}_{n \geq N}$ and $\{u_n\}_{n \geq N}$ by defining $l_i = p_i = u_i$ for $1 \leq i \leq N - 1$. We are interested in the partial sum of these two sequences i.e. $L_n = \sum_{i=1}^n l_i$ and $U_n = \sum_{i=1}^n u_i$. From the previous arguments,

$$\lim_{n \rightarrow \infty} \frac{L_n}{\log n} \leq \lim_{n \rightarrow \infty} \frac{\mathbb{E}S_n}{\log n} \leq \lim_{n \rightarrow \infty} \frac{U_n}{\log n}$$

Actually, the limits of the left and right hand sides are equal to $1/\beta^+$ and $1/\beta^-$ respectively. To see these, we first define two sequence $\{l'_n\}_{n \geq 1}$ and $\{u'_n\}_{n \geq 1}$ such that $l'_{i+1} = f_{\beta^+}(l'_i)$ and $u'_{i+1} = f_{\beta^-}(u'_i)$. Furthermore, $l'_i = l_i$ and $u'_i = u_i$ for all $i \geq N$. In other words, we extend $\{l_n\}_{n \geq N}$ and $\{u_n\}_{n \geq N}$ by two sequences $\{l'_n\}_{n \geq 1}$ and $\{u'_n\}_{n \geq 1}$ constructed completely from recurrence formulas. Note that $\{l'_n\}_{n \geq 1}$ and $\{u'_n\}_{n \geq 1}$ are well-defined because the recurrence formulas f_{β^+} and f_{β^-} are both monotone increasing on $(0, 1/2\beta^+]$ and $l_N, u_N \in (0, 1/4\beta^+]$. It follows that there exist inverse elements $l'_{N-1}, u'_{N-1} \in (0, 1/4\beta^+]$ such that $l_N = f_{\beta^+}(l'_{N-1})$ and $u_N = f_{\beta^-}(u'_{N-1})$. We continue this procedure to get l'_{N-2}, u'_{N-2} such that $l'_{N-1} = f_{\beta^+}(l'_{N-2})$ and $u'_{N-1} = f_{\beta^-}(u'_{N-2})$. Eventually, we can construct $\{l'_n\}_{n \geq 1}$ and $\{u'_n\}_{n \geq 1}$. From [Theorem 7.1](#), the partial sums $L'_n = l'_1 + l'_2 + \dots + l'_n$ and $U'_n = u'_1 + u'_2 + \dots + u'_n$ satisfy

$$\lim_{n \rightarrow \infty} \frac{L'_n}{\log n} = \frac{1}{\beta^+} \text{ and } \lim_{n \rightarrow \infty} \frac{U'_n}{\log n} = \frac{1}{\beta^-}$$

There are only the first N elements in $\{l_n\}_{n \geq 1}$ and $\{u_n\}_{n \geq 1}$ are different from $\{l'_n\}_{n \geq 1}$ and $\{u'_n\}_{n \geq 1}$. The sum of the first N elements divided by $\log n$ goes to 0 as $n \rightarrow \infty$. Consequently,

$$\lim_{n \rightarrow \infty} \frac{L_n}{\log n} = \frac{1}{\beta^+} \text{ and } \lim_{n \rightarrow \infty} \frac{U_n}{\log n} = \frac{1}{\beta^-}$$

This implies that

$$\frac{1}{\beta + \varepsilon} = \frac{1}{\beta^+} \leq \lim_{n \rightarrow \infty} \frac{\mathbb{E}S_n}{\log n} \leq \frac{1}{\beta^-} = \frac{1}{\beta - \varepsilon}$$

and then taking $\varepsilon \rightarrow 0$ yields our desired claim $\lim_{n \rightarrow \infty} \frac{\mathbb{E}S_n}{\log n} = \frac{1}{\beta}$.

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