

Weighted Geometric Mean, Minimum Mediated Set, and Optimal Second-Order Cone Representation

Jie Wang*

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Abstract

We study optimal second-order cone representations for weighted geometric means, which turns out to be closely related to minimum mediated sets. Several lower bounds and upper bounds on the size of optimal second-order cone representations are proved. In the case of bivariate weighted geometric means (equivalently, one dimensional mediated sets), we are able to prove the exact size of an optimal second-order cone representation and give an algorithm to compute one. In the general case, fast heuristic algorithms and traversal algorithms are proposed to compute an approximately optimal second-order cone representation. Finally, applications to polynomial optimization, matrix optimization and quantum information are provided.

Keywords: weighted geometric mean, minimum mediated set, second-order cone representation, polynomial optimization, semidefinite representation

1 Introduction

This paper is concerned with the following problem:

(I) *Given a weighted geometric mean inequality $x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq x_{m+1}$ with weights $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$, $\sum_{i=1}^m \lambda_i = 1$, construct an equivalent representation using as few quadratic inequalities (i.e., $x_i x_j \geq x_k^2$) as possible.*

The study of Problem (I) is motivated by the fact that any solution to Problem (I) immediately gives a second-order cone representation for the region defined by the weighted geometric mean inequality. One advantage of the second-order cone representation is that the related optimization problem can be solved with off-the-shelf second-order cone programming (SOCP) solvers [1],

*wangjie212@amss.ac.cn, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, <https://wangjie212.github.io/jiewang>

e.g., Mosek [2], ECOS [5]. Naturally to achieve the best efficiency, we wish to obtain such a second-order cone representation of as small size as possible.

Using the so-called “tower-of-variable” construction described in [4, Lecture 3.3], one is able to give an equivalent second-order cone representation for a weighted geometric mean inequality, whose size is however typically far from optimal. As far as the author knows, the problem of finding optimal second-order cone representations for general weighted geometric mean inequalities has not been investigated much in the literature, except some special cases that are mentioned below. Assume that in Problem (I) the weights are written as $\lambda_i = \frac{s_i}{t}$, $s_i \in \mathbb{N}$ for $i = 1, \dots, m$ and $t \in \mathbb{N}$. In [15], Morenko et al. solved Problem (I) in the case of $m = 3$ and $t = 2^l$ for some $l \in \mathbb{N}$ when studying p -norm cone programming. In [13], Kian, Berk and Gürlér dealt with Problem (I) in the case of $t = 2^l$ for some $l \in \mathbb{N}$, and proposed a heuristic algorithm for computing an approximately optimal second-order cone representation.

On the other hand, there are plenty of applications of weighted geometric mean inequalities in optimization. For instance:

Second-order cone representations for other types of inequalities.

There are other types of inequalities that can be expressed by weighted geometric mean inequalities, and so the second-order cone representations for weighted geometric mean inequalities can immediately lead to second-order cone representations for these inequalities:

- $x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq x_{m+1}$ ($\lambda_1, \dots, \lambda_m > 0, \sum_{i=1}^m \lambda_i < 1$) $\iff x_1^{\lambda_1} \cdots x_m^{\lambda_m} y^{1-\sum_{i=1}^m \lambda_i} \geq x_{m+1}, y = 1.$
- $x^\lambda \leq y$ ($\lambda > 1$) $\iff y^{\frac{1}{\lambda}} z^{1-\frac{1}{\lambda}} \geq x, z = 1.$
- $x^\lambda \geq y$ ($0 < \lambda < 1$) $\iff x^\lambda z^{1-\lambda} \geq y, z = 1.$
- $x^{-\lambda} \leq y$ ($\lambda > 0$) $\iff x^{\frac{1}{1+\lambda}} y^{\frac{1}{1+\lambda}} \geq z, z = 1.$
- $x_1^{-\lambda_1} \cdots x_m^{-\lambda_m} \leq y$ ($\lambda_1, \dots, \lambda_m > 0$) $\iff x_1^{\frac{\lambda_1}{1+\sum_{i=1}^m \lambda_i}} \cdots x_m^{\frac{\lambda_m}{1+\sum_{i=1}^m \lambda_i}} y^{\frac{1}{1+\sum_{i=1}^m \lambda_i}} \geq z, z = 1.$

p -norm cone programming. Let $p \geq 1$ be a rational number. The p -norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is $\|\mathbf{x}\|_p := \sum_{i=1}^n |x_i|^{\frac{1}{p}}$. The p -norm cone of dimension $n + 1$ is defined by the set

$$\{(\mathbf{x}, z) \in \mathbb{R}^{n+1} \mid z \geq \|\mathbf{x}\|_p\}.$$

Note that the second-order cone is exactly the p -norm cone with $p = 2$. The p -norm cone admits second-order cone representations since the inequality $z \geq \|\mathbf{x}\|_p$ can be rewritten as $z \geq \sum_{i=1}^n |x_i|^p / z^{p-1}$, which is equivalent to

$$\exists (y_i)_{i=1}^n, (w_i)_{i=1}^n \in \mathbb{R}_+^n \text{ s.t. } \begin{cases} z^{1-\frac{1}{p}} y_i^{\frac{1}{p}} \geq w_i, |x_i| \leq w_i, i = 1, \dots, n, \\ \sum_{i=1}^n y_i = z. \end{cases}$$

Power cone programming. Given $\lambda_1, \dots, \lambda_m > 0$ with $\sum_{i=1}^m \lambda_i = 1$, the corresponding *power cone* is defined by the set

$$\{(\mathbf{x}, z) \in \mathbb{R}_+^n \times \mathbb{R} \mid x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq |z|\}.$$

The power cone admits second-order cone representations since the inequality $x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq |z|$ is equivalent to

$$\exists y \geq 0 \text{ s.t. } x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq y, |z| \leq y.$$

Matrix optimization. The second-order cone representation for weighted geometric mean inequalities allows one to give (approximate) semidefinite representations for many matrix functions, as revealed in [8, 9, 18]. These functions include the matrix power function [11], Lieb’s function, the Tsallis entropy, the Tsallis relative entropy [8], the matrix logarithm function [9], which have various applications in matrix optimization and quantum information.

A relevant problem to Problem (I) is

(II) *Given a set of points $\{\alpha_i\}_{i=1}^m \subseteq \mathbb{R}^{m-1}$ forming the vertices of a simplex and a point $\alpha_{m+1} = \sum_{i=1}^m \lambda_i \alpha_i$ with $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$ and $\sum_{i=1}^m \lambda_i = 1$, find as few points as possible (say, $\{\alpha_i\}_{i=m+2}^{m+n}$) such that every point in $\{\alpha_i\}_{i=m+1}^{m+n}$ is an average of two distinct points in $\{\alpha_i\}_{i=1}^{m+n}$.*

Problem (II) arises from the study of nonnegative circuit polynomials [14, 20]. Sums of nonnegative circuit polynomials (SONC) were proposed by Ilman and De Wolff as certificates of polynomial nonnegativity [12], and have been employed to solve sparse polynomial optimization problem in a “degree-free” manner [6, 7, 14]. The set of points $\{\alpha_i\}_{i=m+1}^{m+n}$ considered in Problem (II) is called a (minimum) mediated set. Mediated sets over integers were initially introduced by Reznick in [17] to study agiforms, and were extended to the rational case by the author and Magron [20]. It was proved in [14] that a circuit polynomial is nonnegative if and only if it can be written as a sum of binomial squares supported on a mediated set, with the number of binomial squares equaling the number of points contained in the mediated set. More recent research on mediated sets can be found in [10, 16].

Interestingly, it turns out that solutions to Problem (I) are in a one-to-one correspondence to solutions to Problem (II).

Our main contributions are summarized as follows:

- In Section 3, we prove several lower bounds and upper bounds on the size of optimal second-order cone representations for weighted geometric mean inequalities (equivalently, the size of minimum mediated sets). The lower bounds are shown to be attainable by examples.
- In Section 4, we prove the exact size of an optimal second-order cone representation for bivariate weighted geometric means, which resolves a conjecture proposed in [14]. In addition, we provide a binary tree representation of “successive” one dimensional minimum mediated sets.
- In Section 5, we propose several heuristic algorithms for computing an approximately optimal second-order cone representation of a weighted geometric mean inequality (equivalently, an approximately minimum mediated set) and compare their practical performance. We also propose a brute force algorithm for computing an exact optimal second-order cone representation of a weighted

geometric mean inequality (equivalently, a minimum mediated set). It is demonstrated that the heuristic algorithms produce second-order cone representations of size being equal or close to the optimal one.

- In Section 6, we provide applications of the proposed algorithms to polynomial optimization, matrix optimization and quantum information, and demonstrate their efficiency by numerical experiments.

2 Preliminaries

Let \mathbb{N} , \mathbb{R} , \mathbb{Q} be the set of nonnegative integers, real numbers, rational numbers, respectively. Let \mathbb{N}^* , \mathbb{Q}_+ , \mathbb{R}_+ be the set of positive integers, positive real numbers, positive rational numbers, respectively. For a tuple of integers $(s_1, \dots, s_m) \in \mathbb{N}^m$, we use \hat{s} to denote the sum of its entries, i.e., $\hat{s} := \sum_{i=1}^m s_i$. For a set A , we use $|A|$ to denote the cardinality of A . For a finite set $\mathcal{A} \subseteq \mathbb{N}^n$, we denote by $\text{conv}(\mathcal{A})$ the convex hull of \mathcal{A} , and use $\text{conv}(\mathcal{A})^\circ$ to denote the interior of $\text{conv}(\mathcal{A})$. For two or more integers, we use the parenthesis $()$ to denote their greatest common divisor. Let \mathbf{S}^n , \mathbf{S}_+^n be the set of symmetric matrices, positive semidefinite matrices of size n , respectively.

The n -dimensional *rotated second-order cone* is defined by the set

$$\{(a_i)_{i=1}^n \in \mathbb{R}^n \mid 2a_1a_2 \geq \sum_{i=3}^n a_i^2, a_1 \geq 0, a_2 \geq 0\}.$$

In this paper, we are mostly interested in the 3-dimensional rotated second-order cone which we simply refer to as a second-order cone.

3 Optimal second-order cone representations and minimum mediated sets

In this paper, a *weighted geometric mean inequality* refers to an inequality of the form

$$x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq x_{m+1}, \text{ with } (\lambda_i)_{i=1}^m \in \mathbb{R}_+^m \text{ and } \sum_{i=1}^m \lambda_i = 1,$$

where the variables x_1, \dots, x_{m+1} are assumed to be nonnegative. Throughout the paper, we assume that the weights $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$ are rational numbers. It is known that when the weights are rational, the weighted geometric mean inequality is second-order cone representable [4, Lecture 3.3]. As $(\lambda_i)_{i=1}^m$ are rational numbers by assumption, we can write $\lambda_i = \frac{s_i}{\hat{s}}$, $s_i \in \mathbb{N}$ for $i = 1, \dots, m$. In view of this, we also call $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ with $(s_i)_{i=1}^m \in \mathbb{N}^m$, $(s_1, \dots, s_m) = 1$ a weighted geometric mean inequality.

Definition 1. A second-order cone representation for a weighted geometric mean inequality $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ consists of a set of quadratic inequalities

$x_{i_k} x_{j_k} \geq x_{m+k}^2, k = 1, \dots, n$ such that

$$x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}} \iff \begin{cases} x_{i_1} x_{j_1} \geq x_{m+1}^2, \\ \vdots \\ x_{i_n} x_{j_n} \geq x_{m+n}^2, \\ i_k, j_k \in \{1, 2, \dots, m+n\}, \quad k = 1, \dots, n, \end{cases} \quad (1)$$

where x_{m+2}, \dots, x_{m+n} are auxiliary nonnegative variables. We call n the size of the second-order cone representation. A second-order cone representation of minimum size is called a minimum second-order cone representation whose size is denoted by $L(s_1, \dots, s_m)$.

Remark 2. The second-order cone representation in (1) is uniquely determined by the set of integer triples $\{(i_k, j_k, m+k)\}_{k=1}^n$ which is hereafter referred to as a configuration.

Remark 3. The value of $L(s_1, \dots, s_m)$ clearly does not depend on the ordering of s_1, \dots, s_m .

Example 4. The weighted geometric mean inequality $x_1^3 x_2^8 \geq x_3^{11}$ admits a second-order cone representation: $x_2 x_6 \geq x_3^2, x_1 x_3 \geq x_4^2, x_3 x_4 \geq x_5^2, x_4 x_5 \geq x_6^2$.

A set of points $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{R}^{m-1}$ is called a *trellis* if it comprises the vertex set of a simplex. Given a trellis \mathcal{A} , we say that a set of points \mathcal{B} is an \mathcal{A} -mediated set, if every point $\beta \in \mathcal{B}$ is an average of two distinct points in $\mathcal{A} \cup \mathcal{B}$. An \mathcal{A} -mediated set containing a given point β is called an (\mathcal{A}, β) -mediated set. Given a point $\beta \in \text{conv}(\mathcal{A})^\circ$, a *minimum (\mathcal{A}, β) -mediated set* is an (\mathcal{A}, β) -mediated set with the smallest cardinality. We also say (minimum) mediated sets if there is no need to mention the specific \mathcal{A} and/or β . As $\beta \in \text{conv}(\mathcal{A})^\circ$, there exists a unique tuple of scalars $\lambda_1, \dots, \lambda_m \in \mathbb{R}_+$ (the barycentric coordinate) such that $\sum_{i=1}^m \lambda_i = 1$ and $\beta = \sum_{i=1}^m \lambda_i \alpha_i$. It is easy to see that for an (\mathcal{A}, β) -mediated set to exist, the λ_i 's are necessarily rational numbers.

Example 5. Let $\mathcal{A} = \{\alpha_1 = (4, 2), \alpha_2 = (2, 4), \alpha_3 = (0, 0)\}$, and $\beta_1 = (2, 2), \beta_2 = (1, 2), \beta_3 = (3, 2)$. It is easy to check that $\{\beta_1, \beta_2, \beta_3\}$ is an \mathcal{A} -mediated set (Figure 1).

Assume that $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$ is a trellis and $\alpha_{m+1} = \sum_{i=1}^m \frac{s_i}{\hat{s}} \alpha_i$ with $(s_i)_{i=1}^m \in (\mathbb{N}^*)^m, (s_1, \dots, s_m) = 1$. If $\{\alpha_{m+1}, \dots, \alpha_{m+n}\}$ is an \mathcal{A} -mediated set, then by definition the following system of equations must be satisfied for appropriate i_k, j_k :

$$\begin{cases} \alpha_{i_1} + \alpha_{j_1} = 2\alpha_{m+1}, \\ \vdots \\ \alpha_{i_n} + \alpha_{j_n} = 2\alpha_{m+n}, \\ i_k, j_k \in \{1, 2, \dots, m+n\}, \quad k = 1, \dots, n. \end{cases} \quad (2)$$

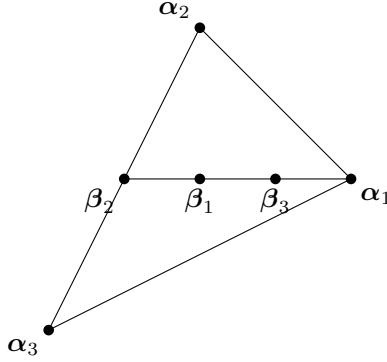


Figure 1: $\{\beta_1, \beta_2, \beta_3\}$ forms an $\{\alpha_1, \alpha_2, \alpha_3\}$ -mediated set.

Comparing (1) with (2), we see that there is a one-to-one correspondence between second-order cone representations for a weighted geometric mean inequality $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ and \mathcal{A} -mediated sets containing a point $\alpha_{m+1} = \sum_{i=1}^m \frac{s_i}{\hat{s}} \alpha_i$. It follows that $L(s_1, \dots, s_m)$ also denotes the cardinality of a minimum $(\mathcal{A}, \alpha_{m+1})$ -mediated set.

We now prove a lower bound on the size of optimal second-order cone representations for a weighted geometric mean inequality in terms of the number of variables involved.

Theorem 6. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$. Then*

$$L(s_1, \dots, s_m) \geq m - 1. \quad (3)$$

Proof. Suppose that $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$ is a trellis and $\alpha_{m+1}, \dots, \alpha_{m+n}$ is a minimum $(\mathcal{A}, \alpha_{m+1})$ -mediated set such that $\sum_{i=1}^m s_i \alpha_i = \hat{s} \alpha_{m+1}$. Let us consider the system of equations (2). Note first that $\{1, \dots, m\} \subseteq \cup_{k=1}^n \{i_k, j_k\}$. Moreover, we must have $\{m+2, \dots, m+n\} \subseteq \cup_{k=1}^n \{i_k, j_k\}$ as if $k \notin \cup_{k=1}^n \{i_k, j_k\}$ for some $k \in \{m+2, \dots, m+n\}$, then we can delete α_k to obtain a smaller $(\mathcal{A}, \alpha_{m+1})$ -mediated set. From this, we get

$$2n \geq m + n - 1,$$

which yields the desired inequality. \square

The lower bound in Theorem 6 is attainable as the following example shows.

Example 7. *The weighted geometric mean inequality $x_1 x_2 x_3 x_4 \geq x_5^4$ admits a second-order cone representation: $x_6 x_7 \geq x_5^2, x_1 x_2 \geq x_6^2, x_3 x_4 \geq x_7^2$.*

In the language of mediated sets, Theorem 6 says that the cardinality of a minimum mediated set is bounded by its dimension (i.e., the dimension of its associated trellis) from below.

From the point of view of mediated sets, we are actually able to give another lower bound on $L(s_1, \dots, s_m)$ in terms of the sum \hat{s} . For the proof, we need the notion of M-matrices and two preliminary results.

Definition 8. Let $C = [c_{ij}]$ be a real matrix. Then C is called an M-matrix if it satisfies

- (1) $c_{ij} \leq 0$ if $i \neq j$;
- (2) $C = tI - B$, where B is a matrix with nonnegative entries, I is an identity matrix, and t is no less than the spectral radius (the maximum of the moduli of the eigenvalues) of B .

A remarkable property of M-matrices is that the determinant of an M-matrix is bounded by the product of its diagonals from above.

Lemma 9. Let $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ be an M-matrix. Then, it holds

$$\det(C) \leq \prod_{i=1}^n c_{ii}. \quad (4)$$

Proof. See Corollary 4.1.2 of [3]. □

The following lemma tells us that a particular class of matrices are M-matrices.

Lemma 10. Let $C = [c_{ij}]$ be a real matrix such that $c_{ii} = 2$ and $c_{ij} \in \{0, -1\}$ if $i \neq j$. Assume that each row of C has at most two -1 's and there is no principal submatrix of C in which each row has exactly two -1 's. Then C is a non-singular M-matrix.

Proof. Let us write $C = 2I - B$, where I is an identity matrix, and $B = [b_{ij}]$ is a matrix with entries being 0 or 1. We finish the proof by showing that the spectral radius of B is less than 2.

Let λ be any eigenvalue of B and $\alpha = (\alpha_i)_i$ a corresponding eigenvector such that $B\alpha = \lambda\alpha$. Let $\xi = \max_i \{|\alpha_i|\}$ and $I = \{i \mid |\alpha_i| = \xi\}$. For $i \in I$, let $T(i) = \{j \mid b_{ij} = 1\}$. Then for each $i \in I$, we have

$$|\lambda|\xi = |\lambda\alpha_i| = \left| \sum_j b_{ij}\alpha_j \right| = \left| \sum_{j \in T(i)} \alpha_j \right| \leq 2\xi.$$

So $|\lambda| \leq 2$ for $\xi > 0$. If $|\lambda| = 2$, then for any $i \in I$, we have $|T(i)| = 2$ and $|\alpha_j| = \xi$ for $j \in T(i)$ which implies $T(i) \subseteq I$. It follows that the principal submatrix of C indexed by I has exactly two -1 's in each row, which is impossible. Thus $|\lambda| < 2$ as desired. □

Now we are ready to prove the promised result.

Theorem 11. Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{R}^{m-1}$ be a trellis and $\beta = \sum_{i=1}^m \frac{s_i}{s} \alpha_i$ with $(s_i)_{i=1}^m \in (\mathbb{N}^*)^m$, $(s_1, \dots, s_m) = 1$. Then for any (\mathcal{A}, β) -mediated set \mathcal{B} , one has $|\mathcal{B}| \geq \lceil \log_2(\hat{s}) \rceil$.

Proof. Assume that $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$ with $\beta_1 = \beta$ is an (\mathcal{A}, β) -mediated set. By definition, for each β_i , we have one of the following three equations holds for suitable $j, k \in \{1, \dots, m\}$:

$$\begin{aligned} 2\beta_i &= \alpha_j + \alpha_k, \\ 2\beta_i - \beta_j &= \alpha_k, \\ 2\beta_i - \beta_j - \beta_k &= \mathbf{0}. \end{aligned} \tag{5}$$

Let us denote the coefficient matrix of (5) by C and let $B = [\beta_1, \dots, \beta_n]^\top$ with β_i 's being viewed as column vectors. Then we can rewrite (5) in matrix form as $CB = A$, where A is a matrix whose row vectors belong to the subspace generated by $\{\mathbf{0}\} \cup \mathcal{A}$. It is clear that C satisfies the conditions of Lemma 10 except the assumption. Suppose by contrary that C has a principal submatrix indexed by I with two -1 's in each row and let $\mathcal{B}' = \{\beta_i\}_{i \in I}$. Then by construction, every point in \mathcal{B}' is an average of two other points in \mathcal{B}' , which is however impossible for a finite set. Therefore by Lemma 10, we deduce that C is M-matrix. In addition by Lemma 9, $\det(C) \leq 2^n$. Solve $CB = A$ for β and we obtain $\beta = \frac{\sum_{i=1}^m r_i \alpha_i}{\det(C)}$ for some $r_i \in \mathbb{N}^*$, $i = 1, \dots, m$. It follows $\hat{s} \leq \det(C) \leq 2^n$ which implies $|\mathcal{B}| = n \geq \lceil \log_2(\hat{s}) \rceil$. \square

From Theorem 11, we immediately obtain the following theorem.

Theorem 12. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$. Then*

$$L(s_1, \dots, s_m) \geq \lceil \log_2(\hat{s}) \rceil. \tag{6}$$

The lower bound in Theorem 12 is attainable as the following example shows.

Example 13. *The weighted geometric mean inequality $x_1 x_2^2 x_3^3 \geq x_4^6$ admits a second-order cone representation: $x_3 x_5 \geq x_4^2, x_2 x_6 \geq x_5^2, x_1 x_5 \geq x_6^2$.*

Remark 14. *Combining Theorems 6 with 12, we get*

$$L(s_1, \dots, s_m) \geq \max\{\lceil \log_2(\hat{s}) \rceil, m - 1\}. \tag{7}$$

Remark 15. *An \mathcal{A} -mediated set \mathcal{B} is said to be isomorphic to an \mathcal{A}' -mediated set \mathcal{B}' if there are one-to-one maps $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{B} \rightarrow \mathcal{B}'$ such that the average relationships (5) are preserved under these maps.*

Suppose that $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{R}^{m-1}$ is a trellis and $\beta = \sum_{i=1}^m \frac{s_i}{\hat{s}} \alpha_i$ with $(s_i)_{i=1}^m \in (\mathbb{N}^)^m, (s_1, \dots, s_m) = 1$. We may define*

$$\beta' = (s_1, \dots, s_{m-1}) \in \mathbb{R}^{m-1},$$

and

$$\alpha'_i = \hat{s} \mathbf{e}_i \in \mathbb{R}^{m-1}, i = 1, \dots, m-1, \alpha'_m = \mathbf{0} \in \mathbb{R}^{m-1},$$

so that

$$\beta' = \sum_{i=1}^m \frac{s_i}{\hat{s}} \alpha'_i.$$

Here, $(\mathbf{e}_i)_{i=1}^{m-1}$ denotes the standard basis of \mathbb{R}^{m-1} . In addition, let $\mathcal{A}' = \{\boldsymbol{\alpha}'_1, \dots, \boldsymbol{\alpha}'_m\}$. It can be seen that any $(\mathcal{A}, \boldsymbol{\beta})$ -mediated set is isomorphic to an $(\mathcal{A}', \boldsymbol{\beta}')$ -mediated set and vice versa. Therefore, to study $(\mathcal{A}, \boldsymbol{\beta})$ -mediated sets, there is no loss of generality in assuming that the trellis \mathcal{A} comprises the vertices of the standard simplex and $\boldsymbol{\beta}$ is a lattice point in the interior of this simplex.

4 A binary tree representation of successive minimum mediated sequences

In this section, we focus particularly on the case of one dimensional mediated sets (equivalently, the case of bivariate weighted geometric mean inequalities). For the sake of conciseness, we use the terminology *mediated sequences* to refer to one dimensional mediated sets. More concretely, given an integer $p > 0$, a set of integers $A \subseteq \mathbb{N}$ is a p -mediated sequence, if every number in A is an average of two distinct numbers in $A \cup \{0, p\}$. A p -mediated sequence containing a given number q is called a (p, q) -mediated sequence. As being a mediated sequence is not changed by a scaling, there is no loss of generality in assuming $(p, q) = 1$. A *minimum* (p, q) -mediated sequence is a (p, q) -mediated sequence with the smallest cardinality. We also say (minimum) mediated sequences if there is no need to mention the specific p, q .

Example 16. *The set $A = \{2, 4, 5, 8\}$ is a minimum $(11, 2)$ -mediated sequence.*

For given p, q , there is an algorithm (Algorithm 1) for computing a minimum (p, q) -mediated sequence.

Algorithm 1

Input: Two integers $0 < q < p$ with $(p, q) = 1$

Output: A minimum (p, q) -mediated sequence

- 1: $l \leftarrow \lceil \log_2(p) \rceil$;
 - 2: $s_1 \leftarrow q, s_2 \leftarrow p - q, s_3 \leftarrow 2^l - p$;
 - 3: $t_1 \leftarrow p, t_2 \leftarrow 0, t_3 \leftarrow q$;
 - 4: **for** $k \leftarrow 1$ to l **do**
 - 5: Find $1 \leq i \neq j \leq 3$ such that $s_i \leq s_j$ are odd numbers;
 - 6: $q_k \leftarrow \frac{t_i + t_j}{2}, t_i \leftarrow q_k, r \leftarrow \{1, 2, 3\} \setminus \{i, j\}$;
 - 7: $s_j \leftarrow \frac{s_j - s_i}{2}, s_r \leftarrow \frac{s_r}{2}$;
 - 8: **end for**
 - 9: **return** $\{q_k\}_{k=1}^l$;
-

Theorem 17. *Algorithm 1 is correct.*

Proof. First note that at Step 5 of Algorithm 1, such i, j always exist as $s_1 + s_2 + s_3 = 2^l$ and s_i is odd throughout the loop. Denote the initial values of $s_1, s_2, s_3, t_1, t_2, t_3$ respectively by $s_1^0, s_2^0, s_3^0, t_1^0, t_2^0, t_3^0$, and the values after the k -th

iteration of the loop respectively by $s_1^k, s_2^k, s_3^k, t_1^k, t_2^k, t_3^k$. We now prove that

$$2^{l-k} = s_1^k + s_2^k + s_3^k \text{ and } 2^{l-k}q = s_1^k t_1^k + s_2^k t_2^k + s_3^k t_3^k \quad (8)$$

hold true for $k = 0, 1, \dots, l$ by induction on k . By initialization, we clearly have $2^l = q + p - q + 2^l - p = s_1^0 + s_2^0 + s_3^0$ and $2^l q = q \cdot p + (p - q) \cdot 0 + (2^l - p) \cdot q = s_1^0 t_1^0 + s_2^0 t_2^0 + s_3^0 t_3^0$. Assume that $2^{l-k} = s_1^k + s_2^k + s_3^k$ and $2^{l-k}q = s_1^k t_1^k + s_2^k t_2^k + s_3^k t_3^k$ are true for some $k \geq 0$. Then, $s_1^{k+1} + s_2^{k+1} + s_3^{k+1} = s_1^k + \frac{s_j^k - s_i^k}{2} + \frac{s_i^k}{2} = \frac{1}{2}(s_1^k + s_2^k + s_3^k) = 2^{l-k-1}$ and $s_1^{k+1} t_1^{k+1} + s_2^{k+1} t_2^{k+1} + s_3^{k+1} t_3^{k+1} = s_i^k \cdot \frac{t_i^k + t_j^k}{2} + \frac{s_j^k - s_i^k}{2} \cdot t_j^k + \frac{s_i^k}{2} \cdot t_r^k = \frac{1}{2}(s_1^k t_1^k + s_2^k t_2^k + s_3^k t_3^k) = 2^{l-k-1}q$. So we complete the induction.

Letting $k = l$ in (8), we obtain $1 = s_1^l + s_2^l + s_3^l$ and $q = s_1^l t_1^l + s_2^l t_2^l + s_3^l t_3^l$. Because s_1^l, s_2^l, s_3^l are nonnegative integers, the equality $1 = s_1^l + s_2^l + s_3^l$ implies $s_i^l = 1, s_j^l = s_r^l = 0$, and hence $q = t_i^l = q_i$. From this and by construction, we see that $\{q_k\}_{k=1}^l$ is indeed a (p, q) -mediated sequence. Moreover, by Theorem 11, $\{q_k\}_{k=1}^l$ is minimum. Thus we prove the correctness of Algorithm 1. \square

Remark 18. Algorithm 1 can be readily adapted to produce an optimal second-order cone representation for the bivariate weighted geometric mean inequality $x_1^q x_2^{p-q} \geq x_3^p$ with $0 < q < p$.

Remark 19. The essential of Algorithm 1 has appeared in the proof of Proposition 5 of [15] in the context of second-order cone representations for trivariate weighted geometric means with $\hat{s} = 2^l$.

Note that for integers $0 < q < p$, $L(q, p - q)$ denotes the cardinality of a minimum (p, q) -mediated sequence. By Theorem 17 we immediately obtain the exact value of $L(q, p - q)$, which resolves a conjecture concerning the value of $L(q, p - q)$ proposed in [14].

Theorem 20. For integers $0 < q < p$ with $(p, q) = 1$, it holds

$$L(q, p - q) = \lceil \log_2(p) \rceil. \quad (9)$$

Theorem 20 allows us to further provide an upper bound on $L(s_1, \dots, s_m)$.

Theorem 21. Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$. Then

$$L(s_1, \dots, s_m) \leq \min_{\sigma \in S_m} \left\{ \sum_{i=1}^{m-1} \left\lceil \log_2 \left(\frac{\sum_{j=i}^m s_{\sigma(j)}}{\left(\sum_{j=i}^m s_{\sigma(j)}, s_{\sigma(i)} \right)} \right) \right\rceil \right\}, \quad (10)$$

where S_m is the symmetry group of $\{1, \dots, m\}$.

Proof. The conclusion follows by iteratively using

$$x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } x_1^{s_1} y^{\sum_{i=2}^m s_i} \geq x_{m+1}^{\hat{s}}, x_2^{s_2} \cdots x_m^{s_m} \geq y^{\sum_{i=2}^m s_i}$$

and applying Theorem 20 to the bivariate weighted geometric mean inequality arising at each iteration. \square

Example 22. By Theorem 12, $L(4, 3, 2) \geq 4$. By Theorem 21, $L(4, 3, 2) \leq 4$. It follows $L(4, 3, 2) = 4$.

We notice that Algorithm 1 can be readily adapted to produce an optimal second-order cone representation for the inequality $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{2^l}$ with $s_1 + s_2 + s_3 = 2^l$, and so obtain the following corollary.

Corollary 23. For integers $s_1, s_2, s_3 \in \mathbb{N}^*$ with $(s_1, s_2, s_3) = 1$, if $s_1 + s_2 + s_3 = 2^l$ for some $l \in \mathbb{N}$, then

$$L(s_1, s_2, s_3) = l. \quad (11)$$

The minimum mediated sequence produced by Algorithm 1 actually has a special structure which leads to the next definition.

Definition 24. For integers $0 < q < p$ with $(p, q) = 1$, a minimum (p, q) -mediated sequence $A = \{q_1, \dots, q_l\}$ with $l = \lceil \log_2(p) \rceil$ is successive if it can be sorted in such a way that

$$(1) \quad q_i = \frac{q_{i-1} + t_i}{2} \text{ for } i = 1, \dots, l,$$

$$(2) \quad q_l = q,$$

where $t_i \in \{0, p, q, q_1, \dots, q_{i-2}\}$ for $i = 1, \dots, l$ and $q_0 \in \{p, q\}$.

As the minimum mediated sequence output by Algorithm 1 is successive by construction, we know that successive minimum (p, q) -mediated sequences exist for any integers $0 < q < p$ with $(p, q) = 1$.

A successive minimum mediated sequence has a distinguished property, that is, it can be represented by a particular ‘‘binary tree’’. To get a quick flavor of this fact, let us begin with an illustrative example. Let $p = 57, q = 11$, and $A = \{34, 17, 37, 27, 22, 11\}$. Noting

$$34 = \frac{p+q}{2}, 17 = \frac{34}{2}, 37 = \frac{17+p}{2}, 27 = \frac{37+17}{2}, 22 = \frac{27+17}{2}, q = \frac{22}{2}, \quad (12)$$

we see that A is a successive minimum (p, q) -mediated sequence. From (12), one can easily get the following continued fraction representation of q :

$$q = \frac{\frac{\frac{p+q}{4} + p}{2} + \frac{p+q}{4} + \frac{p+q}{4}}{4}. \quad (13)$$

The mediated sequence A can be recovered from (13), which is visualized by the binary tree displayed in Figure 2.

Taking inspiration from the above example, we can construct a binary tree representation for any successive minimum (p, q) -mediated sequence $A = \{q_i\}_{i=1}^l$, where A is sorted according to Definition 24. For simplicity, we assume from now on that p, q are both odd. We shall describe the construction in an iterative manner. Let T_1 be the binary tree consisting of a root node labelled by q_1 along

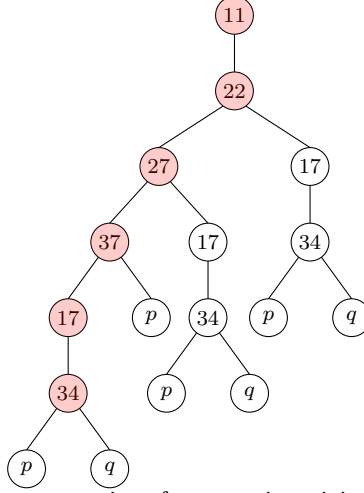


Figure 2: The binary tree representation of a successive minimum (p, q) -mediated sequence with $p = 57, q = 11$.

with two children labelled by p and q , respectively. For $2 \leq i \leq l$, we iteratively define

$$T_i = \begin{cases} \mathcal{C}(q_i, T_{i-1}), & \text{if } q_i = \frac{q_{i-1}}{2}, \\ \mathcal{C}(q_i, T_{i-1}, p), & \text{if } q_i = \frac{q_{i-1} + p}{2}, \\ \mathcal{C}(q_i, T_{i-1}, q), & \text{if } q_i = \frac{q_{i-1} + q}{2}, \\ \mathcal{C}(q_i, T_{i-1}, T_j), & \text{if } q_i = \frac{q_{i-1} + q_j}{2}, \end{cases} \quad (14)$$

where $\mathcal{C}(q_i, T_{i-1})$ denotes the binary tree obtained by connecting T_{i-1} to a root node labelled by q_i such that the root node of T_{i-1} becomes the only child of the root node; $\mathcal{C}(q_i, T_{i-1}, p)$ (resp. $\mathcal{C}(q_i, T_{i-1}, q)$) denotes the binary tree obtained by connecting T_{i-1} and a leaf node labelled by p (resp. q) to a root node labelled by q_i such that T_{i-1} and the leaf node labelled by p (resp. q) become the left and right subtrees of the root node, respectively; $\mathcal{C}(q_i, T_{i-1}, T_j)$ denotes the binary tree obtained by connecting T_{i-1} and T_j to a root node labelled by q_i such that T_{i-1} and T_j become the left and right subtrees of the root node, respectively. We say that T_l is the *binary tree representation* of A . For a binary tree representation T , we can naturally define the height for any node such that the node at the bottom is of height 0. The height of T is the height of the root node. We use $p(T)$ (resp. $q(T)$) to denote the set of heights of leaf nodes labelled by p (resp. q).

Theorem 25. *Suppose that T is the binary tree representation of a successive minimum (p, q) -mediated sequence A . Then the following hold:*

- (1) *The height of T is $\lceil \log_2(p) \rceil$;*
- (2) *Any leaf node of T is labelled by either p or q ;*
- (3) *The root node of T is labelled by q ;*

- (4) The set of root nodes of left subtrees of T coincides with A ;
- (5) Any right subtree of T either is a leaf node or coincides with a left subtree;
- (6) The label of any non-leaf node is the average of labels of its children (the half of the label of its child if there is only one child);
- (7) $\sum_{j \in p(T)} 2^j = q$ and $\sum_{j \in q(T)} 2^j = 2^{\lceil \log_2(p) \rceil} - p$.

Proof. (1)–(6) are immediate from the construction.

Let $l = \lceil \log_2(p) \rceil$ and assume that $A = \{q_i\}_{i=1}^l$ is sorted according to Definition 24. For $i = 1, \dots, l$, let T_i be the left subtree of T with the root node labelled by q_i . We claim that

$$2^i q_i = \left(\sum_{j \in p(T_i)} 2^j \right) p + \left(\sum_{j \in q(T_i)} 2^j \right) q \quad (15)$$

holds true for all i . Let us prove (15) by induction on i . For $i = 1$, we have $2q_1 = p + q$ which is exactly (15). Now assume that (15) is true for $i = k \geq 1$. Then if $q_{k+1} = \frac{q_k}{2}$, we have

$$\begin{aligned} 2^{k+1} q_{k+1} &= 2^k q_k = \left(\sum_{j \in p(T_k)} 2^j \right) p + \left(\sum_{j \in q(T_k)} 2^j \right) q \\ &= \left(\sum_{j \in p(T_{k+1})} 2^j \right) p + \left(\sum_{j \in q(T_{k+1})} 2^j \right) q; \end{aligned}$$

if $q_{k+1} = \frac{q_k + p}{2}$, we have

$$\begin{aligned} 2^{k+1} q_{k+1} &= 2^k q_k + 2^k p = \left(\sum_{j \in p(T_k)} 2^j + 2^k \right) p + \left(\sum_{j \in q(T_k)} 2^j \right) q \\ &= \left(\sum_{j \in p(T_{k+1})} 2^j \right) p + \left(\sum_{j \in q(T_{k+1})} 2^j \right) q; \end{aligned}$$

if $q_{k+1} = \frac{q_k + q}{2}$, we have

$$\begin{aligned} 2^{k+1} q_{k+1} &= 2^k q_k + 2^k q = \left(\sum_{j \in p(T_k)} 2^j \right) p + \left(\sum_{j \in q(T_k)} 2^j + 2^k \right) q \\ &= \left(\sum_{j \in p(T_{k+1})} 2^j \right) p + \left(\sum_{j \in q(T_{k+1})} 2^j \right) q; \end{aligned}$$

if $q_{k+1} = \frac{q_k + q_t}{2}$, we have

$$\begin{aligned} 2^{k+1}q_{k+1} &= 2^k q_k + 2^k q_t \\ &= \left(\sum_{j \in p(T_k)} 2^j + 2^{k-t} \sum_{j \in p(T_t)} 2^j \right) p + \left(\sum_{j \in q(T_k)} 2^j + 2^{k-t} \sum_{j \in q(T_t)} 2^j \right) q \\ &= \left(\sum_{j \in p(T_{k+1})} 2^j \right) p + \left(\sum_{j \in q(T_{k+1})} 2^j \right) q. \end{aligned}$$

Hence (15) is also true for $i = k + 1$ and we complete the induction. Letting $i = l$ in (15) gives $2^{\lceil \log_2(p) \rceil} q = \left(\sum_{j \in p(T)} 2^j \right) p + \left(\sum_{j \in q(T)} 2^j \right) q$, from which we deduce that $\sum_{j \in p(T)} 2^j = q$ and $\sum_{j \in q(T)} 2^j = 2^{\lceil \log_2(p) \rceil} - p$. \square

By virtue of Theorem 25, we are able to enumerate all successive minimum (p, q) -mediated sequences for given p, q via traversing the related binary tree representations. We emphasize that Property (7) stated in Theorem 25 is crucial to reduce the search space of valid binary tree representations.

5 Algorithms

In this section, we study algorithms for computing second-order cone representations of weighted geometric mean inequalities. Unlike the bivariate case treated in the previous section, computing optimal second-order cone representations for general weighted geometric mean inequalities seems a notoriously difficult problem. Therefore, we are mostly interested in efficient heuristic algorithms that can produce approximately optimal second-order cone representations. We will propose two types of fast heuristic algorithms in Section 5.1, and then extend them to certain “traversal-style” algorithms in Section 5.2. For completeness and comparison, a brute force algorithm for computing optimal second-order cone representations is provided in Section 5.3. Finally, we evaluate these algorithms via numerical experiments in Section 5.4.

5.1 Fast heuristic algorithms

Our heuristic algorithms rely on a simple result which is stated in the following lemma.

Lemma 26. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers. Then for a pair $i, j \in \{1, \dots, m\}$ and for any $\gamma \in \mathbb{N}$ with $0 < \gamma \leq \min\{s_i, s_j\}$,*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } x_i^{s_i - \gamma} x_j^{s_j - \gamma} y^{2\gamma} \prod_{\substack{k=1 \\ k \neq i, j}}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}}, x_i x_j \geq y^2.$$

Suppose we are given the inequality $\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}}$ and let $l = \lceil \log_2(\hat{s}) \rceil$. Let us multiply $\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}}$ by $x_{m+1}^{2^{l-\hat{s}}}$ to obtain $\prod_{k=1}^m x_k^{s_k} x_{m+1}^{2^l - \hat{s}} \geq x_{m+1}^{2^l}$ so that the exponent of x_{m+1} is a power of 2. Now we claim that if the exponent of x_{m+1} is a power of 2, then we are able to obtain a second-order cone representation for the weighted geometric mean inequality by iteratively applying Lemma 26 with appropriate i, j and γ as detailed in Algorithm 2. Note that Algorithm 2 terminates when the tuple s_1, \dots, s_m reduces to two nonzero numbers, each being 2^{l-1} .

Algorithm 2

Input: A tuple of positive integers s_1, \dots, s_m

Output: A configuration $\{(i_k, j_k, t_k)\}_k$ that determines a second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$

- 1: $l \leftarrow \lceil \log_2(\hat{s}) \rceil$, $t \leftarrow m + 1$, $m \leftarrow m + 1$, $s_m \leftarrow 2^l - \hat{s}$;
 - 2: $\mathcal{S} \leftarrow \emptyset$;
 - 3: **repeat**
 - 4: Select a pair $i, j \in \{1, \dots, m\}$ and an integer γ satisfying $0 < \gamma \leq \min\{s_i, s_j\}$;
 - 5: $m \leftarrow m + 1$, $s_i \leftarrow s_i - \gamma$, $s_j \leftarrow s_j - \gamma$, $s_m \leftarrow 2\gamma$;
 - 6: $\mathcal{S} \leftarrow \mathcal{S} \cup \{(i, j, m)\}$;
 - 7: **until** $\gamma = 2^{l-1}$
 - 8: **return** $\mathcal{S} \cup \{(i, j, t)\}$;
-

In the following we propose two strategies to guide us to select such pair i, j and the factor γ at Step 4 of Algorithm 2. For an integer r , let $\Omega(r)$ be the set of exponents of 2 involved in the binary representation of r , and let $\Delta(r)$ be the minimal exponent of 2 involved in the binary representation of r . For example, with $r = 7$, one has $\Omega(r) = \{0, 1, 2\}$ and $\Delta(r) = 0$.

Theorem 27. *If we select the pair i, j satisfying $\Omega(s_i) \cap \Omega(s_j) \neq \emptyset$ and let $\gamma = \sum_{k \in \Omega(s_i) \cap \Omega(s_j)} 2^k$ at Step 4, then Algorithm 2 terminates.*

Proof. As $s_1 + \cdots + s_m = 2^l$, we can find a pair i, j such that $\Delta(s_i) = \Delta(s_j)$ and so $\Omega(s_i) \cap \Omega(s_j) \neq \emptyset$. Let us consider the quantity $h := \sum_{i=1}^m |\Omega(s_i)|$. After one iteration, s_i, s_j are replaced by $s_i - \gamma, s_j - \gamma$, and 2γ is added to the tuple. We then see that h decreases by $|\Omega(s_i) \cap \Omega(s_j)| \geq 1$ at each iteration. It follows that the condition $\gamma = 2^{l-1}$ is satisfied in at most $\sum_{i=1}^m |\Omega(s_i)| - 2$ iterations. Therefore, Algorithm 2 terminates. \square

Corollary 28. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers. Then*

$$L(s_1, \dots, s_m) \leq \sum_{i=1}^m |\Omega(s_i)| - 1. \quad (16)$$

Proof. The proof of Theorem 27 immediately implies that Algorithm 2 yields a second-order cone representation of size at most $\sum_{i=1}^m |\Omega(s_i)| - 1$, from which we get the desired inequality. \square

Theorem 29. *If we select the pair i, j such that $\Delta(s_i) = \Delta(s_j) = \min\{\Delta(s_k)\}_{k=1}^m$, and let $\gamma = \min\{s_i, s_j\}$ at Step 4, then Algorithm 2 terminates.*

Proof. As $s_1 + \dots + s_m = 2^l$, we can find a pair i, j such that $\Delta(s_i) = \Delta(s_j) = \min\{\Delta(s_k)\}_{k=1}^m$ and so $\Delta(s_i - s_j) - \Delta(s_j) \geq 1$. Let us consider the quantity $h := \sum_{i=1}^m (l - \Delta(s_i))$ (set $\Delta(0) := l$). After one iteration, s_i, s_j are replaced by $\max\{s_i, s_j\} - \min\{s_i, s_j\}, 0$, and $2 \min\{s_i, s_j\}$ is added to the tuple. We then see that h decreases by $1 + \Delta(s_i - s_j) - \Delta(s_j) \geq 2$ at each iteration. It follows that the condition $\gamma = 2^{l-1}$ is satisfied in at most $\frac{1}{2} \sum_{i=1}^m (l - \Delta(s_i)) - 1$ iterations. Therefore, Algorithm 2 terminates. \square

Corollary 30. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers and let $l = \lceil \log_2(\hat{s}) \rceil$. Then*

$$L(s_1, \dots, s_m) \leq \frac{1}{2} \sum_{i=1}^m (l - \Delta(s_i)). \quad (17)$$

Proof. The proof of Theorem 29 immediately implies that Algorithm 2 yields a second-order cone representation of size at most $\frac{1}{2} \sum_{i=1}^m (l - \Delta(s_i))$, from which we get the desired inequality. \square

Theorems 27 and 29 allow us to find an approximately optimal second-order cone representation for a weighted geometric mean inequality by implementing greedy strategies at each iteration of the loop in Algorithm 2:

- (1) selecting the pair i, j to maximize $|\Omega(s_i) \cap \Omega(s_j)|$ and letting $\gamma = \sum_{k \in \Omega(s_i) \cap \Omega(s_j)} 2^k$ (hereafter referred to as the greedy-common-one strategy), or alternatively,
- (2) selecting the pair i, j to maximize $\Delta(s_i - s_j)$ with $\Delta(s_i) = \Delta(s_j) = \min\{s_k\}_{k=1}^m$ and letting $\gamma = \min\{s_i, s_j\}$ (hereafter referred to as the greedy-power-two strategy).

Remark 31. *Algorithm 2 that employs the greedy-common-one strategy has been used to compute second-order cone representations for weighted geometric mean inequalities with $\hat{s} = 2^l$ in [13].*

The following lemmas whose proofs are straightforward allow us to further enhance the performance of the heuristics.

Lemma 32. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers. Assume $s_m = \max\{s_i\}_{i=1}^m$ and $l = \lceil \log_2(\hat{s}) \rceil$. Then,*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t.} \begin{cases} \prod_{k=1}^{m-1} x_k^{s_k} \geq y^{\frac{\hat{s}}{2}}, x_m y \geq x_{m+1}^2, & \text{if } s_m = \frac{\hat{s}}{2}; \\ \prod_{k=1}^{m-1} x_k^{s_k} x_{m+1}^{2s_m - \hat{s}} \geq y^{s_m}, x_m y \geq x_{m+1}^2, & \text{if } \frac{\hat{s}}{2} < s_m \leq 2^{l-1}; \\ \prod_{k=1}^{m-1} x_k^{s_k} x_m^{s_m - 2^{l-1}} x_{m+1}^{2^l - \hat{s}} \geq y^{2^{l-1}}, x_m y \geq x_{m+1}^2, & \text{if } s_m > 2^{l-1}, \hat{s} < 2^l; \\ \prod_{k=1}^{m-1} x_k^{s_k} x_m^{s_m - 2^{l-1}} \geq y^{2^{l-1}}, x_m y \geq x_{m+1}^2, & \text{if } s_m > 2^{l-1}, \hat{s} = 2^l. \end{cases}$$

Lemma 33. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers and assume $s_1 = s_2$. Then,*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } \prod_{k=3}^m x_k^{s_k} y^{2s_2} \geq x_{m+1}^{\hat{s}}, x_1 x_2 \geq y^2.$$

Lemma 34. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers. Assume $l = \lceil \log_2(\hat{s}) \rceil$ and s_1 is the unique odd number among s_1, \dots, s_m , which satisfies $s_1 \leq 2^l - \hat{s}$. Then,*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } \prod_{k=2}^m x_k^{\frac{s_k}{2}} y^{s_1} \geq x_{m+1}^{\frac{\hat{s}+s_1}{2}}, x_1 x_{m+1} \geq y^2.$$

By invoking any of Lemmas 32–34 if applicable, we get either a reduction of \hat{s} by a factor 2 or a decrease of m by 1 at the price of one quadratic inequality. We thereby embed Lemmas 32–34 into the heuristics as detailed in Algorithm 3.

Before closing this subsection, we prove an upper bound on $L(s_1, \dots, s_m)$ via repeatedly applying Lemma 33.

Theorem 35. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$ and $l = \lceil \log_2(\hat{s}) \rceil$. Let us denote the frequency distribution of $s_1, \dots, s_m, 2^l - \hat{s}$ by r_1, \dots, r_t (values) and k_1, \dots, k_t (frequencies). Then*

$$L(s_1, \dots, s_m) \leq \frac{1}{2} \sum_{\Delta(r_i)=0} k_i + \sum_{p=1}^{l-1} \varphi \left(m+1 - \sum_{\Delta(r_i)<p} \sum_{j=1}^{p-\Delta(r_i)} \varphi^j(k_i) - \sum_{\Delta(r_i)>p} k_i \right), \quad (18)$$

where $\varphi(r) := \lfloor \frac{r}{2} \rfloor$ for $r \in \mathbb{N}$.

Proof. Let us produce a second-order cone representation for $\prod_{k=1}^m x_k^{s_k} x_{m+1}^{2^l - \hat{s}} \geq x_{m+1}^{2^l}$ by using Algorithm 2 that employs the greedy-power-two strategy and Lemma 33. That is, for $p = 0, \dots, l-1$, for each iteration, we select the pairs i, j with $s_i = s_j$ and $\Delta(s_i) = p$, and then select the pairs i, j with $\Delta(s_i) = \Delta(s_j) = p$. When Algorithm 2 comes with $\min\{\Delta(s_i)\}_i = p$, the value of $m+1 - \sum_{\Delta(r_i)<p} \sum_{j=1}^{p-\Delta(r_i)} \varphi^j(k_i) - \sum_{\Delta(r_i)>p} k_i$ denotes the number of s_i with $\Delta(s_i) = p$. We then see that the size of the resulting second-order cone representation is bounded by the right-hand-side of (18), which yields the desired conclusion. \square

5.2 Traversal algorithms

Instead of choosing only the “maximal” pair at each iteration, the traversal algorithms take into account all pairs i, j such that $\Omega(s_i) \cap \Omega(s_j) \neq \emptyset$ (hereafter referred to as the common-one strategy) or $\Delta(s_i) = \Delta(s_j) = \min\{s_k\}_{k=1}^m$ (hereafter referred to as the power-two strategy). This enables us to obtain a possibly smaller second-order cone representation for a weighted geometric mean inequality by spending more time.

Algorithm 3 $\text{Heuristic}(s_1, \dots, s_m; t = m + 1)$

Input: A tuple of nonnegative integers s_1, \dots, s_m and a positive integer t equaling $m + 1$ by default

Output: A configuration $\{(i_k, j_k, t_k)\}_k$ that determines a second-order cone representation for $x_1^{s_1} \dots x_m^{s_m} \geq x_t^{\hat{s}}$

```
1: if  $t = m + 1$  then
2:    $m \leftarrow m + 1, s_m \leftarrow 0$ ;
3: end if
4:  $l \leftarrow \lceil \log_2(\hat{s}) \rceil$ ;
5: if  $\exists i, j \in \{1, \dots, m\}$  such that  $s_i = s_j > 0$  then
6:   if  $s_i \neq 2^{l-1}$  then
7:      $m \leftarrow m + 1, s_m \leftarrow 2s_i, s_i \leftarrow 0, s_j \leftarrow 0$ ;
8:     return  $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(i, j, m)\}$ ;
9:   else
10:    return  $\{(i, j, t)\}$ ;
11:  end if
12: end if
13: Find  $k \in \{1, \dots, m\}$  such that  $s_k = \max\{s_i\}_{i=1}^m$ ;
14: if  $2s_k \geq \hat{s}$  then
15:   if  $2s_k = \hat{s}$  then
16:      $s_k \leftarrow 0$ ;
17:   else if  $s_k \leq 2^{l-1}$  then
18:      $s_t \leftarrow 2s_k - \hat{s}, s_k \leftarrow 0$ ;
19:   else if  $\hat{s} < 2^l$  then
20:      $s_t \leftarrow 2^l - \hat{s}, s_k \leftarrow s_k - 2^{l-1}$ ;
21:   else
22:      $s_k \leftarrow s_k - 2^{l-1}$ ;
23:   end if
24:   return  $\text{Heuristic}(s_1, \dots, s_m; m + 1) \cup \{(k, m + 1, t)\}$ ;
25: end if
26: if  $\arg \min\{\Delta(s_i)\}_{i=1}^m = \{r\}$  and  $s_r \leq 2^l - \hat{s}$  then
27:    $m \leftarrow m + 1, s_m \leftarrow 2s_r, s_r \leftarrow 0$ ;
28:   return  $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(r, t, m)\}$ ;
29: end if
30:  $s_t \leftarrow 2^l - \hat{s}$ ;
31: if  $\exists i \in \{1, \dots, m\}, i \neq t$  such that  $s_t = s_i > 0$  then
32:    $m \leftarrow m + 1, s_m \leftarrow 2s_i, s_i \leftarrow 0, s_t \leftarrow 0$ ;
33:   return  $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(i, t, m)\}$ ;
34: end if
35: Select a pair  $i, j \in \{1, \dots, m\}$  and an integer  $\gamma$  satisfying  $0 < \gamma \leq \min\{s_i, s_j\}$ ;
36:  $m \leftarrow m + 1, s_m \leftarrow 2\gamma, s_i \leftarrow s_i - \gamma, s_j \leftarrow s_j - \gamma$ ;
37: return  $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(i, j, m)\}$ ;
```

5.3 A brute force algorithm

Besides the heuristics, we also propose a brute force algorithm to compute an exact optimal second-order cone representation for a weighted geometric mean inequality.

Suppose that the configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$ determines a second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ (equivalently, an \mathcal{A} -mediated set with $\mathcal{A} = \{\alpha_i\}_{i=1}^m$ being a trellis). As the relation $\sum_{i=1}^m s_i \alpha_i = \hat{s} \alpha_{m+1}$ can be recovered from the system of equations (2), the linear system (19) in variables $\gamma_1, \dots, \gamma_n$ must admit a rational solution; conversely, if the linear system (19) admits a rational solution for some configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$, then this configuration clearly determines an \mathcal{A} -mediated set and hence determines a second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$. Building upon these facts, we give the brute force algorithm for computing optimal second-order cone representations in Algorithm 4.

$$\begin{cases} \sum_{i_t=k} \gamma_t + \sum_{j_t=k} \gamma_t = s_k, & k = 1, \dots, m, \\ \sum_{i_t=k} \gamma_t + \sum_{j_t=k} \gamma_t = 2\gamma_1 - \hat{s}, & k = m+1, \\ \sum_{i_t=k} \gamma_t + \sum_{j_t=k} \gamma_t = 2\gamma_{k-m}, & k = m+2, \dots, m+n, \\ \gamma_t \in \mathbb{Q}, i_t, j_t \in \{1, 2, \dots, m+n\}, & t = 1, \dots, n. \end{cases} \quad (19)$$

Algorithm 4 BruteForce(s_1, \dots, s_m)

Input: A tuple of positive integers s_1, \dots, s_m

Output: A configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$ that determines an optimal second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$

- 1: $n \leftarrow \max\{\lceil \log_2(\hat{s}) \rceil, m-1\}$; # Remark 7
 - 2: Enumerate all legitimate configurations $\{(i_k, j_k, m+k)\}_{k=1}^n$, which are denoted by $\mathcal{T}_{m,n}$;
 - 3: **for** each \mathcal{S} in $\mathcal{T}_{m,n}$ **do**
 - 4: **if** The system (19) is feasible **then**
 - 5: **return** \mathcal{S} ;
 - 6: **end if**
 - 7: **end for**
 - 8: $n \leftarrow n+1$;
 - 9: **goto** Step 2;
-

The most expensive part of Algorithm 4 is Step 2 as for given m, n , the number of legitimate configurations $\{(i_k, j_k, m+k)\}_{k=1}^n$ might be very large. In order to speed up the enumeration, we thereby impose some conditions on legitimate configurations which are stated in the following proposition.

Proposition 36. *There is no loss of generality in assuming that any legitimate configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$ satisfies the following conditions:*

- (1) $\{1, \dots, m\} \cup \{m+2, \dots, m+n\} \subseteq \cup_{k=1}^n \{i_k, j_k\}$;
- (2) $i_k < j_k$;

- (3) $i_k, j_k \neq m + k$;
 - (4) $(i_{k_1}, j_{k_1}) \neq (i_{k_2}, j_{k_2})$ if $k_1 \neq k_2$;
 - (5) $\{i_{k_1}, j_{k_1}, m + k_1\} \neq \{i_{k_2}, j_{k_2}, m + k_2\}$ if $k_1 \neq k_2$;
 - (6) $t_k \geq m + k + 1$ for $k = 1, \dots, n - 1$;
 - (7) $i_1 \leq m, j_1 = m + 2$ or $i_1 = m + 2, j_1 = m + 3$;
 - (8) $i_k \leq t_{k-1}, j_k \leq t_{k-1} + 1$ or $i_k = t_{k-1} + 1, j_k = t_{k-1} + 2$ for $k = 2, \dots, n$,
- where $t_k := \max\{j_1, \dots, j_k\}$ for $k = 1, \dots, n - 1$.

Proof. (1) follows from the proof of Theorem 6. (2)–(5) are immediate from the definition. (6) is due to the fact that we are seeking a configuration of minimum size. (7)–(8) are because we can arbitrarily label the variables x_{m+1}, \dots, x_{m+n} . \square

Table 1 shows the cardinality of $\mathcal{T}_{m,n}$ derived from Proposition 36 for different (m, n) . We can see that $|\mathcal{T}_{m,n}|$ grows rapidly with m, n . However for each (m, n) , the set $\mathcal{T}_{m,n}$ needs to be computed just once and then can be used forever.

Table 1: The number of configurations that satisfy the conditions in Proposition 36.

(m, n)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
$ \mathcal{T}_{m,n} $	3	48	828	17178	419559	18	588	17016	514524

5.4 Numerical experiments

All algorithms discussed above were implemented in the Julia package `MiniSOC`, which is available at <https://github.com/wangjie212/MiniSOC>. The numerical experiments were performed on a desktop computer with Windows 10 system, Intel(R) Core(TM) i9-10900 CPU@2.80GHz and 32G RAM.

5.4.1 Evaluating different heuristic algorithms

To test the performance of different heuristic algorithms, we run them on the partitions s_1, \dots, s_m of the integer $\hat{s} = 83$. The partitions are restricted to be of length $m = 3, 4, 5, 6$ and satisfy $(s_1, \dots, s_m) = 1$. We denote Algorithm 3 implementing the greedy-common-one strategy (resp. the greedy-power-two strategy) by `GreedyCommonone` (resp. `GreedyPowertwo`); we denote the traversal algorithm implementing the common-one strategy (resp. the power-two strategy) by `TraversalCommonone` (resp. `TraversalPowertwo`). The results are reported in Table 2. The data in Table 2 show that: (1) the greedy algorithms are significantly faster than the traversal algorithms; (2) the traversal algorithms may produce second-order cone representations of smaller size than the greedy algorithms; (3) `GreedyPowertwo` not only runs faster than `GreedyCommonone` but also produces second-order cone representations of smaller size.

Table 2: Comparison of heuristic algorithms with $\hat{s} = 83$. For each m , the first column indicates the sum of sizes of second-order cone representations over partitions of length m , and the second column indicates the total running time in seconds. The symbol - means running time > 1 day.

Algorithm	$m = 3$		$m = 4$		$m = 5$		$m = 6$	
GreedyPowertwo	4567	0.001	37996	0.01	196262	0.11	697083	0.40
GreedyCommone	4695	0.025	39625	0.35	204927	2.36	728705	10.3
TraversalPowertwo	4561	0.007	37648	0.25	192654	9.82	681705	90.0
TraversalCommone	4660	0.311	38894	1074	-	-	-	-

We mention that for $m = 3, 4, 5, 6$, the number of different partitions of 83 are 574, 4109, 18487, 58767, respectively, and the average sizes of second-order cone representations produced by the algorithm **GreedyPowertwo** are 8.0, 9.2, 10.6, 11.9, respectively.

To further compare the algorithms **GreedyCommone** with **GreedyPowertwo**, we generate random instances $s_1, \dots, s_m \in (0, 10^t)$ with $m = 10$. The related results are reported in Table 3, which confirms our observation that **GreedyPowertwo** not only runs faster than **GreedyCommone** (by a factor ~ 100) but also produces second-order cone representations of smaller size.

Table 3: Results of **GreedyPowertwo** and **GreedyCommone** on random instances. Each t corresponds to three different random trials. For each algorithm, the first column indicates the size of second-order cone representation, and the second column indicates the running time in seconds.

t	GreedyPowertwo		GreedyCommone	
10	91	0.0002	98	0.02
	90	0.0002	112	0.02
	91	0.0002	114	0.02
12	107	0.0002	135	0.03
	109	0.0003	119	0.03
	108	0.0003	121	0.04
14	133	0.0004	139	0.04
	122	0.0003	155	0.04
	128	0.0003	136	0.05
16	146	0.0004	179	0.07
	140	0.0004	152	0.06
	146	0.0004	195	0.07

5.4.2 Comparison with optimal second-order cone representations

We provide the sizes of optimal second-order cone representations for trivariate weighted geometric mean inequalities $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{\hat{s}}$ with $\hat{s} \leq 15$ in Table 4, which are obtained with the brute force algorithm.

Table 4 shows that the heuristic algorithms and the brute force algorithm produce second-order cone representations of equal size for all instances with

Table 4: Sizes of optimal second-order cone representations for trivariate weighted geometric mean inequalities $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{\hat{s}}$ with $\hat{s} \leq 15$. We exclude the cases $\hat{s} = 4, 8$ in view of Corollary 23.

$\hat{s} = 3$	(1, 1, 1)									
	3									
$\hat{s} = 5$	(2, 2, 1)	(3, 1, 1)								
	4	4								
$\hat{s} = 6$	(3, 2, 1)	(4, 1, 1)								
	3	3								
$\hat{s} = 7$	(3, 2, 2)	(3, 3, 1)	(4, 2, 1)	(5, 1, 1)						
	4	4	3	4						
$\hat{s} = 9$	(4, 3, 2)	(4, 4, 1)	(5, 2, 2)	(5, 3, 1)	(6, 2, 1)	(7, 1, 1)				
	4	5	5	5	4	5				
$\hat{s} = 10$	(4, 3, 3)	(5, 3, 2)	(5, 4, 1)	(6, 3, 1)	(7, 2, 1)	(8, 1, 1)				
	4	4	4	4	4	4				
$\hat{s} = 11$	(4, 4, 3)	(5, 3, 3)	(5, 4, 2)	(5, 5, 1)	(6, 3, 2)	(6, 4, 1)	(7, 2, 2)	(7, 3, 1)	(8, 2, 1)	(9, 1, 1)
	5	5	4	5	4	4	5	5	4	5
$\hat{s} = 12$	(5, 4, 3)	(5, 5, 2)	(6, 5, 1)	(7, 3, 2)	(7, 4, 1)	(8, 3, 1)	(9, 2, 1)	(10, 1, 1)		
	4	4	4	4	4	4	4	4		
$\hat{s} = 13$	(5, 4, 4)	(5, 5, 3)	(6, 4, 3)	(6, 5, 2)	(6, 6, 1)	(7, 3, 3)	(7, 4, 2)	(7, 5, 1)	(8, 3, 2)	(8, 4, 1)
	5	5	4	5	5	5	4	5	4	4
	(9, 2, 2)	(9, 3, 1)	(10, 2, 1)	(11, 1, 1)						
$\hat{s} = 14$	5	5	5	5						
	(5, 5, 4)	(6, 5, 3)	(7, 4, 3)	(7, 5, 2)	(7, 6, 1)	(8, 3, 3)	(8, 5, 1)	(9, 3, 2)	(9, 4, 1)	(10, 3, 1)
	4	4	4	4	4	4	4	5	4	5
$\hat{s} = 15$	(11, 2, 1)	(12, 1, 1)								
	4	4								
	(6, 5, 4)	(7, 4, 4)	(7, 5, 3)	(7, 6, 2)	(7, 7, 1)	(8, 4, 3)	(8, 5, 2)	(8, 6, 1)	(9, 4, 2)	(9, 5, 1)
$\hat{s} = 15$	5	5	6	5	5	4	4	4	4	5
	(10, 3, 2)	(10, 4, 1)	(11, 2, 2)	(11, 3, 1)	(12, 2, 1)	(13, 1, 1)				
	5	4	5	5	4	5				

$m = 3$ and $\hat{s} \leq 15$ but four: (5, 4, 3), (7, 3, 2), (6, 5, 3), (11, 2, 1). For (5, 4, 3), (7, 3, 2) and (11, 2, 1), the heuristic algorithms yield second-order cone representations of size 5, while the brute force algorithm yields second-order cone representations of size 4; for (6, 5, 3), the heuristic algorithms with the (greedy-)common-one strategy yield second-order cone representations of size 6, and the heuristic algorithms with the (greedy-)power-two strategy yield second-order cone representations of size 5, while the brute force algorithm yields second-order cone representations of size 4.

6 Applications

In this section, we give three applications of the proposed algorithm **GreedyPowertwo** in polynomial optimization, matrix optimization, quantum information, respectively. All numerical experiments were performed on a desktop computer with Windows 10 system, Intel(R) Core(TM) i9-10900 CPU@2.80GHz and 32G RAM.

6.1 SONC optimization

Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$ be the ring of real n -variate polynomials. For $\boldsymbol{\alpha} = (\alpha_i)_i \in \mathbb{N}^n$, $\mathbf{x}^\boldsymbol{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Suppose that $\mathcal{A} = \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\} \subseteq (2\mathbb{N})^{m-1}$ is a trellis. A polynomial $f = \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d \mathbf{x}^\boldsymbol{\beta} \in \mathbb{R}[\mathbf{x}]$ is called a *circuit polynomial* if $c_i > 0$ for $i = 1, \dots, m$ and $\boldsymbol{\beta} \in \text{conv}(\mathcal{A})^\circ \cap \mathbb{N}^{m-1}$ [12]. The nonnegativity of a circuit polynomial f on \mathbb{R}^n can be easily verified by

$$f \geq 0 \iff \begin{cases} \prod_{i=1}^m (c_i/\lambda_i)^{\lambda_i} \geq d, & \text{if } \boldsymbol{\beta} \in (2\mathbb{N})^n, \\ \prod_{i=1}^m (c_i/\lambda_i)^{\lambda_i} \geq |d|, & \text{if } \boldsymbol{\beta} \notin (2\mathbb{N})^n, \end{cases} \quad (20)$$

where $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$ is the barycentric coordinate of $\boldsymbol{\beta}$ with respect to \mathcal{A} satisfying $\boldsymbol{\beta} = \sum_{i=1}^m \lambda_i \boldsymbol{\alpha}_i$ and $\sum_{i=1}^m \lambda_i = 1$. Note that (20) admits a second-order cone representation as the inequality $\prod_{i=1}^m (c_i/\lambda_i)^{\lambda_i} \geq d$ (resp. $|d|$) is equivalent to

$$\prod_{i=1}^m c_i^{\lambda_i} \geq y, d \prod_{i=1}^m \lambda_i^{\lambda_i} = y \quad (\text{resp. } |d| \prod_{i=1}^m \lambda_i^{\lambda_i} \leq y). \quad (21)$$

One can certify the nonnegativity of a polynomial f by decomposing it into a *sum of nonnegative circuit polynomials (SONC)*. Furthermore, one can provide a lower bound on the global minimum of a polynomial f by solving the following SONC optimization problem:

$$\begin{cases} \sup & \gamma \\ \text{s.t.} & f - \gamma \text{ is a SONC.} \end{cases} \quad (22)$$

By (21), (22) can be modeled as a second-order cone program.

We take 20 randomly generated polynomials from the database provided by Seidler and de Wolff in [19], and solve the related SONC optimization problem

(22) with the SOCP solver ECOS¹. For each instance, we use two algorithms to generate the required second-order cone representations: **GreedyPowertwo** and the one proposed in the paper [13]. The results are reported in Table 5. It is evident that the approach with **GreedyPowertwo** is more efficient than the one with the algorithm from [13], sometimes by a order of magnitude.

Table 5: Results for the SONC optimization problem (22). n, d, t denote the number of variables, the degree, the number of terms of the polynomial, respectively; the column labelled by “opt” indicates the optimum; the columns labelled by “W” and “K-B-G” indicate the running time in seconds of the approaches with **GreedyPowertwo** and the algorithm from [13], respectively.

(n, d, t)	opt	W	K-B-G
(10, 20, 30)	0.6965	0.04	0.12
(10, 20, 100)	3.3169	0.36	0.80
(10, 20, 300)	31.696	1.78	3.33
(10, 30, 30)	3.3130	0.06	0.24
(10, 30, 100)	15.308	0.32	1.08
(10, 30, 300)	3.3077	2.07	4.17
(10, 40, 30)	0.4694	0.06	0.33
(10, 40, 100)	5.4251	0.51	1.65
(10, 40, 300)	38.662	1.18	5.58
(10, 50, 30)	1.5630	0.06	0.34
(10, 50, 100)	0.1972	0.38	2.31
(10, 50, 300)	7.0048	2.44	7.31
(10, 60, 30)	3.3131	0.08	0.59
(10, 60, 100)	2.5232	0.59	3.04
(10, 60, 300)	23.417	2.13	9.53
(20, 30, 50)	0.6998	0.11	2.29
(20, 30, 100)	4.9266	0.20	4.10
(20, 40, 50)	4.1268	0.07	1.78
(20, 40, 100)	3.6426	0.21	5.45
(20, 40, 200)	13.250	0.75	19.0

¹The computation was performed with the Julia package **SONCSOCP** which is available at <https://github.com/wangjie212/SONCSOCP>.

6.2 Matrix optimization

6.2.1 Semidefinite representation for matrix geometric mean

For positive definite matrices $A, B \in \mathbf{S}_+^n$ and $\lambda \in [0, 1]$, the λ -weighted geometric mean of A and B is defined by

$$G_\lambda(A, B) = A \#_\lambda B := A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\lambda A^{\frac{1}{2}}.$$

It was shown in [8, 18] that the second-order cone representation for a scalar geometric mean can be lifted to a semidefinite representation for the corresponding matrix geometric mean. That is, Algorithm 1 can be readily adapted to produce a semidefinite representation for the matrix geometric mean G_λ when λ is a rational number. Moreover, the size of the semidefinite representation can be deduced from Theorem 20.

Theorem 37. *Let $\lambda = \frac{q}{p}$ with $p, q \in \mathbb{N}^*$, $0 < q < p$ and $(p, q) = 1$. Then G_λ admits a semidefinite representation with $\lceil \log_2(p) \rceil$ linear matrix inequalities of size $2n \times 2n$ and one linear matrix inequality of size $n \times n$.*

Remark 38. *The semidefinite representation for G_λ provided in [8, 18] needs as many as $2 \lceil \log_2(p) \rceil - 1$ linear matrix inequalities of size $2n \times 2n$ and one linear matrix inequality of size $n \times n$.*

6.2.2 Semidefinite representation for the multivariate generalization of Lieb's function

Given $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$ with $\sum_{i=1}^m \lambda_i = 1$, the multivariate generalization of Lieb's function is defined by

$$(A_1, \dots, A_m) \in \mathbf{S}_+^{n_1} \times \dots \times \mathbf{S}_+^{n_m} \mapsto A_1^{\lambda_1} \otimes \dots \otimes A_m^{\lambda_m} \in \mathbf{S}_+^{n_1 \cdots n_m}, \quad (23)$$

where \otimes denotes the Kronecker product of matrices. We are able to give a semidefinite representation for (23) by iteratively using

$$A_1^{\lambda_1} \otimes \dots \otimes A_m^{\lambda_m} \succeq T \iff \exists S \in \mathbf{S}_+^{n_1 n_2} \text{ s.t. } \begin{cases} A_1^{\frac{\lambda_1}{\lambda_1 + \lambda_2}} \otimes A_2^{\frac{\lambda_2}{\lambda_1 + \lambda_2}} \succeq S, \\ S^{\lambda_1 + \lambda_2} \otimes A_3^{\lambda_3} \otimes \dots \otimes A_m^{\lambda_m} \succeq T, \end{cases}$$

and applying Theorem 37 to

$$A^{1-\lambda} \otimes B^\lambda = (A \otimes I) \#_\lambda (I \otimes B), \quad (24)$$

where I denotes the identity matrix of appropriate size.

Let us consider the following trace optimization problem:

$$\begin{cases} \max_{w_1, w_2, w_3} & \text{tr} \left((w_1 A_1 + A_2)^{\lambda_1} \otimes (w_2 A_3 + A_4)^{\lambda_2} \otimes (w_3 A_5 + A_6)^{\lambda_3} \right) \\ \text{s.t.} & w_1 + w_2 + w_3 = 1, \\ & w_1, w_2, w_3 \geq 0, \end{cases} \quad (25)$$

where $(\lambda_i)_{i=1}^3 \in \mathbb{Q}_+^3$ with $\sum_{i=1}^3 \lambda_i = 1$, and $A_1, \dots, A_6 \in \mathbf{S}_+^n$ are randomly generated positive definite matrices. By introducing a matrix variable T , this problem is equivalent to

$$\begin{cases} \max_{w_1, w_2, w_3} & \text{tr}(T) \\ \text{s.t.} & (w_1 A_1 + A_2)^{\lambda_1} \otimes (w_2 A_3 + A_4)^{\lambda_2} \otimes (w_3 A_5 + A_6)^{\lambda_3} \succeq T, \\ & w_1 + w_2 + w_3 = 1, \\ & w_1, w_2, w_3 \geq 0. \end{cases} \quad (26)$$

We can further convert (26) into a semidefinite program (SDP) using the semidefinite representation for (23). In Table 6, we present the results of solving (25) with different $(\lambda_1, \lambda_2, \lambda_3)$ and $n = 2, 3^2$. Here `Mosek` serves as the SDP solver.

Table 6: Results for the trace optimization problem (23). For each n , “opt” denotes the optimum, and “time” denotes the running time in seconds.

$(\lambda_1, \lambda_2, \lambda_3)$	$n = 2$		$n = 3$	
	opt	time	opt	time
$(\frac{4}{9}, \frac{3}{9}, \frac{2}{9})$	4.656	0.03	10.46	2.38
$(\frac{8}{15}, \frac{4}{15}, \frac{3}{15})$	4.621	0.03	8.976	2.07
$(\frac{25}{44}, \frac{10}{44}, \frac{9}{44})$	5.054	0.05	9.474	3.13
$(\frac{37}{83}, \frac{24}{83}, \frac{22}{83})$	5.065	0.06	10.39	4.31
$(\frac{72}{169}, \frac{47}{169}, \frac{40}{169})$	4.579	0.07	10.03	5.57
$(\frac{159}{278}, \frac{64}{278}, \frac{55}{278})$	4.972	0.07	9.801	7.12
$(\frac{212}{453}, \frac{133}{453}, \frac{108}{453})$	4.039	0.07	9.188	5.92
$(\frac{391}{882}, \frac{280}{882}, \frac{211}{882})$	4.755	0.08	9.654	7.08

6.3 Quantum information

Building on second-order cone representations for bivariate weighted geometric means, Fawzi and Saunderson provided semidefinite representations for several matrix functions arising from quantum information [8]. We record their results below.

Suppose that $A \in \mathbf{S}_+^n$, $B \in \mathbf{S}_+^m$ are positive definite matrices and λ is a rational number in $(0, 1)$. The semidefinite representation for Lieb’s function $F_\lambda(A, B) := \text{tr}(K^\top A^{1-\lambda} K B^\lambda)$ ($K \in \mathbb{R}^{n \times m}$ are fixed) can be obtained via

$$\text{tr}(K^\top A^{1-\lambda} K B^\lambda) \geq t \iff \exists T \in \mathbf{S}_+^{nm} \text{ s.t. } \begin{cases} A^{1-\lambda} \otimes B^\lambda \succeq T, \\ \text{vec}(K)^\top T \text{vec}(K) \geq t, \end{cases} \quad (27)$$

where $\text{vec}(K)$ is a column vector of size nm obtained by concatenating the rows of K . The semidefinite representation for the Tsallis entropy $S_\lambda(A) :=$

²The script is available at <https://github.com/wangjie212/MiniSOC>.

$\frac{1}{\lambda}\text{tr}(A^{1-\lambda} - A)$ can be obtained via

$$\frac{1}{\lambda}\text{tr}(A^{1-\lambda} - A) \geq t \iff \exists T \in \mathbf{S}^n \text{ s.t. } \begin{cases} A \#_{\lambda} I \succeq T, \\ \frac{1}{\lambda}\text{tr}(T - A) \geq t, \end{cases} \quad (28)$$

where I is the identity matrix of appropriate size. The semidefinite representation for the Tsallis relative entropy $S_{\lambda}(A\|B) := \frac{1}{\lambda}\text{tr}(A - A^{1-\lambda}B^{\lambda})$ can be obtained via

$$\frac{1}{\lambda}\text{tr}(A - A^{1-\lambda}B^{\lambda}) \leq t \iff \exists s \in \mathbb{R} \text{ s.t. } \begin{cases} \text{tr}(A^{1-\lambda}B^{\lambda}) \geq s, \\ \frac{1}{\lambda}(\text{tr}(A) - s) \leq t. \end{cases} \quad (29)$$

Now let us consider the following maximum entropy optimization problem [8]:

$$\begin{cases} \max_{w_i} & S_{\lambda}(\sum_{i=1}^m w_i A_i) \\ \text{s.t.} & \sum_{i=1}^m w_i = 1, \\ & w_i \geq 0, i = 1, \dots, m, \end{cases} \quad (30)$$

where $A_1, \dots, A_m \in \mathbf{S}_+^n$ are fixed positive semidefinite matrices of trace one. In Table 7 we present numerical results of solving (30) with $m = 10$, $n = 20, 40$ and different λ^3 . The results were obtained with the solver **Mosek**. For each instance, we use two algorithms to generate the required second-order cone representations: **GreedyPowertwo** and the one proposed in the paper [8]. As we can see from the table, the approach with **GreedyPowertwo** is more efficient than the one with the algorithm from [8] by a factor ~ 2 .

Table 7: Results for the maximum entropy optimization problem (30). The column labelled by “opt” indicates the optimum; the columns labelled by “W” and “F-S” indicate the running time in seconds of the approaches with **GreedyPowertwo** and the algorithm from [8], respectively.

λ	$n = 20$			$n = 40$		
	opt	W	F-S	opt	W	F-S
$\frac{5}{27}$	3.9230	0.87	1.84	5.2130	21.5	52.7
$\frac{12}{41}$	4.7037	0.86	1.51	6.5351	20.9	41.6
$\frac{34}{63}$	7.3642	1.01	1.73	11.530	29.3	39.8
$\frac{21}{107}$	4.0125	1.42	2.77	5.3271	32.9	80.4
$\frac{79}{168}$	6.4613	1.21	2.25	9.7798	33.3	60.6
$\frac{43}{213}$	4.0429	1.73	3.12	5.3920	49.3	91.0
$\frac{135}{422}$	4.9303	1.77	3.33	6.9332	49.0	97.1
$\frac{341}{745}$	6.3203	1.65	3.63	9.4727	45.1	98.9

³The script is available at <https://github.com/wangjie212/MiniSOC>.

7 Conclusion and discussion

In this paper, we have studied the optimal size of second-order cone representations for a weighted geometric mean inequality and the minimum cardinality of mediated sets containing a given point. Fast heuristic algorithms have been proposed to compute approximately optimal second-order cone representations for weighted geometric mean inequalities. We conclude the paper by listing some problems for future research:

(1) Is there a polynomial time algorithm for computing an optimal second-order cone representation for a weighted geometric mean inequality? Or is this an NP-hard problem?

(2) We have proved $L(s_1, \dots, s_m) \geq m - 1$ in Theorem 6. On the other hand, we observed that for the tested examples the lower bound $m - 1$ is attained only if \hat{s} is even. Is this always true? Can we prove $L(s_1, \dots, s_m) \geq m$ given that \hat{s} is odd?

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